EXTREMES OF VECTOR-VALUED GAUSSIAN PROCESSES: EXACT ASYMPTOTICS

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Abstract: Let $\{X_i(t), t \ge 0\}, 1 \le i \le n$ be mutually independent centered Gaussian processes with almost surely continuous sample paths. We derive the exact asymptotics of

$$\mathbb{P}\left(\exists_{t\in[0,T]}\forall_{i=1,\dots,n}X_i(t) > u\right)$$

as $u \to \infty$, for both locally stationary X_i's and X_i's with a non-constant generalized variance function. Additionally, we analyze properties of multidimensional counterparts of the Pickands and Piterbarg constants, that appear in the derived asymptotics. Important by-products of this contribution are the vector-process extensions of the Piterbarg inequality, the Borell-TIS inequality, the Slepian lemma and the Pickands-Piterbarg lemma which are the main pillars of the extremal theory of vector-valued Gaussian processes.

Key Words: Gaussian process; conjunction; extremes; double-sum method; Slepian lemma; Borell-TIS inequality; Piterbarg inequality; generalized Pickands constant; generalized Piterbarg constant; Pickands-Piterbarg lemma.

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1. Introduction

Consider a vector-valued Gaussian process $\{\mathbf{X}(t), t \ge 0\}$, where $\mathbf{X}(t) = (X_1(t), \dots, X_n(t))$ with $\{X_i(t), t \ge 0\}, 1 \le i \le n$, $n \in \mathbb{N}$, being independent centered Gaussian processes with almost surely (a.s.) continuous sample paths. In this paper we focus on the asymptotic behaviour of the probability that $\mathbf{X}$ enters the upper orthant $\{(x_1, \dots, x_n) : x_i > u, i \in \{1, \dots, n\}\}$ over a fixed time interval $[0, T]$, i.e.,

$$\mathbb{P}\left(\exists_{t\in[0,T]}\forall_{i=1,\dots,n}X_i(t) > u\right) \tag{1}$$

as $u \to \infty$.

One of important motivations to analyze (1) is its connection with the *conjunction* problem for Gaussian processes. The set of conjunctions $C_{T,u}$ on the fixed time interval $[0, T]$ with respect to some threshold u is defined as

$$C_{T,u} := \{t \in [0, T] : \min_{1\le i\le n} X_i(t) > u\}$$

see e.g., the seminal contribution [32]. One of the key properties of $C_{T,u}$, that recently focused substantial attention, is the probability that $C_{T,u}$ is non-empty

$$p_{T,u} := \mathbb{P}\left(C_{T,u} \ne \phi\right) = \mathbb{P}\left(\sup_{t\in[0,T]} \min_{1\le i\le n} X_i(t) > u\right). \tag{2}$$

Clearly, $p_{T,u}$ is equivalent to (1), implying that one can view at (1) as at the probability of extremal behaviour of the process $\{\min_{1\le i\le n} X_i(t), t \ge 0\}$. Typically, in applications such as the analysis of functional magnetic resonance imaging (fMRI) data, X_i's are assumed to be real-valued Gaussian random fields. We refer to, e.g., [2, 7, 32], for approximations of $p_{T,u}$ in the case of smooth Gaussian random fields. Results for non-Gaussian random fields and general stationary processes can be found in [3, 12].

In the special case when $n = 1$, then (1) reduces to the the tail asymptotics of supremum of a centered Gaussian process. One of the techniques that was found to be particularly successful in finding exact asymptotic behaviour of supremas of Gaussian processes is the *double-sum method*. This method was originally introduced for the stationary case in seminal papers of J. Pickands III [27, 28]. Later, it was extended to non-stationary Gaussian processes (and

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fields) including locally stationary Gaussian process and Gaussian process with a non-constant variance function. For a complete survey on related results we refer to [29, 30].

The main goal of this contribution is to derive exact asymptotics of (1) for large classes of non-stationary Gaussian processes X_i's, providing multidimensional counterparts of the seminal Pickands' and Piterbarg-Prishyaznyuk's results, respectively; see e.g., Theorem D2 and Theorem D3 in [29]. The proofs of our main results are based on an extension of the double-sum technique applied to the analysis of (1). Remarkably, the relation between (1) and (2) also implies the applicability of the double-sum method to non-Gaussian processes, as, e.g., the process $\{\min_{1\le i\le n} X_i(t), t\ge 0\}$. Interestingly, in the obtained asymptotics, there appear multidimensional counterparts of the classical Pickands and Piterbarg constants (see Sections 2 and 3). We analyze properties of these new constants in Section 3.

In the literature there are few results on extremes of non-smooth vector-valued Gaussian processes; see [4, 15, 22, 34] and the references therein. In Section 5 we shall present some extensions (tailored for our use) of the Slepian lemma, the Borell-TIS inequality and the Piterbarg inequality for vector-valued Gaussian random fields. These results are of independent interest given their crucial role in the theory of Gaussian processes and random fields; see e.g., [1, 8, 26, 29] and the references therein.

The organization of the paper: Basic notation and some preliminary results are presented in Section 2. In Section 3 we analyze properties of vector-valued Pickands and Piterbarg constants. The main results of the paper, concerning the asymptotics of (1) for both locally stationary X_i's and X_i's with a non-constant generalized variance function, are displayed in Section 4. All the proofs are relegated to Section 5.

2. Notation and preliminaries

We shall use some standard notation which is common when dealing with vectors. All the operations on vectors are meant componentwise, for instance, for any given $\boldsymbol{x}=(x_1,\dots,x_n)\in\mathbb{R}^n$ and $\boldsymbol{y}=(y_1,\dots,y_n)\in\mathbb{R}^n$, we write $\boldsymbol{x}>\boldsymbol{y}$ if and only if $x_i>y_i$ for all $1\le i\le n$, write $1/\boldsymbol{x}=(1/x_1,\dots,1/x_n)$ if $x_i\neq 0, 1\le i\le n$, and write $\boldsymbol{xy}=(x_1y_1,\dots,x_ny_n)$. Further we set $\boldsymbol{0}:=(0,\dots,0)\in\mathbb{R}^n$ and $\boldsymbol{1}:=(1,\dots,1)\in\mathbb{R}^n$.

We use the notation $f(u)=h(u)(1+o(1))$ if $\lim_{u\to\infty}\frac{f(u)}{h(u)}=1$ and write $f(u)=o(h(u))$ if $\lim_{u\to\infty}\frac{f(u)}{h(u)}=0$. By $\Psi(\cdot)$ we denote the survival function of an $N(0,1)$ random variable, and $\Gamma(\cdot)$ denotes the Euler Gamma function.

We shall refer to $\{\boldsymbol{X}(t), t\ge 0\}$ as a centered n-dimensional *vector-valued* Gaussian process, where $\boldsymbol{X}(t)=(X_1(t),\dots,X_n(t))$ with X_i's being independent centered Gaussian processes with a.s. continuous sample paths. Since n hereafter is always fixed we shall occasionally omit "n-dimensional", mentioning simply that $\boldsymbol{X}$ is a centered vector-valued Gaussian process. Define next

$$\sigma^2_{\boldsymbol{X}}(\cdot)=(\sigma^2_{X_1}(\cdot),\dots,\sigma^2_{X_n}(\cdot)),\quad R_{\boldsymbol{X}}(\cdot,\cdot)=(R_{X_1}(\cdot,\cdot),\dots,R_{X_n}(\cdot,\cdot)),$$

with $\sigma^2_{X_i}(t)=\mathrm{Var}(X_i(t))$ and $R_{X_i}(s,t)=\mathrm{Cov}(X_i(s),X_i(t))$.

Let in the following $\{B_{i,\kappa}(t), t\in\mathbb{R}\}, 1\le i\le n$ be n mutually independent standard fractional Brownian motions (fBm's) defined on $\mathbb{R}$ with common Hurst index $\kappa/2\in(0,1]$, and set $\boldsymbol{B}_\kappa(t)=(B_{1,\kappa}(t),\dots,B_{n,\kappa}(t))$.

A key step in the investigation of the tail asymptotics of supremum of Gaussian processes is the derivation of the tail asymptotic behaviour of the supremum taken over "short intervals". For the stationary case this is achieved by the so-called Pickands lemma. The non-stationary case is covered by the so-called Piterbarg lemma (see [10, 11, 23] for similar terminology and related results). Before deriving an extension of these classical results for the vector-valued Gaussian processes, we need to introduce some further notation.

Let $\{Y(t), t\in\mathbb{R}\}$ be a centered Gaussian process with a.s. continuous sample paths such that $Y(0)=0$ a.s., and let $d\colon\mathbb{R}\to\mathbb{R}$ be a continuous function such that $d(0)=0$. Further, denote S_1,S_2 to be two non-negative constants satisfying $\max(S_1,S_2)>0$.

Let $\{X_u(t), t\in[-S_1,S_2]\}, u>0$ be a family of centered Gaussian processes with a.s. continuous sample paths that satisfies

P1: $\sigma^2_{X_u}(0)=1$ for all u large and $\lim_{u\to\infty}u^2(1-\sigma_{X_u}(t))=d(t)$ uniformly with respect to $t\in[-S_1,S_2]$;

P2: $\lim_{u\to\infty} u^2\mathrm{Var}(X_u(t)-X_u(s)) = 2\mathrm{Var}(Y(t)-Y(s))$ for all $t,s\in[-S_1,S_2]$;

P3: there exist $G, u_0>0$ and $\gamma\in(0,2]$ such that $u^2\mathrm{Var}(X_u(t)-X_u(s))\le G|t-s|^\gamma$ holds for all $u\ge u_0$ and $s,t\in[-S_1,S_2]$.

We write $X_u\in\mathcal{P}(Y,d)$ if $\{X_u\}_{u>0}$ satisfies **P1-P3**.

Introduce next some further notation which is related to vector version of the Pickands and Piterbarg constants. Consider $\{\boldsymbol{Y}(t), t\in\mathbb{R}\}$, with $\boldsymbol{Y}(t)=(Y_1(t),\ldots,Y_n(t))$, where $\{Y_i(t), t\in\mathbb{R}\}$ are mutually independent Gaussian processes with a.s. continuous sample paths such that $Y_i(0)=0$ a.s., and let $\boldsymbol{d}(t)=(d_1(t),\ldots,d_n(t))$ with $d_i(\cdot)$ being continuous functions such that $d_i(0)=0$. We define

$$\begin{aligned}\mathcal{H}_{\boldsymbol{Y},\boldsymbol{d}}[-S_1,S_2] &:= \int_{\mathbb{R}^n} e^{\sum_{i=1}^n w_i}\mathbb{P}\Big(\exists_{t\in[-S_1,S_2]}\ \sqrt{2}\boldsymbol{Y}(t)-\sigma^2_{\boldsymbol{Y}}(t)-\boldsymbol{d}(t)>\boldsymbol{w}\Big)d\boldsymbol{w}\\ &= \int_{\mathbb{R}^n} e^{\sum_{i=1}^n w_i}\mathbb{P}\left(\sup_{t\in[-S_1,S_2]}\min_{1\le i\le n}\Big(\sqrt{2}Y_i(t)-\sigma^2_{Y_i}(t)-d_i(t)-w_i\Big)>0\right)d\boldsymbol{w}\in(0,\infty).\end{aligned}$$

In the special case of $\boldsymbol{Y}(t)=\boldsymbol{B}_\kappa(t)$ being an n-dimensional vector-valued fBm process with independent coordinates we set

$$\mathcal{H}_{\boldsymbol{B}_\kappa}(S_2) := \mathcal{H}_{\boldsymbol{B}_\kappa,\boldsymbol{0}}[0,S_2].$$

The above defined constants play significant role in the following multidimensional extension of the Pickands-Piterbarg lemma (compare with, e.g., [10, 14, 29]).

Proposition 2.1. *Let $\{\boldsymbol{X}_u(t), t\in[-S_1,S_2]\}, u>0$ be a family of centered vector-valued Gaussian process with independent coordinates $X_{i,u}\in\mathcal{P}(Y_i,d_i)$ for some $Y_i, d_i, 1\le i\le n$. If $\boldsymbol{f}(\cdot)$ is an n-dimensional vector function such that $\lim_{u\to\infty}\boldsymbol{f}(u)/u=\boldsymbol{c}>\boldsymbol{0}$, then as $u\to\infty$*

$$\mathbb{P}\Big(\exists_{t\in[-S_1,S_2]}\ \boldsymbol{X}_u(t)>\boldsymbol{f}(u)\Big)=\mathcal{H}_{\boldsymbol{c}\boldsymbol{Y},\boldsymbol{c}^2\boldsymbol{d}}[-S_1,S_2]\prod_{i=1}^n\Psi(f_i(u))(1+o(1)).$$

The proof of Proposition 2.1 is given in Section 5.1.

