Asymptotics of Maxima of Strongly Dependent Gaussian Processes

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Abstract: Let $\{X_n(t), t \in [0,\infty)\}, n \in \mathbb{N}$ be a sequence of centered dependent stationary Gaussian processes. The limit distribution of $\sup_{t\in[0,T(n)]} |X_n(t)|$ is established as $r_n(t)$, the correlation function of $\{X_n(t), t \in [0,\infty)\}, n \in \mathbb{N}$, satisfies the local and long range strong dependence conditions, which extends the results obtained by Seleznjev (1991).
Key Words: Stationary Gaussian process; strong dependence; Berman condition; limit theorems; Pickands constant.
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1 Introduction

Let $\{X(t), t \in [0,\infty)\}$ be a standard (mean zero and unit variance) stationary Gaussian process with continuous sample paths, and let $\{r(t), t \geq 0\}$ denote its correlation function. Assume that the correlation function $r(t)$ of the process satisfies

$$r(t) = 1 - |t|^\alpha + o(|t|^\alpha) \quad \text{as} \quad t \to 0, \quad \text{and} \quad r(t) < 1 \quad \text{for} \quad t > 0 \tag{1.1}$$

for some $\alpha \in (0,2]$, and further assume

$$r(t)\log t \to 0, \quad \text{as} \quad t \to \infty. \tag{1.2}$$

For the study of the asymptotic properties of the supremum of Gaussian processes the local condition (1.1) is a standard one, whereas the condition (1.2) is the weak dependence condition, or the so-called Berman's condition, see e.g., Piterbarg (1996). Under these two conditions on the correlation function $r(t)$, it is well-known (see e.g., Leadbetter et al. (1983) or Berman (1992)) that

$$\lim_{T\to\infty} \sup_{x\in\mathbb{R}} \left| \mathbb{P}\left\{ a_T \left(\sup_{t\in[0,T]} X(t) - b_T \right) \leq x \right\} - \exp(-e^{-x}) \right| = 0, \tag{1.3}$$

where

$$a_T = \sqrt{2\log T}, \quad b_T = \sqrt{2\log T} + \frac{\log(\mathcal{H}_\alpha (2\pi)^{-1/2} (2\log T)^{-1/2+1/\alpha})}{\sqrt{2\log T}}. \tag{1.4}$$

Here $\mathcal{H}_\alpha$ denotes the Pickands constant defined by $\mathcal{H}_\alpha = \lim_{\lambda\to\infty} \lambda^{-1}\mathcal{H}_\alpha(\lambda)$, where

$$\mathcal{H}_\alpha(\lambda) = \mathbb{E}\left\{ \exp\left(\max_{t\in[0,\lambda]} \sqrt{2} B_{\alpha/2}(t) - t^\alpha \right) \right\}$$

and B_α is a fractional Brownian motion (a mean zero Gaussian process with stationary increments such that $\mathbb{E}\left\{B_\alpha^2(t)\right\} = |t|^{2\alpha}, t \in \mathbb{R}$). It is also well-known that $0 < \mathcal{H}_\alpha < \infty$, see e.g., Berman (1992), and Piterbarg (1996).
In this paper, the following Pickands exact asymptotics plays a curial role in deriving the limit relation of (1.3). Specifically, for some fixed constant $h > 0$

$$\mathbb{P}\left\{ \sup_{t\in[0,h]} X(t) > u \right\} \;=\; h\mu(u)(1 + o(1)), \quad \text{as} \quad u \to \infty, \tag{1.5}$$

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provided that the correlation function $r(t)$ satisfies (1.1) and

$$\mu(u) = \mathcal{H}_\alpha u^{2/\alpha}\Psi(u), \tag{1.6}$$

where $\Psi(\cdot)$ is the survival function of a standard Gaussian random variable. For more details see Leadbetter et al. (1983) and Piterbarg (1996). A correct proof of Pickand's theorem (see Pickands (1969)) was given in Piterbarg (1972); for the main properties of Pickands and related constants, see Adler (1990), Berman (1992), Shao (1996), Dieker (2005), Dębicki and Kisowski (2009) and Albin and Choi (2010).

A uniform version of (1.5) for stationary Gaussian processes has been established by Seleznjev (1991), where the author investigated the limit distribution of the error of approximation of Gaussian stationary periodic processes by random trigonometric polynomials in the uniform metric. Next, we formulate the aforementioned result.

