A bivariate Laguerre expansions approach for joint ruin probabilities in a two-dimensional insurance risk process

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Abstract

In this paper, we consider a two-dimensional insurance risk model where each business line faces not only stand-alone claims but also common shocks that induce dependent losses to both lines simultaneously. The joint ruin probability is analyzed, and it is shown that under some model assumptions it can be expressed in terms of a bivariate Laguerre series with the initial surplus levels of the two business lines as arguments. Our approach is based on utilizing various attractive properties of Laguerre functions to solve a partial-integro differential equation satisfied by the joint ruin probability, so that continuum operations such as convolutions and partial differentiation are translated to lattice operations on the Laguerre coefficients. For computational purposes, the bivariate Laguerre series needs to be truncated, and the corresponding Laguerre coefficients can be obtained through a system of linear equations. The computational procedure is easy to implement, and a numerical example is provided that illustrates its excellent performance. Finally, the results are also applied to address a related capital allocation problem.

Keywords: Bivariate risk process; Common shock; Bivariate Laguerre series; Dependence; Capital allocation.

1 Introduction

In this paper, we study a bivariate insurance risk model with common shocks (see e.g. Chan et al. (2003) and Gong et al. (2012)). Denoting the surplus process of the i-th business line by $\{U_i(t)\}_{t\geq 0}$ (for $i = 1, 2$), the dynamics are described by

$$U_i(t) = u_i + c_i t - \sum_{k=1}^{N_i(t)} Y_{i,k} - \sum_{k=1}^{N_{12}(t)} Z_{i,k}, \qquad t \geq 0,$$

where $u_i = U_i(0) \geq 0$ is the initial surplus level, $c_i > 0$ is the incoming premium rate, $\{N_i(t)\}_{t\geq 0}$ is a Poisson process with rate $\lambda_i > 0$, and $\{Y_{i,k}\}_{k=1}^{\infty}$ is a sequence of independent and identically distributed (i.i.d.) positive random variables with density f_i and survival function $\overline{F}_i$. Moreover, $\{N_{12}(t)\}_{t\geq 0}$ is a Poisson process with rate $\lambda_{12} > 0$, which represents a 'common shock' component that impacts both

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lines of business at the same time, and $\{(Z_{1,k}, Z_{2,k})\}_{k=1}^{\infty}$ is a sequence of i.i.d. positive bivariate random vectors with joint density g_{12} and joint survival function $\overline{G}_{12}$. Also define g_i to be the marginal density of the i.i.d. sequence $\{Z_{i,k}\}_{k=1}^{\infty}$ for $i = 1, 2$. In Badila et al. (2014)'s terminology, the process $\{N_i(t)\}_{t\geq 0}$ counts the number of claims dedicated to line i only while $\{N_{12}(t)\}_{t\geq 0}$ is concerned with simultaneous claim arrivals. We assume that the processes $\{N_1(t)\}_{t\geq 0}$, $\{N_2(t)\}_{t\geq 0}$ and $\{N_{12}(t)\}_{t\geq 0}$ and the sequences $\{Y_{1,k}\}_{k=1}^{\infty}$, $\{Y_{2,k}\}_{k=1}^{\infty}$ and $\{(Z_{1,k}, Z_{2,k})\}_{k=1}^{\infty}$ are all mutually independent. Note that each surplus process follows the classical compound Poisson risk model. In particular, for $i = 1, 2$, the i-th business line has Poisson rate $\lambda_i + \lambda_{12}$ and i.i.d. claim amounts with common density

$$h_i(x) = \frac{\lambda_i}{\lambda_i + \lambda_{12}} f_i(x) + \frac{\lambda_{12}}{\lambda_i + \lambda_{12}} g_i(x), \qquad x \geq 0. \tag{1.1}$$

In addition, the ruin time of the i-th line is $\tau_i = \inf\{t \geq 0 | U_i(t) < 0\}$, and to ensure that each risk process has a positive survival probability we assume the premium income satisfies $c_i = (1 + \theta_i)(\lambda_i E[Y_{i,1}] + \lambda_{12} E[Z_{i,1}])$ for some $\theta_i > 0$ (and this is known as the positive security loading condition).

Unlike univariate risk models, there are possibly different definitions of ruin for the bivariate risk processes $\{(U_1(t), U_2(t))\}_{t\geq 0}$ (e.g. Chan et al. (2003)). We shall consider two of these definitions in this paper, including (i) $\tau_{\text{or}} = \inf\{t \geq 0 | \min\{U_1(t), U_2(t)\} < 0\} = \min(\tau_1, \tau_2)$: the first time when $\{U_1(t)\}_{t\geq 0}$ or $\{U_2(t)\}_{t\geq 0}$ is below zero; and (ii) $\tau_{\text{and}} = \max(\tau_1, \tau_2)$: the first time both $\{U_1(t)\}_{t\geq 0}$ and $\{U_2(t)\}_{t\geq 0}$ have ruined (but not necessarily simultaneously). These respectively resemble the 'joint life' status and the 'last survivor' status in life contingencies. The corresponding ruin probabilities are denoted by $\psi_{\text{or}}(u_1, u_2) = \Pr\{\tau_{\text{or}} < \infty | (U_1(0), U_2(0)) = (u_1, u_2)\}$ and $\psi_{\text{and}}(u_1, u_2) = \Pr\{\tau_{\text{and}} < \infty | (U_1(0), U_2(0)) = (u_1, u_2)\}$, and they are related via

$$\psi_{\text{and}}(u_1, u_2) = \psi_1(u_1) + \psi_2(u_2) - \psi_{\text{or}}(u_1, u_2), \qquad u_1, u_2 \geq 0, \tag{1.2}$$

where $\psi_i(u_i) = \Pr\{\tau_i < \infty | U_i(0) = u_i\}$ is the univariate ruin probability for line i.

The analysis of bivariate or more generally multivariate risk processes poses a great challenge to researchers, and exact solutions to ruin-related quantities such as ruin probabilities are rarely available. Some exceptions are e.g. Avram et al. (2008a,b) and Badescu et al. (2011), where the two lines of business are engaged in a proportional reinsurance contract and therefore the proportionality of the claim amounts across the two lines allows the bivariate problems to be reduced to simpler univariate problems. These were further generalized by Badila et al. (2014, 2015) in a multivariate setting where the claims of different business lines are ordered. In most other works in multivariate risk theory, researchers usually resort to bounds, asymptotic results or numerical approximations for the ruin probabilities. While early simple bounds are available in e.g. Chan et al. (2003), Cai and Li (2005, 2007) and Li et al. (2007), various results can be found e.g. in Collamore (1996, 1998) regarding Cramér type asymptotics and in Li et al. (2007, Section 4), Chen et al. (2011), and Cojocaru (2017) for heavy-tailed asymptotics. Concerning approximations, recursive integral formulas were derived by Dang et al. (2009) which were further generalized and probabilistically interpreted by Gong et al. (2012) who also transformed these to recursive sums that are easier to compute. Moreover, approximations of continuous-time bivariate models via their discrete-time counterparts were developed by Yuen et al. (2006), Castañer et al. (2013), and Liu and Cheung (2015). Multi-dimensional Brownian motion risk models have recently been analyzed by e.g. Dębicki et al. (2020), and while such models are of interest in their own right they can also be used to approximate multivariate Markovian models (see Delsing et al. (2020)). For dividend strategies in two-dimensional risk processes, see e.g. Czarna and Palmowski (2011), Liu and Cheung (2015), Albrecher et al. (2019), Gu et al. (2018), Azcue et al. (2019), and Grandits (2019). In addition, applications of bivariate ruin probabilities in reinsurance were considered by e.g. Kaishev and Dimitrova (2006), Dimitrova and Kaishev (2010), and Castañer et al. (2013). For ruin functions defined via hitting events of a rare or remote set see e.g. Collamore (1996, 1998), Hult et al. (2005), Blanchet and Liu (2014), Liu and Woo (2014) and Pan and Borovkov (2019), and a connection to fluid models was exploited in Rabehasaina (2009). We refer to Asmussen and Albrecher (2010, Chapter XIII.9) for a comprehensive overview of multivariate risk models.

Among the challenges of multivariate ruin problems is that the ruin-related quantities often satisfy a partial integro-differential equation (PIDE) for which exact or explicit solutions are difficult to obtain. In this paper, we shall investigate bivariate Laguerre series as a tool in this direction. As seen in e.g. Keilson and Nunn (1979) and Sumita and Kijima (1985), Laguerre series possess various nice properties concerning differentiation and convolutions. In particular, it turns out that these continuum operations on the Laguerre functions can be mapped to lattice operations on the Laguerre coefficients which are computationally more attractive. While Laguerre expansions have successfully been applied to one-dimensional problems in the context of insurance risk in the past (see e.g. Albrecher et al. (2001), Goffard et al. (2016), Zhang and Su (2018), Asmussen et al. (2019), Avram et al. (2019), Cheung and Zhang (2021)), the present approach seems to be the first of its kind to explore the feasibility of Laguerre expansions in multivariate ruin theory.

We would like to note that with the multivariate duality established in Badila et al. (2014, Section 2), the results developed in this paper are also directly applicable to two parallel M/G/1 queues with simultaneous arrivals. Specifically, in the coupled queueing system, the i-th queue (for $i = 1, 2$) has dedicated arrivals at Poisson rate λ_i where the k-th arrival has service requirement $Y_{i,k}$. Moreover, simultaneous arrivals occur at rate λ_{12} with the k-th arrival bringing in respective service requirements of $Z_{1,k}$ and $Z_{2,k}$ to the two queues. If the i-th server handles workload at a constant rate of c_i (where the loading condition $\theta_i > 0$ in risk theory is equivalent to that the server can handle the traffic), then the survival probability $1 - \psi_{\mathrm{or}}(u_1, u_2) = \Pr\{\tau_1 = \infty, \tau_2 = \infty | (U_1(0), U_2(0)) = (u_1, u_2)\}$ corresponds to the probability that the steady-state workloads in the two queues are no larger than u_1 and u_2 respectively.

The rest of this paper is organized as follows. Section 2 reviews some basic properties of bivariate Laguerre series along with the notion of rapidly decreasing functions and Schwartz functions. The main results are presented in Section 3. First, a PIDE for the ruin probability ψ_{and} is derived. In order to solve it using bivariate Laguerre series, it is proved that ψ_{and} is a Schwartz function under some easily verifiable conditions on the (joint) survival functions $\overline{F}_1$, $\overline{F}_2$ and $\overline{G}_{12}$. Then, the Laguerre coefficients are shown to satisfy a countable system of linear equations which will be truncated for computation. Section 4 provides an example which not only demonstrates the excellent numerical performance of the proposed approach but also illustrates an application in capital allocation. Some technical proofs are deferred to an Appendix.

2 Preliminaries on Laguerre series

We start by reviewing some facts and properties of Laguerre series, and the exposition of this section closely follows Cheung et al. (2021). See e.g. Keilson and Nunn (1979) and Sumita and Kijima (1985) for more details. For each $k \in \mathbb{N}_0$ (where $\mathbb{N}_0$ is the set of non-negative integers), the Laguerre function is defined as

$$\varphi_k(x) = L_k(x) e^{-\frac{x}{2}}, \qquad x \geq 0, \tag{2.1}$$

where

$$L_k(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{x^j}{j!}, \qquad x \geq 0,$$

is the k-th Laguerre polynomial. The Laguerre functions are known to be uniformly bounded such that $\sup_{x\geq 0} |\varphi_k(x)| \leq 1$ for $k \in \mathbb{N}_0$. Let $\mathbb{L}^2(\mathbb{R}_+)$ be the class of square-integrable functions on the positive half-line $\mathbb{R}_+$, and define the scalar product and $\mathbb{L}^2$-norm on $\mathbb{L}^2(\mathbb{R}_+)$ as

$$\langle a_1, a_2 \rangle = \int_0^\infty a_1(x) a_2(x)\, dx \quad \text{and} \quad \|a_1\| = \sqrt{\int_0^\infty [a_1(x)]^2\, dx}, \qquad \forall\ a_1, a_2 \in \mathbb{L}^2(\mathbb{R}_+),$$

respectively. The collection $\{\varphi_k\}_{k=0}^\infty$ is known to form a complete orthonormal basis of $\mathbb{L}^2(\mathbb{R}_+)$ satisfying (i) $\|\varphi_k\| = 1$ for each $k \in \mathbb{N}_0$; and (ii) $\langle \varphi_j, \varphi_k \rangle = 0$ for $k \neq j$. Consequently, every $a \in \mathbb{L}^2(\mathbb{R}_+)$ can be represented as

$$a(x) = \sum_{k=0}^\infty \Theta_{a,k}\, \varphi_k(x), \qquad x \geq 0, \tag{2.2}$$

where

$$\Theta_{a,k} = \langle a, \varphi_k \rangle = \int_0^\infty a(x) \varphi_k(x)\, dx, \qquad k \in \mathbb{N}_0, \tag{2.3}$$

with the convention $\Theta_{a,k} = 0$ for $k < 0$. The constants $\{\Theta_{a,k}\}_{k=0}^\infty$ are called Laguerre dagger coefficients or Laguerre coefficients in short. Then, the Laguerre sharp coefficients are defined as the difference $\Theta_{a,k}^{\#} = \Theta_{a,k} - \Theta_{a,k-1}$ for $k \in \mathbb{N}_0$.

Analogously, let $\mathbb{L}^2(\mathbb{R}_+^2)$ be the class of square-integrable functions on the positive orthant $\mathbb{R}_+^2$ (i.e. $a(\cdot,\cdot)$ such that $\int_0^\infty \int_0^\infty [a(x_1, x_2)]^2\, dx_1\, dx_2 < \infty$). Every function $a \in \mathbb{L}^2(\mathbb{R}_+^2)$ can then be developed on the Laguerre basis. In order to utilize some nice properties of bivariate Laguerre series, the notion of rapidly decreasing functions and Schwartz functions is given below.

Definition 1 ***(Rapidly decreasing function)***

(a) A univariate function $a(\cdot)$ on $\mathbb{R}_+$ is rapidly decreasing if $\sup_{x\in\mathbb{R}_+} |x^k a(x)| < \infty$ for any $k \in \mathbb{N}_0$.

(b) A bivariate function $a(\cdot,\cdot)$ on $\mathbb{R}_+^2$ is rapidly decreasing if $\sup_{(x_1,x_2)\in\mathbb{R}_+^2} |x_1^k x_2^l a(x_1, x_2)| < \infty$ for any $k, l \in \mathbb{N}_0$.

Definition 2 ***(Schwartz space)*** *A Schwartz function is a function whose derivatives are rapidly decreasing. The Schwartz space is the set of Schwartz functions and is defined as follows.*

(a) In the univariate case, the Schwartz space is given by

$$C_\downarrow^\infty(\mathbb{R}_+) = \left\{ a : \mathbb{R}_+ \to \mathbb{R} \text{ such that } \sup_{x\in\mathbb{R}_+} \left| x^k \frac{d^i}{dx^i} a(x) \right| < \infty \text{ for all } i, k \in \mathbb{N}_0 \right\}.$$

(b) In the bivariate case, the Schwartz space is given by

$$C_\downarrow^\infty(\mathbb{R}_+^2) = \left\{ a : \mathbb{R}_+^2 \to \mathbb{R} \text{ such that } \sup_{(x_1,x_2)\in\mathbb{R}_+^2} \left| x_1^k x_2^l \frac{\partial^{i+j}}{\partial x_1^i \partial x_2^j} a(x_1, x_2) \right| < \infty \text{ for all } i, j, k, l \in \mathbb{N}_0 \right\}.$$

It is clear that $C_{\downarrow}^{\infty}(\mathbb{R}_+) \subset \mathbb{L}^2(\mathbb{R}_+)$ and $C_{\downarrow}^{\infty}(\mathbb{R}_+^2) \subset \mathbb{L}^2(\mathbb{R}_+^2)$. From Sumita and Kijima (1985), every $a \in C_{\downarrow}^{\infty}(\mathbb{R}_+^2)$ admits the representation

$$a(x_1,x_2) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Theta_{a,m,n}\,\varphi_m(x_1)\varphi_n(x_2), \qquad x_1,x_2 \geq 0, \tag{2.4}$$

where

$$\Theta_{a,m,n} = \int_0^{\infty}\int_0^{\infty} a(x_1,x_2)\varphi_m(x_1)\varphi_n(x_2)\,dx_1\,dx_2, \qquad m,n \in \mathbb{N}_0, \tag{2.5}$$

with the convention $\Theta_{a,m,n} = 0$ if $m < 0$ or $n < 0$. For later use, the Laguerre sharp coefficients are defined as the difference $\Theta^{\#}_{a,m,n} = \Theta_{a,m,n} - \Theta_{a,m-1,n} - \Theta_{a,m,n-1} + \Theta_{a,m-1,n-1}$ for $m,n \in \mathbb{N}_0$. Like the univariate counterpart, bivariate Laguerre series possess various nice operational properties, many of which are summarized in Sumita and Kijima (1985, Section 4). Note that Property 1 below concerns functions defined on $\mathbb{R}_+^2$, whereas in the former reference the case $\mathbb{R}^2$ is treated (for a proof of this adaptation see Cheung et al. (2021)).