Let $\boldsymbol{X}$ be a centered vector-valued Gaussian processes with independent coordinates X_i's which are stationary Gaussian processes with unit variance and correlation functions $r_i(\cdot), 1\le i\le n$ satisfying

$$r_i(t)=1-a_i|t|^{\kappa_i}+o(|t|^{\kappa_i})\quad t\to0,\quad\text{and } r_i(t)<1,\ \forall t\neq0, \tag{3}$$

where $\kappa_i\in(0,2]$, $a_i>0$, $1\le i\le n$. Let $\kappa=\min_{1\le i\le n}\kappa_i$, and denote $\boldsymbol{a}=(a_1 1_{\{\kappa_1=\kappa\}},\ldots,a_n 1_{\{\kappa_n=\kappa\}})$ with $1_{\{\cdot\}}$ denoting the indicator function. Hereafter we write $\boldsymbol{X}\in\mathcal{S}(\boldsymbol{a},\kappa)$ if (3) is satisfied by the vector-valued Gaussian process $\boldsymbol{X}$.

As a straightforward implication of Proposition 2.1 we obtain the following corollary.

Corollary 2.2. *Consider a centered vector-valued stationary Gaussian process $\boldsymbol{X}\in\mathcal{S}(\boldsymbol{a},\kappa)$. For any $\beta_i\ge\kappa$ and $\boldsymbol{b}(t)$ such that $b_i(t)=\underline{b}_i|t|^{\beta_i}1_{\{t\le0\}}+\overline{b}_i|t|^{\beta_i}1_{\{t>0\}}$ $1\le i\le n$, define $Z_i(t)=\frac{X_i(t)}{1+b_i(t)}, t\in\mathbb{R}, 1\le i\le n$. If $\boldsymbol{f}(\cdot)$ is an n-dimensional vector function such that $\lim_{u\to\infty}\boldsymbol{f}(u)/u=\boldsymbol{c}>\boldsymbol{0}$, then for any non-negative constants S_1,S_2 satisfying* $\max(S_1,S_2)>0$

$$\mathbb{P}\Big(\exists_{t\in[-S_1u^{-2/\kappa},S_2u^{-2/\kappa}]}\ \boldsymbol{Z}(t)>\boldsymbol{f}(u)\Big)=\mathcal{H}_{\boldsymbol{c}\sqrt{\boldsymbol{a}}\boldsymbol{B}_\kappa,\boldsymbol{c}^2\boldsymbol{d}}[-S_1,S_2]\prod_{i=1}^n\Psi(f_i(u))(1+o(1)) \tag{4}$$

holds as $u\to\infty$, where $\boldsymbol{d}(t)=(d_1(t),\ldots,d_n(t))$ with $d_i(t)=b_i(t)1_{\{\beta_i=\kappa\}}$.

Next we introduce multidimensional counterparts of the Pickands constant, defined as

$$\mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa} := \lim_{S\to\infty}S^{-1}\mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa}(S) \tag{5}$$

for $\kappa \in (0,2]$ and $\boldsymbol{C} \ge \mathbf{0}$, $\boldsymbol{C} \neq \mathbf{0}$. Note that if $n=1$ and $C_1 \neq 0$, then $\mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa} = C_1^{2/\kappa}\mathcal{H}_\kappa^*$, where

$$\mathcal{H}_\kappa^* = \lim_{T\to\infty} T^{-1}\mathbb{E}\left(\exp\left(\sup_{t\in[0,T]}(\sqrt{2}B_{1,\kappa}(t) - t^\kappa)\right)\right)$$

is the classical Pickands constant; see e.g., [29] and the recent contributions [19, 20, 33]. The existence and finiteness of $\mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa}$ follow by Fekete's Lemma, since by Lemma 3.1 displayed in Section 3, $\mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa}(S)$ is sub-additive. Furthermore, Proposition 3.2 below shows that $\mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa} > 0$.

Finally we introduce multidimensional counterparts of Piterbarg constants. For $\kappa \in (0,2]$ let $\underline{\boldsymbol{d}} = (\underline{d}_1, \ldots, \underline{d}_n)$, $\overline{\boldsymbol{d}} = (\overline{d}_1, \ldots, \overline{d}_n)$ be such that $\sum_{i=1}^n \underline{d}_i > 0$ and $\sum_{i=1}^n \overline{d}_i > 0$, and let $\boldsymbol{d}(t) = (d_1(t), \ldots, d_n(t))$ with $d_i(t) = \underline{d}_i |t|^\kappa 1_{\{t\le 0\}} + \overline{d}_i |t|^\kappa 1_{\{t>0\}}$. We define, for $\boldsymbol{C} \ge 0$, $\boldsymbol{C} \neq \mathbf{0}$,

$$\begin{aligned}
\mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa}^{\underline{\boldsymbol{d}}} &:= \lim_{S\to\infty} \mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa,\boldsymbol{d}}[-S,0] \\
\mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa}^{\overline{\boldsymbol{d}}} &:= \lim_{S\to\infty} \mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa,\boldsymbol{d}}[0,S] \\
\mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa}^{\underline{\boldsymbol{d}},\overline{\boldsymbol{d}}} &:= \lim_{S\to\infty} \mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa,\boldsymbol{d}}[-S,S].
\end{aligned}$$

In Theorem 4.3 we shall prove that the above generalized Piterbarg constants exist and are both positive and finite.

3. Estimates of the generalized Pickands and Piterbarg constants

In this section we provide some estimates of the above defined multidimensional counterparts of Pickands and Piterbarg constants. We begin with the subadditivity property of $\mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa}(S)$.

Lemma 3.1. *Let $\kappa \in (0,2]$ and $\boldsymbol{C} \ge \mathbf{0}$, $\boldsymbol{C} \neq \mathbf{0}$. Then for all $S \in \mathbb{N}$*

$$\mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa}(S) \;\le\; S\ \mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa}(1) \in (0,\infty). \tag{6}$$

The proof of Lemma 3.1 is given in Section 5.2.

Clearly, from the subadditivity of $\mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa}(\cdot)$ we obtain that $\mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa}$ exists and is finite. In the next proposition we confirm that $\mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa}$ is strictly positive by establishing a positive lower bound.

Proposition 3.2. *If $\kappa \in (0,2]$ and $\boldsymbol{C} \ge \mathbf{0}$, $\boldsymbol{C} \neq \mathbf{0}$, then*

$$\mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa} \ge \frac{\left(\sum_{i=1}^n C_i^2\right)^{1/\kappa}}{4^{1+1/\kappa}\Gamma(1/\kappa+1)}.$$

Proposition 3.3. *For each $n \in \mathbb{N}$ we have*

$$\mathcal{H}_{\boldsymbol{B}_1} \;\le\; n\left(\frac{n}{n-1}\left(2+\sqrt{\frac{2}{\pi e}}\right)\right)^{n-1} \qquad \text{and} \qquad \mathcal{H}_{\boldsymbol{B}_2} \;\le\; n\left(\frac{n}{n-1}\right)^{n-1}\frac{1}{\sqrt{\pi}},$$

where $n/(n-1)$ is set to be 1 for $n=1$.

We conclude this section with lower bounds for the generalized Piterbarg constants $\mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa}^{\overline{\boldsymbol{d}}}, \mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa}^{\underline{\boldsymbol{d}},\overline{\boldsymbol{d}}}$.

Proposition 3.4. *For any $\kappa \in (0,2]$, $\boldsymbol{C} \ge \mathbf{0}, \boldsymbol{C} \neq \mathbf{0}$ and $\underline{\boldsymbol{d}}, \overline{\boldsymbol{d}}$ satisfying $\sum_{i=1}^n \underline{d}_i > 0, \sum_{i=1}^n \overline{d}_i > 0$ we have*

$$\mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa}^{\overline{\boldsymbol{d}}} \ge \left(e\kappa \sum_{i=1}^n \max(0,\overline{d}_i)\right)^{-1/\kappa} \mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa}$$

and

$$\mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa}^{\underline{\boldsymbol{d}},\overline{\boldsymbol{d}}} \ge\; 2\,(e\kappa)^{-1/\kappa}\left(\sum_{i=1}^n \left(\max(0,\underline{d}_i) + \max(0,\overline{d}_i)\right)\right)^{-1/\kappa} \mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa}.$$

We note that the lower bounds above are new even for the case $n=1$.

4. Main Results

In this section we derive the asymptotics of (1) for $\boldsymbol{X}$ with locally stationary coordinates (see e.g., [5, 6, 24, 29] for locally stationary Gaussian processes) in Theorem 4.1 and for a large class of $\boldsymbol{X}$ with a non-constant generalized variance function in Theorem 4.3. These results provide multidimensional counterparts of Pickands theorem and Piterbarg-Prishyaznyuk theorem.

4.1. **Locally stationary coordinates.** Let $\{\boldsymbol{X}(t), t \in [0,T]\}$ be a centered vector-valued Gaussian process with independent coordinates X_i's which are locally stationary Gaussian processes with a.s. continuous sample paths, unit variance and correlation functions $r_i(\cdot,\cdot), 1 \le i \le n$ satisfying

$$r_i(t, t+h) = 1 - a_i(t)\,|h|^{\kappa_i} + o(|h|^{\kappa_i}), \quad h \to 0 \tag{7}$$

uniformly with respect to $t \in [0,T]$, where $\kappa_i \in (0,2]$, and $a_i(t), 1 \le i \le n$ are positive continuous functions on $[0,T]$, and further

$$r_i(s,t) < 1, \ \forall s,t \in [0,T] \text{ and } s \neq t. \tag{8}$$

Let in the following $\boldsymbol{a}(t) = (a_1(t)1_{\{\kappa_1=\kappa\}}, ..., a_n(t)1_{\{\kappa_n=\kappa\}}), t \in [0,T]$. Recall that we set $\kappa = \min_{1\le i\le n} \kappa_i$.
Note that $\boldsymbol{X} \in \mathcal{S}(\boldsymbol{a}, \kappa)$ is a particular example of the above defined vector-valued Gaussian processes.

Theorem 4.1. *Let $\boldsymbol{X}$ be a centered vector-valued Gaussian process with independent locally stationary coordinates satisfying (7) and (8). If $\boldsymbol{f}(\cdot)$ is an n-dimensional vector function such that $\lim_{u\to\infty} \boldsymbol{f}(u)/u = \boldsymbol{c} > \boldsymbol{0}$, then*

$$\mathbb{P}\left(\exists_{t\in[0,T]}\ \boldsymbol{X}(t) > \boldsymbol{f}(u)\right) \ = \ \int_0^T \mathcal{H}_{\boldsymbol{c}\sqrt{\boldsymbol{a}(t)}\boldsymbol{B}_\kappa} dt\ u^{\frac{2}{\kappa}} \prod_{i=1}^n \Psi(f_i(u))(1+o(1)), \quad u \to \infty. \tag{9}$$

The special case of Theorem 4.1 for $\boldsymbol{X} \in \mathcal{S}(\boldsymbol{a}, \kappa)$ has been derived in [9]. A straightforward comparison of Theorem 4.1 with Theorem 1.1 in [9] implies the following proposition.

Proposition 4.2. *If $\boldsymbol{C} \ge \boldsymbol{0}$, $\boldsymbol{C} \neq \boldsymbol{0}$ and $\kappa \in (0,2]$, then*

$$\mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa} = \lim_{u\downarrow 0} \frac{1}{u}\mathbb{P}\left(\max_{k\ge 1} \mathcal{Z}_{\boldsymbol{C}\boldsymbol{B}_\kappa}(uk) \le 0\right) \in (0,\infty),$$

with

$$\mathcal{Z}_{\boldsymbol{C}\boldsymbol{B}_\kappa}(t) := \min_{1\le i\le n}\left(\sqrt{2}C_i B_{i,\kappa}(t) - C_i^2 t^\kappa + E_i\right), \quad t \ge 0,$$

where E_i's are mutually independent unit mean exponential random variables being further independent of $B_{i,\kappa}$'s.

4.2. **General non-stationary coordinates.** Let $\{\boldsymbol{X}(t), t \in [0,T]\}$ be a centered vector-valued non-stationary Gaussian process with a non-constant generalized variance function. The following set of conditions constitutes a vector-valued counterpart of Piterbarg-type conditions on X_i's (see e.g., [29] for the original Piterbarg's conditions imposed on Gaussian processes or fields with a non-constant variance function):
Assumption I: The following *generalized variance* function

$$g(t) = \sum_{i=1}^n \frac{1}{\sigma^2_{X_i}(t)}$$

attains its minimum over $[0,T]$ at the unique point $t = t_0 \in [0,T]$.
Assumption II: There exist $\alpha_i \in (0,2]$, $a_i > 0$, $1 \le i \le n$ such that

$$\mathrm{Cov}\left(\frac{X_i(t)}{\sigma_{X_i}(t)}, \frac{X_i(s)}{\sigma_{X_i}(s)}\right) = 1 - a_i\,|t-s|^{\alpha_i} - o(|t-s|^{\alpha_i}), \quad 1 \le i \le n$$

holds as $t, s \to t_0$.
Assumption III: There exist some $\beta > 0$, $\underline{\boldsymbol{b}} = (\underline{b}_1, \dots, \underline{b}_n)$ and $\overline{\boldsymbol{b}} = (\overline{b}_1, \dots, \overline{b}_n)$ such that

$$1 - \frac{\sigma_{X_i}(t_0+t)}{\sigma_{X_i}(t_0)} = \underline{b}_i\,|t|^\beta\, 1_{\{t\le 0\}} + \overline{b}_i\,|t|^\beta\, 1_{\{t>0\}} + o(|t|^\beta), \quad 1 \le i \le n$$

holds as $t \to 0$.