Theorem A. *Let* $\{X_n(t), t \in [0,\infty)\}, n \in \mathbb{N}$ *be standard stationary Gaussian processes with a.s. continuous sample paths and correlation function* $r_n(t)$. *Let* $T(n) > 0, u_n, n \ge 1$ *be constants such that* $\lim_{n\to\infty} \min(T(n), u_n) = \infty$. Suppose further that

(A1). $r_n(t) = 1 - c_n|t|^\alpha + \varepsilon_n(t)|t|^\alpha$, $0 < \alpha \le 2$, where $c_n \to 1$ as $n \to \infty$ and $\varepsilon_n(t) \to 0$ as $t \to 0$, uniformly in n;

(A2). for any $\varepsilon > 0$, there exists $\gamma > 0$ such that $\sup\{|r_n(t)|, T \ge |t| \ge \varepsilon, n \in \mathbb{N}\} < \gamma < 1$;

(A3). $r_n(t)\log(t) \to 0$ as $t \to \infty$, uniformly in n.

(i). If (A1) and (A2) hold, then for any fixed $h > 0$ and $\mu(\cdot)$ defined in (1.6)

$$\lim_{n\to\infty} \frac{\mathbb{P}\left\{\sup_{t\in[0,h]} |X_n(t)| > u_n\right\}}{2h\mu(u_n)} = 1.$$

(ii). If additionally $\lim_{n\to\infty} T(n)\mu(u_n) = \theta \in (0,\infty]$ and (A3) hold, then

$$\lim_{n\to\infty} \mathbb{P}\left\{\sup_{t\in[0,T(n)]} |X_n(t)| \le u_n\right\} = e^{-2\theta},$$

where we set $e^{-2\theta} = 0$ if $\theta = \infty$.

(iii). If instead of Assumptions (A1)-(A3), the correlation functions $r_n(t)$ are such that

$$1 - r_n(t) \le |t|^\alpha, \quad t \in [0, T(n)],$$

with $\alpha \in (0,2]$ and $T(n) \ge T_0 > 0$ for all large n, then

$$\lim_{n\to\infty} \mathbb{P}\left\{\sup_{t\in[0,T(n)]} |X_n(t)| \le u_n\right\} = 1,$$

provided that $\lim_{n\to\infty} T(n)\mu(u_n) = 0$.

(iv). Let $a_{T(n)}, b_{T(n)}$ be defined as in (1.4). If (A1), (A2) and (A3) hold, then

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| \mathbb{P}\left\{ a_{T(n)}\left(\sup_{t\in[0,T(n)]} |X_n(t)| - b_{T(n)}\right) \le x\right\} - \exp(-2e^{-x})\right| = 0.$$

The above result has been extended by Seleznjev (1996) to a certain class of non-stationary Gaussian processes. For further extensions and related studies, we refer to Hüsler (1999), Hüsler et al. (2003) and Seleznjev (2006).

With impetus from Seleznjev (1991), in this paper we present the corresponding version of Theorem A for a sequence of strongly dependent stationary Gaussian processes (see definition below).

The paper is organized as follows. Section 2 displays the main result, followed then by Section 3 where we present the proofs.

2 Main Results

In this section, we extend Theorem A to a sequence of strongly dependent stationary Gaussian processes. A sequence of standard stationary Gaussian process $\{X_n(t), t \in [0,\infty)\}, n \in \mathbb{N}$ is called strongly dependent if the correlation function $r_n(t)$ satisfies one of the following assumptions:

(B1). $r_n(t)\log t \to r \in (0,\infty)$ as $t \to \infty$, uniformly in n;

(B2). $r_n(t)\log t \to \infty$ as $t \to \infty$, uniformly in n.

Indeed, Assumptions (B1) and (B2) are natural extensions of Assumption (A3). For related studies on extremes for strongly dependent Gaussian process, we refer to Mital and Ylvisaker (1975), Piterbarg (1996), Ho and McCormick (1999) and Stamatovic and Stamatovic (2010).

Let in the following φ and Φ denote the probability density function and the distribution function of a standard Gaussian random variable $\mathcal{W}$, respectively, and set

$$\Lambda_r(x) \quad = \quad \mathbb{E}\left\{[\Lambda(x+r)]^{e^{\sqrt{2r}\mathcal{W}}+e^{-\sqrt{2r}\mathcal{W}}}\right\}, \quad x \in \mathbb{R}, \tag{2.7}$$

with $\Lambda(x) = \exp(-\exp(-x)), x \in \mathbb{R}$ the unit Gumbel distribution function.