Property 1: Partial differentiation. Suppose that $a \in C_{\downarrow}^{\infty}(\mathbb{R}_+^2)$. Then we have

$$\frac{\partial}{\partial x_1}a(x_1,x_2) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Theta_{a,1',m,n}\,\varphi_m(x_1)\varphi_n(x_2), \qquad x_1,x_2 \geq 0,$$

where the Laguerre sharp coefficients $\Theta^{\#}_{a,1',m,n} = \Theta_{a,1',m,n} - \Theta_{a,1',m-1,n} - \Theta_{a,1',m,n-1} + \Theta_{a,1',m-1,n-1}$ are given by

$$\Theta^{\#}_{a,1',m,n} = \begin{cases} -\frac{1}{2}\Theta_{a,0,0} - \sum_{i=1}^{\infty}\Theta_{a,i,0}, & m=0; n=0,\\ -\frac{1}{2}\Theta_{a,0,n} - \sum_{i=1}^{\infty}\Theta_{a,i,n} + \frac{1}{2}\Theta_{a,0,n-1} + \sum_{i=1}^{\infty}\Theta_{a,i,n-1}, & m=0; n\geq 1,\\ \frac{1}{2}(\Theta_{a,m,0} + \Theta_{a,m-1,0}), & m\geq 1; n=0,\\ \frac{1}{2}(\Theta_{a,m,n} + \Theta_{a,m-1,n} - \Theta_{a,m,n-1} - \Theta_{a,m-1,n-1}), & m\geq 1; n\geq 1. \end{cases}$$

Similarly, we have

$$\frac{\partial}{\partial x_2}a(x_1,x_2) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Theta_{a,2',m,n}\,\varphi_m(x_1)\varphi_n(x_2), \qquad x_1,x_2 \geq 0,$$

where the Laguerre sharp coefficients $\Theta^{\#}_{a,2',m,n} = \Theta_{a,2',m,n} - \Theta_{a,2',m-1,n} - \Theta_{a,2',m,n-1} + \Theta_{a,2',m-1,n-1}$ are given by

$$\Theta^{\#}_{a,2',m,n} = \begin{cases} -\frac{1}{2}\Theta_{a,0,0} - \sum_{j=1}^{\infty}\Theta_{a,0,j}, & m=0; n=0,\\ \frac{1}{2}(\Theta_{a,0,n} + \Theta_{a,0,n-1}), & m=0; n\geq 1,\\ -\frac{1}{2}\Theta_{a,m,0} - \sum_{j=1}^{\infty}\Theta_{a,m,j} + \frac{1}{2}\Theta_{a,m-1,0} + \sum_{j=1}^{\infty}\Theta_{a,m-1,j}, & m\geq 1; n=0,\\ \frac{1}{2}(\Theta_{a,m,n} + \Theta_{a,m,n-1} - \Theta_{a,m-1,n} - \Theta_{a,m-1,n-1}), & m\geq 1; n\geq 1. \end{cases}$$

Property 2: Convolution in one argument. If $a \in C_{\downarrow}^{\infty}(\mathbb{R}_+^2)$ and $b \in C_{\downarrow}^{\infty}(\mathbb{R}_+)$, then

$$\int_0^{x_1} a(x_1-y_1,x_2)b(y_1)\,dy_1 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Theta_{a_1*b,m,n}\,\varphi_m(x_1)\varphi_n(x_2), \qquad x_1,x_2 \geq 0,$$

where the Laguerre sharp coefficients $\Theta^{\#}_{a_1*b,m,n} = \Theta_{a_1*b,m,n} - \Theta_{a_1*b,m-1,n} - \Theta_{a_1*b,m,n-1} + \Theta_{a_1*b,m-1,n-1}$ are given by

$$\Theta^{\#}_{a_1*b,m,n} = \sum_{k=0}^{m} \Theta^{\#}_{b,m-k}\, \Theta^{\#}_{a,k,n},$$

and in addition one has

$$\int_0^{x_2} a(x_1, x_2 - y_2) b(y_2)\, dy_2 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Theta_{a_2*b,m,n}\, \varphi_m(x_1)\varphi_n(x_2), \qquad x_1, x_2 \geq 0,$$

where the Laguerre sharp coefficients $\Theta^{\#}_{a_2*b,m,n} = \Theta_{a_2*b,m,n} - \Theta_{a_2*b,m-1,n} - \Theta_{a_2*b,m,n-1} + \Theta_{a_2*b,m-1,n-1}$ are given by

$$\Theta^{\#}_{a_2*b,m,n} = \sum_{k=0}^{n} \Theta^{\#}_{b,n-k}\, \Theta^{\#}_{a,m,k}.$$

Here it is understood that $\{\Theta^{\#}_{b,k} = \Theta_{b,k} - \Theta_{b,k-1}\}_{k=0}^{\infty}$ are the Laguerre sharp coefficients of b.

Property 3: Convolution in both arguments. If $a, b \in C^{\infty}_{\downarrow}(\mathbb{R}^2_+)$, then

$$\int_0^{x_1}\int_0^{x_2} a(x_1 - y_1, x_2 - y_2) b(y_1, y_2)\, dy_2\, dy_1 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Theta_{a*b,m,n}\, \varphi_m(x_1)\varphi_n(x_2), \qquad x_1, x_2 \geq 0,$$

where the Laguerre sharp coefficients $\Theta^{\#}_{a*b,m,n} = \Theta_{a*b,m,n} - \Theta_{a*b,m-1,n} - \Theta_{a*b,m,n-1} + \Theta_{a*b,m-1,n-1}$ are given by

$$\Theta^{\#}_{a*b,m,n} = \sum_{i=0}^{m}\sum_{j=0}^{n} \Theta^{\#}_{b,m-i,n-j}\Theta^{\#}_{a,i,j}.$$

Here it is understood that $\{\Theta^{\#}_{b,i,j} = \Theta_{b,i,j} - \Theta_{b,i-1,j} - \Theta_{b,i,j-1} + \Theta_{b,i-1,j-1}\}_{i,j=0}^{\infty}$ are the Laguerre sharp coefficients of b.

3 Main results on the ruin probability

3.1 Partial integro-differential equation (PIDE)

In order to express a function as a bivariate Laguerre series, it has to be square-integrable. However, because $\psi_{\text{or}}(u_1, u_2) \geq \psi_{\text{or}}(\infty, u_2) = \psi_2(u_2)$, one has that

$$\int_0^{\infty}\int_0^{\infty} [\psi_{\text{or}}(u_1, u_2)]^2\, du_1\, du_2 \geq \int_0^{\infty}\int_0^{\infty} [\psi_2(u_2)]^2\, du_1\, du_2 = +\infty,$$

i.e. ψ_{or} is not square-integrable. Consequently, we shall work with ψ_{and} whose square-integrability can be proved as follows under mild conditions. For line i ($i = 1, 2$), denote M_i as the maximum aggregate loss, which is well known to follow a compound geometric distribution (where the primary geometric distribution has 'failure' probability $1/(1+\theta_i)$ and the secondary distribution is the equilibrium distribution of (1.1)). Because $\{\tau_{\text{and}} < \infty\} = \{\tau_1 < \infty, \tau_2 < \infty\} = \{M_1 > u_1, M_2 > u_2\}$, one easily checks that

$$\int_0^{\infty}\int_0^{\infty} [\psi_{\text{and}}(u_1, u_2)]^2\, du_1\, du_2 \leq \int_0^{\infty}\int_0^{\infty} \psi_{\text{and}}(u_1, u_2)\, du_1\, du_2$$

$$
\begin{aligned}
&= \int_0^\infty \int_0^\infty \Pr\{M_1 > u_1, M_2 > u_2\}\, du_1\, du_2 \\
&= E[M_1 M_2] \\
&\le \{E[M_1^2]\}^{\frac{1}{2}} \{E[M_2^2]\}^{\frac{1}{2}},
\end{aligned}
$$

where the last two lines follow from the joint moment formula in e.g. Lo (2019, Proposition 3.1) and the Cauchy-Schwarz inequality respectively. For $E[M_1^2]$ and $E[M_2^2]$ to be finite, we shall require the second moment of the equilibrium distribution of (1.1) to be finite for $i = 1, 2$, which in turn requires the third moments of $Y_{1,1}$, $Y_{2,1}$, $Z_{1,1}$ and $Z_{2,1}$ to be finite. In fact, throughout the paper we shall make the following stronger assumption that will be useful in Section 3.2 for another purpose and in particular entails that all moments of $Y_{1,1}$, $Y_{2,1}$, $Z_{1,1}$ and $Z_{2,1}$ are finite (see e.g. Klugman et al. (2013, Theorem 11.3)).

Assumption 1. *For each $i = 1, 2$, the adjustment coefficient exists for the univariate surplus process $\{U_i(t)\}_{t\ge 0}$. That is, there exists a positive value R_i inside the radius of convergence of the moment generating function corresponding to the density h_i defined in* (1.1) *such that*

$$
c_i R_i = (\lambda_i + \lambda_{12}) \left(\int_0^\infty e^{R_i x} h_i(x)\, dx - 1 \right).
$$

As a result, Lundberg's upper bound

$$
\psi_i(u) \le e^{-R_i u}, \qquad u \ge 0, \tag{3.1}
$$

holds true.

Next, we present the following lemma with regard to a PIDE satisfied by the ruin probability ψ_{and}.

Lemma 1 ***(PIDE satisfied by ψ_{and})*** *The bivariate ruin probability ψ_{and} satisfies the PIDE*

$$
\begin{aligned}
&c_1 \frac{\partial}{\partial u_1} \psi_{\text{and}}(u_1, u_2) + c_2 \frac{\partial}{\partial u_2} \psi_{\text{and}}(u_1, u_2) - (\lambda_1 + \lambda_2 + \lambda_{12}) \psi_{\text{and}}(u_1, u_2) \\
&+ \lambda_1 \int_0^{u_1} \psi_{\text{and}}(u_1 - y_1, u_2) f_1(y_1)\, dy_1 + \lambda_2 \int_0^{u_2} \psi_{\text{and}}(u_1, u_2 - y_2) f_2(y_2)\, dy_2 \\
&+ \lambda_{12} \int_0^{u_1} \int_0^{u_2} \psi_{\text{and}}(u_1 - y_1, u_2 - y_2) g_{12}(y_1, y_2)\, dy_2\, dy_1 = \gamma(u_1, u_2), \qquad u_1, u_2 \ge 0,
\end{aligned} \tag{3.2}
$$

where

$$
\begin{aligned}
\gamma(u_1, u_2) = &- \lambda_1 \psi_2(u_2) \overline{F}_1(u_1) - \lambda_2 \psi_1(u_1) \overline{F}_2(u_2) - \lambda_{12} \int_0^{u_1} \int_{u_2}^\infty \psi_1(u_1 - y_1) g_{12}(y_1, y_2)\, dy_2\, dy_1 \\
&- \lambda_{12} \int_{u_1}^\infty \int_0^{u_2} \psi_2(u_2 - y_2) g_{12}(y_1, y_2)\, dy_2\, dy_1 - \lambda_{12} \overline{G}_{12}(u_1, u_2)
\end{aligned} \tag{3.3}
$$

only consists of the univariate ruin probabilities and known functions corresponding to the claim distributions.

Proof. We proceed by considering a time interval $(0, \epsilon]$ for some small $\epsilon > 0$ and analyzing all possible events in relation to the ruin definition $\tau_{\text{and}} = \max(\tau_1, \tau_2)$ as follows.

1. With probability $1 - (\lambda_1 + \lambda_2 + \lambda_{12})\epsilon + o(\epsilon)$, there is no claim event at all. At time ϵ, the surplus levels of lines 1 and 2 are $u_1 + c_1\epsilon$ and $u_2 + c_2\epsilon$ respectively.

2. An event from the process $\{N_1(t)\}_{t\geq 0}$ occurs with probability $\lambda_1\epsilon+o(\epsilon)$, causing a claim in line 1 but not in line 2. Depending on the resulting claim amount y_1 to line 1, this is further separated into two cases.
   a. If $y_1 \leq u_1 + c_1\epsilon$, then line 1 survives the claim at time ϵ with new surplus level $u_1 + c_1\epsilon - y_1$. The bivariate process $\{(U_1(t), U_2(t))\}_{t\geq 0}$ has ruin probability $\psi_{\text{and}}(u_1 + c_1\epsilon - y_1, u_2 + c_2\epsilon)$ at time ϵ.
   b. If $y_1 > u_1 + c_1\epsilon$, then line 1 ruins at time ϵ. Whether ruin will occur in the bivariate risk process will depend on whether line 2, possessing $u_2 + c_2\epsilon$ at time ϵ, will ruin in the future, and this will happen with probability $\psi_2(u_2 + c_2\epsilon)$.
3. An event from the process $\{N_2(t)\}_{t\geq 0}$ occurs with probability $\lambda_2\epsilon+o(\epsilon)$. The possibilities are similar to point 2 above.
4. A common shock event occurs from the process $\{N_{12}(t)\}_{t\geq 0}$ with probability $\lambda_{12}\epsilon+o(\epsilon)$, causing claims of amounts y_1 and y_2 in lines 1 and 2 respectively. This further consists of four cases depending on the values of (y_1, y_2) which follow the joint density g_{12}.
   a. If $y_1 \leq u_1 + c_1\epsilon$ and $y_2 \leq u_2 + c_2\epsilon$, then both lines survive time ϵ with new surplus levels $u_1 + c_1\epsilon - y_1$ and $u_2 + c_2\epsilon - y_2$ respectively.
   b. If $y_1 \leq u_1 + c_1\epsilon$ and $y_2 > u_2 + c_2\epsilon$, then line 1 survives but line 2 ruins at time ϵ, and the ruin probability of the bivariate process is equivalent to $\psi_1(u_1 + c_1\epsilon - y_1)$.
   c. If $y_1 > u_1 + c_1\epsilon$ and $y_2 \leq u_2 + c_2\epsilon$, then this is just an opposite situation compared to point 4b above.
   d. If $y_1 > u_1 + c_1\epsilon$ and $y_2 > u_2 + c_2\epsilon$, then ruin occurs at time ϵ in the bivariate process.
5. Two or more claim events occur. This happens with negligible probability $o(\epsilon)$.

Consolidating all the above, we arrive at

$$\begin{aligned}
\psi_{\text{and}}(u_1, u_2) = {} & [1 - (\lambda_1 + \lambda_2 + \lambda_{12})\epsilon]\psi_{\text{and}}(u_1 + c_1\epsilon, u_2 + c_2\epsilon) \\
& + \lambda_1\epsilon\left(\int_0^{u_1+c_1\epsilon} \psi_{\text{and}}(u_1 + c_1\epsilon - y_1, u_2 + c_2\epsilon) f_1(y_1)\, dy_1 + \psi_2(u_2 + c_2\epsilon)\overline{F}_1(u_1 + c_1\epsilon)\right) \\
& + \lambda_2\epsilon\left(\int_0^{u_2+c_2\epsilon} \psi_{\text{and}}(u_1 + c_1\epsilon, u_2 + c_2\epsilon - y_2) f_2(y_2)\, dy_2 + \psi_1(u_1 + c_1\epsilon)\overline{F}_2(u_2 + c_2\epsilon)\right) \\
& + \lambda_{12}\epsilon\Bigg(\int_0^{u_1+c_1\epsilon}\int_0^{u_2+c_2\epsilon} \psi_{\text{and}}(u_1 + c_1\epsilon - y_1, u_2 + c_2\epsilon - y_2) g_{12}(y_1, y_2)\, dy_2\, dy_1 \\
& \qquad + \int_0^{u_1+c_1\epsilon}\int_{u_2+c_2\epsilon}^{\infty} \psi_1(u_1 + c_1\epsilon - y_1) g_{12}(y_1, y_2)\, dy_2\, dy_1 \\
& \qquad + \int_{u_1+c_1\epsilon}^{\infty}\int_0^{u_2+c_2\epsilon} \psi_2(u_2 + c_2\epsilon - y_2) g_{12}(y_1, y_2)\, dy_2\, dy_1 + \overline{G}_{12}(u_1 + c_1\epsilon, u_2 + c_2\epsilon)\Bigg) \\
& + o(\epsilon).
\end{aligned}$$

Letting $\epsilon \to 0$ followed by rearrangements gives rise to the PIDE (3.2) with $\gamma(u_1, u_2)$ given by (3.3). Finally, note that the differentiability of $\psi_{\text{and}}(u_1, u_2)$ is guaranteed by the same argument as in Asmussen and Albrecher (2010, Remark VIII.1.11). Another way to derive this PIDE is the (essentially equivalent) generator approach (cf. Asmussen and Albrecher (2010, Ch.II)). ■

3.2 ψ_{and} as a Schwartz function

To solve the PIDE (3.2), we would like to ensure that the ruin probability ψ_{and} is a Schwartz function so that the nice properties of bivariate Laguerre series discussed in Section 2 can be applied. We start with the following simple lemma which shows that ψ_{and} is rapidly decreasing.

Lemma 2 *Under Assumption 1, there exist constants $r_1, r_2 > 0$ such that*

$$\psi_{\text{and}}(u_1, u_2) \le e^{-r_1 u_1 - r_2 u_2}, \qquad u_1, u_2 \ge 0. \tag{3.4}$$

Specifically, one can choose $r_1 = r_2 = \min(R_1, R_2)/2$. Consequently, ψ_{and} is rapidly decreasing.

Proof. First, by the definition $\psi_{\text{and}}(u_1, u_2) = \Pr\{\tau_1 < \infty, \tau_2 < \infty | (U_1(0), U_2(0)) = (u_1, u_2)\}$ of the bivariate ruin probability, it is clear that $\psi_{\text{and}}(u_1, u_2)$ cannot be larger than the univariate ruin probabilities $\psi_1(u_1) = \Pr\{\tau_1 < \infty | U_1(0) = u_1\}$ and $\psi_2(u_2) = \Pr\{\tau_2 < \infty | U_2(0) = u_2\}$. We divide $\mathbb{R}_+^2$ into two regions, namely $0 \le u_2 \le u_1$ and $0 \le u_1 < u_2$. Using the Lundberg bound (3.1) with $i = 1$ in the first region gives

$$\psi_{\text{and}}(u_1, u_2) \le \psi_1(u_1) \le e^{-R_1 u_1} \le e^{-R_1\left(\frac{u_1+u_2}{2}\right)}, \qquad 0 \le u_2 \le u_1,$$

and similarly for the second region we have

$$\psi_{\text{and}}(u_1, u_2) \le \psi_2(u_2) \le e^{-R_2 u_2} \le e^{-R_2\left(\frac{u_1+u_2}{2}\right)}, \qquad 0 \le u_1 < u_2,$$

from which the result follows. Finally, it is clear that (3.4) implies that ψ_{and} is rapidly decreasing according to Definition 1(b). ■

In order to prove that the derivatives of ψ_{and} are also rapidly decreasing, we will need two more assumptions on the claim distributions.