Note in passing that Assumption III implies that

$$g(t_0+t)-g(t_0) = 2\left(\underline{\theta} 1_{\{t\le 0\}} + \overline{\theta} 1_{\{t>0\}}\right)|t|^{\beta} + o(|t|^{\beta}), \quad t \to 0, \tag{10}$$

which combined with Assumption I implies

$$\underline{\theta} := \sum_{i=1}^{n} \frac{\underline{b}_i}{\sigma^2_{X_i}(t_0)} \ge 0, \quad \overline{\theta} := \sum_{i=1}^{n} \frac{\overline{b}_i}{\sigma^2_{X_i}(t_0)} \ge 0.$$

Assumption IV: There exist some positive constants G, γ and ρ such that

$$\max_{1\le i\le n} \mathbb{E}\left((X_i(t)-X_i(s))^2\right) \le G\,|t-s|^{\gamma}$$

holds for all $s,t \in (t_0-\rho, t_0+\rho)\cap[0,T]$.

Theorem 4.3. *Let $\boldsymbol{X}$ be a centered vector-valued Gaussian process that satisfies Assumptions I–IV with the parameters therein. Denote $\alpha = \min_{1\le i\le n}\alpha_i$, $\boldsymbol{a} = (a_1 1_{\{\alpha_1=\alpha\}}, \ldots, a_n 1_{\{\alpha_n=\alpha\}})$, and let $\boldsymbol{c}=(c_1,\ldots,c_n)$ with $c_i = \frac{1}{\sigma_{X_i}(t_0)}, 1\le i\le n$. Suppose that $\underline{\theta}>0$ and $\overline{\theta}>0$.*

i) If $\alpha<\beta$, then

$$\mathbb{P}\left(\sup_{t\in[0,T]}\min_{1\le i\le n} X_i(t) > u\right) = \mathcal{H}_{\boldsymbol{c}\sqrt{\boldsymbol{a}}\boldsymbol{B}_\alpha}\Theta\,\Gamma\left(\frac{1}{\beta}+1\right)u^{\frac{2}{\alpha}-\frac{2}{\beta}}\prod_{i=1}^{n}\Psi(c_i u)(1+o(1)), \quad u\to\infty,$$

where

$$\Theta = \begin{cases} \overline{\theta}^{-\frac{1}{\beta}}, & t_0 = 0 \\ \underline{\theta}^{-\frac{1}{\beta}} + \overline{\theta}^{-\frac{1}{\beta}}, & t_0\in(0,T) \\ \underline{\theta}^{-\frac{1}{\beta}}, & t_0 = T. \end{cases}$$

ii) If $\alpha=\beta$, then

$$\mathbb{P}\left(\sup_{t\in[0,T]}\min_{1\le i\le n} X_i(t) > u\right) = \widehat{\mathcal{H}}\prod_{i=1}^{n}\Psi(c_i u)(1+o(1)), \quad u\to\infty,$$

where

$$\widehat{\mathcal{H}} = \begin{cases} \mathcal{H}^{\boldsymbol{c}^2\overline{\boldsymbol{b}}}_{\boldsymbol{c}\sqrt{\boldsymbol{a}}\boldsymbol{B}_\alpha}, & t_0=0 \\ \mathcal{H}^{\boldsymbol{c}^2\underline{\boldsymbol{b}},\boldsymbol{c}^2\overline{\boldsymbol{b}}}_{\boldsymbol{c}\sqrt{\boldsymbol{a}}\boldsymbol{B}_\alpha}, & t_0\in(0,T) \\ \mathcal{H}^{\boldsymbol{c}^2\underline{\boldsymbol{b}}}_{\boldsymbol{c}\sqrt{\boldsymbol{a}}\boldsymbol{B}_\alpha}, & t_0=T. \end{cases}$$

iii) If $\alpha>\beta$, then

$$\mathbb{P}\left(\sup_{t\in[0,T]}\min_{1\le i\le n} X_i(t) > u\right) = \prod_{i=1}^{n}\Psi(c_i u)(1+o(1)), \quad u\to\infty.$$

Remarks: *a) For $n=1$, the above theorem reduces to the classical result for non-stationary Gaussian processes (see e.g.,* [29, 17]*).*

b) Let $\boldsymbol{X}$ be a centered vector-valued Gaussian process with independent coordinates X_i's which are copies of a Gaussian process X, and let $\{X_{r:n}(t), t\ge 0\}, 1\le r\le n$ be the order statistics processes of $\{X_i(t), t\ge0\}, 1\le i\le n$, i.e., we define

$$X_{1:n}(t) := \max_{1\le i\le n} X_i(t) \ge X_{2:n}(t) \ge \ldots \ge X_{n:n}(t) = \min_{1\le i\le n} X_i(t), \quad t\ge 0.$$

Under the assumptions of Theorem 4.1 or Theorem 4.3, with similar arguments as in [9] *we obtain*

$$\mathbb{P}\left(\sup_{t\in[0,T]} X_{r:n}(t) > u\right) = \frac{n!}{(n-r)!r!}\mathbb{P}\left(\sup_{t\in[0,T]}\min_{1\le i\le r} X_i(t) > u\right)(1+o(1)), \quad u\to\infty.$$

5. Proofs

Before proceeding to the proofs of Theorem 4.1 and Theorem 4.3, we present four lemmas that will play important roles in further analysis and being also of some independent interest. We begin with a vector version of the Slepian lemma, then give the vector-valued counterparts of the Borell-TIS inequality and the Piterbarg inequality, respectively. Below we write $\mathcal{T}$ for a compact set in $\mathbb{R}^k, k \ge 1$ and denote by $|x|$ the Euclidean norm of $x \in \mathbb{R}^k$.

Lemma 5.1. *(Slepian Lemma) Let $\{\boldsymbol{Y}(t), t \in \mathcal{T}\}$ and $\{\boldsymbol{Z}(t), t \in \mathcal{T}\}$ be two centered separable vector-valued Gaussian processes with independent coordinates. If for all $s, t \in \mathcal{T}$*

$$\sigma^2_{\boldsymbol{Y}}(t) = \sigma^2_{\boldsymbol{Z}}(t), \quad R_{\boldsymbol{Y}}(t,s) \ge R_{\boldsymbol{Z}}(t,s),$$

then for any $\boldsymbol{u} \in \mathbb{R}^n$ we have

$$\mathbb{P}\Big(\exists_{t\in\mathcal{T}}\ \boldsymbol{Y}(t) > \boldsymbol{u}\Big) \ \le\ \mathbb{P}\Big(\exists_{t\in\mathcal{T}}\ \boldsymbol{Z}(t) > \boldsymbol{u}\Big). \tag{11}$$

Proof: The claim for any finite set $\mathcal{T}$ follows by a direct application of Gordon's inequality (see [21]). If $\mathcal{T}$ is a given compact set of $\mathbb{R}^k$, then the proof can be easily established using standard arguments that make use of the separability assumption; see e.g., [1]. □

Set in the following $\tau^2_{\mathcal{T}} = \inf_{t\in\mathcal{T}} \sum_{i=1}^n \frac{1}{\sigma^2_{X_i}(t)}$.

Lemma 5.2. *(Borell-TIS inequality) Let $\{\boldsymbol{X}(t), t \in \mathcal{T}\}$ be a centered vector-valued Gaussian process with independent coordinates which have a.s. continuous sample paths. If $\tau_{\mathcal{T}} > 0$, then there exists some positive constant μ such that for $u > \mu$*

$$\mathbb{P}\left(\exists_{t\in\mathcal{T}}\boldsymbol{X}(t) > u\boldsymbol{1}\right) \le \exp\left(-\frac{(u-\mu)^2}{2}\tau^2_{\mathcal{T}}\right).$$

Proof: It follows that

$$\mathbb{P}\left(\exists_{t\in\mathcal{T}}\boldsymbol{X}(t) > u\boldsymbol{1}\right) \quad\le\quad \mathbb{P}\left(\sup_{t\in\mathcal{T}} Y(t) > u\right), \tag{12}$$

where (set $A(t) = \sum_{i=1}^n \prod_{j=1,j\ne i}^n \sigma^2_{Z_j}(t), t \in \mathcal{T}$)

$$Y(t) = \sum_{i=1}^n \left(\frac{\prod_{j=1,j\ne i}^n \sigma^2_{Z_j}(t)}{A(t)}\right) X_i(t), \quad t \in \mathcal{T}.$$

Since further

$$\mathrm{Var}\left(Y(t)\right) = \left(\sum_{i=1}^n \frac{1}{\sigma^2_{X_i}(t)}\right)^{-1}$$

the claim follows from the Borell-TIS inequality for one-dimensional Gaussian processes (e.g., [1]) with

$$\mu = \mathbb{E}\left(\sup_{t\in\mathcal{T}} Y(t)\right) < \infty$$

and thus the proof is complete. □

Lemma 5.3. *(Piterbarg inequality) Under the conditions of Lemma 5.2, if further Assumption IV holds, then for all u large*

$$\mathbb{P}\left(\exists_{t\in\mathcal{T}}\boldsymbol{X}(t) > u\boldsymbol{1}\right) \le C mes(\mathcal{T})\ u^{\frac{2}{\nu}-1} \exp\left(-\frac{u^2}{2}\tau^2_{\mathcal{T}}\right), \tag{13}$$

where C is some positive constant not depending on u.

Proof: We use the same notation as in the proof of Lemma 5.2. In the light of (12) and Theorem 8.1 in [29], it suffices to show that

$$\mathbb{E}\left((Y(t)-Y(s))^2\right) \le L\,|t-s|^{\nu}, \quad \forall s,t \in \mathcal{T} \tag{14}$$

holds for some positive constant L, which is a direct consequence of Assumption IV. □

The last lemma below concerns the asymptotics of a probability of double events; it is crucial when dealing with the double sum term in the proof of our main results.

Lemma 5.4. *Consider a centered vector-valued stationary Gaussian process $\boldsymbol{X} \in \mathcal{S}(\boldsymbol{a},\kappa)$. Suppose that for those X_i's with $\kappa_i = \kappa$ there exists some global constant $\varepsilon > 0$ such that*

$$1 - \frac{a_i}{2} t^{\kappa} \ge R_{X_i}(t,0) \ge 1 - 2a_i t^{\kappa}$$

holds for all $t \in [0,\varepsilon]$. Assume further that $\boldsymbol{f}(\cdot), \boldsymbol{h}(\cdot)$ are two n-dimensional vector functions such that $\lim_{u\to\infty} \boldsymbol{f}(u)/u = \boldsymbol{c}_1 > \boldsymbol{0}$ and $\lim_{u\to\infty} \boldsymbol{h}(u)/u = \boldsymbol{c}_2 > \boldsymbol{0}$. Then there exist two positive constants F, G such that for all $t_0 > S > 1$

$$\begin{aligned}
&\mathbb{P}\left(\exists_{t\in[0,S]u^{-2/\kappa}}\ \boldsymbol{X}(t) > \boldsymbol{f}(u), \exists_{t\in[t_0,t_0+S]u^{-2/\kappa}}\ \boldsymbol{X}(t) > \boldsymbol{h}(u)\right)\\
&\le \quad F S^{2n} \exp\left(-G(t_0-S)^{\kappa}\right) \prod_{i=1}^{n} \Psi\left(\frac{f_i(u)+h_i(u)}{2}\right)
\end{aligned}$$

holds for all u large.

Proof: First note that if $\kappa_i = \kappa$, in view of the proof of Lemma 6.3 in [29] (or Lemma 5 in [25]) we obtain that

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\in[0,S]u^{-2/\kappa}} X_i(t) > f_i(u), \sup_{t\in[t_0,t_0+S]u^{-2/\kappa}} X_i(t) > h_i(u)\right)\\
&\le F_i S^2 \exp\left(-G_i(t_0-S)^{\kappa}\right) \Psi\left(\frac{f_i(u)+h_i(u)}{2}\right)
\end{aligned}$$

holds with some positive constants F_i, G_i. Further, if $\kappa_i > \kappa$, then there exist some positive constant L and sufficiently small $\varepsilon_1 > 0$ such that

$$r_i(t) \ge e^{-Lt^{\kappa}}$$

is valid for all $t \in [0,\varepsilon_1]$.