Next, we state our main results.

Theorem 2.1. *Let $\{X_n(t), t \in [0,\infty)\}, n \in \mathbb{N}$ be a standard stationary Gaussian processes with a.s. continuous sample paths and correlation function $r_n(t)$ satisfying (A1),(A2) and (B1).*
(i). If $\lim_{n\to\infty} T(n)\mu(u_n) = \theta \in (0,\infty]$, then

$$\lim_{n\to\infty} \mathbb{P}\left\{\sup_{t\in[0,T(n)]} |X_n(t)| \le u_n\right\} \quad = \quad \Lambda_r(-\log\theta), \tag{2.8}$$

where $\Lambda_r(-\log\theta) =: 0$ if $\theta = \infty$.
(ii). Let $a_{T(n)}, b_{T(n)}$ be defined as in (1.4), for $x \in \mathbb{R}$ we have

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| \mathbb{P}\left\{a_{T(n)}\left(\sup_{t\in[0,T(n)]} |X_n(t)| - b_{T(n)}\right) \le x\right\} - \Lambda_r(x)\right| = 0. \tag{2.9}$$

Remarks 2.1. *(a) From the proof of Theorem 2.1, it follows that both (2.8) and (2.10) can be shown to hold also for $r = 0$, retrieving thus the result of Theorem A.*
(b) Assertion (iii) of Theorem A still holds under the conditions of Theorem 2.1.

Theorem 2.2. *Let $\{X_n(t), t \in [0,\infty)\}, n \in \mathbb{N}$ be a standard stationary Gaussian processes with a.s. continuous sample paths and correlation function $r_n(t)$ satisfying (A1) with $0 < \alpha \le 1$, (A2) and (B2). Assume that $r_n(t)$ is convex for $t \ge 0$ and $r_n(t) = o(1)$ uniformly in n. If further $r_n(t)\log t$ is monotone for large t, then with $b_{T(n)}$ as in (1.4), we have*

$$\lim_{n\to\infty} \sup_{x\in(0,\infty)} \left| \mathbb{P}\left\{r_n^{-1/2}(T(n))\left(\sup_{t\in[0,T(n)]} |X_n(t)| - (1-r_n(T(n)))^{1/2} b_{T(n)}\right) \le x\right\} - 2\Phi(x) + 1\right| = 0. \tag{2.10}$$

Remarks 2.2. *Theorem 2.2 is a uniform version of Theorem 3.1 of Mittal and Ylvisaker (1975).*

3 Further Results and Proofs

We begin with some auxiliary lemmas needed for the proofs of Theorem 2.1 and 2.2.

For given $\varepsilon > 0$, we divide interval $[0, T(n)]$ onto intervals of length 1, and split each of them onto subintervals I_j^ε, I_j of length ε, $1-\varepsilon$, $j = 1, 2, \cdots, [T(n)]$, respectively, where $[x]$ denotes the integral part of x. It can be easily seen that a possible remaining interval with length smaller than 1 plays no role in our consideration. We denote this interval with J.

Let $\{X_n^{(i)}(t), t \geq 0\}$, $i = 1, 2, \cdots$ be independent copies of $\{X_n(t), t \geq 0\}$ and $\{\eta_n(t), t \geq 0\}$ be such that $\eta_n(t) = X_n^{(j)}(t)$ for $t \in I_j$. Let $\rho(T(n)) := r/\log T(n)$ and define

$$\xi_n(t) = (1 - \rho(T(n)))^{1/2}\eta_n(t) + \rho^{1/2}(T(n))\mathcal{W}, \quad t \in \cup_{j=1}^{[T(n)]} I_j,$$

where $\mathcal{W}$ is a standard Gaussian random variable independent of $\{\eta_n(t), t \geq 0\}$. Note that $\{\xi_n(t), t \in \cup_{j=1}^{[T(n)]} I_j\}$ is a standard non-stationary Gaussian process with correlation function $\varrho_n(\cdot, \cdot)$ which is given by

$$\varrho_n(t, s) = \begin{cases} r_n(t, s) + (1 - r_n(t, s))\rho(T(n)), & t \in I_j, s \in I_i, i = j, \\ \rho(T(n)), & t \in I_j, s \in I_i, i \neq j. \end{cases}$$

In the sequel, assume that a, u_n, v_n are positive constants, and set

$$q := q(u_n) = a u_n^{-2/\alpha}, \quad \mu(u_n) := \mathcal{H}_\alpha u_n^{2/\alpha}\Psi(u_n), \quad \delta(a) := 1 - \frac{\mathcal{H}_\alpha(a)}{\mathcal{H}_\alpha}.$$

Further, $C_1 - C_6$ shall denote positive constants whose values may vary from place to place.