Assumption 2. *For $i = 1, 2$, the survival function $\overline{F}_i$ of the claim amounts specific to line i is infinitely differentiable, and for any $k \in \mathbb{N}_0$ it satisfies*

$$|\overline{F}_i^{(k)}(x)| \le A_{ik} e^{-\alpha_i x}, \qquad x \ge 0, \tag{3.5}$$

where A_{ik}'s and α_i's are positive constants. This implies $\overline{F}_i \in C_\downarrow^\infty(\mathbb{R}_+)$ for $i = 1, 2$.

Assumption 3. *The joint survival function $\overline{G}_{12}$ of the claim amounts resulting from common shocks is infinitely differentiable, and for any $i, j \in \mathbb{N}_0$ it satisfies*

$$\left| \frac{\partial^{i+j}}{\partial x_1^i \partial x_2^j} \overline{G}_{12}(x_1, x_2) \right| \le B_{ij} e^{-\beta_1 x_1 - \beta_2 x_2}, \qquad x_1, x_2 \ge 0, \tag{3.6}$$

where B_{ij}'s, β_1 and β_2 are positive constants. This implies $\overline{G}_{12} \in C_\downarrow^\infty(\mathbb{R}_+^2)$.

Moreover, bounds on the derivatives of γ defined by (3.3) are needed, and these are presented in the following lemma, with the proof provided in Appendix A.1.

Lemma 3 ***(Bound showing γ is a Schwartz function)*** *Under Assumptions 1-3, for any given $i, j \in \mathbb{N}_0$ one has*

$$\left| \frac{\partial^{i+j}}{\partial u_1^i \partial u_2^j} \gamma(u_1, u_2) \right| \le K_{ij}(u_1 + u_2 + 1) e^{-R_1^* u_1 - R_2^* u_2}, \qquad u_1, u_2 \ge 0, \tag{3.7}$$

where K_{ij}'s, R_1^* and R_2^* are positive constants. Specifically, one can take $R_k^* = \min(R_k, \alpha_k, \beta_k)$ for $k = 1, 2$. Consequently, γ is a Schwartz function.

Now we have the necessary ingredients to state the following lemma. Its proof is quite technical and delegated to Appendix A.2.

Lemma 4 ***(Bound showing ψ_{and} is a Schwartz function)*** *Under Assumptions 1-3, for any given $i, j \in \mathbb{N}_0$ one has*

$$\left| \frac{\partial^{i+j}}{\partial u_1^i \partial u_2^j} \psi_{\text{and}}(u_1, u_2) \right| \leq H_{ij}(u_1+1)^{i+j}(u_2+1)^{i+j} e^{-R_1^{**}u_1 - R_2^{**}u_2}, \qquad u_1, u_2 \geq 0, \tag{3.8}$$

where H_{ij}'s, R_1^{**} and R_2^{**} are positive constants. Specifically, one can take $R_k^{**} = \min(r_k, \alpha_k, \beta_k)$ for $k = 1, 2$. Consequently, ψ_{and} is a Schwartz function.

Remark 1 Note that both the classes of combinations of exponentials and Erlang mixtures satisfy Assumption 2, and can be used to model the dedicated claim amounts $\{Y_{i,k}\}_{k=1}^\infty$ for business line i $(i = 1, 2)$. For a combination of exponentials, the density is

$$f_i(x) = \sum_{j=1}^{n_i} q_{ij} \nu_{ij} e^{-\nu_{ij} x}, \qquad x \geq 0, \tag{3.9}$$

where ν_{ij}'s are positive (and distinct, without loss of generality) and q_{ij}'s are non-zero such that $\sum_{j=1}^{n_i} q_{ij} = 1$. On the other hand, a mixed Erlang distribution has density

$$f_i(x) = \sum_{j=1}^{n_i} q_j \frac{\nu^j x^{j-1} e^{-\nu x}}{(j-1)!}, \qquad x \geq 0,$$

where $\nu > 0$, and $\{q_j\}_{j=1}^{n_i}$ constitute a probability mass function. These two classes are known to be dense in the set of positive continuous distributions. Regarding the simultaneous claims of the two business lines, Assumption 3 is satisfied by the class of bivariate mixed Erlang distributions. This class is dense in the set of positive continuous bivariate distributions and has joint density

$$g_{12}(x_1, x_2) = \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \omega_{ij} \frac{\nu_1^i x_1^{i-1} e^{-\nu_1 x_1}}{(i-1)!} \frac{\nu_2^j x_2^{j-1} e^{-\nu_2 x_2}}{(j-1)!}, \qquad x_1, x_2 \geq 0,$$

where $\nu_1, \nu_2 > 0$ and ω_{ij}'s form a bivariate probability mass function. Interested readers are referred to e.g. Dufresne (2007), Willmot and Woo (2007, 2015), Lee and Lin (2010, 2012) for the nice analytic properties and fitting of the afore-mentioned classes of distributions. In Section 4 we shall provide another class of joint distributions in terms of a copula that satisfies Assumption 3. □

3.3 An exact formula for ψ_{and}

Since the claim densities f_1 and f_2 are square-integrable (as implied by (3.5) in Assumption 2), they admit the Laguerre series representation (2.2) with Laguerre coefficients $\{\Theta_{f_1,k}\}_{k=0}^\infty$ and $\{\Theta_{f_2,k}\}_{k=0}^\infty$ computable via (2.3). Moreover, with $g_{12}, \gamma, \psi_{\text{and}} \in C_\downarrow^\infty(\mathbb{R}_+^2)$ under Assumptions 1-3, these functions can be represented in the form of (2.4). Because the claim density g_{12} arising from common shocks is known and γ defined in (3.3) is also expressed in terms of known functions (including the univariate ruin probabilities), their Laguerre coefficients $\{\Theta_{g_{12},m,n}\}_{m,n=0}^{\infty}$ and $\{\Theta_{\gamma,m,n}\}_{m,n=0}^{\infty}$ can be evaluated via (2.5). On the other hand, for the bivariate Laguerre series of the joint ruin probability

$$\psi_{\text{and}}(u_1,u_2)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Theta_{\psi_{\text{and}},m,n}\,\varphi_m(u_1)\varphi_n(u_2),\qquad u_1,u_2\geq 0, \tag{3.10}$$

the Laguerre coefficients $\{\Theta_{\psi_{\text{and}},m,n}\}_{m,n=0}^{\infty}$ are unknown and need to be determined. As we have shown that all the functions involved in the PIDE are Schwartz functions under Assumptions 1-3, this can be done by applying the three properties in Section 2 to the PIDE (3.2) followed by matching the coefficients of $\varphi_m(u_1)\varphi_n(u_2)$. With the obvious notation one has that

$$c_1\Theta_{\psi_{\text{and}},1',m,n}+c_2\Theta_{\psi_{\text{and}},2',m,n}-(\lambda_1+\lambda_2+\lambda_{12})\Theta_{\psi_{\text{and}},m,n}+\lambda_1\Theta_{\psi_{\text{and}},1*f_1,m,n}+\lambda_2\Theta_{\psi_{\text{and}},2*f_2,m,n}+\lambda_{12}\Theta_{\psi_{\text{and}}*g_{12},m,n}$$
$$=\Theta_{\gamma,m,n},\qquad m,n\in\mathbb{N}_0.$$

The above relation is also true for the Laguerre sharp coefficients, i.e.

$$c_1\Theta^{\#}_{\psi_{\text{and}},1',m,n}+c_2\Theta^{\#}_{\psi_{\text{and}},2',m,n}-(\lambda_1+\lambda_2+\lambda_{12})\Theta^{\#}_{\psi_{\text{and}},m,n}+\lambda_1\Theta^{\#}_{\psi_{\text{and}},1*f_1,m,n}+\lambda_2\Theta^{\#}_{\psi_{\text{and}},2*f_2,m,n}+\lambda_{12}\Theta^{\#}_{\psi_{\text{and}}*g_{12},m,n}$$
$$=\Theta^{\#}_{\gamma,m,n},\qquad m,n\in\mathbb{N}_0.$$

The equation is written down explicitly according to four cases below.

1. $m=0$ and $n=0$:

$$\left(-\frac{c_1+c_2}{2}-(\lambda_1+\lambda_2+\lambda_{12})+\lambda_1\Theta_{f_1,0}+\lambda_2\Theta_{f_2,0}+\lambda_{12}\Theta_{g_{12},0,0}\right)\Theta_{\psi_{\text{and}},0,0}-c_1\sum_{i=1}^{\infty}\Theta_{\psi_{\text{and}},i,0}-c_2\sum_{j=1}^{\infty}\Theta_{\psi_{\text{and}},0,j}$$
$$=\Theta_{\gamma,0,0}. \tag{3.11}$$

2. $m=0$ and $n\geq 1$:

$$\left(\frac{-c_1+c_2}{2}-(\lambda_1+\lambda_2+\lambda_{12})+\lambda_1\Theta_{f_1,0}\right)\Theta_{\psi_{\text{and}},0,n}+\left(\frac{c_1+c_2}{2}+(\lambda_1+\lambda_2+\lambda_{12})-\lambda_1\Theta_{f_1,0}\right)\Theta_{\psi_{\text{and}},0,n-1}$$
$$+\sum_{j=0}^{n}(\lambda_2\Theta^{\#}_{f_2,n-j}+\lambda_{12}\Theta^{\#}_{g_{12},0,n-j})(\Theta_{\psi_{\text{and}},0,j}-\Theta_{\psi_{\text{and}},0,j-1})-c_1\sum_{i=1}^{\infty}\Theta_{\psi_{\text{and}},i,n}+c_1\sum_{i=1}^{\infty}\Theta_{\psi_{\text{and}},i,n-1}$$
$$=\Theta_{\gamma,0,n}-\Theta_{\gamma,0,n-1}. \tag{3.12}$$

3. $m\geq 1$ and $n=0$:

$$\left(\frac{c_1-c_2}{2}-(\lambda_1+\lambda_2+\lambda_{12})+\lambda_2\Theta_{f_2,0}\right)\Theta_{\psi_{\text{and}},m,0}+\left(\frac{c_1+c_2}{2}+(\lambda_1+\lambda_2+\lambda_{12})-\lambda_2\Theta_{f_2,0}\right)\Theta_{\psi_{\text{and}},m-1,0}$$
$$+\sum_{i=0}^{m}(\lambda_1\Theta^{\#}_{f_1,m-i}+\lambda_{12}\Theta^{\#}_{g_{12},m-i,0})(\Theta_{\psi_{\text{and}},i,0}-\Theta_{\psi_{\text{and}},i-1,0})-c_2\sum_{j=1}^{\infty}\Theta_{\psi_{\text{and}},m,j}+c_2\sum_{j=1}^{\infty}\Theta_{\psi_{\text{and}},m-1,j}$$
$$=\Theta_{\gamma,m,0}-\Theta_{\gamma,m-1,0}. \tag{3.13}$$

4. $m\geq 1$ and $n\geq 1$:

$$\frac{c_1}{2}(\Theta_{\psi_{\text{and}},m,n}+\Theta_{\psi_{\text{and}},m-1,n}-\Theta_{\psi_{\text{and}},m,n-1}-\Theta_{\psi_{\text{and}},m-1,n-1})$$

$$
\begin{aligned}
&+ \frac{c_2}{2}(\Theta_{\psi_{\text{and}},m,n} + \Theta_{\psi_{\text{and}},m,n-1} - \Theta_{\psi_{\text{and}},m-1,n} - \Theta_{\psi_{\text{and}},m-1,n-1}) \\
&- (\lambda_1 + \lambda_2 + \lambda_{12})(\Theta_{\psi_{\text{and}},m,n} - \Theta_{\psi_{\text{and}},m-1,n} - \Theta_{\psi_{\text{and}},m,n-1} + \Theta_{\psi_{\text{and}},m-1,n-1}) \\
&+ \lambda_1 \sum_{k=0}^{m} \Theta^{\#}_{f_1,m-k}(\Theta_{\psi_{\text{and}},k,n} - \Theta_{\psi_{\text{and}},k-1,n} - \Theta_{\psi_{\text{and}},k,n-1} + \Theta_{\psi_{\text{and}},k-1,n-1}) \\
&+ \lambda_2 \sum_{k=0}^{n} \Theta^{\#}_{f_2,n-k}(\Theta_{\psi_{\text{and}},m,k} - \Theta_{\psi_{\text{and}},m-1,k} - \Theta_{\psi_{\text{and}},m,k-1} + \Theta_{\psi_{\text{and}},m-1,k-1}) \\
&+ \lambda_{12} \sum_{i=0}^{m}\sum_{j=0}^{n} \Theta^{\#}_{g_{12},m-i,n-j}(\Theta_{\psi_{\text{and}},i,j} - \Theta_{\psi_{\text{and}},i-1,j} - \Theta_{\psi_{\text{and}},i,j-1} + \Theta_{\psi_{\text{and}},i-1,j-1}) \\
&= \Theta_{\gamma,m,n} - \Theta_{\gamma,m-1,n} - \Theta_{\gamma,m,n-1} + \Theta_{\gamma,m-1,n-1}.
\end{aligned} \tag{3.14}
$$

We summarize the results in the following proposition.

Proposition 1 ***(Ruin probability*** ψ_{and}***)*** *Under Assumptions 1-3, the exact solution to* ψ_{and} *is given by* (3.10)*, where the Laguerre coefficients* $\{\Theta_{\psi_{\text{and}},m,n}\}_{m,n=0}^{\infty}$ *are the solution of the (countably infinite) linear system of equations consisting of* (3.11)*-*(3.14)*. Here* $\{\Theta_{f_1,k}\}_{k=0}^{\infty}$*,* $\{\Theta_{f_2,k}\}_{k=0}^{\infty}$ *and* $\{\Theta_{g_{12},m,n}\}_{m,n=0}^{\infty}$ *are the Laguerre coefficients of the claim densities, and* $\{\Theta_{\gamma,m,n}\}_{m,n=0}^{\infty}$ *are the Laguerre coefficients of the known function* (3.3)*.*

Although the above result is exact, for the purposes of implementation we need to proceed by truncating (3.10) and writing

$$
\widetilde{\psi}_{\text{and}}(u_1, u_2) = \sum_{m=0}^{M}\sum_{n=0}^{N} \widetilde{\Theta}_{\psi_{\text{and}},m,n}\, \varphi_m(u_1)\varphi_n(u_2), \tag{3.15}
$$

where $\widetilde{\Theta}_{\psi_{\text{and}},m,n}$'s are solved from a truncated version of the system (3.11)-(3.14) (i.e. the upper limits of the summation over i and j in (3.11)-(3.13) are replaced by M and N respectively). In particular, the truncated system consists of a total of $(M+1)(N+1)$ equations: we get 1 equation from (3.11); N equations from (3.12) (i.e. $m = 0; n = 1, 2, \ldots, N$); M equations from (3.13) (i.e. $m = 1, 2, \ldots, M; n = 0$); and MN equations from (3.14) (i.e. $m = 1, 2, \ldots, M; n = 1, 2, \ldots, N$). This results in an approximation where the performance improves as M and N increase. It is important to note that once the coefficients $\{\widetilde{\Theta}_{\psi_{\text{and}},m,n} : 0 \le m \le M; 0 \le n \le N\}$ have been obtained for a given pair of (M, N), the approximation (3.15) is valid for all $u_1, u_2 \ge 0$. This greatly facilitates the plotting of $\widetilde{\psi}_{\text{and}}(u_1, u_2)$ and can be convenient for optimization with respect to u_1 and/or u_2 (see Section 4.3).

3.4 A note on scaling

As in the case of univariate Laguerre expansion, the number of terms required for the bivariate series to converge depends on the unit used. In our case, each Laguerre series is expanded as a function of the surplus level which is measured in monetary units. Suppose that the surplus levels of business lines 1 and 2 are measured in the same unit of s. For example, s may represent 1,000,000 if money is quoted in millions, or s may mean 1 AUD if money is expressed in terms of Australian dollars. If we change the unit of line i to s_i^*, the resulting surplus process will be denoted by $\{U_i^*(t)\}_{t\ge0}$. Such a surplus process then has initial surplus $u_i^* = u_i s/s_i^*$ and premium rate $c_i^* = c_i s/s_i^*$. Moreover, if the scale parameters of $Y_{i,k}$ and $Z_{i,k}$ are μ_{Y_1} and μ_{Z_1} respectively, then the surplus process $\{U_i^*(t)\}_{t\ge0}$ has claim amounts

$Y^*_{i,k} = Y_{i,k}s/s^*_i$ and $Z^*_{i,k} = Z_{i,k}s/s^*_i$ with respective scale parameters $\mu_{Y^*_1} = \mu_{Y_1}s/s^*_i$ and $\mu_{Z^*_1} = \mu_{Z_1}s/s^*_i$. In addition, the marginal density f_i is scaled to give $f^*_i(x) = (s^*_i/s)f_i((s^*_i/s)x)$ for $i = 1, 2$ and the joint density g_{12} is replaced by $g^*_{12}(x_1, x_2) = (s^*_1 s^*_2/s^2)g_{12}((s^*_1/s)x_1, (s^*_2/s)x_2)$. Then the ruin probability ψ_{and} of the original bivariate process $\{(U_1(t), U_2(t))\}_{t\geq 0}$ is linked to the ruin probability ψ^*_{and} of the rescaled bivariate process $\{(U^*_1(t), U^*_2(t))\}_{t\geq 0}$ as $\psi_{\text{and}}(u_1, u_2) = \psi^*_{\text{and}}(u^*_1, u^*_2)$. When approximating $\psi^*_{\text{and}}(u^*_1, u^*_2)$ using truncation at (M^*, N^*) (in the same way as in (3.15)), some numerical tests can be done and s^*_1 and s^*_2 can be manually selected (via a trial-and-error approach) such that the series converges fast already for small values of M^* and N^*. To get a feeling of what s^*_1 and s^*_2 should approximately be, one can first develop f^*_1, f^*_2 and g^*_{12} on the Laguerre basis under different choices of s^*_1 and s^*_2 and check the number of terms required for their series to converge satisfactorily. This can help rule out pairs of s^*_1 and s^*_2 that are likely to require unreasonably large values of M^* and N^* for $\psi^*_{\text{and}}(u^*_1, u^*_2)$ to converge. Then, the same check can be performed for the univariate ruin probabilities ψ^*_1 and ψ^*_2 concerning the scaled univariate processes $\{U^*_1(t)\}_{t\geq 0}$ and $\{U^*_2(t)\}_{t\geq 0}$ and subsequently for γ^* (which is the corresponding γ in (3.3) for the process $\{(U^*_1(t), U^*_2(t))\}_{t\geq 0}$ and depends on ψ^*_1 and ψ^*_2), leading to reasonably good choices of s^*_1 and s^*_2.