Let $\{\xi(t), t \ge 0\}$ be a stationary Gaussian process with a.s. continuous sample paths and correlation function $r_{\xi}(t) = e^{-Lt^{\kappa}}, t \ge 0$. By the Slepain lemma (cf. Theorem C.1 in [29] or Lemma 5.1) we have

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\in[0,Su^{-2/\kappa}]} X_i(t) > f_i(u), \sup_{t\in[t_0,t_0+S]u^{-2/\kappa}} X_i(t) > h_i(u)\right)\\
&\le \mathbb{P}\left(\sup_{t\in[0,Su^{-2/\kappa}]} X_i(t) > \max(f_i(u), h_i(u))\right)\\
&\le \mathbb{P}\left(\sup_{t\in[0,Su^{-2/\kappa}]} \xi(t) > \frac{f_i(u)+h_i(u)}{2}\right).
\end{aligned}$$

Consequently, the Pickands lemma (cf. Lemma D.1 in [29] or Corollary 2.2) implies

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\in[0,Su^{-2/\kappa}]} X_i(t) > f_i(u), \sup_{t\in[t_0,t_0+S]u^{-2/\kappa}} X_i(t) > h_i(u)\right)\\
&\le F_i S^2 \Psi\left(\frac{f_i(u)+h_i(u)}{2}\right) \le F_i S^2 \exp\left(-G_i(t_0-S)^{\kappa}\right) \Psi\left(\frac{f_i(u)+h_i(u)}{2}\right)
\end{aligned}$$

for all u sufficiently large, with $G_i = 0$ and some $F_i > 1$. Moreover since in view of the independence of X_i's

$$\mathbb{P}\left(\exists_{t\in[0,S]u^{-2/\kappa}}\ \boldsymbol{X}(t) > \boldsymbol{f}(u), \exists_{t\in[t_0,t_0+S]u^{-2/\kappa}}\ \boldsymbol{X}(t) > \boldsymbol{h}(u)\right)$$

$$\le \mathbb{P}\left(\bigcap_{i=1}^{n}\left\{\sup_{t\in[0,S]u^{-2/\kappa}} X_i(t) > f_i(u)\right\}, \bigcap_{i=1}^{n}\left\{\sup_{t\in[t_0,t_0+S]u^{-2/\kappa}} X_i(t) > h_i(u)\right\}\right)$$
$$= \prod_{i=1}^{n}\mathbb{P}\left(\sup_{t\in[0,S]u^{-2/\kappa}} X_i(t) > f_i(u), \sup_{t\in[t_0,t_0+S]u^{-2/\kappa}} X_i(t) > h_i(u)\right)$$

the claim follows by choosing $F = \prod_{i=1}^n F_i > 0$, $G = \sum_{i=1}^n G_i > 0$. This completes the proof. □

5.1. **Proof of Proposition 2.1.** The idea of the proof is based on a multidimensional modification of the proof of Theorem D.1 in [29]. We shall present only the main steps that lead to the claim. For all $u > 0$ we have

$$\mathbb{P}\left(\exists_{t\in[0,T]}\ \boldsymbol{X}_u(t) > \boldsymbol{f}(u)\right) = \int_{\mathbb{R}^n}\prod_{i=1}^{n}\left(\frac{1}{\sqrt{2\pi}}e^{-v_i^2/2}\right)\mathbb{P}\left(\exists_{t\in[0,T]}\ \boldsymbol{X}_u(t) > \boldsymbol{f}(u)\ \Big|\ \boldsymbol{X}_u(0) = \boldsymbol{v}\right)d\boldsymbol{v}$$
$$= \prod_{i=1}^{n}\left(\Psi(f_i(u))\right)\int_{\mathbb{R}^n} e^{\sum_{i=1}^n\left(w_i - w_i^2/(2f_i^2(u))\right)}\mathbb{P}\left(\exists_{t\in[0,T]}\ \boldsymbol{X}_u(t) > \boldsymbol{f}(u)\ \Big|\ \boldsymbol{X}_u(0) = \boldsymbol{f}(u) - \frac{\boldsymbol{w}}{\boldsymbol{f}(u)}\right)d\boldsymbol{w}.$$

Consider the family $\boldsymbol{\chi}_u(t) = (\chi_{1,u}(t), \ldots, \chi_{n,u}(t))$ indexed by u, where

$$\chi_{i,u}(t) := \left(f_i(u)\left(X_{i,u}(t) - f_i(u)\right) + w_i\ \Big|\ X_{i,u}(0) = f_i(u) - \frac{w_i}{f_i(u)}\right),$$

and observe that

$$\mathbb{P}\left(\exists_{t\in[0,T]}\ \boldsymbol{X}_u(t) > \boldsymbol{f}(u)\ \Big|\ \boldsymbol{X}_u(0) = \boldsymbol{f}(u) - \frac{\boldsymbol{w}}{\boldsymbol{f}(u)}\right) = \mathbb{P}\left(\exists_{t\in[0,T]}\ \boldsymbol{\chi}_u(t) > \boldsymbol{w}\right).$$

By **P1-P2** for any $w \in \mathbb{R}$

$$\mathbb{E}\left(\chi_{i,u}(t)\right) \to -\mathrm{Var}(Y_i(t)) - d_i(t), \quad u \to \infty$$

holds uniformly with respect to $t \in [0, T]$. Moreover,

$$\mathbb{E}\left((\chi_{i,u}(t) - \chi_{i,u}(s))^2\right) \to 2\mathrm{Var}(Y_i(t) - Y_i(s)), \quad u \to \infty$$

for all $s, t \in [0, T]$. Hence

$$\lim_{u\to\infty}\mathbb{P}\left(\exists_{t\in[0,T]}\ \boldsymbol{\chi}_u(t) > \boldsymbol{w}\right) = \mathbb{P}\left(\exists_{t\in[0,T]}\ \left(\sqrt{2}\boldsymbol{Y}(t) - \mathrm{Var}(\boldsymbol{Y}(t)) - \boldsymbol{d}(t)\right) > \boldsymbol{w}\right)$$

for each $w \in \mathbb{R}$. The remaining part of the proof follows line-by-line the same reasoning as the corresponding proof of Lemma D.1 in [29], where **P3** is used for the tightness of $\chi_{i,u}$'s; see also Proposition 9.7 in [30] and Lemma 2 in [18]. This completes the proof. □

5.2. **Proof of Lemma 3.1.** It suffices to suppose that in Corolarry 2.2 we have $b_1(t) = \cdots = b_n(t) = 0$ (so $Z_i(\cdot) = X_i(\cdot)$ is stationary) and note that

$$\begin{aligned}\mathbb{P}\left(\exists_{t\in[0,Su^{-2/\kappa}]}\ \boldsymbol{X}(t) > \boldsymbol{f}(u)\right) &\le \sum_{k=1}^{S}\mathbb{P}\left(\exists_{t\in[k-1,k]u^{-2/\kappa}}\ \boldsymbol{X}(t) > \boldsymbol{f}(u)\right)\\ &= S\mathbb{P}\left(\exists_{t\in[0,u^{-2/\kappa}]}\ \boldsymbol{X}(t) > \boldsymbol{f}(u)\right)\end{aligned}$$

is valid for all $u > 0$. □

5.3. **Proof of Proposition 3.2.** The idea of the proof is based on a multidimensional modification of a technique developed in Lemma 16 and Corollary 17 in [16] and in Lemma 7 in [31]. For a fixed $a > 0$ and a positive integer N, using Bonferroni's inequality, we obtain

$$\begin{aligned}\mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa}(aN) &= \int_{\mathbb{R}^n} e^{\sum_{i=1}^n w_i}\mathbb{P}\left(\exists_{t\in[0,aN]}\left\{\sqrt{2}\boldsymbol{C}\boldsymbol{B}_\kappa(t) - \boldsymbol{C}^2t^\kappa > \boldsymbol{w}\right\}\right)d\boldsymbol{w}\\ &\ge \int_{\mathbb{R}^n} e^{\sum_{i=1}^n w_i}\mathbb{P}\left(\exists_{1\le k\le N}\left\{\sqrt{2}\boldsymbol{C}\boldsymbol{B}_\kappa(ak) - \boldsymbol{C}^2(ak)^\kappa > \boldsymbol{w}\right\}\right)d\boldsymbol{w}\\ &\ge \sum_{k=1}^{N}\int_{\mathbb{R}^n} e^{\sum_{i=1}^n w_i}\mathbb{P}\left(\sqrt{2}\boldsymbol{C}\boldsymbol{B}_\kappa(ak) - \boldsymbol{C}^2(ak)^\kappa) > \boldsymbol{w}\right)d\boldsymbol{w}\end{aligned}$$

$$
\begin{aligned}
& -\sum_{k=1}^{N-1}\sum_{l=k+1}^{N}\int_{\mathbb{R}^n} e^{\sum_{i=1}^n w_i}\mathbb{P}\left(\sqrt{2}\boldsymbol{C}(\boldsymbol{B}_\kappa(ak)+\boldsymbol{B}_\kappa(al))-\boldsymbol{C}^2((ak)^\kappa+(al)^\kappa)>2\boldsymbol{w}\right)d\boldsymbol{w}\\
=\ & \sum_{k=1}^{N}\int_{\mathbb{R}^n} e^{\sum_{i=1}^n w_i}\ \prod_{i=1}^{n}\mathbb{P}\left(\sqrt{2}C_iB_{i,\kappa}(ak)-C_i^2(ak)^\kappa>w_i\right)d\boldsymbol{w}\\
& -\sum_{k=1}^{N-1}\sum_{l=k+1}^{N}\int_{\mathbb{R}^n} e^{\sum_{i=1}^n w_i}\prod_{i=1}^{n}\mathbb{P}\left(\sqrt{2}C_i(B_{i,\kappa}(ak)+B_{i,\kappa}(al))-C_i^2((ak)^\kappa+(al)^\kappa)>2w_i\right)d\boldsymbol{w}\\
=\ & \sum_{k=1}^{N}\prod_{i=1}^{n}\int_{\mathbb{R}} e^{w_i}\mathbb{P}\left(\sqrt{2}C_iB_{i,\kappa}(ak)-C_i^2(ak)^\kappa>w_i\right)dw_i\\
& -\sum_{k=1}^{N-1}\sum_{l=k+1}^{N}\prod_{i=1}^{n}\int_{\mathbb{R}} e^{w_i}\mathbb{P}\left(\frac{\sqrt{2}C_i(B_{i,\kappa}(ak)+B_{i,\kappa}(al))-C_i^2((ak)^\kappa+(al)^\kappa)}{2}>w_i\right)dw_i\\
=\ & \sum_{k=1}^{N}\prod_{i=1}^{n}\mathbb{E}\left(\exp\left(\sqrt{2}C_iB_{i,\kappa}(ak)-C_i^2(ak)^\kappa\right)\right)\\
& -\sum_{k=1}^{N-1}\sum_{l=k+1}^{N}\prod_{i=1}^{n}\mathbb{E}\left(\exp\left(\frac{\sqrt{2}C_i(B_{i,\kappa}(ak)+B_{i,\kappa}(al))-C_i^2((ak)^\kappa+(al)^\kappa)}{2}\right)\right).
\end{aligned}
$$

Since $\exp\left(\sqrt{2}C_iB_{i,\kappa}(ak)-C_i^2(ak)^\kappa\right)$ is log-normal distributed

$$
\begin{aligned}
\mathbb{E}\left(\exp\left(\sqrt{2}C_iB_{i,\kappa}(ak)-C_i^2(ak)^\kappa\right)\right) &= 1,\\
\mathbb{E}\left(\exp\left(\frac{\sqrt{2}C_i(B_{i,\kappa}(ak)+B_{i,\kappa}(al))-C_i^2((ak)^\kappa+(al)^\kappa)}{2}\right)\right) &= \exp\left(-C_i^2(a(l-k))^\kappa/4\right)
\end{aligned}
$$

implying

$$
\begin{aligned}
\mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa}(aN) &\geq N\ -\sum_{k=1}^{N-1}\sum_{l=k+1}^{N}\exp\left(-\frac{\sum_{i=1}^n C_i^2(a(l-k))^\kappa}{4}\right)\\
&\geq N\left(1-\sum_{k=1}^{N}\exp\left(-\frac{C^2(ak)^\kappa}{4}\right)\right),
\end{aligned}
$$

where $C^2=\sum_{i=1}^n C_i^2$. From the definition of $\mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa}$, for any $a>0$

$$
\begin{aligned}
\mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa} &= \lim_{N\to\infty}\frac{\mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa}(aN)}{aN} \geq \frac{1}{a}\left(1-\sum_{k=1}^{\infty}\exp\left(-\frac{C^2a^\kappa}{4}k^\kappa\right)\right)\\
&\geq \frac{1}{a}\left(1-\int_0^\infty\exp\left(-\frac{C^2a^\kappa}{4}x^\kappa\right)dx\right)\\
&= \frac{1}{a}\left(1-\frac{\Gamma(1/\kappa)}{\kappa\left(C^2a^\kappa/4\right)^{1/\kappa}}\right)\\
&= \frac{1}{a}\left(1-\frac{1}{a}\frac{\Gamma(1/\kappa)}{\kappa\left(C^2/4\right)^{1/\kappa}}\right).
\end{aligned}
$$