Lemma 3.1. *If the Assumptions (A1) and (A2) hold, then for each interval I of fixed length $h > 0$*

$$0 \leq \mathbb{P}\left\{\max_{jq \in I} |X_n(jq)| \leq u_n\right\} - \mathbb{P}\left\{\sup_{s \in I} |X_n(s)| \leq u_n\right\} \leq 2h\delta(a)\mu(u_n) + o(\mu(u_n)) \tag{3.11}$$

and

$$0 \leq \mathbb{P}\left\{\max_{jq \in I} X_n(jq) \leq u_n\right\} - \mathbb{P}\left\{\sup_{s \in I} X_n(s) \leq u_n\right\} \leq h\delta(a)\mu(u_n) + o(\mu(u_n)), \tag{3.12}$$

where $\delta(a) \to 0$ as $a \downarrow 0$.

Proof. Both claims above are established in the proof of Theorem 1 of Seleznjev (1991). □

Lemma 3.2. *Suppose that (A1) and (A2) hold. If $T(n)\mu(u_n) = O(1)$ and $T(n)\mu(v_n) = O(1)$, then*

$$\mathbb{P}\left\{\sup_{s \in [0, T(n)]} |X_n(s)| \leq u_n\right\} - \mathbb{P}\left\{\sup_{s \in \cup I_j} |X_n(s)| \leq u_n\right\} \quad \to \quad 0 \tag{3.13}$$

and

$$\mathbb{P}\left\{-v_n \leq \inf_{s \in [0,1]} X_n(s), \sup_{s \in [0,1]} X_n(s) \leq u_n\right\} - \mathbb{P}\left\{-v_n \leq \inf_{s \in I_1} X_n(s), \sup_{s \in I_1} X_n(s) \leq u_n\right\} \quad \to \quad 0 \tag{3.14}$$

as $n \to \infty$ and $\varepsilon \downarrow 0$.

Proof. By the stationarity of $\{X_n(t), t \in [0, T(n)]\}$ and Theorem A (i) we obtain

$$\begin{aligned} &\left|\mathbb{P}\left\{\sup_{s \in [0, T(n)]} |X_n(s)| \leq u_n\right\} - \mathbb{P}\left\{\sup_{s \in \cup I_j} |X_n(s)| \leq u_n\right\}\right| \\ &\leq \sum_{j=1}^{[T(n)]} \mathbb{P}\left\{\max_{s \in I_j^\varepsilon} |X_n(s)| > u_n\right\} + \mathbb{P}\left\{\max_{s \in J} |X_n(s)| > u_n\right\} \\ &\leq 2([T(n)]\varepsilon + 1)\mu(u_n)(1 + o(1)) \\ &= O(1)\varepsilon(1 + o(1)) \\ &\to 0 \end{aligned}$$

as $u \to \infty$ and $\varepsilon \downarrow 0$, which completes the proof of (3.13). Note in passing that

$$\left|\mathbb{P}\left\{-v_n \leq \inf_{s \in [0,1]} X_n(s), \sup_{s \in [0,1]} X_n(s) \leq u_n\right\} - \mathbb{P}\left\{-v_n \leq \inf_{s \in I_1} X_n(s), \sup_{s \in I_1} X_n(s) \leq u_n\right\}\right|$$

$$\begin{aligned}
&\leq \left|\mathbb{P}\left\{\sup_{s\in[0,1]} X_n(s)\leq u_n\right\}-\mathbb{P}\left\{\sup_{s\in I_1} X_n(s)\leq u_n\right\}\right|\\
&+ \left|\mathbb{P}\left\{\inf_{s\in[0,1]} X_n(s)\geq -v_n\right\}-\mathbb{P}\left\{\inf_{s\in I_1} X_n(s)\geq -v_n\right\}\right|.
\end{aligned}$$