4 A numerical illustration

4.1 FGM copula for common shocks

To apply Proposition 1, we require the Laguerre coefficients $\{\Theta_{f_1,k}\}^\infty_{k=0}$, $\{\Theta_{f_2,k}\}^\infty_{k=0}$, $\{\Theta_{g_{12},m,n}\}^\infty_{m,n=0}$ and $\{\Theta_{\gamma,m,n}\}^\infty_{m,n=0}$. In this entire Section 4, it will be assumed that the claim amounts dedicated to line i ($i = 1, 2$) are distributed as a combination of exponentials with density (3.9). In that case, the evaluation of the Laguerre coefficients relies on the fact that the Laplace transform of the Laguerre function (2.1) has the pleasant form

$$\int_0^\infty e^{-ax}\varphi_k(x)\,dx = \frac{\left(a - \frac{1}{2}\right)^k}{\left(a + \frac{1}{2}\right)^{k+1}}, \qquad \Re(a) \geq 0. \tag{4.1}$$

(Cf. Keilson and Nunn (1979, Equation (1.8).) Therefore, by virtue of (2.3) and (4.1), the Laguerre coefficients of f_i can be obtained as

$$\Theta_{f_i,k} = \sum_{j=1}^{n_i} q_{ij}\nu_{ij}\frac{\left(\nu_{ij} - \frac{1}{2}\right)^k}{\left(\nu_{ij} + \frac{1}{2}\right)^{k+1}}.$$

The simultaneous claims arising from common shocks are also assumed to be combinations of exponentials within each line of business. For $i = 1, 2$, similar to (3.9) the marginal density of $Z_{i,k}$ is correspondingly

$$g_i(x) = \sum_{j=1}^{m_i} q^*_{ij}\eta_{ij}e^{-\eta_{ij}x}, \qquad x \geq 0, \tag{4.2}$$

where η_{ij}'s are positive and distinct, and q^*_{ij}'s are non-zero with $\sum_{j=1}^{m_i} q^*_{ij} = 1$. The dependency between these claim amounts across the two lines are modelled by a Farlie-Gumbel-Morgenstern (FGM) copula of the form

$$C(x_1, x_2) = x_1x_2 + \omega x_1x_2(1 - x_1)(1 - x_2), \qquad 0 \leq x_1, x_2 \leq 1, \tag{4.3}$$

with correspodning copula density $c(x_1, x_2) = (\partial^2/\partial x_1\partial x_2)C(x_1, x_2) = 1 + \omega(1 - 2x_1)(1 - 2x_2)$ (see e.g. Nelsen (2006)). Here ω is the dependence parameter satisfying $-1 \leq \omega \leq 1$, and the Kendall's tau and Spearman's rho of the FGM copula are $2\omega/9$ and $\omega/3$ respectively. The use of a copula to model dependence is particularly useful for numerical demonstration because one can easily alter the degree of dependence while keeping the marginal distributions fixed, and therefore any change of the results can be attributed to the effect of dependence. Interested readers are also referred to e.g. Cossette et al. (2010, 2013) and Bargès et al. (2011) for applications of the FGM copula in risk theory.

Denoting the survival function corresponding to g_i by $\overline{G}_i$, under the FGM copula the joint density of $(Z_{1,k}, Z_{2,k})$ is given by

$$\begin{aligned}
&g_{12}(x_1,x_2)\\
&= c(1-\overline{G}_1(x_1), 1-\overline{G}_2(x_2))g_1(x_1)g_2(x_2)\\
&= g_1(x_1)g_2(x_2) + \omega[2\overline{G}_1(x_1)g_1(x_1) - g_1(x_1)][2\overline{G}_2(x_2)g_2(x_2) - g_2(x_2)]\\
&= \left(\sum_{i=1}^{m_1} q_{1i}^*\eta_{1i}e^{-\eta_{1i}x_1}\right)\left(\sum_{k=1}^{m_2} q_{2k}^*\eta_{2k}e^{-\eta_{2k}x_2}\right)\\
&\quad + \omega\left(2\sum_{i=1}^{m_1}\sum_{j=1}^{m_1} q_{1i}^*q_{1j}^*\eta_{1i}e^{-(\eta_{1i}+\eta_{1j})x_1} - \sum_{i=1}^{m_1} q_{1i}^*\eta_{1i}e^{-\eta_{1i}x_1}\right)\left(2\sum_{k=1}^{m_2}\sum_{l=1}^{m_2} q_{2k}^*q_{2l}^*\eta_{2k}e^{-(\eta_{2k}+\eta_{2l})x_2} - \sum_{k=1}^{m_2} q_{2k}^*\eta_{2k}e^{-\eta_{2k}x_2}\right).
\end{aligned}$$

Thus, with the help of (2.5) and (4.1), the Laguerre coefficients of g_{12} can be evaluated explicitly as

$$\begin{aligned}
\Theta_{g_{12},m,n} = &\left(\sum_{i=1}^{m_1} q_{1i}^*\eta_{1i}\frac{\left(\eta_{1i}-\frac{1}{2}\right)^m}{\left(\eta_{1i}+\frac{1}{2}\right)^{m+1}}\right)\left(\sum_{k=1}^{m_2} q_{2k}^*\eta_{2k}\frac{\left(\eta_{2k}-\frac{1}{2}\right)^n}{\left(\eta_{2k}+\frac{1}{2}\right)^{n+1}}\right)\\
&+\omega\left(2\sum_{i=1}^{m_1}\sum_{j=1}^{m_1} q_{1i}^*q_{1j}^*\eta_{1i}\frac{\left(\eta_{1i}+\eta_{1j}-\frac{1}{2}\right)^m}{\left(\eta_{1i}+\eta_{1j}+\frac{1}{2}\right)^{m+1}} - \sum_{i=1}^{m_1} q_{1i}^*\eta_{1i}\frac{\left(\eta_{1i}-\frac{1}{2}\right)^m}{\left(\eta_{1i}+\frac{1}{2}\right)^{m+1}}\right)\\
&\quad\times\left(2\sum_{k=1}^{m_2}\sum_{l=1}^{m_2} q_{2k}^*q_{2l}^*\eta_{2k}\frac{\left(\eta_{2k}+\eta_{2l}-\frac{1}{2}\right)^n}{\left(\eta_{2k}+\eta_{2l}+\frac{1}{2}\right)^{n+1}} - \sum_{k=1}^{m_2} q_{2k}^*\eta_{2k}\frac{\left(\eta_{2k}-\frac{1}{2}\right)^n}{\left(\eta_{2k}+\frac{1}{2}\right)^{n+1}}\right).
\end{aligned}$$

Next, we would like to calculate the Laguerre coefficients $\{\Theta_{\gamma,m,n}\}_{m,n=0}^{\infty}$ pertaining to the function γ defined in (3.3). To this end, we require the univariate ruin probabilities ψ_1 and ψ_2. Under (3.9) and (4.2), it is easily seen from (1.1) that the claim density h_i for business line i (when viewed as a univariate risk process) still remains a combination of exponentials. However, the number of distinct exponential terms in the combintation depends on whether some $\{\nu_{ij}\}_{j=1}^{n_i}$ overlap with $\{\eta_{ij}\}_{j=1}^{m_i}$. Suppose that, for fixed $i=1,2$, there are s_i distinct values in the set containing $\{\nu_{ij}\}_{j=1}^{n_i}$ and $\{\eta_{ij}\}_{j=1}^{m_i}$. From Dufresne and Gerber (1988), the univariate ruin probability can be represented as

$$\psi_i(u) = \sum_{j=1}^{s_i} V_{ij}e^{-\kappa_{ij}u}, \qquad u \geq 0, \tag{4.4}$$

where $\{-\kappa_{ij}\}_{j=1}^{s_i}$ are the roots of the Lundberg equation with negative real parts (which are typically distinct), and $\{V_{ij}\}_{j=1}^{s_i}$ are constants that can be evaluated explicitly. The roots $\{-\kappa_{ij}\}_{j=1}^{s_i}$ are known to be distinct from $\{\nu_{ij}\}_{j=1}^{n_i}$ and $\{\eta_{ij}\}_{j=1}^{m_i}$. Then, one can proceed to evaluate (3.3) and apply (2.5) and (4.1) to get $\Theta_{\gamma,m,n}$. Assuming that, for fixed $b=1,2$, none of $\{\kappa_{ba}\}_{a=1}^{s_b}$ coincide with $\{\eta_{bi}+\eta_{bj}\}_{i,j=1}^{m_b}$, we omit the straightforward details to arrive at

$\Theta_{\gamma,m,n}$

$$
\begin{aligned}
=& -\lambda_1\left(\sum_{i=1}^{n_1} q_{1i}\frac{\left(\nu_{1i}-\frac{1}{2}\right)^m}{\left(\nu_{1i}+\frac{1}{2}\right)^{m+1}}\right)\left(\sum_{j=1}^{s_2} V_{2j}\frac{\left(\kappa_{2j}-\frac{1}{2}\right)^n}{\left(\kappa_{2j}+\frac{1}{2}\right)^{n+1}}\right)-\lambda_2\left(\sum_{l=1}^{s_1} V_{1l}\frac{\left(\kappa_{1l}-\frac{1}{2}\right)^m}{\left(\kappa_{1l}+\frac{1}{2}\right)^{m+1}}\right)\left(\sum_{k=1}^{n_2} q_{2k}\frac{\left(\nu_{2k}-\frac{1}{2}\right)^n}{\left(\nu_{2k}+\frac{1}{2}\right)^{n+1}}\right)\\
&-\lambda_{12}\sum_{a=1}^{s_1} V_{1a}\Bigg\{\left[\sum_{i=1}^{m_1}\frac{q^*_{1i}\eta_{1i}}{\eta_{1i}-\kappa_{1a}}\left(\frac{\left(\kappa_{1a}-\frac{1}{2}\right)^m}{\left(\kappa_{1a}+\frac{1}{2}\right)^{m+1}}-\frac{\left(\eta_{1i}-\frac{1}{2}\right)^m}{\left(\eta_{1i}+\frac{1}{2}\right)^{m+1}}\right)\right]\left(\sum_{k=1}^{m_2} q^*_{2k}\frac{\left(\eta_{2k}-\frac{1}{2}\right)^n}{\left(\eta_{2k}+\frac{1}{2}\right)^{n+1}}\right)\\
&\qquad+\omega\left[2\sum_{i=1}^{m_1}\sum_{j=1}^{m_1}\frac{q^*_{1i}q^*_{1j}\eta_{1i}}{\eta_{1i}+\eta_{1j}-\kappa_{1a}}\left(\frac{\left(\kappa_{1a}-\frac{1}{2}\right)^m}{\left(\kappa_{1a}+\frac{1}{2}\right)^{m+1}}-\frac{\left(\eta_{1i}+\eta_{1j}-\frac{1}{2}\right)^m}{\left(\eta_{1i}+\eta_{1j}+\frac{1}{2}\right)^{m+1}}\right)-\sum_{i=1}^{m_1}\frac{q^*_{1i}\eta_{1i}}{\eta_{1i}-\kappa_{1a}}\left(\frac{\left(\kappa_{1a}-\frac{1}{2}\right)^m}{\left(\kappa_{1a}+\frac{1}{2}\right)^{m+1}}-\frac{\left(\eta_{1i}-\frac{1}{2}\right)^m}{\left(\eta_{1i}+\frac{1}{2}\right)^{m+1}}\right)\right]\\
&\qquad\times\left[2\sum_{k=1}^{m_2}\sum_{l=1}^{m_2}\frac{q^*_{2k}q^*_{2l}\eta_{2k}}{\eta_{2k}+\eta_{2l}}\frac{\left(\eta_{2k}+\eta_{2l}-\frac{1}{2}\right)^n}{\left(\eta_{2k}+\eta_{2l}+\frac{1}{2}\right)^{n+1}}-\sum_{k=1}^{m_2} q^*_{2k}\frac{\left(\eta_{2k}-\frac{1}{2}\right)^n}{\left(\eta_{2k}+\frac{1}{2}\right)^{n+1}}\right]\Bigg\}\\
&-\lambda_{12}\sum_{a=1}^{s_2} V_{2a}\Bigg\{\left(\sum_{i=1}^{m_1} q^*_{1i}\frac{\left(\eta_{1i}-\frac{1}{2}\right)^m}{\left(\eta_{1i}+\frac{1}{2}\right)^{m+1}}\right)\left[\sum_{k=1}^{m_2}\frac{q^*_{2k}\eta_{2k}}{\eta_{2k}-\kappa_{2a}}\left(\frac{\left(\kappa_{2a}-\frac{1}{2}\right)^n}{\left(\kappa_{2a}+\frac{1}{2}\right)^{n+1}}-\frac{\left(\eta_{2k}-\frac{1}{2}\right)^n}{\left(\eta_{2k}+\frac{1}{2}\right)^{n+1}}\right)\right]\\
&\qquad+\omega\left[2\sum_{i=1}^{m_1}\sum_{j=1}^{m_1}\frac{q^*_{1i}q^*_{1j}\eta_{1i}}{\eta_{1i}+\eta_{1j}}\frac{\left(\eta_{1i}+\eta_{1j}-\frac{1}{2}\right)^m}{\left(\eta_{1i}+\eta_{1j}+\frac{1}{2}\right)^{m+1}}-\sum_{i=1}^{m_1} q^*_{1i}\frac{\left(\eta_{1i}-\frac{1}{2}\right)^m}{\left(\eta_{1i}+\frac{1}{2}\right)^{m+1}}\right]\\
&\qquad\times\left[2\sum_{k=1}^{m_2}\sum_{l=1}^{m_2}\frac{q^*_{2k}q^*_{2l}\eta_{2k}}{\eta_{2k}+\eta_{2l}-\kappa_{2a}}\left(\frac{\left(\kappa_{2a}-\frac{1}{2}\right)^n}{\left(\kappa_{2a}+\frac{1}{2}\right)^{n+1}}-\frac{\left(\eta_{2k}+\eta_{2l}-\frac{1}{2}\right)^n}{\left(\eta_{2k}+\eta_{2l}+\frac{1}{2}\right)^{n+1}}\right)-\sum_{k=1}^{m_2}\frac{q^*_{2k}\eta_{2k}}{\eta_{2k}-\kappa_{2a}}\left(\frac{\left(\kappa_{2a}-\frac{1}{2}\right)^n}{\left(\kappa_{2a}+\frac{1}{2}\right)^{n+1}}-\frac{\left(\eta_{2k}-\frac{1}{2}\right)^n}{\left(\eta_{2k}+\frac{1}{2}\right)^{n+1}}\right)\right]\Bigg\}\\
&-\lambda_{12}\Bigg\{\left(\sum_{i=1}^{m_1} q^*_{1i}\frac{\left(\eta_{1i}-\frac{1}{2}\right)^m}{\left(\eta_{1i}+\frac{1}{2}\right)^{m+1}}\right)\left(\sum_{k=1}^{m_2} q^*_{2k}\frac{\left(\eta_{2k}-\frac{1}{2}\right)^n}{\left(\eta_{2k}+\frac{1}{2}\right)^{n+1}}\right)\\
&\quad+\omega\left[2\sum_{i=1}^{m_1}\sum_{j=1}^{m_1}\frac{q^*_{1i}q^*_{1j}\eta_{1i}}{\eta_{1i}+\eta_{1j}}\frac{\left(\eta_{1i}+\eta_{1j}-\frac{1}{2}\right)^m}{\left(\eta_{1i}+\eta_{1j}+\frac{1}{2}\right)^{m+1}}-\sum_{i=1}^{m_1} q^*_{1i}\frac{\left(\eta_{1i}-\frac{1}{2}\right)^m}{\left(\eta_{1i}+\frac{1}{2}\right)^{m+1}}\right]\left[2\sum_{k=1}^{m_2}\sum_{l=1}^{m_2}\frac{q^*_{2k}q^*_{2l}\eta_{2k}}{\eta_{2k}+\eta_{2l}}\frac{\left(\eta_{2k}+\eta_{2l}-\frac{1}{2}\right)^n}{\left(\eta_{2k}+\eta_{2l}+\frac{1}{2}\right)^{n+1}}-\sum_{k=1}^{m_2} q^*_{2k}\frac{\left(\eta_{2k}-\frac{1}{2}\right)^n}{\left(\eta_{2k}+\frac{1}{2}\right)^{n+1}}\right]\Bigg\}.
\end{aligned}
$$

We now have all components in the linear system (3.11)-(3.14), so that Proposition 1 and the subsequent comments regarding truncation can be applied to approximate ψ_{and} by $\widetilde{\psi}_{\text{and}}$ in (3.15).