The maximum over $a>0$ of $f(a)=\frac{1}{a}\left(1-\frac{c}{a}\right)$ is attained at $a^*=2c$ with $f(a^*)=\frac{1}{4c}$. Consequently, setting $c=\frac{\Gamma(1/\kappa)}{\kappa(C^2/4)^{1/\kappa}}$ we obtain

$$
\mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa} \geq \frac{\kappa\left(C^2/4\right)^{1/\kappa}}{4\Gamma(1/\kappa)}
$$

establishing the claim. □

5.4. **Proof of Proposition 3.3.** In view of Lemma 2.2 and Lemma 2.3 in [13], we have for any $T>0$

$$\begin{aligned}\mathcal{H}_{B_1}(T) &= (2+T)\,\Phi\left(\sqrt{T/2}\right)+\sqrt{T/\pi}\exp\left(-T/4\right) \le 2+\sqrt{\frac{2}{\pi e}}+T,\\ \mathcal{H}_{B_2}(T) &= 1+\frac{T}{\sqrt{\pi}}.\end{aligned}$$

Hence the case $n=1$ is clear. Next, for $n\ge 2$, from the subadditivity of $\mathcal{H}_{\mathbf{B}_\kappa}(\cdot)$ and the independence of $B_{i,\kappa}(\cdot)$ we have

$$\mathcal{H}_{\mathbf{B}_\kappa} = \inf_{T>0}\frac{\mathcal{H}_{\mathbf{B}_\kappa}(T)}{T} \le \inf_{T>0}\frac{\left(\prod_{i=1}^n \mathcal{H}_{B_{i,\kappa}}(T)\right)}{T} = \inf_{T>0}\frac{\left(\mathcal{H}_{B_\kappa}(T)\right)^n}{T}.$$

Therefore, for $\kappa=1$, $\mathcal{H}_{\mathbf{B}_\kappa}\le \min_{x>0}\frac{(c+x)^n}{x}$ with $c=2+\sqrt{\frac{2}{\pi e}}$, and the minimum is attained at $x^*=\frac{c}{n-1}$. For $\kappa=2$, $\mathcal{H}_{\mathbf{B}_\kappa}\le\min_{x>0}\frac{(1+cx)^n}{x}$ with $c=\frac{1}{\sqrt{\pi}}$, and the minimum is attained for $x^*=\frac{1}{(n-1)c}$. This completes the proof. □

5.5. **Proof of Proposition 3.4.** It is sufficient to show the proof for $\mathcal{H}^{\overline{\boldsymbol{d}}}_{\boldsymbol{C}\boldsymbol{B}_\kappa}$. By definition for any $T>0$ we have

$$\begin{aligned}\lim_{S\to\infty}\mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa,\boldsymbol{d}}[0,S] &\ge \mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa,\boldsymbol{d}}[0,T]\\ &= \int_{\mathbb{R}^n} e^{\sum_{i=1}^n w_i}\mathbb{P}\Big(\exists_{t\in[0,T]}\ \sqrt{2}\boldsymbol{C}\boldsymbol{B}_\kappa(t)-\boldsymbol{C}^2\sigma^2_{\boldsymbol{B}_\kappa}(t)-\overline{\boldsymbol{d}}t^\kappa>\boldsymbol{w}\Big)d\boldsymbol{w}\\ &\ge \int_{\mathbb{R}^n} e^{\sum_{i=1}^n w_i}\mathbb{P}\Big(\exists_{t\in[0,T]}\ \sqrt{2}\boldsymbol{C}\boldsymbol{B}_\kappa(t)-\boldsymbol{C}^2\sigma^2_{\boldsymbol{B}_\kappa}(t)>\boldsymbol{w}+\max(\boldsymbol{0},\overline{\boldsymbol{d}})T^\kappa\Big)d\boldsymbol{w}\\ &= e^{-d^+T^\kappa}\mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa,\boldsymbol{0}}[0,T],\quad d^+:=\sum_{i=1}^n\max(0,\overline{d}_i).\end{aligned}$$

Since $\mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa,\boldsymbol{0}}[0,T]=\mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa}(T)$ is subadditive, Fekete's Lemma implies

$$\begin{aligned}\lim_{S\to\infty}\mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa,\boldsymbol{d}}[0,S] &\ge \sup_{T>0}\left(e^{-d^+T^\kappa}\mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa,\boldsymbol{0}}[0,T]\right)\\ &\ge \sup_{T>0}Te^{-d^+T^\kappa}\inf_{T>0}\frac{\mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa}(T)}{T}\\ &= \sup_{T>0}\left(Te^{-d^+T^\kappa}\right)\lim_{T\to\infty}\frac{\mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa}(T)}{T}\\ &= \left(d^+e\kappa\right)^{-1/\kappa}\mathcal{H}_{\boldsymbol{C}\boldsymbol{B}_\kappa}\end{aligned}$$

establishing the proof. □

5.6. **Proof of Theorem 4.1.** The complete proof consists of two steps. In Step 1 we show the claim for $\boldsymbol{X}$ with stationary coordinates, and then in Step 2 we show the proof for $\boldsymbol{X}$ with locally stationary coordinates.

Step 1. Stationary coordinates, i.e., $\boldsymbol{X}\in\mathcal{S}(\boldsymbol{a},\kappa)$.

First let $S>1$ and denote for $u>0$

$$\Delta_k=[kSu^{-\frac{2}{\kappa}},(k+1)Su^{-\frac{2}{\kappa}}],k\in\mathbb{N}_0,\qquad N(u)=\left\lfloor Tu^{2/\kappa}S^{-1}\right\rfloor+1.$$

Here $\lfloor\cdot\rfloor$ denotes the ceiling function. By Bonferroni's inequality and the stationarity of $\boldsymbol{X}$ for sufficiently large u we have

$$\begin{aligned}\mathbb{P}\left(\exists_{t\in[0,T]}\ \boldsymbol{X}(t)>\boldsymbol{f}(u)\right) &\le \sum_{k=0}^{N(u)}\mathbb{P}\left(\exists_{t\in\Delta_k}\ \boldsymbol{X}(t)>\boldsymbol{f}(u)\right)\\ &= N(u)\mathbb{P}\left(\exists_{t\in\Delta_0}\ \boldsymbol{X}(t)>\boldsymbol{f}(u)\right).\end{aligned}$$

Thus, by Corollary 2.2 we obtain

$$\limsup_{u\to\infty}\frac{\mathbb{P}\left(\exists_{t\in[0,T]}\ \boldsymbol{X}(t)>\boldsymbol{f}(u)\right)}{Tu^{2/\kappa}\prod_{i=1}^n\Psi(f_i(u))} \le \frac{\mathcal{H}_{\boldsymbol{c}\sqrt{\boldsymbol{a}}\boldsymbol{B}_\kappa}(S)}{S}. \tag{15}$$

Again by Bonferroni's inequality

$$\mathbb{P}\left(\exists_{t\in[0,T]}\ \boldsymbol{X}(t) > \boldsymbol{f}(u)\right) \ge \sum_{k=0}^{N(u)-1} \mathbb{P}\left(\exists_{t\in\Delta_k}\ \boldsymbol{X}(t) > \boldsymbol{f}(u)\right) - \Sigma(u)$$

holds, where

$$\Sigma(u) = \sum_{0\le k<l\le N(u)} \mathbb{P}\left(\exists_{t\in\Delta_k}\ \boldsymbol{X}(t) > \boldsymbol{f}(u), \exists_{t\in\Delta_l}\ \boldsymbol{X}(t) > \boldsymbol{f}(u)\right).$$

Similarly to the proof of (15) we obtain

$$\liminf_{u\to\infty} \frac{\sum_{k=0}^{N(u)-1} \mathbb{P}\left(\exists_{t\in\Delta_k}\ \boldsymbol{X}(t) > \boldsymbol{f}(u)\right)}{Tu^{2/\kappa}\prod_{i=1}^n \Psi(f_i(u))} \ \ge\ \frac{\mathcal{H}_{\boldsymbol{c}\sqrt{\boldsymbol{a}}\boldsymbol{B}_\kappa}(S)}{S}. \tag{16}$$

Next we shall focus on the double sum term $\Sigma(u)$. We choose some small positive ε such that the assumptions in Lemma 5.4 are satisfied. We divide $\Sigma(u)$ into three parts, say, $\Sigma_1(u)$ the sum over indexes k,l such that $(l-k-1)Su^{2/\kappa} > \varepsilon$, $\Sigma_2(u)$ the sum over indexes $l > k+1$ and $(l-k-1)Su^{2/\kappa} \le \varepsilon$, and $\Sigma_3(u)$ the sum over indexes $l = k+1$.
For the summand of $\Sigma_1(u)$ similarly as in the proof of Lemma 5.4 we have

$$\begin{aligned}\mathbb{P}\left(\exists_{t\in\Delta_k}\boldsymbol{X}(t) > \boldsymbol{f}(u), \exists_{t\in\Delta_l}\boldsymbol{X}(t) > \boldsymbol{f}(u)\right) &\le \prod_{i=1}^n \mathbb{P}\left(\sup_{t\in\Delta_k} X_i(t) > f_i(u), \sup_{t\in\Delta_l} X_i(t) > f_i(u)\right)\\ &\le \prod_{i=1}^n \mathbb{P}\left(\sup_{(t,s)\in\Delta_k\times\Delta_l} X_i(t)+X_i(s) > 2f_i(u)\right).\end{aligned}$$

Further since $r_i(t) < 1$ for any $t\neq 0$, then

$$\delta_i = \max_{(t,s)\in\Delta_k\times\Delta_l} r(s-t) < 1,$$

which yields

$$\mathrm{Var}(X_i(t)+X_i(s)) = 2(1+r_i(s-t)) < 2(1+\delta_i) < 4.$$

Therefore from the Borell-TIS inequality for all sufficiently large u

$$\mathbb{P}\left(\exists_{t\in\Delta_k}\boldsymbol{X}(t) > \boldsymbol{f}(u), \exists_{t\in\Delta_l}\boldsymbol{X}(t) > \boldsymbol{f}(u)\right) \ \le\ \prod_{i=1}^n \exp\left(-\frac{(f_i(u)-m_i)^2}{1+\delta_i}\right)$$

holds for some positive constants $m_i, 1\le i\le n$. Consequently,

$$\limsup_{u\to\infty} \frac{\Sigma_1(u)}{Tu^{2/\kappa}\prod_{i=1}^n \Psi(f_i(u))} = 0. \tag{17}$$

For the summand of $\Sigma_2(u)$, we get from Lemma 5.4 that for all sufficiently large u

$$\mathbb{P}\left(\exists_{t\in\Delta_k}\boldsymbol{X}(t) > \boldsymbol{f}(u), \exists_{t\in\Delta_l}\boldsymbol{X}(t) > \boldsymbol{f}(u)\right) \le FS^{2n}\exp\left(-G((l-k-1)S)^\kappa\right)\prod_{i=1}^n \Psi\left(f_i(u)\right)$$

holds with some positive constants F, G. Thus, for sufficiently large u

$$\Sigma_2(u) \le (N(u)+1)\sum_{l=1}^\infty FS^{2n}\exp\left(-G(lS)^\kappa\right)\prod_{i=1}^n \Psi(f_i(u))$$

is valid. Note that for any $\theta, G > 0$ and $S > (\theta G/2)^{-1/\theta}$ we have

$$\sum_{k=1}^\infty e^{-G(kS)^\theta} \ \le\ 2e^{-GS^\theta}.$$

Consequently, for large enough S

$$\limsup_{u\to\infty} \frac{\Sigma_2(u)}{Tu^{2/\kappa}\prod_{i=1}^n \Psi(f_i(u))} \ \le\ 2FS^{2n-1}e^{-GS^\kappa}. \tag{18}$$

For the summand of $\Sigma_3(u)$, by the stationarity of $\boldsymbol{X}$ (set $\boldsymbol{X}_u(t) = \boldsymbol{X}(tu^{-2/\kappa})$) we have

$$\mathbb{P}\left(\exists_{t\in\Delta_k}\ \boldsymbol{X}(t) > \boldsymbol{f}(u), \exists_{t\in\Delta_l}\ \boldsymbol{X}(t) > \boldsymbol{f}(u)\right)$$

$$
\begin{aligned}
&= \mathbb{P}\left(\exists_{t\in[0,S]}\ \boldsymbol{X}_u(t) > \boldsymbol{f}(u), \exists_{t\in[S,2S]}\ \boldsymbol{X}_u(t) > \boldsymbol{f}(u)\right)\\
&= \mathbb{P}\left(\exists_{t\in[0,S]}\ \boldsymbol{X}_u(t) > \boldsymbol{f}(u), \left\{\exists_{t\in[S,S+\sqrt{S}]}\ \boldsymbol{X}_u(t) > \boldsymbol{f}(u)\right\} \cup \left\{\exists_{t\in[S+\sqrt{S},2S]}\ \boldsymbol{X}_u(t) > \boldsymbol{f}(u)\right\}\right)\\
&\le \mathbb{P}\left(\exists_{t\in[0,S]}\ \boldsymbol{X}_u(t) > \boldsymbol{f}(u), \exists_{t\in[S+\sqrt{S},2S+\sqrt{S}]}\ \boldsymbol{X}_u(t) > \boldsymbol{f}(u)\right) + \mathbb{P}\left(\exists_{t\in[S,S+\sqrt{S}]}\ \boldsymbol{X}_u(t) > \boldsymbol{f}(u)\right).
\end{aligned}
$$