The proof of (3.14) is similar to that of (3.13), and therefore omitted. □

Lemma 3.3. *Under the assumptions of Lemma 3.2 we have*

$$\mathbb{P}\left\{\sup_{s\in\cup I_j}|X_n(s)|\leq u_n\right\}-\mathbb{P}\left\{\max_{kq\in\cup I_j}|X_n(kq)|\leq u_n\right\} \quad\to\quad 0 \tag{3.15}$$

and

$$\mathbb{P}\left\{-v_n\leq\inf_{s\in I_1}X_n(s),\sup_{s\in I_1}X_n(s)\leq u_n\right\}-\mathbb{P}\left\{-v_n\leq\min_{kq\in I_1}X_n(kq),\max_{kq\in I_1}X_n(kq)\leq u_n\right\} \quad\to\quad 0 \tag{3.16}$$

as $n\to\infty$ and $a\downarrow 0$.

Proof. By Lemma 3.2

$$\begin{aligned}
&\left|\mathbb{P}\left\{\sup_{s\in\cup I_j}|X_n(s)|\leq u_n\right\}-\mathbb{P}\left\{\sup_{kq\in\cup I_j}|X_n(kq)|\leq u_n\right\}\right|\\
&\leq T(n)\max_j\left(\mathbb{P}\left\{\max_{kq\in I_j}|X_n(kq)|\leq u_n\right\}-\mathbb{P}\left\{\sup_{s\in I_j}|X_n(s)|\leq u_n\right\}\right)\\
&\leq 2(1-\varepsilon)[T(n)]\mu(u_n)\delta(a)+T(n)o(\mu(u_n))\\
&=2(1-\varepsilon)O(1)\delta(a)+o(1)\\
&\to 0
\end{aligned}$$

as $n\to\infty$ and $a\downarrow 0$. Hence the first claim follows. Note that

$$\begin{aligned}
&\left|\mathbb{P}\left\{-v_n\leq\inf_{s\in I_1}X_n(s),\sup_{s\in I_1}X_n(s)\leq u_n\right\}-\mathbb{P}\left\{-v_n\leq\min_{kq\in I_1}X_n(kq),\max_{kq\in I_1}X_n(kq)\leq u_n\right\}\right|\\
&\leq \left|\mathbb{P}\left\{\max_{kq\in I_1}X_n(kq)\leq u_n\right\}-\mathbb{P}\left\{\sup_{s\in I_1}X_n(s)\leq u_n\right\}\right|+\left|\mathbb{P}\left\{\min_{kq\in I_1}X_n(kq)\geq -v_n\right\}-\mathbb{P}\left\{\inf_{s\in I_1}X_n(s)\geq -v_n\right\}\right|.
\end{aligned}$$

We omit the proof of (3.16) since it is similar to that of (3.15). □

Lemma 3.4. *Suppose that (A1),(A2) and (B1) hold. If $T(n)\mu(u_n)=O(1)$, then*

$$\lim_{n\to\infty}\left|\mathbb{P}\left\{\max_{kq\in\cup I_j}|X_n(kq)|\leq u_n\right\}-\mathbb{P}\left\{\max_{kq\in\cup I_j}|\xi_n(kq)|\leq u_n\right\}\right| \quad=\quad 0. \tag{3.17}$$

Proof. Applying the generalized Berman inequality (cf. Theorem 1.2 of Piterbarg (1996)), we have (set next $T:=T(n)$)

$$\begin{aligned}
&\left|\mathbb{P}\left\{\max_{kq\in\cup I_j}|X_n(kq)|\leq u_n\right\}-\mathbb{P}\left\{\max_{kq\in\cup I_j}|\xi_n(kq)|\leq u_n\right\}\right|\\
&\leq \sum_{kq\in I_i,\,lq\in I_j}\frac{4}{2\pi}|r_n(kq,lq)-\varrho_n(kq,lq)|\int_0^1\frac{1}{\sqrt{1-r^{(h)}(kq,lq)}}\exp\left(-\frac{u_n^2}{1+r^{(h)}(kq,lq)}\right)dh\\
&\leq \sum_{\substack{kq\in I_i,\,lq\in I_i,\\ i\in\{1,2,\cdots,[T(n)]\}}}\mathbb{A}(n,k,l,q)+\sum_{\substack{kq\in I_i,\,lq\in I_j,i\neq j\\ i,j\in\{1,2,\cdots,[T(n)]\}}}\mathbb{A}(n,k,l,q),
\end{aligned} \tag{3.18}$$