4.2 Bivariate ruin probabilities and effect of scaling on convergence

In the numerical example, we first assume the Poisson arrival rates $\lambda_1 = \lambda_2 = 1$ and $\lambda_{12} = 0.2$. For both business lines ($i = 1, 2$) the dedicated claim amounts $\{Y_{i,k}\}_{k=1}^{\infty}$ are assumed to be exponential with density

$$f_i(x) = 5e^{-5x}, \qquad x \geq 0,$$

so that the mean is 0.2 and the variance is 0.04 (with a coefficient of variation of 1). For common shocks, the resulting claim amounts $\{Z_{1,k}\}_{k=1}^{\infty}$ and $\{Z_{2,k}\}_{k=1}^{\infty}$ of the two lines follow the marginal densities

$$
\begin{aligned}
g_1(x) &= 2\left(\frac{3}{2}e^{-\frac{3}{2}x}\right) + (-1)\left(3e^{-3x}\right), \qquad x \geq 0,\\
g_2(x) &= \frac{1}{3}\left(\frac{1}{2}e^{-\frac{1}{2}x}\right) + \frac{2}{3}\left(2e^{-2x}\right), \qquad x \geq 0,
\end{aligned}
$$

which are of the form (4.2). Thus, $Z_{1,k}$ and $Z_{2,k}$ have the same mean of 1 but possess different variances of 0.56 and 2 respectively (implying respective coefficients of variation of 0.75 and 1.41). Assuming that the security loading factors for business lines 1 and 2 are $\theta_1 = 0.1$ and $\theta_2 = 0.2$, the premium rates are calculated to be $c_1 = 0.44$ and $c_2 = 0.48$. Note that under the above setting, common shocks occur less frequently than claims dedicated to specific lines but produce larger claims on average. Moreover, a higher security loading is applied for line 2 which faces claims with higher variability when a common shock strikes. Regarding the dependence between $Z_{1,k}$ and $Z_{2,k}$, three cases of the FGM copula (4.3) will be considered, namely (i) $\omega = -1$; (ii) $\omega = 0$; and (iii) $\omega = 1$, which correspond to negative dependence, independence and positive dependence respectively.

	$s/s_1^*=s/s_2^*=1$					$s/s_1^*=s/s_2^*=1/3$					$s/s_1^*=s/s_2^*=3$				
$\psi_{\text{and}}(2,2)$	$N^*=10$	$N^*=20$	$N^*=30$	$N^*=40$	$N^*=50$	$N^*=10$	$N^*=20$	$N^*=30$	$N^*=40$	$N^*=50$	$N^*=10$	$N^*=20$	$N^*=30$	$N^*=40$	$N^*=50$
$M^*=10$	0.346166	0.346155	0.346153	0.346153	0.346153	0.342574	0.346365	0.345570	0.345409	0.345462	0.338664	0.343701	0.343837	0.343845	0.343845
$M^*=20$	0.346164	0.346152	0.346150	0.346150	0.346150	0.343615	0.347263	0.346532	0.346367	0.346414	0.339799	0.345772	0.346065	0.346085	0.346085
$M^*=30$	0.346163	0.346151	0.346149	0.346149	0.346149	0.343355	0.347021	0.346280	0.346116	0.346164	0.339835	0.345819	0.346121	0.346144	0.346145
$M^*=40$	0.346162	0.346151	0.346149	0.346149	0.346149	0.343265	0.346953	0.346205	0.346040	0.346089	0.339839	0.345823	0.346126	0.346148	0.346149
$M^*=50$	0.346162	0.346151	0.346149	0.346149	0.346149	0.343289	0.346974	0.346227	0.346061	0.346111	0.339839	0.345823	0.346125	0.346148	0.346149

Table 1: Approximated values of $\psi_{\text{and}}(2,2)$ for selected (M^*, N^*) when $\omega = -1$

	$s/s_1^*=s/s_2^*=1$					$s/s_1^*=s/s_2^*=1/3$					$s/s_1^*=s/s_2^*=3$				
$\psi_{\text{and}}(2,2)$	$N^*=10$	$N^*=20$	$N^*=30$	$N^*=40$	$N^*=50$	$N^*=10$	$N^*=20$	$N^*=30$	$N^*=40$	$N^*=50$	$N^*=10$	$N^*=20$	$N^*=30$	$N^*=40$	$N^*=50$
$M^*=10$	0.360181	0.360178	0.360176	0.360176	0.360176	0.357045	0.360407	0.359678	0.359530	0.359581	0.351460	0.357344	0.357499	0.357507	0.357507
$M^*=20$	0.360175	0.360173	0.360171	0.360171	0.360171	0.357972	0.361206	0.360539	0.360385	0.360430	0.352605	0.359689	0.360070	0.360096	0.360097
$M^*=30$	0.360174	0.360172	0.360170	0.360170	0.360170	0.357717	0.360972	0.360295	0.360143	0.360188	0.352641	0.359737	0.360133	0.360164	0.360165
$M^*=40$	0.360174	0.360172	0.360170	0.360170	0.360170	0.357628	0.360904	0.360221	0.360067	0.360114	0.352646	0.359742	0.360137	0.360168	0.360170
$M^*=50$	0.360174	0.360172	0.360170	0.360170	0.360170	0.357650	0.360923	0.360241	0.360088	0.360134	0.352645	0.359741	0.360137	0.360168	0.360170

Table 2: Approximated values of $\psi_{\text{and}}(2,2)$ for selected (M^*, N^*) when $\omega = 0$

	$s/s_1^*=s/s_2^*=1$					$s/s_1^*=s/s_2^*=1/3$					$s/s_1^*=s/s_2^*=3$				
$\psi_{\text{and}}(2,2)$	$N^*=10$	$N^*=20$	$N^*=30$	$N^*=40$	$N^*=50$	$N^*=10$	$N^*=20$	$N^*=30$	$N^*=40$	$N^*=50$	$N^*=10$	$N^*=20$	$N^*=30$	$N^*=40$	$N^*=50$
$M^*=10$	0.375049	0.375058	0.375056	0.375056	0.375056	0.372398	0.375287	0.374629	0.374494	0.374542	0.364944	0.371787	0.371962	0.371970	0.371970
$M^*=20$	0.375039	0.375048	0.375046	0.375046	0.375046	0.373219	0.376001	0.375402	0.375260	0.375302	0.366073	0.374435	0.374927	0.374961	0.374962
$M^*=30$	0.375039	0.375048	0.375046	0.375046	0.375046	0.372967	0.375776	0.375166	0.375026	0.375069	0.366110	0.374484	0.374996	0.375038	0.375041
$M^*=40$	0.375039	0.375048	0.375046	0.375046	0.375046	0.372877	0.375707	0.375092	0.374951	0.374994	0.366114	0.374489	0.375001	0.375044	0.375046
$M^*=50$	0.375039	0.375048	0.375046	0.375046	0.375046	0.372898	0.375725	0.375111	0.374969	0.375013	0.366114	0.374488	0.375001	0.375043	0.375046

Table 3: Approximated values of $\psi_{\text{and}}(2,2)$ for selected (M^*, N^*) when $\omega = 1$

As discussed in Section 3.4, the above notation implicitly assumes that a certain monetary unit, say s, is applied to the two business lines. One may want to check whether the use of different monetary units can make the bivariate Laguerre series (3.15) converge with a smaller number of terms. For illustrative purposes, we shall consider three sets of monetary units (s_1^*, s_2^*) such that (i) $s/s_1^* = s/s_2^* = 1$ (i.e. without scaling); (ii) $s/s_1^* = s/s_2^* = 1/3$; and (iii) $s/s_1^* = s/s_2^* = 3$. The number of terms required for the series to converge will be examined. Tables 1-3 show the approximated values of $\psi_{\text{and}}(2,2)$ rounded to six decimal places for various pairs of (M^*, N^*) (which are the truncation points in (3.15) but for the scaled process) under three different dependence parameters respectively. Starting with Table 1 ($\omega = -1$), we first look at the scaling $s/s_1^* = s/s_2^* = 1$ and observe that the approximated ruin probability converges when (i) N^* increases across a row (keeping M^* fixed); and (ii) M^* increases down a column (keeping N^* fixed). In particular, the use of $(M^*, N^*) = (30, 30)$ leads to converging result of 0.346149 that is correct up to at least six decimal places. It is noteworthy that $(M^*, N^*) = (10, 10)$ is sufficient to produce a satisfactory result of 0.346166 correct at the third decimal, which is arguably good enough for decision making purposes. In contrast, when the scaling $s/s_1^* = s/s_2^*$ is chosen to be 1/3 or 3, a larger number of terms is needed for the series to converge. Specifically, when $s/s_1^* = s/s_2^* = 1/3$, the ruin probability does not fully converge even when $(M^*, N^*) = (50, 50)$, and this appears to be the worst option among the three choices of scaling factors as far as convergence is concerned. When $s/s_1^* = s/s_2^* = 3$, accuracy at the sixth decimal place is achieved with the truncation points $(M^*, N^*) = (40, 50)$. Such performance is still good but not as superior as that produced by $s/s_1^* = s/s_2^* = 1$. Nevertheless, the above results demonstrate that it can be important to take into account scaling during computation. Moving to Table 2 ($\omega = 0$) and Table 3 ($\omega = 1$), similar pattern can be observed such that $s/s_1^* = s/s_2^* = 1$ leads to the best convergence results. Comparing across Tables 1-3, the number of terms required for a given accuracy is insensitive to the dependence parameter ω of the FGM copula. This intuitively makes sense because ω is unitless.

$\psi_{\text{and}}(u_1, u_2)$	$u_2 = 0$	$u_2 = 2$	$u_2 = 4$	$u_2 = 6$	$u_2 = 8$	$u_2 = 10$
$u_1 = 0$	0.768375	0.472728	0.337113	0.241968	0.173667	0.124595
$u_1 = 2$	0.534529	0.346149	0.252221	0.184192	0.134135	0.097431
$u_1 = 4$	0.375081	0.250588	0.185965	0.137953	0.101872	0.074922
$u_1 = 6$	0.262287	0.178943	0.134757	0.101321	0.075769	0.056381
$u_1 = 8$	0.183091	0.126778	0.096574	0.073427	0.055511	0.041742
$u_1 = 10$	0.127675	0.089382	0.068703	0.052716	0.040223	0.030525

Table 4: Values of $\psi_{\text{and}}(u_1, u_2)$ for selected (u_1, u_2) when $\omega = -1$

$\psi_{\text{and}}(u_1, u_2)$	$u_2 = 0$	$u_2 = 2$	$u_2 = 4$	$u_2 = 6$	$u_2 = 8$	$u_2 = 10$
$u_1 = 0$	0.771137	0.477053	0.340563	0.244614	0.175656	0.126072
$u_1 = 2$	0.541171	0.360170	0.264642	0.194333	0.142091	0.103524
$u_1 = 4$	0.380708	0.264254	0.199266	0.149538	0.111418	0.082522
$u_1 = 6$	0.266610	0.190161	0.146355	0.111916	0.084850	0.063854
$u_1 = 8$	0.186292	0.135403	0.105855	0.082207	0.063273	0.048310
$u_1 = 10$	0.130003	0.095805	0.075808	0.059615	0.046473	0.035934

Table 5: Values of $\psi_{\text{and}}(u_1, u_2)$ for selected (u_1, u_2) when $\omega = 0$

$\psi_{\text{and}}(u_1, u_2)$	$u_2 = 0$	$u_2 = 2$	$u_2 = 4$	$u_2 = 6$	$u_2 = 8$	$u_2 = 10$
$u_1 = 0$	0.774192	0.481611	0.344037	0.247185	0.177533	0.127432
$u_1 = 2$	0.548281	0.375046	0.277508	0.204590	0.149961	0.109429
$u_1 = 4$	0.386447	0.278507	0.213201	0.161614	0.121277	0.090281
$u_1 = 6$	0.270898	0.201657	0.158491	0.123105	0.094463	0.071752
$u_1 = 8$	0.189414	0.144120	0.115517	0.091523	0.071605	0.055405
$u_1 = 10$	0.132247	0.102230	0.083165	0.066942	0.053233	0.041862

Table 6: Values of $\psi_{\text{and}}(u_1, u_2)$ for selected (u_1, u_2) when $\omega = 1$

Since the scaling $s/s_1^* = s/s_2^* = 1$ leads to superior performance in terms of convergence, this will be applied in the remainder of our numerical illustration. To ensure the ruin probability values $\psi_{\text{and}}(u_1, u_2)$ are of high accuracy, we shall apply the truncation points $(M^*, N^*) = (50, 50)$ for the rest of the paper. Tables 4-6 show the resulting values for various pairs of (u_1, u_2) when $\omega = -1, 0, 1$ respectively, and all these values have converged up to at least six decimal places (upon checking against results calculated using larger values of M^* and N^*). As expected, within each of these tables, the ruin probability $\psi_{\text{and}}(u_1, u_2)$ is decreasing in the initial surplus levels u_1 and u_2. Meanwhile, for a fixed pair of (u_1, u_2), comparing across Tables 4-6 reveals that $\psi_{\text{and}}(u_1, u_2)$ increases as the dependence parameter ω increases from -1 to 0 and then to 1. This can be explained intuitively as follows. When $\omega > 0$ (resp. $\omega < 0$), the claims $Z_{1,k}$ and $Z_{2,k}$ incurred by the two business lines as a result of the k-th common shock are positively (resp. negatively) dependent, implying there is a higher (resp. lower) chance that both lines face large claims and experience ruin. Thus, $\psi_{\text{and}}(u_1, u_2)$ is larger when $\omega = 1$ and smaller when $\omega = -1$, with the independent case $\omega = 0$ lying between these two cases. We have also further tested the accuracy of our algorithm using larger values of u_1 and u_2 with $(u_1, u_2) = (20, 20), (30, 30), (40, 40)$, and the results presented in Table 7 are correct up to at least six significant figures. It is confirmed that the use of $(M^*, N^*) = (50, 50)$ in our approach works uniformly well at least for $0 \le u_1, u_2 \le 40$ in this example. For larger pairs of initial surplus levels like $(u_1, u_2) = (50, 50)$, the ruin probability $\psi_{\text{and}}(u_1, u_2)$ is essentially zero at the sixth decimal place.

Next, we shall use the exact univariate ruin probability (4.4) for $i = 1, 2$ and utilize the relation (1.2) to calculate the bivariate ruin probability $\psi_{\text{or}}(u_1, u_2)$, and the results are summarized in Tables 8-10.

$\psi_{\text{and}}(u_1, u_2)$	$(u_1, u_2) = (20, 20)$	$(u_1, u_2) = (30, 30)$	$(u_1, u_2) = (40, 40)$
$\omega = -1$	0.00158443	0.0000845617	0.00000457669
$\omega = 0$	0.00220892	0.000140694	0.00000912874
$\omega = 1$	0.00299008	0.000222413	0.0000169034

Table 7: Values of $\psi_{\text{and}}(u_1, u_2)$ for larger (u_1, u_2)

Again, $\psi_{\text{or}}(u_1, u_2)$ is decreasing in both u_1 and u_2. However, as opposed to the analysis of $\psi_{\text{and}}(u_1, u_2)$, it is noted that (for fixed (u_1, u_2)) the ruin probability $\psi_{\text{or}}(u_1, u_2)$ decreases in ω. Indeed, when the claims arising from a given common shock are negatively dependent, it is more likely that the claim amounts go in the opposite direction. This in turn means there is less chance that the claims in both lines are small, resulting in a higher ruin probability $\psi_{\text{or}}(u_1, u_2)$ (which takes into account sample paths for which line 1 or line 2 ruins).

$\psi_{\text{or}}(u_1, u_2)$	$u_2 = 0$	$u_2 = 2$	$u_2 = 4$	$u_2 = 6$	$u_2 = 8$	$u_2 = 10$
$u_1 = 0$	0.974050	0.940777	0.930242	0.923410	0.918854	0.915784
$u_1 = 2$	0.915816	0.775277	0.723056	0.689108	0.666307	0.650869
$u_1 = 4$	0.887277	0.682850	0.601324	0.547359	0.510581	0.485390
$u_1 = 6$	0.869346	0.623771	0.521808	0.453267	0.405960	0.373207
$u_1 = 8$	0.857650	0.585043	0.469099	0.390268	0.335326	0.296953
$u_1 = 10$	0.849869	0.559242	0.433773	0.347782	0.287417	0.244973

Table 8: Values of $\psi_{\text{or}}(u_1, u_2)$ for selected (u_1, u_2) when $\omega = -1$

$\psi_{\text{or}}(u_1, u_2)$	$u_2 = 0$	$u_2 = 2$	$u_2 = 4$	$u_2 = 6$	$u_2 = 8$	$u_2 = 10$
$u_1 = 0$	0.971287	0.936451	0.926793	0.920764	0.916864	0.914306
$u_1 = 2$	0.909175	0.761256	0.710636	0.678968	0.658351	0.644776
$u_1 = 4$	0.881650	0.669184	0.588022	0.535774	0.501035	0.477790
$u_1 = 6$	0.865024	0.612553	0.510210	0.442672	0.396880	0.365734
$u_1 = 8$	0.854449	0.576418	0.459818	0.381489	0.327564	0.290386
$u_1 = 10$	0.847541	0.552819	0.426668	0.340883	0.281167	0.239564

Table 9: Values of $\psi_{\text{or}}(u_1, u_2)$ for selected (u_1, u_2) when $\omega = 0$

$\psi_{\text{or}}(u_1, u_2)$	$u_2 = 0$	$u_2 = 2$	$u_2 = 4$	$u_2 = 6$	$u_2 = 8$	$u_2 = 10$
$u_1 = 0$	0.968232	0.931893	0.923319	0.918193	0.914987	0.912947
$u_1 = 2$	0.902065	0.746380	0.697769	0.668710	0.650480	0.638872
$u_1 = 4$	0.875910	0.654931	0.574088	0.523698	0.491176	0.470031
$u_1 = 6$	0.860735	0.601057	0.498074	0.431483	0.387266	0.357836
$u_1 = 8$	0.851327	0.567701	0.450155	0.372172	0.319232	0.283291
$u_1 = 10$	0.845296	0.546394	0.419311	0.333556	0.274406	0.233636

Table 10: Values of $\psi_{\text{or}}(u_1, u_2)$ for selected (u_1, u_2) when $\omega = 1$