Applying Lemma 5.4 with $t_0 = S+\sqrt{S}$ and Pickands lemma (see Corollary 2.2) to the last two terms above, respectively, we obtain that for sufficiently large u, S

$$
\begin{aligned}
&\mathbb{P}\left(\exists_{t\in\Delta_k}\ \boldsymbol{X}(t) > \boldsymbol{f}(u), \exists_{t\in\Delta_l}\ \boldsymbol{X}(t) > \boldsymbol{f}(u)\right)\\
&\quad \le \quad FS^{2n}\exp\left(-G\sqrt{S^\kappa}\right)\prod_{i=1}^n \Psi(f_i(u)) + F\mathcal{H}_{\boldsymbol{c}\sqrt{\boldsymbol{a}}\boldsymbol{B}_\kappa}(\sqrt{S})\prod_{i=1}^n \Psi(f_i(u))\\
&\quad \overset{(6)}{\le} \quad FS^{2n}\exp\left(-G\sqrt{S^\kappa}\right)\prod_{i=1}^n \Psi(f_i(u)) + F\mathcal{H}_{\boldsymbol{c}\sqrt{\boldsymbol{a}}\boldsymbol{B}_\kappa}(1)\sqrt{S}\prod_{i=1}^n \Psi(f_i(u))\\
&\quad \le \quad F_1\left(S^{2n}\exp\left(-G\sqrt{S^\kappa}\right) + \sqrt{S}\right)\prod_{i=1}^n \Psi(f_i(u)),
\end{aligned}
$$

with some constant $F_1 > 0$. Therefore

$$
\Sigma_3(u) \ \le\ (N(u)+1)F_1\left(S^{2n}\exp\left(-G\sqrt{S^\kappa}\right) + \sqrt{S}\right)\prod_{i=1}^n \Psi(f_i(u))
$$

for sufficiently large u, and thus

$$
\limsup_{u\to\infty}\frac{\Sigma_3(u)}{Tu^{2/\kappa}\prod_{i=1}^n \Psi(f_i(u))} \ \le\ F_1\left(S^{2n-1}\exp\left(-G\sqrt{S^\kappa}\right) + S^{-\frac{1}{2}}\right). \tag{19}
$$

Consequently, it follows from (15–19) that for any sufficiently large S_1, S_2

$$
\begin{aligned}
\frac{\mathcal{H}_{\boldsymbol{c}\sqrt{\boldsymbol{a}}\boldsymbol{B}_\kappa}(S_1)}{S_1} \quad &\ge \quad \limsup_{u\to\infty}\frac{\mathbb{P}\left(\exists_{t\in[0,T]}\ \boldsymbol{X}(t) > \boldsymbol{f}(u)\right)}{Tu^{2/\kappa}\prod_{i=1}^n \Psi(f_i(u))}\\
&\ge \quad \liminf_{u\to\infty}\frac{\mathbb{P}\left(\exists_{t\in[0,T]}\ \boldsymbol{X}(t) > \boldsymbol{f}(u)\right)}{Tu^{2/\kappa}\prod_{i=1}^n \Psi(f_i(u))}\\
&\ge \quad \frac{\mathcal{H}_{\boldsymbol{c}\sqrt{\boldsymbol{a}}\boldsymbol{B}_\kappa}(S_2)}{S_2} - F_1\left(S_2^{2n-1}\exp\left(-G\sqrt{S_2^\kappa}\right) + S_2^{-\frac{1}{2}}\right) - 2FS_2^{2n-1}e^{-GS_2^\kappa}.
\end{aligned} \tag{20}
$$

Hence, the claim of the theorem follows from (20) by letting $S_1, S_2 \to \infty$.

For Step 2, we point out that a close observation of (20) shows that

$$
\frac{\mathcal{H}_{\boldsymbol{c}\sqrt{\boldsymbol{a}}\boldsymbol{B}_\kappa}(S_1)}{S_1} \to \mathcal{H}_{\boldsymbol{c}\sqrt{\boldsymbol{a}}\boldsymbol{B}_\kappa}, \quad S_1 \to \infty \tag{21}
$$

uniformly with respect to $\boldsymbol{a} \in [\boldsymbol{\nu}, \boldsymbol{\mu}] := \prod_{i=1}^n [\nu_i, \mu_i]$, with $\boldsymbol{\nu} \ge \boldsymbol{0}$, $\boldsymbol{\nu} \ne \boldsymbol{0}$ and $\nu_i < \mu_i < \infty, 1 \le i \le n$.

Step 2. Locally stationary coordinates.

We consider only the case where $\kappa = \kappa_1 = \cdots = \kappa_n$; the same approach applies for the general case. It follows from (7) that for any $\varepsilon > 0$ there is some small $\delta_0 > 0$ such that for all $1 \le i \le n$

$$
(1-\varepsilon)a_i(t)\,|h|^\kappa \le 1 - r_i(t, t+h) \le (1+\varepsilon)a_i(t)\,|h|^\kappa \tag{22}
$$

hold for all $t, t+h \in [0,T]$ satisfying $|h| \le \delta_0$. Now let $\lambda \in (0, \delta_0)$ be any small constant and denote $\lambda_k = k\lambda, k \in \mathbb{N}_0$. Clearly

$$
\begin{aligned}
\sum_{k=0}^{\lfloor T/\lambda\rfloor+1}\mathbb{P}\left(\exists_{t\in[\lambda_k,\lambda_{k+1}]}\ \boldsymbol{X}(t) > \boldsymbol{f}(u)\right) \quad &\ge \quad \mathbb{P}\left(\exists_{t\in[0,T]}\ \boldsymbol{X}(t) > \boldsymbol{f}(u)\right)\\
&\ge \quad \sum_{k=0}^{\lfloor T/\lambda\rfloor}\mathbb{P}\left(\exists_{t\in[\lambda_k,\lambda_{k+1}]}\ \boldsymbol{X}(t) > \boldsymbol{f}(u)\right) - \Sigma_4(u),
\end{aligned}
$$

with

$$\Sigma_4(u) = \sum_{0\le k<l\le \lfloor T/\lambda\rfloor} \mathbb{P}\left(\exists_{t\in[\lambda_k,\lambda_{k+1}]} \boldsymbol{X}(t) > \boldsymbol{f}(u), \exists_{t\in[\lambda_l,\lambda_{l+1}]} \boldsymbol{X}(t) > \boldsymbol{f}(u)\right).$$

Next, for any fixed $k \in \mathbb{N}_0$, define centered stationary Gaussian processes $\{\xi_i^{\varepsilon\pm}(t), t \ge 0\}$ with unit variance and correlation functions

$$r_{\xi_i^{\varepsilon\pm}}(t) = \exp(-(1\pm\varepsilon)a_i(\lambda_k)\,|t|^{\kappa}), t\ge 0, \quad 1\le i\le n,$$

and let $\boldsymbol{\xi}^{\varepsilon\pm}(t) = (\xi_1^{\varepsilon\pm}(t),\cdots,\xi_n^{\varepsilon\pm}(t)), t\ge 0$. In view of (22) and Lemma 5.1 we have

$$\mathbb{P}\left(\exists_{t\in[\lambda_k,\lambda_{k+1}]} \boldsymbol{\xi}^{\varepsilon-}(t) > \boldsymbol{f}(u)\right) \le \mathbb{P}\left(\exists_{t\in[\lambda_k,\lambda_{k+1}]} \boldsymbol{X}(t) > \boldsymbol{f}(u)\right) \le \mathbb{P}\left(\exists_{t\in[\lambda_k,\lambda_{k+1}]} \boldsymbol{\xi}^{\varepsilon+}(t) > \boldsymbol{f}(u)\right).$$

Then applying the results in Step 1 for vector-valued stationary Gaussian process we conclude that for λ sufficiently small

$$\begin{aligned}\limsup_{u\to\infty} \frac{\sum_{k=0}^{\lfloor T/\lambda\rfloor+1}\mathbb{P}\left(\exists_{t\in[\lambda_k,\lambda_{k+1}]} \boldsymbol{X}(t) > \boldsymbol{f}(u)\right)}{u^{\frac{2}{\kappa}}\prod_{i=1}^n \Psi(f_i(u))} &\le \sum_{k=0}^{\lfloor T/\lambda\rfloor+1} \mathcal{H}_{\boldsymbol{c}\sqrt{(1+\varepsilon)\boldsymbol{a}(\lambda_k)}\boldsymbol{B}_\kappa}\,\lambda\\ &\le (1+\varepsilon)\int_0^T \mathcal{H}_{\boldsymbol{c}\sqrt{(1+\varepsilon)\boldsymbol{a}(t)}\boldsymbol{B}_\kappa} dt, \qquad (23)\end{aligned}$$

where the last inequality follows from the fact that $\mathcal{H}_{\boldsymbol{c}\sqrt{(1+\varepsilon)\boldsymbol{a}(t)}\boldsymbol{B}_\kappa}$ is continuous with respect to $t\in[0,T]$ which is due to (21) and some elementary derivations. Similarly, we have for λ sufficiently small

$$\liminf_{u\to\infty} \frac{\sum_{k=0}^{\lfloor T/\lambda\rfloor}\mathbb{P}\left(\exists_{t\in[\lambda_k,\lambda_{k+1}]} \boldsymbol{X}(t) > \boldsymbol{f}(u)\right)}{u^{\frac{2}{\kappa}}\prod_{i=1}^n \Psi(f_i(u))} \ge (1-\varepsilon)\int_0^T \mathcal{H}_{\boldsymbol{c}\sqrt{(1-\varepsilon)\boldsymbol{a}(t)}\boldsymbol{B}_\kappa} dt. \qquad (24)$$

Furthermore, similar arguments as in the proof of Theorem 7.1 in [29] show that

$$\limsup_{u\to\infty} \frac{\Sigma_4(u)}{u^{\frac{2}{\kappa}}\prod_{i=1}^n \Psi(f_i(u))} = 0.$$

Consequently, the claim follows by letting $\varepsilon\to 0$ in (23) and (24). This completes the proof. □

5.7. **Proof of Theorem 4.3.** We only give the proof for the case that $t_0\in(0,T)$, $\alpha=\alpha_1=\cdots=\alpha_n$ and $\overline{\boldsymbol{b}}>\boldsymbol{0}, \underline{\boldsymbol{b}}>\boldsymbol{0}$. The proofs of the other cases follow by similar arguments and are therefore omitted.

Let $\delta(u) = (\ln u/u)^{2/\beta}$, and denote $D_u = [-\delta(u),\delta(u)]$ for u large. In the following, all formulas are meant for large enough u. With these notation we have

$$\begin{aligned}P_1(u) := \mathbb{P}\left(\sup_{t\in(t_0+D_u)}\min_{1\le i\le n} X_i(t) > u\right) &\le \mathbb{P}\left(\sup_{t\in[0,T]}\min_{1\le i\le n} X_i(t) > u\right)\\ &\le P_1(u) + \mathbb{P}\left(\sup_{t\in[0,T]/(t_0+D_u)}\min_{1\le i\le n} X_i(t) > u\right)\\ &=: P_1(u)+P_2(u).\end{aligned}$$

Next, we shall derive the exact asymptotics of $P_1(u)$ as $u\to\infty$, and show that

$$P_2(u) = o(P_1(u)), \qquad u\to\infty \qquad (25)$$

implying thus

$$\mathbb{P}\left(\sup_{t\in[0,T]}\min_{1\le i\le n} X_i(t) > u\right) = P_1(u)(1+o(1)), \qquad u\to\infty.$$

Now we focus on the the asymptotics of $P_1(u)$ as $u\to\infty$. For any small enough $\varepsilon>0$ define

$$Z_i^{\varepsilon\pm}(t) = \frac{\sigma_{X_i}(t_0)}{1+d_i^{\varepsilon\mp}(t)}\eta_i^{\varepsilon\pm}(t), \quad t\in\mathbb{R}, \quad 1\le i\le n,$$

where

$$d_i^{\varepsilon\mp}(t) = (\underline{b}_i\mp\varepsilon)|t|^{\beta}1_{\{t\le 0\}} + (\overline{b}_i\mp\varepsilon)|t|^{\beta}1_{\{t>0\}}, \quad t\in\mathbb{R}, \qquad (26)$$

and $\{\eta_i^{\varepsilon\pm}(t), t\in\mathbb{R}\}$ are centered stationary Gaussian processes with unit variance and correlation functions

$$r_{\eta_i^{\varepsilon\pm}}(t)=\exp\left(-a_i^{\varepsilon\pm}|t|^{\alpha}\right),\ t\ge 0,\quad a_i^{\varepsilon\pm}=a_i\pm\varepsilon.$$

In view of Assuptions II–III and Lemma 5.1, we have that for any small enough $\varepsilon>0$

$$\mathbb{P}\left(\sup_{t\in D_u}\min_{1\le i\le n} Z_i^{\varepsilon-}(t)>u\right)\le P_1(u)\le \mathbb{P}\left(\sup_{t\in D_u}\min_{1\le i\le n} Z_i^{\varepsilon+}(t)>u\right) \tag{27}$$

holds for all u sufficiently large. In the following, we shall show that the above upper and lower bounds for $P_1(u)$ are asymptotically equivalent as $u\to\infty$ and $\varepsilon\to 0$.