where $\varphi(x,y,r^{(h)})$ is a Gaussian two-dimensional density with the covariance $r^{(h)}$, the variance equal to one and zero mean and

$$r^{(h)}(kq,lq)=hr_n(kq,lq)+(1-h)\varrho_n(kq,lq),\quad h\in[0,1].$$

In the following part of the proof, let $\varpi_n(kq,lq) = \max\{|r_n(kq,lq)|, |\varrho_n(kq,lq)|\}$ and $\vartheta(t) = \sup_{t<|kq-lq|\le T}\{\varpi_n(kq,lq)\}$. By Assumption (A2) and the definition of $\varrho_n(t,s)$, we have $\vartheta(\varepsilon) = \sup_{\varepsilon<|kq-lq|\le T}\{\varpi_n(kq,lq); n \in \mathbb{N}\} < 1$ for sufficiently large T. Further, let β be such that $0 < \beta < \frac{1-\vartheta(\varepsilon)}{1+\vartheta(\varepsilon)}$ for all sufficiently large T.

Next, we estimate the upper bound of (3.18) in the case that kq and lq belong to the same interval I. Note that in this case, $\varrho_n(kq,lq) = r_n(kq,lq) + (1-r_n(kq,lq))\rho(T) \sim r_n(kq,lq)$ for sufficiently large T. Split the first term of (3.18) into two parts as

$$\sum_{\substack{kq\in I_i,\ lq\in I_i, i\in\{1,2,\cdots,[T(n)]\} \\ 0<|kq-lq|\le\varepsilon}} \mathbb{A}(n,k,l,q) + \sum_{\substack{kq\in I_i,\ lq\in I_i, i\in\{1,2,\cdots,[T(n)]\} \\ \varepsilon<|kq-lq|\le 1-\varepsilon}} \mathbb{A}(n,k,l,q) =: J_{n1} + J_{n2}. \tag{3.19}$$

The Assumption (A1) implies for all $|t| \le \varepsilon < 2^{-1/\alpha}$

$$1 - r_n(t) \le 2|t|^\alpha.$$

From the assumption that $T\mu(u_n) = T(n)\mu(u_n) = O(1)$, we have

$$u_n \sim (2\log T)^{1/2}, \quad e^{-\frac{u_n^2}{2}} \sim (2\pi)^{1/2} H_\alpha^{-1} u_n^{1-2/\alpha} T^{-1} O(1). \tag{3.20}$$

Consequently, with $q := a u_n^{-2/\alpha} \sim a(\log T)^{-1/\alpha}$ we obtain

$$\begin{aligned}
J_{n1} &\le C_1 \sum_{\substack{kq\in I_i,\ lq\in I_i, i\in\{1,2,\cdots,[T(n)]\} \\ 0<|kq-lq|\le\varepsilon}} |r_n(kq,lq) - \varrho_n(kq,lq)| \frac{1}{\sqrt{1-\varrho_n(kq,lq)}} \exp\left(-\frac{u_n^2}{1+\varrho_n(kq,lq)}\right) \\
&\le C_1 \sum_{\substack{kq\in I_i,\ lq\in I_i, i\in\{1,2,\cdots,[T(n)]\} \\ 0<|kq-lq|\le\varepsilon}} |(1-r_n(kq,lq))\rho(T)| \frac{1}{\sqrt{1-r_n(kq,lq)}} \exp\left(-\frac{u_n^2}{1+r_n(kq,lq)}\right) \\
&\le C_1\rho(T) \sum_{\substack{kq\in I_i,\ lq\in I_i, i\in\{1,2,\cdots,[T(n)]\} \\ 0<|kq-lq|\le\varepsilon}} \sqrt{1-r_n(kq,lq)} \exp\left(-\frac{u_n^2}{1+r_n(kq,lq)}\right) \\
&\le C_1\rho(T)\frac{T}{q} \sum_{0<kq\le\varepsilon} \sqrt{1-r_n(kq)} \exp\left(-\frac{u_n^2}{2}\right) \exp\left(-\frac{(1-r_n(kq))u_n^2}{2(1+r_n(kq))}\right) \\
&\le C_1\rho(T)\frac{T}{q} T^{-1}(\log T)^{1/2-1/\alpha} \sum_{0<kq\le\varepsilon} (kq)^{\alpha/2} \exp\left(-\frac{1}{8}|kq|^\alpha\right) \\
&\le C_1(\log T)^{-1/2},
\end{aligned} \tag{3.21}$$