4.3 A capital allocation problem

If the two business lines with surplus processes $\{U_1(t)\}_{t\geq 0}$ and $\{U_2(t)\}_{t\geq 0}$ belong to the same company, then the formulas for ψ_{and} and ψ_{or} can also be used to analyze a capital allocation problem, see e.g. Gong et al. (2012, Section 6.3). Concretely, one can identify the best split of a capital amount u^* between the two lines so as to minimize a bivariate ruin probability. In this subsection, we shall minimize $\psi_{\text{or}}(u_1, u_2)$ with respect to $u_1 \geq 0$ and $u_2 \geq 0$ subject to the constraint $u_1 + u_2 = u^*$ (see Remark 2 for discussion

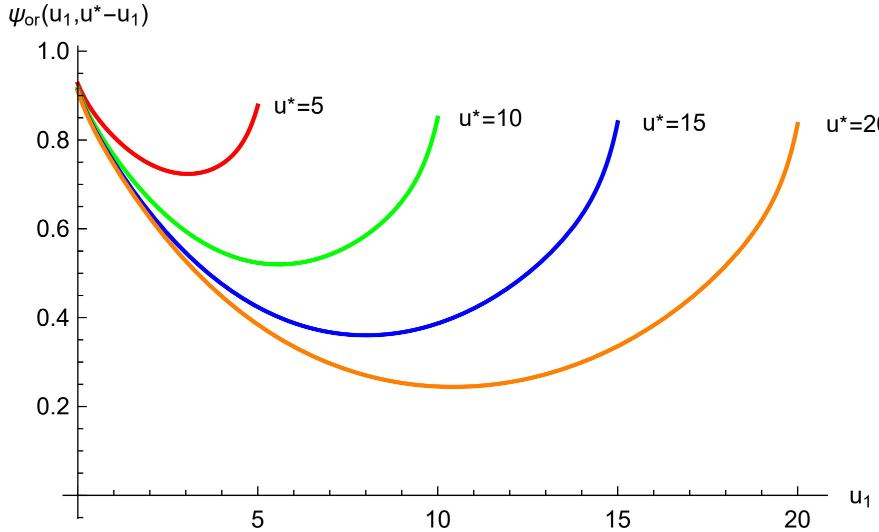

Figure 1: Plot of $\psi_{\text{or}}(u_1, u^* - u_1)$ against u_1 for $\omega = -1$ and $\lambda_{12} = 0.2$

	$u^* = 5$	$u^* = 10$	$u^* = 15$	$u^* = 20$
$\omega = -1$	3.055 (61.11%) **0.723676**	5.580 (55.80%) **0.519907**	8.005 (53.37%) **0.360336**	10.423 (52.12%) **0.244203**
$\omega = 0$	3.041 (60.81%) **0.709365**	5.558 (55.58%) **0.508019**	7.981 (53.21%) **0.352024**	10.399 (52.99%) **0.238867**
$\omega = 1$	3.024 (60.49%) **0.694293**	5.533 (55.33%) **0.495530**	7.954 (53.03%) **0.343136**	10.373 (51.86%) **0.233015**

Table 11: Optimal u_1, ratio of optimal u_1 to u^* (in parentheses) and minimized $\psi_{\text{or}}(u_1, u^* - u_1)$ (in boldface) when $\lambda_{12} = 0.2$

on $\psi_{\text{and}}(u_1, u_2)$). This is equivalent to minimizing $\psi_{\text{or}}(u_1, u^* - u_1)$ with respect to u_1 in the domain $0 \leq u_1 \leq u^*$.

First, Figure 1 plots $\psi_{\text{or}}(u_1, u^* - u_1)$ against u_1 (for $0 \leq u_1 \leq u^*$) under $\omega = -1$ (i.e. negative dependence). For each available capital $u^* = 5, 10, 15, 20$ to be allocated, it can be seen that, as u_1 increases, $\psi_{\text{or}}(u_1, u^* - u_1)$ first decreases and then increases such that there is a distinctive minimum ruin probability achieved at an optimal value of u_1. The plots of the cases $\omega = 0$ and $\omega = 1$ are almost identical and are omitted here. Instead, the optimal u_1, the ratio of the optimal u_1 to the available capital u^* and the corresponding minimized ruin probability $\psi_{\text{or}}(u_1, u^* - u_1)$ when $\omega = -1, 0, 1$ are provided in Table 11. For each ω and u^*, Table 11 shows that neither business line is allocated significantly more capital than the other under the optimal allocation. According to the definition of τ_{or}, ruin is said to occur when any of the line is ruined. In general, if significantly more capital is allocated to one line, although its own probability can be reduced the ruin probability of the other line can be noticeably higher, leading to a higher bivariate ruin probability $\psi_{\text{or}}(u_1, u_2)$. This can also be seen from the simple lower bound $\psi_{\text{or}}(u_1, u_2) \geq \max(\psi_1(u_1), \psi_2(u_2))$ (see e.g. Chan et al. (2003, Equation (2.1))). In our setting, the insurance portfolios of the two business lines are of very similar size in terms of claim frequencies and claim amounts, where line 1 has a smaller security loading factor of $\theta_1 = 0.1$ compared to line 2's $\theta_2 = 0.2$. Consequently, it is not surprising that the minimum $\psi_{\text{or}}(u_1, u^* - u_1)$ is attained when slightly more capital is allocated to line 1.

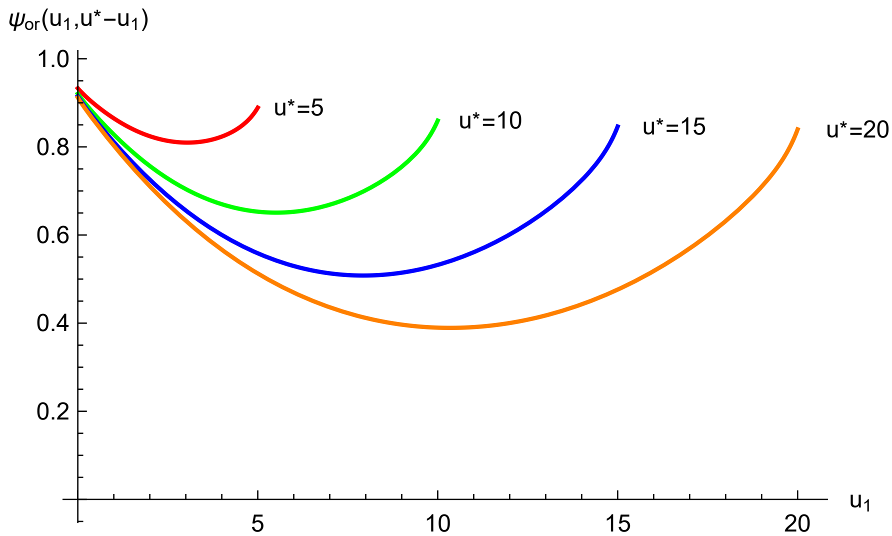

Figure 2: Plot of $\psi_{\text{or}}(u_1, u^* - u_1)$ against u_1 for $\omega = -1$ and $\lambda_{12} = 1$

	$u^* = 5$	$u^* = 10$	$u^* = 15$	$u^* = 20$
$\omega = -1$	3.044 (60.89%) **0.809724**	5.513 (55.13%) **0.650779**	7.928 (52.85%) **0.508052**	10.340 (51.70%) **0.389289**
$\omega = 0$	3.024 (60.48%) **0.794778**	5.480 (54.80%) **0.635503**	7.890 (52.60%) **0.494907**	10.301 (51.50%) **0.378926**
$\omega = 1$	3.002 (60.03%) **0.778604**	5.446 (54.46%) **0.619197**	7.850 (52.33%) **0.480796**	10.257 (51.28%) **0.367646**

Table 12: Optimal u_1, ratio of optimal u_1 to u^* (in parentheses) and minimized $\psi_{\text{or}}(u_1, u^* - u_1)$ (in boldface) when $\lambda_{12} = 1$

Finally, we consider the situation where $\lambda_{12} = 1$ (instead of $\lambda_{12} = 0.2$) so that common shocks occur more frequently while keeping all other assumptions on the claims processes unchanged. To maintain the loading factors for the two lines at $\theta_1 = 0.1$ and $\theta_2 = 0.2$, the premium rates are updated to $c_1 = 1.32$ and $c_2 = 1.44$. We found that the scaling $s/s_1^* = s/s_2^* = 1$ applied for the case $\lambda_{12} = 0.2$ remains a good choice (because λ_{12} is related to time unit but not monetary unit), and again the use of truncation points $(M^*, N^*) = (50, 50)$ yields excellent performance. When $\omega = -1$, the results are provided in Figure 2 and Table 12 in the same manner as in Figure 1 and Table 11. For each ω and u^*, we observe that the optimal values of u_1 in Tables 11 and 12 are very close but the minimized values of $\psi_{\text{or}}(u_1, u^* - u_1)$ are significantly higher in Table 12. Clearly, the same logic used to explain the allocation of capital when $\lambda_{12} = 0.2$ is still applicable to the current case of $\lambda_{12} = 1$. To explain the latter observation, it is important to note that a higher λ_{12} leads to a higher expected aggregate claim per unit time for both business lines. Although the premium rates in the present case have been increased accordingly compared to the case of $\lambda_{12} = 0.2$, the same amount of capital u^* available for allocation becomes less sufficient to prevent ruin.

Remark 2 Instead of working with $\psi_{\text{or}}(u_1, u^* - u_1)$, one may attempt to minimize $\psi_{\text{and}}(u_1, u^* - u_1)$ with respect to u_1 for $0 \leq u_1 \leq u^*$ as far as capital allocation is concerned. However, it is found that $\psi_{\text{and}}(u_1, u^* - u_1)$ is typically minimized when u_1 is close to zero or u^*, i.e. it is optimal to allocate almost all available capital to one of the two lines. This is because ψ_{and} only takes into account sample paths where both business lines ruin, and consequently it is sufficient to ensure that one of the lines survives with high probability if we would like to minimize ψ_{and}, and to do so we simply invest almost everything in one line. This can also be seen from the obvious upper bound $\psi_{\text{and}}(u_1, u_2) \leq \min(\psi_1(u_1), \psi_2(u_2))$. $\square$

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A Appendix

A.1 Proof of Lemma 3: Bounds for $|\frac{\partial^{i+j}}{\partial u_1^i \partial u_2^j}\gamma(u_1,u_2)|$ when $i,j \in \mathbb{N}_0$

We start by rewriting (3.3) as

$$\begin{aligned}\gamma(u_1,u_2) =& -\lambda_1\psi_2(u_2)\overline{F}_1(u_1) - \lambda_2\psi_1(u_1)\overline{F}_2(u_2) + \lambda_{12}\int_0^{u_1}\psi_1(u_1-y_1)\left(\frac{\partial}{\partial y_1}\overline{G}_{12}(y_1,u_2)\right)dy_1 \\ &+ \lambda_{12}\int_0^{u_2}\psi_2(u_2-y_2)\left(\frac{\partial}{\partial y_2}\overline{G}_{12}(u_1,y_2)\right)dy_2 - \lambda_{12}\overline{G}_{12}(u_1,u_2).\end{aligned}$$

Then for $i,j \in \mathbb{N}_0$ after some algebra one finds

$$\begin{aligned}&\frac{\partial^{i+j}}{\partial u_1^i \partial u_2^j}\gamma(u_1,u_2) \\ &= -\lambda_1\psi_2^{(j)}(u_2)\overline{F}_1^{(i)}(u_1) - \lambda_2\psi_1^{(i)}(u_1)\overline{F}_2^{(j)}(u_2) + \lambda_{12}\sum_{k=0}^{i-1}\psi_1^{(k)}(u_1)\left(\frac{\partial^{i-k+j}}{\partial y_1^{i-k}\partial u_2^j}\overline{G}_{12}(y_1,u_2)\Big|_{y_1=0}\right) \\ &+ \lambda_{12}\int_0^{u_1}\psi_1(u_1-y_1)\left(\frac{\partial^{i+1+j}}{\partial y_1^{i+1}\partial u_2^j}\overline{G}_{12}(y_1,u_2)\right)dy_1 + \lambda_{12}\sum_{l=0}^{j-1}\psi_2^{(l)}(u_2)\left(\frac{\partial^{i+j-l}}{\partial u_1^i\partial y_2^{j-l}}\overline{G}_{12}(u_1,y_2)\Big|_{y_2=0}\right) \\ &+ \lambda_{12}\int_0^{u_2}\psi_2(u_2-y_2)\left(\frac{\partial^{i+j+1}}{\partial u_1^i\partial y_2^{j+1}}\overline{G}_{12}(u_1,y_2)\right)dy_2 - \lambda_{12}\frac{\partial^{i+j}}{\partial u_1^i\partial u_2^j}\overline{G}_{12}(u_1,u_2). \qquad \text{(A.1)}\end{aligned}$$

To bound the above expression, apart from the bounds (3.1), (3.5) and (3.6) from Assumptions 1-3, we also need bounds on the derivatives of the univariate ruin probabilities ψ_1 and ψ_2. To this end, Cheung and Zhang (2021, Lemma 4.1) is applicable if the derivatives of the densities h_1 and h_2 defined via (1.1) can be bounded by functions that decay exponentially. This can be readily verified to be true under Assumptions 2 and 3. For example, for h_1 one has that, for any $k \in \mathbb{N}_0$,

$$h_1^{(k)}(x) = -\frac{\lambda_1}{\lambda_1+\lambda_{12}}\overline{F}_1^{(k+1)}(x) - \frac{\lambda_{12}}{\lambda_1+\lambda_{12}}\frac{\partial^{k+1}}{\partial x^{k+1}}\overline{G}_{12}(x,0).$$

The use of (3.5) and (3.6) leads to the upper bound

$$|h_1^{(k)}(x)| \leq \left(\frac{\lambda_1}{\lambda_1+\lambda_{12}}A_{1,k+1} + \frac{\lambda_{12}}{\lambda_1+\lambda_{12}}B_{k+1,0}\right)e^{-[\min(\alpha_1,\beta_1)]x}.$$

Similar bounds are also available for $h_2^{(k)}$. Consequently, from Cheung and Zhang (2021, Lemma 4.1) we have for $i=1,2$ and any $k \in \mathbb{N}_0$ that

$$|\psi_i^{(k)}(u)| \leq C_{ik}(u+1)e^{-R_i^* u}, \qquad u \geq 0, \tag{A.2}$$

where $C_{ik}>0$ is a constant and we can take $R_i^* = \min(R_i, \alpha_i, \beta_i)$. Utilizing (3.1), (3.5), (3.6) and (A.2), we can now upper bound (A.1) by

$$\begin{aligned}
&\left|\frac{\partial^{i+j}}{\partial u_1^i \partial u_2^j}\gamma(u_1,u_2)\right| \\
&\leq \lambda_1 C_{2j}(u_2+1)e^{-R_2^* u_2}A_{1i}e^{-\alpha_1 u_1} + \lambda_2 C_{1i}(u_1+1)e^{-R_1^* u_1}A_{2j}e^{-\alpha_2 u_2} + \lambda_{12}\sum_{k=0}^{i-1} C_{1k}(u_1+1)e^{-R_1^* u_1}B_{i-k,j}e^{-\beta_2 u_2} \\
&\quad + \lambda_{12}\int_0^{u_1} e^{-R_1(u_1-y_1)}B_{i+1,j}e^{-\beta_1 y_1-\beta_2 u_2}\,dy_1 + \lambda_{12}\sum_{l=0}^{j-1} C_{2l}(u_2+1)e^{-R_2^* u_2}B_{i,j-l}e^{-\beta_1 u_1} \\
&\quad + \lambda_{12}\int_0^{u_2} e^{-R_2(u_2-y_2)}B_{i,j+1}e^{-\beta_1 u_1-\beta_2 y_2}\,dy_2 + \lambda_{12}B_{ij}e^{-\beta_1 u_1-\beta_2 u_2}.
\end{aligned} \tag{A.3}$$

Note that the first integral term can be upper bounded by

$$\begin{aligned}
\int_0^{u_1} e^{-R_1(u_1-y_1)}B_{i+1,j}e^{-\beta_1 y_1-\beta_2 u_2}\,dy_1 &\leq B_{i+1,j}\int_0^{u_1} e^{-[\min(R_1,\beta_1)](u_1-y_1)}e^{-[\min(R_1,\beta_1)]y_1-\beta_2 u_2}\,dy_1 \\
&= B_{i+1,j}u_1 e^{-[\min(R_1,\beta_1)]u_1-\beta_2 u_2},
\end{aligned}$$

and similarly for the second integral term one has

$$\int_0^{u_2} e^{-R_2(u_2-y_2)}B_{i,j+1}e^{-\beta_1 u_1-\beta_2 y_2}\,dy_2 \leq B_{i,j+1}u_2 e^{-\beta_1 u_1-[\min(R_2,\beta_2)]u_2}.$$

By incorporating these two inequalities into (A.3) and recalling that $R_k^* = \min(R_k,\alpha_k,\beta_k)$ for $k=1,2$, one asserts that (3.7) holds true, where one can take

$$K_{ij} = \lambda_1 C_{2j}A_{1i} + \lambda_2 C_{1i}A_{2j} + \lambda_{12}\sum_{k=0}^{i-1} C_{1k}B_{i-k,j} + \lambda_{12}B_{i+1,j} + \lambda_{12}\sum_{l=0}^{j-1} C_{2l}B_{i,j-l} + \lambda_{12}B_{i,j+1} + \lambda_{12}B_{ij}.$$

Note that (3.7) implies that γ is a Schwartz function according to Definition 2(b).