Next we introduce some notation. Let T_1 be any positive constant. For the case that $\alpha\le\beta$, we can split the interval D_u into several sub-intervals of side lengths $T_1u^{-2/\alpha}$. Specifically, let

$$\triangle_k=\left[kT_1u^{-\frac{2}{\alpha}},(k+1)T_1u^{-\frac{2}{\alpha}}\right],\quad k\in\mathbb{Z},\quad N(u)=\left\lfloor T_1^{-1}(\ln u)^{\frac{2}{\beta}}u^{\frac{2}{\alpha}-\frac{2}{\beta}}\right\rfloor+1$$

and note that

$$\bigcup_{k=-N(u)+1}^{N(u)-1}\triangle_k\subset D_u\subset\bigcup_{k=-N(u)}^{N(u)}\triangle_k.$$

We deal with the three cases i) $\alpha<\beta$, ii) $\alpha=\beta$ and iii) $\alpha>\beta$ one-by-one, using different techniques.

Case i) $\alpha<\beta$: *Upper bound.* Using Bonferroni inequality we have

$$P_1(u)\le\mathbb{P}\left(\sup_{t\in D_u}\min_{1\le i\le n} Z_i^{\varepsilon+}(t)>u\right)\le\sum_{k=-N(u)}^{N(u)}\mathbb{P}\left(\sup_{t\in\triangle_k}\min_{1\le i\le n} Z_i^{\varepsilon+}(t)>u\right)$$

and

$$\begin{aligned}\sum_{k=0}^{N(u)}\mathbb{P}\left(\sup_{t\in\triangle_k}\min_{1\le i\le n} Z_i^{\varepsilon+}(t)>u\right) &\le \sum_{k=0}^{N(u)}\mathbb{P}\left(\exists_{t\in\triangle_k}\boldsymbol{\eta}^{\varepsilon+}(t)>\boldsymbol{f}^{\varepsilon-}(k,u)\right)\\ &= \sum_{k=0}^{N(u)}\mathbb{P}\left(\exists_{t\in[0,T_1u^{-\frac{2}{\alpha}}]}\boldsymbol{\eta}^{\varepsilon+}(t)>\boldsymbol{f}^{\varepsilon-}(k,u)\right),\end{aligned}$$

where $\boldsymbol{\eta}^{\varepsilon+}(t)=(\eta_1^{\varepsilon\pm}(t),\cdots,\eta_n^{\varepsilon\pm}(t)), t\ge 0$, $\boldsymbol{f}^{\varepsilon\pm}(k,u)=(f_1^{\varepsilon\pm}(k,u),\cdots,f_n^{\varepsilon\pm}(k,u))$ with

$$f_i^{\varepsilon\pm}(k,u)=\frac{1}{\sigma_{X_i}(t_0)}\left(1+(\underline{b}_i\pm\varepsilon)(|k|T_1u^{-\frac{2}{\alpha}})^{\beta}1_{\{k\le 0\}}+(\overline{b}_i\pm\varepsilon)(kT_1u^{-\frac{2}{\alpha}})^{\beta}1_{\{k>0\}}\right)u,\quad 1\le i\le n.$$

Recall that we set $\boldsymbol{c}=(c_1,\cdots,c_n)$ with $c_i=\frac{1}{\sigma_{X_i}(t_0)}, 1\le i\le n$. Applying Corollary 2.2 we obtain

$$\sum_{k=0}^{N(u)}\mathbb{P}\left(\exists_{t\in[0,T_1u^{-\frac{2}{\alpha}}]}\boldsymbol{\eta}^{\varepsilon+}(t)>\boldsymbol{f}^{\varepsilon-}(k,u)\right)=\mathcal{H}_{\boldsymbol{c}\sqrt{\boldsymbol{a}^{\varepsilon+}}\boldsymbol{B}_\alpha}(T_1)\sum_{k=0}^{N(u)}\prod_{i=1}^{n}\Psi(f_i^{\varepsilon-}(k,u))(1+o(1))$$

as $u\to\infty$. Since further, with $\overline{\theta}_{\varepsilon\pm}:=\sum_{i=1}^n c_i^2(\overline{b}_i\pm\varepsilon)$,

$$\begin{aligned}\sum_{k=0}^{N(u)}\prod_{i=1}^{n}\Psi(f_i^{\varepsilon-}(k,u)) &= \sum_{k=0}^{N(u)}\prod_{i=1}^{n}\left(\frac{1}{\sqrt{2\pi}f_i^{\varepsilon-}(k,u)}\exp\left(-\frac{(f_i^{\varepsilon-}(k,u))^2}{2}\right)\right)(1+o(1))\\ &= (2\pi)^{-\frac{n}{2}}\left(\prod_{i=1}^{n}\sigma_{X_i}(t_0)\right)u^{-n}\sum_{k=0}^{N(u)}\exp\left(-\sum_{i=1}^{n}\frac{(1+2(\overline{b}_i-\varepsilon)(kT_1u^{-\frac{2}{\alpha}})^{\beta})u^2}{2\sigma^2_{X_i}(t_0)}\right)(1+o(1))\\ &= (2\pi)^{-\frac{n}{2}}\left(\prod_{i=1}^{n}\sigma_{X_i}(t_0)\right)u^{-n}\exp\left(-\frac{u^2}{2}g(t_0)\right)\sum_{k=0}^{N(u)}\exp\left(-\overline{\theta}_{\varepsilon-}(kT_1u^{\frac{2}{\beta}-\frac{2}{\alpha}})^{\beta}\right)(1+o(1))\\ &= T_1^{-1}(2\pi)^{-\frac{n}{2}}\left(\prod_{i=1}^{n}\sigma_{X_i}(t_0)\right)u^{\frac{2}{\alpha}-\frac{2}{\beta}-n}\exp\left(-\frac{u^2}{2}g(t_0)\right)\int_0^{\infty}\exp\left(-\overline{\theta}_{\varepsilon-}x^{\beta}\right)dx(1+o(1))\end{aligned}$$

we conclude that

$$\sum_{k=0}^{N(u)} \mathbb{P}\left(\sup_{t\in\triangle_k}\min_{1\le i\le n} Z_i^{\varepsilon+}(t)>u\right) \quad\le\quad \frac{\mathcal{H}_{\boldsymbol{c}\sqrt{\boldsymbol{a}^{\varepsilon+}}\boldsymbol{B}_\alpha}(T_1)}{T_1\overline{\theta}_{\varepsilon-}^{1/\beta}}\varphi^*(u)(1+o(1)),$$

where

$$\begin{aligned}\varphi^*(u) \quad&:=\quad (2\pi)^{-\frac{n}{2}}\left(\prod_{i=1}^n \sigma_{X_i}(t_0)\right)\Gamma\left(\frac{1}{\beta}+1\right)u^{\frac{2}{\alpha}-\frac{2}{\beta}-n}\exp\left(-\frac{u^2}{2}g(t_0)\right)\\ &=\quad \Gamma\left(\frac{1}{\beta}+1\right)u^{\frac{2}{\alpha}-\frac{2}{\beta}}\prod_{i=1}^n\Psi(c_iu)(1+o(1)).\end{aligned}$$

Similarly, we can find an upper bound for $\sum_{k=-N(u)}^{0}\mathbb{P}\left(\sup_{t\in\triangle_k}\min_{1\le i\le n} Z_i^{\varepsilon+}(t)>u\right)$, which leads to

$$\sum_{k=-N(u)}^{N(u)} \mathbb{P}\left(\sup_{t\in\triangle_k}\min_{1\le i\le n} Z_i^{\varepsilon+}(t)>u\right) \quad\le\quad \frac{\mathcal{H}_{\boldsymbol{c}\sqrt{\boldsymbol{a}^{\varepsilon+}}\boldsymbol{B}_\alpha}(T_1)}{T_1}\Theta_{\varepsilon-}\varphi^*(u)(1+o(1)), \tag{28}$$

where $\Theta_{\varepsilon-}=\left(\overline{\theta}_{\varepsilon-}\right)^{-1/\beta}+\left(\underline{\theta}_{\varepsilon-}\right)^{-1/\beta}$, with $\underline{\theta}_{\varepsilon\pm}=\sum_{i=1}^n c_i^2(\underline{b}_i\pm\varepsilon)$.

Lower bound. Applying again the Bonferroni inequality we have

$$P_1(u)\ge\mathbb{P}\left(\sup_{t\in D_u}\min_{1\le i\le n} Z_i^{\varepsilon-}(t)>u\right)\ge\sum_{k=-N(u)+1}^{N(u)-1}\mathbb{P}\left(\sup_{t\in\triangle_k}\min_{1\le i\le n} Z_i^{\varepsilon-}(t)>u\right)-\Sigma_1(u),$$

where

$$\Sigma_1(u)=\sum\sum_{-N(u)\le k<l\le N(u)}\mathbb{P}\left(\sup_{t\in\triangle_k}\min_{1\le i\le n} Z_i^{\varepsilon-}(t)>u,\sup_{t\in\triangle_l}\min_{1\le i\le n} Z_i^{\varepsilon-}(t)>u\right).$$

With similar arguments as for the derivation of (28) we obtain that

$$\sum_{k=-N(u)+1}^{N(u)-1} \mathbb{P}\left(\sup_{t\in\triangle_k}\min_{1\le i\le n} Z_i^{\varepsilon-}(t)>u\right) \quad\ge\quad \frac{\mathcal{H}_{\boldsymbol{c}\sqrt{\boldsymbol{a}^{\varepsilon-}}\boldsymbol{B}_\alpha}(T_1)}{T_1}\Theta_{\varepsilon+}\varphi^*(u)(1+o(1)), \tag{29}$$

where $\Theta_{\varepsilon+}=\left(\overline{\theta}_{\varepsilon+}\right)^{-1/\beta}+\left(\underline{\theta}_{\varepsilon+}\right)^{-1/\beta}$.

Next we consider the double sum term $\Sigma_1(u)=:\Sigma_2(u)+\Sigma_3(u)$ where $\Sigma_2(u)$ is the sum over indexes $l=k+1$, and $\Sigma_3(u)$ is the sum over indexes $l>k+1$. It follows that

$$\begin{aligned}\Sigma_2(u)\quad&=\quad\sum_{k=-N(u)}^{N(u)}\mathbb{P}\left(\sup_{t\in\triangle_k}\min_{1\le i\le n} Z_i^{\varepsilon-}(t)>u\right)+\sum_{k=-N(u)}^{N(u)}\mathbb{P}\left(\sup_{t\in\triangle_{k+1}}\min_{1\le i\le n} Z_i^{\varepsilon-}(t)>u\right)\\ &\qquad-\sum_{k=-N(u)}^{N(u)}\mathbb{P}\left(\sup_{t\in\triangle_k\cup\triangle_{k+1}}\min_{1\le i\le n} Z_i^{\varepsilon-}(t)>u\right).\end{aligned}$$

Thus, we have from (28) and (29) that

$$\limsup_{u\to\infty}\frac{\Sigma_2(u)}{\varphi^*(u)}\quad\le\quad 2\Theta_{\varepsilon-}\frac{\mathcal{H}_{\boldsymbol{c}\sqrt{\boldsymbol{a}^{\varepsilon+}}\boldsymbol{B}_\alpha}(T_1)}{T_1}-2\Theta_{\varepsilon+}\frac{\mathcal{H}_{\boldsymbol{c}\sqrt{\boldsymbol{a}^{\varepsilon-}}\boldsymbol{B}_\alpha}(2T_1)}{2T_1}. \tag{30}$$

Moreover

$$\begin{aligned}\Sigma_3(u)\quad&=\quad\sum_{k=0}^{N(u)}\sum_{l=k+2}^{N(u)}\mathbb{P}\left(\sup_{t\in\triangle_k}\min_{1\le i\le n} Z_i^{\varepsilon-}(t)>u,\sup_{t\in\triangle_l}\min_{1\le i\le n} Z_i^{\varepsilon-}(t)>u\right)\\ &\qquad+\sum_{k=-N(u)}^{-1}\sum_{l=k+2}^{N(u)}\mathbb{P}\left(\sup_{t\in\triangle_k}\min_{1\le i\le n} Z_i^{\varepsilon-}(t)>u,\sup_{t\in\triangle_l}\min_{1\le i\le n} Z_i^{\varepsilon-}(t)>u\right)\\ &=:\quad\Sigma_{3,1}(u)+\Sigma_{3,2}(u)\end{aligned}$$