which implies $\lim_{n\to\infty} J_{n1} = 0$. By (3.20) for large T we have

$$\begin{aligned}
J_{n2} &\le C_2 \sum_{\substack{kq\in I_i,\ lq\in I_i, i\in\{1,2,\cdots,[T(n)]\} \\ \varepsilon<|kq-lq|\le 1-\varepsilon}} |r_n(kq,lq) - \varrho_n(kq,lq)| \exp\left(-\frac{u_n^2}{1+\varpi_n(kq,lq)}\right) \\
&\le C_2 \sum_{\substack{kq\in I_i,\ lq\in I_i, i\in\{1,2,\cdots,[T(n)]\} \\ \varepsilon<|kq-lq|\le 1-\varepsilon}} \exp\left(-\frac{u_n^2}{1+\vartheta(\varepsilon)}\right) \\
&\le C_2\frac{T}{q} \sum_{\varepsilon<kq\le 1-\varepsilon} \exp\left(-\frac{u_n^2}{1+\vartheta(\varepsilon)}\right) \\
&\le C_2\frac{T}{q^2} \left(\exp\left(-\frac{u_n^2}{2}\right)\right)^{\frac{2}{1+\vartheta(\varepsilon)}} \\
&\le C_2 T^{-\frac{1-\vartheta(\varepsilon)}{1+\vartheta(\varepsilon)}} (\log T)^{\frac{2\vartheta(\varepsilon)+\alpha}{\alpha(1+\vartheta(\varepsilon))}}.
\end{aligned} \tag{3.22}$$

Hence since $\vartheta(\varepsilon) < 1$, then $\lim_{n\to\infty} J_{n2} = 0$.

We continue with an estimate for the upper bound of (3.18) where $kq \in I_i$ and $lq \in I_j$, $i \ne j$. Note that in this case, the distance between any two intervals I_i and I_j is large than ε. Split the second term of (3.18) as

$$\sum_{\substack{kq\in I_i,\ lq\in I_j, i\ne j\in\{1,2,\cdots,[T(n)]\} \\ \varepsilon<|kq-lq|\le T^\beta}} \mathbb{A}(n,k,l,q) + \sum_{\substack{kq\in I_i,\ lq\in I_j, i\ne j\in\{1,2,\cdots,[T(n)]\} \\ T^\beta<|kq-lq|\le T}} \mathbb{A}(n,k,l,q) =: I_{n1} + I_{n2}. \tag{3.23}$$

Similarly to the derivation of (3.22), we have

$$\begin{aligned}
I_{n1} &\leq C_3 \sum_{\substack{kq\in I_i,\ lq\in I_j, i\neq j\in\{1,2,\cdots,[T(n)]\}\\ \varepsilon<|kq-lq|\leq T^\beta}} |r_n(kq,lq)-\varrho_n(kq,lq)| \exp\left(-\frac{u_n^2}{1+\varpi_n(kq,lq)}\right)\\
&\leq C_3 \sum_{\substack{kq\in I_i,\ lq\in I_j, i\neq j\in\{1,2,\cdots,[T(n)]\}\\ \varepsilon<|kq-lq|\leq T^\beta}} \exp\left(-\frac{u_n^2}{1+\vartheta(\varepsilon)}\right)\\
&\leq C_3 \frac{T}{q} \sum_{\varepsilon<kq\leq T^\beta} \exp\left(-\frac{u_n^2}{1+\vartheta(\varepsilon)}\right)\\
&\leq C_3 \frac{T^{1+\beta}}{q^2} \left(\exp\left(-\frac{u_n^2}{2}\right)\right)^{\frac{2}{1+\vartheta(\varepsilon)}}\\
&\leq C_3 T^{\beta-\frac{1-\vartheta(\varepsilon)}{1+\vartheta(\varepsilon)}} (\log T)^{\frac{2\vartheta(\varepsilon)+\alpha}{\alpha(1+\vartheta(\varepsilon))}}. \qquad (3.24)
\end{aligned}$$