A.2 Proof of Lemma 4: ψ_{and} is a Schwartz function

Part 1: Bounds for $|\frac{\partial^i}{\partial u_1^i}\psi_{\text{and}}(u_1,u_2)|$ when $i \in \mathbb{N}_0$

It is instructive to note that the result of Lemma 2 implies that (3.8) holds true when $i = j = 0$. We start the proof by following essentially the same analysis as in Lemma 1, but instead of considering a small time interval we condition on the first time t when a claim event occurs. This leads to

$$\begin{aligned}\psi_{\text{and}}(u_1,u_2) = &\int_0^\infty \lambda_1 e^{-(\lambda_1+\lambda_2+\lambda_{12})t} \int_0^{u_1+c_1t} \psi_{\text{and}}(u_1+c_1t-y_1, u_2+c_2t) f_1(y_1)\, dy_1\, dt \\ &+ \int_0^\infty \lambda_2 e^{-(\lambda_1+\lambda_2+\lambda_{12})t} \int_0^{u_2+c_2t} \psi_{\text{and}}(u_1+c_1t, u_2+c_2t-y_2) f_2(y_2)\, dy_2\, dt \\ &+ \int_0^\infty \lambda_{12} e^{-(\lambda_1+\lambda_2+\lambda_{12})t} \int_0^{u_1+c_1t}\int_0^{u_2+c_2t} \psi_{\text{and}}(u_1+c_1t-y_1, u_2+c_2t-y_2) g_{12}(y_1,y_2)\, dy_2\, dy_1\, dt \\ &- \int_0^\infty e^{-(\lambda_1+\lambda_2+\lambda_{12})t}\gamma(u_1+c_1t, u_2+c_2t)\, dt. \qquad \text{(A.4)}\end{aligned}$$

Our goal here is to express the higher order derivatives in terms of lower order ones by differentiating the above equation with respect to u_1. By a change of variable, we first rewrite the second integral in (A.4) as

$$\begin{aligned}&\int_0^\infty \lambda_2 e^{-(\lambda_1+\lambda_2+\lambda_{12})t} \int_0^{u_2+c_2t} \psi_{\text{and}}(u_1+c_1t, u_2+c_2t-y_2) f_2(y_2)\, dy_2\, dt \\ &= \frac{\lambda_2}{c_1}\int_{u_1}^\infty e^{-\frac{\lambda_1+\lambda_2+\lambda_{12}}{c_1}(x-u_1)} \int_0^{u_2+\frac{c_2}{c_1}(x-u_1)} \psi_{\text{and}}(x,y_2) f_2\left(u_2+\frac{c_2}{c_1}(x-u_1)-y_2\right) dy_2\, dx.\end{aligned}$$

Its first derivative is thus

$$\begin{aligned}&\frac{\partial}{\partial u_1}\int_0^\infty \lambda_2 e^{-(\lambda_1+\lambda_2+\lambda_{12})t} \int_0^{u_2+c_2t} \psi_{\text{and}}(u_1+c_1t, y_2) f_2(u_2+c_2t-y_2)\, dy_2\, dt \\ &= -\frac{\lambda_2}{c_1}\int_0^{u_2} \psi_{\text{and}}(u_1,y_2) f_2(u_2-y_2)\, dy_2 \\ &\quad + \frac{\lambda_2}{c_1}\int_{u_1}^\infty \frac{\lambda_1+\lambda_2+\lambda_{12}}{c_1} e^{-\frac{\lambda_1+\lambda_2+\lambda_{12}}{c_1}(x-u_1)} \int_0^{u_2+\frac{c_2}{c_1}(x-u_1)} \psi_{\text{and}}(x,y_2) f_2\left(u_2+\frac{c_2}{c_1}(x-u_1)-y_2\right) dy_2\, dx \\ &\quad + \frac{\lambda_2}{c_1}\int_{u_1}^\infty e^{-\frac{\lambda_1+\lambda_2+\lambda_{12}}{c_1}(x-u_1)} \left(-\frac{c_2}{c_1}\right)\Bigg[\psi_{\text{and}}\left(x, u_2+\frac{c_2}{c_1}(x-u_1)\right) f_2(0) \\ &\qquad\qquad + \int_0^{u_2+\frac{c_2}{c_1}(x-u_1)} \psi_{\text{and}}(x,y_2) f_2'\left(u_2+\frac{c_2}{c_1}(x-u_1)-y_2\right) dy_2\Bigg] dx \\ &= -\frac{\lambda_2}{c_1}\int_0^{u_2} \psi_{\text{and}}(u_1,y_2) f_2(u_2-y_2)\, dy_2 \\ &\quad + \frac{\lambda_1+\lambda_2+\lambda_{12}}{c_1}\int_0^\infty \lambda_2 e^{-(\lambda_1+\lambda_2+\lambda_{12})t} \int_0^{u_2+c_2t} \psi_{\text{and}}(u_1+c_1t, u_2+c_2t-y_2) f_2(y_2)\, dy_2\, dt \\ &\quad - \frac{c_2}{c_1}\int_0^\infty \lambda_2 e^{-(\lambda_1+\lambda_2+\lambda_{12})t}\left(\psi_{\text{and}}(u_1+c_1t, u_2+c_2t) f_2(0) + \int_0^{u_2+c_2t} \psi_{\text{and}}(u_1+c_1t, u_2+c_2t-y_2) f_2'(y_2)\, dy_2\right) dt.\end{aligned}$$

Using the above result, differentiating (A.4) leads to

$$\frac{\partial}{\partial u_1}\psi_{\text{and}}(u_1,u_2)$$

$$
\begin{aligned}
&= \int_0^\infty \lambda_1 e^{-(\lambda_1+\lambda_2+\lambda_{12})t} \left(\psi_{\text{and}}(u_1+c_1t, u_2+c_2t) f_1(0) + \int_0^{u_1+c_1t} \psi_{\text{and}}(u_1+c_1t-y_1, u_2+c_2t) f_1'(y_1)\, dy_1 \right) dt \\
&\quad - \frac{\lambda_2}{c_1} \int_0^{u_2} \psi_{\text{and}}(u_1, y_2) f_2(u_2-y_2)\, dy_2 \\
&\quad + \frac{\lambda_1+\lambda_2+\lambda_{12}}{c_1} \int_0^\infty \lambda_2 e^{-(\lambda_1+\lambda_2+\lambda_{12})t} \int_0^{u_2+c_2t} \psi_{\text{and}}(u_1+c_1t, u_2+c_2t-y_2) f_2(y_2)\, dy_2\, dt \\
&\quad - \frac{c_2}{c_1} \int_0^\infty \lambda_2 e^{-(\lambda_1+\lambda_2+\lambda_{12})t} \left(\psi_{\text{and}}(u_1+c_1t, u_2+c_2t) f_2(0) + \int_0^{u_2+c_2t} \psi_{\text{and}}(u_1+c_1t, u_2+c_2t-y_2) f_2'(y_2)\, dy_2 \right) dt \\
&\quad + \int_0^\infty \lambda_{12} e^{-(\lambda_1+\lambda_2+\lambda_{12})t} \Bigg[\int_0^{u_2+c_2t} \psi_{\text{and}}(u_1+c_1t, u_2+c_2t-y_2) g_{12}(0, y_2)\, dy_2 \\
&\qquad\qquad + \int_0^{u_1+c_1t} \int_0^{u_2+c_2t} \psi_{\text{and}}(u_1+c_1t-y_1, u_2+c_2t-y_2) \left(\frac{\partial}{\partial y_1} g_{12}(y_1, y_2) \right) dy_2\, dy_1 \Bigg] dt \\
&\quad - \int_0^\infty e^{-(\lambda_1+\lambda_2+\lambda_{12})t} \left(\frac{\partial}{\partial u_1} \gamma(u_1+c_1t, u_2+c_2t) \right) dt. \qquad \text{(A.5)}
\end{aligned}
$$

Because $f_i(x_i) = -\overline{F}_i(x_i)$ and $g_{12}(y_1, y_2) = \frac{\partial^2}{\partial y_1 \partial y_2} \overline{G}_{12}(y_1, y_2)$, the right-hand side of the above equation only consists of functions whose bounds are available from (3.4)-(3.7). It is observed that certain types of integrals appear repeatedly, and we would like to bound these as, for $k \in \mathbb{N}_0$,

$$
\begin{aligned}
&\left| \int_0^\infty e^{-(\lambda_1+\lambda_2+\lambda_{12})t} \int_0^{u_1+c_1t} \psi_{\text{and}}(u_1+c_1t-y_1, u_2+c_2t) f_1^{(k)}(y_1)\, dy_1\, dt \right| \\
&\le \int_0^\infty e^{-(\lambda_1+\lambda_2+\lambda_{12})t} \int_0^{u_1+c_1t} e^{-r_1(u_1+c_1t-y_1)-r_2(u_2+c_2t)} A_{1,k+1} e^{-\alpha_1 y_1}\, dy_1\, dt \\
&\le A_{1,k+1} \int_0^\infty e^{-(\lambda_1+\lambda_2+\lambda_{12})t-r_2(u_2+c_2t)} \int_0^{u_1+c_1t} e^{-[\min(r_1,\alpha_1)](u_1+c_1t-y_1)} e^{-[\min(r_1,\alpha_1)]y_1}\, dy_1\, dt \\
&= A_{1,k+1} \int_0^\infty e^{-(\lambda_1+\lambda_2+\lambda_{12})t-[\min(r_1,\alpha_1)](u_1+c_1t)-r_2(u_2+c_2t)} (u_1+c_1t)\, dt \\
&\le D_{1k}(u_1+1) e^{-[\min(r_1,\alpha_1)]u_1-r_2u_2}, \qquad \text{(A.6)}
\end{aligned}
$$

for some obvious choice of $D_{1k} > 0$. Similarly, one has for $k \in \mathbb{N}_0$ that

$$
\left| \int_0^\infty e^{-(\lambda_1+\lambda_2+\lambda_{12})t} \int_0^{u_2+c_2t} \psi_{\text{and}}(u_1+c_1t, u_2+c_2t-y_2) f_2^{(k)}(y_2)\, dy_2\, dt \right| \le D_{2k}(u_2+1) e^{-r_1u_1-[\min(r_2,\alpha_2)]u_2}, \qquad \text{(A.7)}
$$

for some $D_{2k} > 0$. In addition, we also have, for $i \in \mathbb{N}_0$,

$$
\begin{aligned}
&\left| \int_0^\infty e^{-(\lambda_1+\lambda_2+\lambda_{12})t} \int_0^{u_1+c_1t} \int_0^{u_2+c_2t} \psi_{\text{and}}(u_1+c_1t-y_1, u_2+c_2t-y_2) \left(\frac{\partial^i}{\partial y_1^i} g_{12}(y_1, y_2) \right) dy_2\, dy_1\, dt \right| \\
&\le \int_0^\infty e^{-(\lambda_1+\lambda_2+\lambda_{12})t} \int_0^{u_1+c_1t} \int_0^{u_2+c_2t} e^{-r_1(u_1+c_1t-y_1)-r_2(u_2+c_2t-y_2)} B_{i+1,1} e^{-\beta_1y_1-\beta_2y_2}\, dy_2\, dy_1\, dt \\
&\le B_{i+1,1} \int_0^\infty e^{-(\lambda_1+\lambda_2+\lambda_{12})t} \int_0^{u_1+c_1t} \int_0^{u_2+c_2t} e^{-[\min(r_1,\beta_1)](u_1+c_1t-y_1)-[\min(r_2,\beta_2)](u_2+c_2t-y_2)} \\
&\qquad\qquad \times e^{-[\min(r_1,\beta_1)]y_1-[\min(r_2,\beta_2)]y_2}\, dy_2\, dy_1\, dt \\
&= B_{i+1,1} \int_0^\infty e^{-(\lambda_1+\lambda_2+\lambda_{12})t-[\min(r_1,\beta_1)](u_1+c_1t)-[\min(r_1,\beta_1)](u_2+c_2t)} (u_1+c_1t)(u_2+c_2t)\, dt
\end{aligned}
$$

$$\leq E_i(u_1+1)(u_2+1)e^{-[\min(r_1,\beta_1)]u_1-[\min(r_2,\beta_2)]u_2}, \tag{A.8}$$

and

$$\begin{aligned}
&\left|\int_0^\infty e^{-(\lambda_1+\lambda_2+\lambda_{12})t}\left(\frac{\partial^i}{\partial u_1^i}\gamma(u_1+c_1t,u_2+c_2t)\right)dt\right| \\
&\leq \int_0^\infty e^{-(\lambda_1+\lambda_2+\lambda_{12})t}K_{i0}(u_1+c_1t+u_2+c_2t+1)e^{-R_1^*(u_1+c_1t)-R_2^*(u_2+c_2t)}\,dt \\
&= K_i^*(u_1+u_2+1)e^{-R_1^*u_1-R_2^*u_2},
\end{aligned} \tag{A.9}$$

for some $E_i, K_i^* > 0$. With the help of (3.4) and (A.6)-(A.9), we can upper bound (A.5) as

$$\begin{aligned}
&\left|\frac{\partial}{\partial u_1}\psi_{\text{and}}(u_1,u_2)\right| \\
&\leq \int_0^\infty \lambda_1 e^{-(\lambda_1+\lambda_2+\lambda_{12})t}e^{-r_1(u_1+c_1t)-r_2(u_2+c_2t)}A_{11}\,dt + \lambda_1 D_{11}(u_1+1)e^{-[\min(r_1,\alpha_1)]u_1-r_2u_2} \\
&\quad + \frac{\lambda_2}{c_1}\int_0^{u_2} e^{-r_1u_1-r_2y_2}A_{21}e^{-\alpha_2(u_2-y_2)}\,dy_2 + \frac{\lambda_1+\lambda_2+\lambda_{12}}{c_1}\lambda_2 D_{20}(u_2+1)e^{-r_1u_1-[\min(r_2,\alpha_2)]u_2} \\
&\quad + \frac{c_2}{c_1}\int_0^\infty \lambda_2 e^{-(\lambda_1+\lambda_2+\lambda_{12})t}e^{-r_1(u_1+c_1t)-r_2(u_2+c_2t)}A_{21}\,dt + \frac{c_2}{c_1}\lambda_2 D_{21}(u_2+1)e^{-r_1u_1-[\min(r_2,\alpha_2)]u_2} \\
&\quad + \int_0^\infty \lambda_{12}e^{-(\lambda_1+\lambda_2+\lambda_{12})t}\int_0^{u_2+c_2t} e^{-r_1(u_1+c_1t)-r_2(u_2+c_2t-y_2)}B_{11}e^{-\beta_2y_2}\,dy_2\,dt \\
&\quad + \lambda_{12}E_1(u_1+1)(u_2+1)e^{-[\min(r_1,\beta_1)]u_1-[\min(r_2,\beta_2)]u_2} + K_1^*(u_1+u_2+1)e^{-R_1^*u_1-R_2^*u_2} \\
&\leq A_{11}e^{-r_1u_1-r_2u_2}\int_0^\infty \lambda_1 e^{-(\lambda_1+\lambda_2+\lambda_{12}+r_1c_1+r_2c_2)t}\,dt + \lambda_1 D_{11}(u_1+1)e^{-[\min(r_1,\alpha_1)]u_1-r_2u_2} \\
&\quad + \frac{\lambda_2}{c_1}A_{21}u_2e^{-r_1u_1-[\min(r_2,\alpha_2)]u_2} + \frac{\lambda_1+\lambda_2+\lambda_{12}}{c_1}\lambda_2 D_{20}(u_2+1)e^{-r_1u_1-[\min(r_2,\alpha_2)]u_2} \\
&\quad + \frac{c_2}{c_1}A_{21}e^{-r_1u_1-r_2u_2}\int_0^\infty \lambda_2 e^{-(\lambda_1+\lambda_2+\lambda_{12}+r_1c_1+r_2c_2)t}\,dt + \frac{c_2}{c_1}\lambda_2 D_{21}(u_2+1)e^{-r_1u_1-[\min(r_2,\alpha_2)]u_2} \\
&\quad + \lambda_{12}B_{11}\int_0^\infty e^{-(\lambda_1+\lambda_2+\lambda_{12})t-r_1(u_1+c_1t)-[\min(r_2,\beta_2)](u_2+c_2t)}(u_2+c_2t)\,dt \\
&\quad + \lambda_{12}E_1(u_1+1)(u_2+1)e^{-[\min(r_1,\beta_1)]u_1-[\min(r_2,\beta_2)]u_2} + K_1^*(u_1+u_2+1)e^{-R_1^*u_1-R_2^*u_2} \\
&\leq H_{10}(u_1+1)(u_2+1)e^{-R_1^{**}u_1-R_2^{**}u_2},
\end{aligned}$$

for some $H_{10} > 0$, where $R_k^{**} = \min(R_k^*, r_k, \alpha_k, \beta_k) = \min(r_k, \alpha_k, \beta_k)$ for $k = 1, 2$ by recalling that $r_1 = r_2 = \min(R_1, R_2)/2$ from Lemma 2 and $R_k^* = \min(R_k, \alpha_k, \beta_k)$ for $k = 1, 2$ from Lemma 3. Thus (3.8) is valid when $i = 1$ and $j = 0$. Now we would like to show by induction that (3.8) (with $j = 0$ fixed) holds true for $i = I + 1$ by assuming that (3.8) is true for $i = 0, 1, \ldots, I$ for some $I > 0$. Now, we apply the operator $\frac{\partial^I}{\partial u_1^I}$ to (A.5) to get

$$\begin{aligned}
&\frac{\partial^{I+1}}{\partial u_1^{I+1}}\psi_{\text{and}}(u_1,u_2) \\
&= \int_0^\infty \lambda_1 e^{-(\lambda_1+\lambda_2+\lambda_{12})t}\left[\sum_{k=0}^{I}\left(\frac{\partial^k}{\partial u_1^k}\psi_{\text{and}}(u_1+c_1t,u_2+c_2t)\right)f_1^{(I-k)}(0)\right.
\end{aligned}$$

$$
+ \int_0^{u_1+c_1 t} \psi_{\text{and}}(u_1+c_1t-y_1, u_2+c_2t) f_1^{(I+1)}(y_1)\, dy_1 \Big] \, dt
$$

$$
- \frac{\lambda_2}{c_1} \int_0^{u_2} \left(\frac{\partial^I}{\partial u_1^I} \psi_{\text{and}}(u_1, y_2) \right) f_2(u_2 - y_2)\, dy_2
$$

$$
+ \frac{\lambda_1+\lambda_2+\lambda_{12}}{c_1} \int_0^\infty \lambda_2 e^{-(\lambda_1+\lambda_2+\lambda_{12})t} \int_0^{u_2+c_2t} \left(\frac{\partial^I}{\partial u_1^I} \psi_{\text{and}}(u_1+c_1t, u_2+c_2t-y_2) \right) f_2(y_2)\, dy_2\, dt
$$

$$
- \frac{c_2}{c_1} \int_0^\infty \lambda_2 e^{-(\lambda_1+\lambda_2+\lambda_{12})t} \Bigg[\left(\frac{\partial^I}{\partial u_1^I} \psi_{\text{and}}(u_1+c_1t, u_2+c_2t) \right) f_2(0)
$$

$$
+ \int_0^{u_2+c_2t} \left(\frac{\partial^I}{\partial u_1^I} \psi_{\text{and}}(u_1+c_1t, u_2+c_2t-y_2) \right) f_2'(y_2)\, dy_2 \Bigg] \, dt
$$

$$
+ \int_0^\infty \lambda_{12} e^{-(\lambda_1+\lambda_2+\lambda_{12})t} \Bigg[\sum_{k=0}^{I} \int_0^{u_2+c_2t} \left(\frac{\partial^k}{\partial u_1^k} \psi_{\text{and}}(u_1+c_1t, u_2+c_2t-y_2) \right) \left(\frac{\partial^{I-k}}{\partial y_1^{I-k}} g_{12}(y_1, y_2) \Big|_{y_1=0} \right) dy_2
$$

$$
+ \int_0^{u_1+c_1t} \int_0^{u_2+c_2t} \psi_{\text{and}}(u_1+c_1t-y_1, u_2+c_2t-y_2) \left(\frac{\partial^{I+1}}{\partial y_1^{I+1}} g_{12}(y_1, y_2) \right) dy_2\, dy_1 \Bigg] \, dt
$$

$$
- \int_0^\infty e^{-(\lambda_1+\lambda_2+\lambda_{12})t} \left(\frac{\partial^{I+1}}{\partial u_1^{I+1}} \gamma(u_1+c_1t, u_2+c_2t) \right) dt.
$$