An application of Lemma 5.4 gives that

$$\begin{aligned}\Sigma_{3,1}(u) &\le \sum_{k=0}^{N(u)}\sum_{l=k+2}^{N(u)} \mathbb{P}\left(\exists_{t\in\triangle_k}\boldsymbol{\eta}^{\varepsilon-}(t) > \boldsymbol{f}^{\varepsilon+}(k,u), \exists_{t\in\triangle_l}\boldsymbol{\eta}^{\varepsilon-}(t) > \boldsymbol{f}^{\varepsilon+}(l,u)\right)\\ &\le \sum_{k=0}^{N(u)}\sum_{l=k+2}^{N(u)} FT_1^{2n}\exp\left(-G((l-k-1)T_1)^{\alpha}\right)\prod_{i=1}^{n}\Psi\left(\frac{f_i^{\varepsilon+}(k,u)+f_i^{\varepsilon+}(l,u)}{2}\right)\end{aligned}$$

holds with some positive constants F, G for any u sufficiently large. Using the same reasoning as in (28) and noting that $\overline{\theta}_{\varepsilon\pm} > 0, \underline{\theta}_{\varepsilon\pm} > 0$ for sufficiently small ε, we conclude that

$$\limsup_{u\to\infty}\frac{\Sigma_{3,1}(u)}{\varphi^*(u)} \le F_1 T_1^{2n-1}\sum_{l=1}^{\infty}\exp\left(-G(lT_1)^{\alpha}\right),$$

with some $F_1 > 0$. Similar arguments apply also for $\Sigma_{3,2}(u)$ and thus we have

$$\limsup_{T_1\to\infty}\limsup_{u\to\infty}\frac{\Sigma_3(u)}{\varphi^*(u)} = 0. \tag{31}$$

Consequently, by letting $\varepsilon \to 0$ and $T_1 \to \infty$ we obtain from (28–31) that

$$P_1(u) = \mathcal{H}_{\boldsymbol{c}\sqrt{\boldsymbol{a}}\boldsymbol{B}_\alpha}\left(\underline{\theta}^{-\frac{1}{\beta}} + \overline{\theta}^{-\frac{1}{\beta}}\right)\Gamma\left(\frac{1}{\beta}+1\right)u^{\frac{2}{\alpha}-\frac{2}{\beta}}\prod_{i=1}^{n}\Psi\left(c_i u\right)(1+o(1)).$$

Case ii) $\alpha = \beta$: *Upper bound.* Bonferroni inequality implies

$$P_1(u) \le \mathbb{P}\left(\sup_{t\in D_u}\min_{1\le i\le n} Z_i^{\varepsilon+}(t) > u\right) \le \mathbb{P}\left(\sup_{t\in\triangle_{-1}\cup\triangle_0}\min_{1\le i\le n} Z_i^{\varepsilon+}(t) > u\right) + \Sigma_4(u) + \Sigma_5(u),$$

where

$$\Sigma_4(u) = \sum_{k=1}^{N(u)}\mathbb{P}\left(\sup_{t\in\triangle_k}\min_{1\le i\le n} Z_i^{\varepsilon+}(t) > u\right), \quad \Sigma_5(u) = \sum_{k=-N(u)}^{-2}\mathbb{P}\left(\sup_{t\in\triangle_k}\min_{1\le i\le n} Z_i^{\varepsilon+}(t) > u\right).$$

It follows from Corollary 2.2 that

$$\mathbb{P}\left(\sup_{t\in\triangle_{-1}\cup\triangle_0}\min_{1\le i\le n} Z_i^{\varepsilon+}(t) > u\right) = \mathcal{H}_{\boldsymbol{c}\sqrt{\boldsymbol{a}^{\varepsilon+}}\boldsymbol{B}_\alpha, \boldsymbol{c}^2\boldsymbol{d}^{\varepsilon-}}[-T_1, T_1]\prod_{i=1}^{n}\Psi\left(c_i u\right)(1+o(1))$$

as $u \to \infty$, where $\boldsymbol{d}^{\varepsilon\pm}(t) = (d_1^{\varepsilon\pm}(t), \cdots, d_n^{\varepsilon\pm}(t))$ with $d_i^{\varepsilon\pm}(t)$ given as in (26). Moreover, the same arguments as in the derivation of (28) yield

$$\begin{aligned}\Sigma_4(u) &\le \mathcal{H}_{\boldsymbol{c}\sqrt{\boldsymbol{a}^{\varepsilon+}}\boldsymbol{B}_\alpha}(T_1)\sum_{k=1}^{N(u)}\prod_{i=1}^{n}\Psi(f_i^{\varepsilon-}(k,u))(1+o(1))\\ &\le \mathcal{H}_{\boldsymbol{c}\sqrt{\boldsymbol{a}^{\varepsilon+}}\boldsymbol{B}_\alpha}(T_1)\prod_{i=1}^{n}\Psi\left(c_i u\right)\sum_{k=1}^{\infty}\exp\left(-\overline{\theta}_{\varepsilon-}(kT_1)^{\beta}\right)(1+o(1))\end{aligned}$$

and similarly

$$\Sigma_5(u) \le \mathcal{H}_{\boldsymbol{c}\sqrt{\boldsymbol{a}^{\varepsilon+}}\boldsymbol{B}_\alpha}(T_1)\prod_{i=1}^{n}\Psi\left(c_i u\right)\sum_{k=1}^{\infty}\exp\left(-\underline{\theta}_{\varepsilon-}(kT_1)^{\beta}\right)(1+o(1)).$$

Lower bound. Let T_2 be any positive constant. We have by Corollary 2.2 that

$$\begin{aligned}P_1(u) &\ge \mathbb{P}\left(\sup_{t\in\left[-T_2u^{-\frac{2}{\alpha}}, T_2u^{-\frac{2}{\alpha}}\right]}\min_{1\le i\le n} Z_i^{\varepsilon-}(t) > u\right)\\ &= \mathcal{H}_{\boldsymbol{c}\sqrt{\boldsymbol{a}^{\varepsilon-}}\boldsymbol{B}_\alpha, \boldsymbol{c}^2\boldsymbol{d}^{\varepsilon+}}[-T_2, T_2]\prod_{i=1}^{n}\Psi\left(c_i u\right)(1+o(1))\end{aligned}$$

as $u \to \infty$. From the upper and lower bounds obtained above we conclude that

$$\begin{aligned}\mathcal{H}_{\boldsymbol{c}\sqrt{\boldsymbol{a}^{\varepsilon-}}\boldsymbol{B}_\alpha,\boldsymbol{c}^2\boldsymbol{d}^{\varepsilon+}}[-T_2,T_2] &\le \liminf_{u\to\infty}\frac{P_1(u)}{\prod_{i=1}^n\Psi(c_iu)}\\ &\le \limsup_{u\to\infty}\frac{P_1(u)}{\prod_{i=1}^n\Psi(c_iu)}\\ &\le \mathcal{H}_{\boldsymbol{c}\sqrt{\boldsymbol{a}^{\varepsilon+}}\boldsymbol{B}_\alpha,\boldsymbol{c}^2\boldsymbol{d}^{\varepsilon-}}[-T_1,T_1]+\mathcal{H}_{\boldsymbol{c}\sqrt{\boldsymbol{a}^{\varepsilon+}}\boldsymbol{B}_\alpha}(T_1)\sum_{k=1}^\infty\exp\left(-\underline{\theta}_{\varepsilon-}(kT_1)^\beta\right)\\ &\quad+\mathcal{H}_{\boldsymbol{c}\sqrt{\boldsymbol{a}^{\varepsilon+}}\boldsymbol{B}_\alpha}(T_1)\sum_{k=1}^\infty\exp\left(-\underline{\theta}_{\varepsilon+}(kT_1)^\beta\right).\end{aligned}$$

In the light of (6), using that $\underline{\theta}>0,\overline{\theta}>0$, and letting $\varepsilon\to0,T_2\to\infty$ on the left-hand side of the last equation we obtain that $\mathcal{H}^{\boldsymbol{c}^2\underline{\boldsymbol{b}},\boldsymbol{c}^2\overline{\boldsymbol{b}}}_{\boldsymbol{c}\sqrt{\boldsymbol{a}}\boldsymbol{B}_\alpha}<\infty$. Similarly, letting $\varepsilon\to0,T_1\to\infty$ on the right-hand side implies $\mathcal{H}^{\boldsymbol{c}^2\underline{\boldsymbol{b}},\boldsymbol{c}^2\overline{\boldsymbol{b}}}_{\boldsymbol{c}\sqrt{\boldsymbol{a}}\boldsymbol{B}_\alpha}>0$. Therefore, we conclude that

$$P_1(u)=\mathcal{H}^{\boldsymbol{c}^2\underline{\boldsymbol{b}},\boldsymbol{c}^2\overline{\boldsymbol{b}}}_{\boldsymbol{c}\sqrt{\boldsymbol{a}}\boldsymbol{B}_\alpha}\prod_{i=1}^n\Psi(c_iu)(1+o(1)),\quad u\to\infty.$$

Case iii) $\alpha>\beta$:

Upper bound. Since $\alpha>\beta$, we have that

$$\begin{aligned}P_1(u) &\le \mathbb{P}\left(\sup_{t\in D_u}\min_{1\le i\le n}Z_i^{\varepsilon+}(t)>u\right)\le\mathbb{P}\left(\sup_{t\in\triangle_{-1}\cup\triangle_0}\min_{1\le i\le n}Z_i^{\varepsilon+}(t)>u\right)\\ &\le \mathbb{P}\left(\sup_{t\in\triangle_{-1}\cup\triangle_0}\min_{1\le i\le n}W_i^{\varepsilon+}(t)>u\right),\end{aligned}$$

where

$$W_i^{\varepsilon+}(t)=\frac{\sigma_{X_i}(t_0)}{1+d_i^{\varepsilon-}(t)}V_i(t),\quad t\in\mathbb{R},$$

with $V_i(t)$ being a centered stationary Gaussian process with covariance $r_{V_i}(t)=\exp(-|t|^\beta),t\ge0$. Thus, by the proof of Case *ii)*

$$P_1(u)\le\mathcal{H}_{\boldsymbol{c}\boldsymbol{B}_\beta,\boldsymbol{c}^2\boldsymbol{d}^{\varepsilon-}}[-T_1,T_1]\prod_{i=1}^n\Psi(c_iu)(1+o(1)),\quad u\to\infty.$$

Lower bound. It follows easily that

$$P_1(u)\ge\mathbb{P}\left(\min_{1\le i\le n}X_i(t_0)>u\right)=\prod_{i=1}^n\Psi(c_iu).$$

Letting $T_1\to0$ we have from the above upper and lower bounds that

$$P_1(u)=\prod_{i=1}^n\Psi(c_iu)(1+o(1)),\quad u\to\infty.$$

Asymptotics of $P_2(u)$: In order to complete the proof we show that (25) is valid. To this end, we shall derive an adequate upper bound for $P_2(u)$ by utilizing the generalized Borell-TIS and Piterbarg inequalities given in Lemma 5.2 and Lemma 5.3, respectively.

By Assumption I and Assumption III we can choose some small $\varepsilon>0$ such that

$$\inf_{t\in[0,T]\setminus[t_0-\varepsilon,t_0+\varepsilon]}g(t)\ >\ g(t_0).$$

Clearly

$$P_2(u)\le\mathbb{P}\left(\sup_{t\in[0,T]\setminus[t_0-\varepsilon,t_0+\varepsilon]}\min_{1\le i\le n}X_i(t)>u\right)+\mathbb{P}\left(\sup_{t\in[t_0-\varepsilon,t_0+\varepsilon]\setminus(t_0+D_u)}\min_{1\le i\le n}X_i(t)>u\right).$$

Further, by the generalized Borell-TIS inequality in Lemma 5.2 for all sufficiently large u we have

$$\mathbb{P}\left(\sup_{t\in[0,T]\setminus[t_0-\varepsilon,t_0+\varepsilon]}\min_{1\le i\le n}X_i(t)>u\right)\le \exp\left(-\frac{(u-\mu)^2}{2}\inf_{t\in[0,T]\setminus[t_0-\varepsilon,t_0+\varepsilon]}g(t)\right) \tag{32}$$

for some $\mu>0$. Moreover, in view of the generalized Piterbarg inequality in Lemma 5.3 we obtain for all sufficiently large u

$$\mathbb{P}\left(\sup_{t\in[t_0-\varepsilon,t_0+\varepsilon]\setminus(t_0+D_u)}\min_{1\le i\le n}X_i(t)>u\right)\le Q_1u^{\frac{2}{\min(\gamma,\alpha)}-1}\exp\left(-\frac{u^2}{2}\inf_{t\in[t_0-\varepsilon,t_0+\varepsilon]\setminus(t_0+D_u)}g(t)\right)$$

for some $Q_1>0$. In addition (recall that $\underline{\theta}>0$, $\overline{\theta}>0$), we have that

$$g(t_0+t)\ge g(t_0)+Q_2(\delta(u))^\beta$$

holds for all $t\in[-\varepsilon,\varepsilon]\setminus D_u$. Therefore, for all u large

$$\mathbb{P}\left(\sup_{t\in[t_0-\varepsilon,t_0+\varepsilon]\setminus D_u}\min_{1\le i\le n}X_i(t)>u\right)\le Q_1u^{\frac{2}{\min(\gamma,\alpha)}-1}\exp\left(-\frac{u^2}{2}g(t_0)\right)\exp\left(-\frac{Q_2}{2}(\ln u)^2\right). \tag{33}$$

Consequently, the claim in (25) follows immediately from (32), (33) and the asymptotics of $P_1(u)$ obtained above. Thus the proof is complete. □

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