Utilizing the induction assumption along with (3.5), (A.6), (A.8) and (A.9) gives rise to

$$
\left| \frac{\partial^{I+1}}{\partial u_1^{I+1}} \psi_{\text{and}}(u_1, u_2) \right|
$$

$$
\leq \int_0^\infty \lambda_1 e^{-(\lambda_1+\lambda_2+\lambda_{12})t} \sum_{k=0}^{I} H_{k0}(u_1+c_1t+1)^k (u_2+c_2t+1)^k e^{-R_1^{**}(u_1+c_1t)-R_2^{**}(u_2+c_2t)} A_{1,I-k+1}\, dt
$$

$$
+ \lambda_1 D_{1,I+1}(u_1+1) e^{-[\min(r_1,\alpha_1)]u_1 - r_2 u_2} + \frac{\lambda_2}{c_1} \int_0^{u_2} H_{I0}(u_1+1)^I (y_2+1)^I e^{-R_1^{**}u_1 - R_2^{**}y_2} A_{21} e^{-\alpha_2(u_2-y_2)}\, dy_2
$$

$$
+ \frac{\lambda_1+\lambda_2+\lambda_{12}}{c_1} \int_0^\infty \lambda_2 e^{-(\lambda_1+\lambda_2+\lambda_{12})t}
$$

$$
\times \int_0^{u_2+c_2t} H_{I0}(u_1+c_1t+1)^I (u_2+c_2t-y_2+1)^I e^{-R_1^{**}(u_1+c_1t)-R_2^{**}(u_2+c_2t-y_2)} A_{21} e^{-\alpha_2 y_2}\, dy_2\, dt
$$

$$
+ \frac{c_2}{c_1} \int_0^\infty \lambda_2 e^{-(\lambda_1+\lambda_2+\lambda_{12})t} \Bigg(H_{I0}(u_1+c_1t+1)^I (u_2+c_2t+1)^I e^{-R_1^{**}(u_1+c_1t)-R_2^{**}(u_2+c_2t)} A_{21}
$$

$$
+ \int_0^{u_2+c_2t} H_{I0}(u_1+c_1t+1)^I (u_2+c_2t-y_2+1)^I e^{-R_1^{**}(u_1+c_1t)-R_2^{**}(u_2+c_2t-y_2)} A_{22} e^{-\alpha_2 y_2}\, dy_2 \Bigg) dt
$$

$$
+ \int_0^\infty \lambda_{12} e^{-(\lambda_1+\lambda_2+\lambda_{12})t}
$$

$$
\times \sum_{k=0}^{I} \int_0^{u_2+c_2t} H_{k0}(u_1+c_1t+1)^k (u_2+c_2t-y_2+1)^k e^{-R_1^{**}(u_1+c_1t)-R_2^{**}(u_2+c_2t-y_2)} B_{I-k+1,1} e^{-\beta_2 y_2}\, dy_2\, dt
$$

$$
+ \lambda_{12} E_{I+1}(u_1+1)(u_2+1) e^{-[\min(r_1,\beta_1)]u_1 - [\min(r_2,\beta_2)]u_2} + K_{I+1}^*(u_1+u_2+1) e^{-R_1^* u_1 - R_2^* u_2}. \tag{A.10}
$$

Note that integrals of similar form appear repeatedly and these can be dealt with as follows. First, for

$k = 0, 1, \ldots, I$ one has

$$
\begin{aligned}
&\int_0^\infty e^{-(\lambda_1+\lambda_2+\lambda_{12})t}(u_1+c_1t+1)^k(u_2+c_2t+1)^k e^{-R_1^{**}(u_1+c_1t)-R_2^{**}(u_2+c_2t)}\,dt \\
&\le e^{-R_1^{**}u_1-R_2^{**}u_2}\int_0^\infty e^{-(\lambda_1+\lambda_2+\lambda_{12})t}(u_1+c_1t+1)^I(u_2+c_2t+1)^I e^{-R_1^{**}c_1t-R_2^{**}c_2t}\,dt \\
&= e^{-R_1^{**}u_1-R_2^{**}u_2}\int_0^\infty e^{-(\lambda_1+\lambda_2+\lambda_{12})t}\left[\sum_{m=0}^{I}\binom{I}{m}(u_1+1)^m(c_1t)^{I-m}\right]\left[\sum_{n=0}^{I}\binom{I}{n}(u_2+1)^n(c_2t)^{I-n}\right]e^{-R_1^{**}c_1t-R_2^{**}c_2t}\,dt \\
&\le L(u_1+1)^I(u_2+1)^I e^{-R_1^{**}u_1-R_2^{**}u_2}, \qquad \text{(A.11)}
\end{aligned}
$$

for some $L > 0$. Second, we recall $R_2^{**} = \min(r_2, \alpha_2, \beta_2)$ and find that

$$
\begin{aligned}
\int_0^{u_2}(u_1+1)^I(y_2+1)^I e^{-R_1^{**}u_1-R_2^{**}y_2}e^{-\alpha_2(u_2-y_2)}\,dy_2 &\le (u_1+1)^I(u_2+1)^I e^{-R_1^{**}u_1}\int_0^{u_2}e^{-R_2^{**}y_2}e^{-R_2^{**}(u_2-y_2)}\,dy_2 \\
&= (u_1+1)^I(u_2+1)^I e^{-R_1^{**}u_1-R_2^{**}u_2}u_2 \\
&\le (u_1+1)^I(u_2+1)^{I+1}e^{-R_1^{**}u_1-R_2^{**}u_2}. \qquad \text{(A.12)}
\end{aligned}
$$

Third, letting a below be either α_2 or β_2, similar to the above inequalities we look at, for $k = 0, 1, \ldots, I$,

$$
\begin{aligned}
&\int_0^{u_2+c_2t}(u_1+c_1t+1)^k(u_2+c_2t-y_2+1)^k e^{-R_1^{**}(u_1+c_1t)-R_2^{**}(u_2+c_2t-y_2)}e^{-ay_2}\,dy_2 \\
&\le (u_1+c_1t+1)^I(u_2+c_2t+1)^{I+1}e^{-R_1^{**}(u_1+c_1t)-R_2^{**}(u_2+c_2t)}, \qquad \text{(A.13)}
\end{aligned}
$$

and therefore

$$
\begin{aligned}
&\int_0^\infty e^{-(\lambda_1+\lambda_2+\lambda_{12})t}\int_0^{u_2+c_2t}(u_1+c_1t+1)^k(u_2+c_2t-y_2+1)^k e^{-R_1^{**}(u_1+c_1t)-R_2^{**}(u_2+c_2t-y_2)}e^{-ay_2}\,dy_2\,dt \\
&\le L_a^*(u_1+1)^I(u_2+1)^{I+1}e^{-R_1^{**}u_1-R_2^{**}u_2}, \qquad \text{(A.14)}
\end{aligned}
$$

for some $L_a^* > 0$. We can now incorporate (A.11)-(A.14) into (A.10) to confirm that (3.8) is true for $i = I + 1$ (with $j = 0$ fixed), and the induction on i is complete.

Part 2: Bounds for $|\frac{\partial^{i+j}}{\partial u_1^i \partial u_2^j}\psi_{\text{and}}(u_1,u_2)|$ when $j \ge 1$

We shall now derive bounds for $|\frac{\partial^{i+j}}{\partial u_1^i \partial u_2^j}\psi_{\text{and}}(u_1,u_2)|$ when $i \in \mathbb{N}_0$ and $j \ge 1$. To this end, we can make use of the PIDE (3.2). In particular, applying the operator $\frac{\partial^{i+j-1}}{\partial u_1^i \partial u_2^{j-1}}$ to (3.2) followed by rearrangements yields

$$
\begin{aligned}
&\frac{\partial^{i+j}}{\partial u_1^i \partial u_2^j}\psi_{\text{and}}(u_1,u_2) \\
&= -\frac{c_1}{c_2}\frac{\partial^{i+j}}{\partial u_1^{i+1}\partial u_2^{j-1}}\psi_{\text{and}}(u_1,u_2) + \frac{\lambda_1+\lambda_2+\lambda_{12}}{c_2}\frac{\partial^{i+j-1}}{\partial u_1^i \partial u_2^{j-1}}\psi_{\text{and}}(u_1,u_2) \\
&\quad - \frac{\lambda_1}{c_2}\left[\sum_{k=0}^{i-1}\left(\frac{\partial^{k+j-1}}{\partial u_1^k \partial u_2^{j-1}}\psi_{\text{and}}(u_1,u_2)\right)f_1^{(i-1-k)}(0) + \int_0^{u_1}\left(\frac{\partial^{j-1}}{\partial u_2^{j-1}}\psi_{\text{and}}(u_1-y_1,u_2)\right)f_1^{(i)}(y_1)\,dy_1\right] \\
&\quad - \frac{\lambda_2}{c_2}\left[\sum_{l=0}^{j-2}\left(\frac{\partial^{i+l}}{\partial u_1^i \partial u_2^l}\psi_{\text{and}}(u_1,u_2)\right)f_2^{(j-2-l)}(0) + \int_0^{u_2}\left(\frac{\partial^{i}}{\partial u_1^i}\psi_{\text{and}}(u_1,u_2-y_2)\right)f_2^{(j-1)}(y_2)\,dy_2\right]
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\lambda_{12}}{c_2}\left[\sum_{k=0}^{i-1}\sum_{l=0}^{j-2}\left(\frac{\partial^{k+l}}{\partial u_1^k \partial u_2^l}\psi_{\text{and}}(u_1,u_2)\right)\left(\frac{\partial^{i-1-k+j-2-l}}{\partial y_1^{i-1-k}\partial y_2^{j-2-l}}g_{12}(y_1,y_2)\bigg|_{y_1,y_2=0}\right)\right.\\
&\qquad + \sum_{k=0}^{i-1}\int_0^{u_2}\left(\frac{\partial^k}{\partial u_1^k}\psi_{\text{and}}(u_1,u_2-y_2)\right)\left(\frac{\partial^{i-1-k+j-1}}{\partial y_1^{i-1-k}\partial y_2^{j-1}}g_{12}(y_1,y_2)\bigg|_{y_1=0}\right)dy_2\\
&\qquad + \sum_{l=0}^{j-2}\int_0^{u_1}\left(\frac{\partial^l}{\partial u_2^l}\psi_{\text{and}}(u_1-y_1,u_2)\right)\left(\frac{\partial^{i+j-2-l}}{\partial y_1^{i}\partial y_2^{j-2-l}}g_{12}(y_1,y_2)\bigg|_{y_2=0}\right)dy_1\\
&\qquad \left. + \int_0^{u_1}\int_0^{u_2}\psi_{\text{and}}(u_1-y_1,u_2-y_2)\left(\frac{\partial^{i+j-1}}{\partial y_1^{i}\partial y_2^{j-1}}g_{12}(y_1,y_2)\right)dy_2\,dy_1\right] + \frac{1}{c_2}\frac{\partial^{i+j-1}}{\partial u_1^i\partial u_2^{j-1}}\gamma(u_1,u_2).
\end{aligned}
\tag{A.15}
$$

From the above equation, one observes that bounds involving the j-th derivatives of $\psi_{\text{and}}(u_1,u_2)$ with respect to u_2 are obtainable from those for the $(j-1)$-th derivatives. As a result, inductively (on j) we can deduce that bounds for the derivatives of $\psi_{\text{and}}(u_1,u_2)$ with respect to u_1 only, which are available from Part 1, will suffice. Starting with the case $j=1$, (A.15) implies that, for all $i \in \mathbb{N}_0$,

$$
\begin{aligned}
&\left|\frac{\partial^{i+1}}{\partial u_1^i \partial u_2}\psi_{\text{and}}(u_1,u_2)\right|\\
&\le \frac{c_1}{c_2}\left|\frac{\partial^{i+1}}{\partial u_1^{i+1}}\psi_{\text{and}}(u_1,u_2)\right| + \frac{\lambda_1+\lambda_2+\lambda_{12}}{c_2}\left|\frac{\partial^{i}}{\partial u_1^{i}}\psi_{\text{and}}(u_1,u_2)\right|\\
&\quad + \frac{\lambda_1}{c_2}\left(\sum_{k=0}^{i-1}\left|\frac{\partial^k}{\partial u_1^k}\psi_{\text{and}}(u_1,u_2)\right||f_1^{(i-1-k)}(0)| + \int_0^{u_1}\psi_{\text{and}}(u_1-y_1,u_2)|f_1^{(i)}(y_1)|\,dy_1\right)\\
&\quad + \frac{\lambda_2}{c_2}\int_0^{u_2}\left|\frac{\partial^i}{\partial u_1^i}\psi_{\text{and}}(u_1,u_2-y_2)\right|f_2(y_2)\,dy_2 + \frac{\lambda_{12}}{c_2}\left(\sum_{k=0}^{i-1}\int_0^{u_2}\left|\frac{\partial^k}{\partial u_1^k}\psi_{\text{and}}(u_1,u_2-y_2)\right|\left|\frac{\partial^{i-1-k}}{\partial y_1^{i-1-k}}g_{12}(y_1,y_2)\bigg|_{y_1=0}\right|dy_2\right.\\
&\qquad \left. + \int_0^{u_1}\int_0^{u_2}\psi_{\text{and}}(u_1-y_1,u_2-y_2)\left|\frac{\partial^i}{\partial y_1^i}g_{12}(y_1,y_2)\right|dy_2\,dy_1\right) + \frac{1}{c_2}\left|\frac{\partial^i}{\partial u_1^i}\gamma(u_1,u_2)\right|.
\end{aligned}
\tag{A.16}
$$

The integrals appearing above can be bounded as follows. First, following the steps in obtaining (A.6), it is found that

$$
\int_0^{u_1}\psi_{\text{and}}(u_1-y_1,u_2)|f_1^{(i)}(y_1)|\,dy_1 \le A_{1,i+1}u_1 e^{-[\min(r_1,\alpha_1)]u_1 - r_2u_2}.
$$

Second, similar to (A.13) one has

$$
\int_0^{u_2}\left|\frac{\partial^i}{\partial u_1^i}\psi_{\text{and}}(u_1,u_2-y_2)\right|f_2(y_2)\,dy_2 \le H_{i0}A_{21}(u_1+1)^i(u_2+1)^{i+1}e^{-R_1^{**}u_1-R_2^{**}u_2},
$$

and, for $k=0,1,\ldots,i-1$ (which is an empty set when $i=0$),

$$
\int_0^{u_2}\left|\frac{\partial^k}{\partial u_1^k}\psi_{\text{and}}(u_1,u_2-y_2)\right|\left|\frac{\partial^{i-1-k}}{\partial y_1^{i-1-k}}g_{12}(y_1,y_2)\bigg|_{y_1=0}\right|dy_2 \le H_{k0}B_{i-k,1}(u_1+1)^{i-1}(u_2+1)^{i}e^{-R_1^{**}u_1-R_2^{**}u_2}.
$$

Third, from the steps of (A.8) we have

$$
\int_0^{u_1}\int_0^{u_2}\psi_{\text{and}}(u_1-y_1,u_2-y_2)\left|\frac{\partial^i}{\partial y_1^i}g_{12}(y_1,y_2)\right|dy_2\,dy_1 \le B_{i+1,1}u_1u_2e^{-[\min(r_1,\beta_1)]u_1-[\min(r_2,\beta_2)]u_2}.
$$

Consequently, with the further help of (3.8) at $j = 0$ (proved in Part 1) and (3.7), it is confirmed from (A.16) that (3.8) is valid for all $i \in \mathbb{N}_0$ and $j = 1$. In the inductive step, it is assumed that, for some $J > 0$, (3.8) is true for all $i \in \mathbb{N}_0$ and $j = J$. Then one can use (A.15) at $j = J + 1$ to show that (3.8) is true for all $i \in \mathbb{N}_0$ and $j = J + 1$ to complete the induction on j. Since the algebra involved is rather repetitive, the details are omitted.

Having shown that (3.8) is valid for $i, j \in \mathbb{N}_0$, it is obvious that ψ_{and} is a Schwartz function according to Definition 2(b).