Thus, $\lim_{n\to\infty} I_{n1}=0$, since $\beta<\frac{1-\vartheta(\varepsilon)}{1+\vartheta(\varepsilon)}$. Further, Assumption (B1) implies that there exists a positive constant K such that $\varpi_n(kq)\leq K/\log T^\beta$ for $kq>T^\beta$. Using (3.20) again, for $q=au_n^{-2/\alpha}\sim a(\log T)^{-1/\alpha}$ we have

$$\begin{aligned}
\frac{T^2}{q^2\log T}\exp\left(-\frac{u_n^2}{1+\vartheta(T^\beta)}\right) &\leq \frac{T^2}{q^2\log T}\exp\left(-\frac{u_n^2}{1+K/\log T^\beta}\right)\\
&\leq C_4 \exp\left(\frac{2K\log T}{K+\beta\log T}-(1-2/\alpha)\frac{K\log\log T}{K+\beta\log T}\right)\\
&= O(1).
\end{aligned}$$

Hence, following the argument of the proof of Lemma 6.4.1 of Leadbetter et al. (1983) we may further write

$$\begin{aligned}
I_{n2} &\leq C_5 \sum_{\substack{kq\in I_i,\ lq\in I_j, i\neq j\in\{1,2,\cdots,[T(n)]\}\\ T^\beta<|kq-lq|\leq T}} |r_n(kq,lq)-\varrho_n(kq,lq)| \exp\left(-\frac{u_n^2}{1+\varpi_n(kq,lq)}\right)\\
&\leq C_5 \sum_{\substack{kq\in I_i,\ lq\in I_j, i\neq j\in\{1,2,\cdots,[T(n)]\}\\ T^\beta<|kq-lq|\leq T}} |r_n(kq,lq)-\rho(T)| \exp\left(-\frac{u_n^2}{1+\vartheta(T^\beta)}\right)\\
&= C_5 \frac{q\log T}{T}\sum_{T^\beta<kq\leq T} |r_n(kq)-\rho(T)| \frac{T^2}{q^2\log T}\exp\left(-\frac{u_n^2}{1+\vartheta(T^\beta)}\right)\\
&\leq C_5 \frac{q\log T}{T}\sum_{T^\beta<kq\leq T} |r_n(kq)-\rho(T)|\\
&\leq C_5 \frac{q}{\beta T}\sum_{T^\beta<kq\leq T} |r_n(kq)\log kq-r| + C_6 r\frac{q}{T}\sum_{T^\beta<kq\leq T} |1-\frac{\log T}{\log kq}|. \qquad (3.25)
\end{aligned}$$

By Assumption (B1), the first term of the right hand-side of (3.25) tends to 0. Furthermore, the second term therein also tends to 0, which follows by an integral estimate as in the proof of Lemma 6.4.1 of Leadbetter et al. (1983). Consequently, the proof is established by (3.18)-(3.19) and (3.21)-(3.25). □

Lemma 3.5. *Suppose that (A1) and (A2) hold. If* $T(n)\mu(u_n)=O(1)$ *and* $T(n)\mu(v_n)=O(1)$*, then*

$$\mathbb{P}\left\{\sup_{s\in[0,1]} X_n(s)>u_n,\ \inf_{s\in[0,1]} X_n(s)<-v_n\right\} = o(\mu(u_n)+\mu(v_n)), \quad n\to\infty. \qquad (3.26)$$

Proof. The proof is similar to that of Lemma 11.1.4 in Leadbetter et al. (1983). □

Proof of Theorem 2.1. We only prove case (i), since case (ii) is a special case of (i).
(1). Case $\theta\in(0,\infty)$. The definition of $\{\xi_n(t), t\in\cup_{j=1}^{[T(n)]} I_j\}$ implies

$$\mathbb{P}\left\{\max_{kq\in\cup I_j}|\xi_n(kq)|\leq u_n\right\} = \mathbb{P}\left\{\max_{kq\in\cup I_j}|(1-\rho(T(n)))^{1/2}\eta_n(kq)+\rho^{1/2}(T(n))\mathcal{W}|\leq u_n\right\}$$

$$
\begin{aligned}
&= \mathbb{P}\left\{-u_n \le (1-\rho(T(n)))^{1/2}\eta_n(kq) + \rho^{1/2}(T(n))\mathcal{W} \le u_n, kq \in \cup I_j\right\} \\
&= \int_{-\infty}^{+\infty} \mathbb{P}\left\{\frac{-u_n - \rho^{1/2}(T(n))z}{(1-\rho(T(n)))^{1/2}} \le \eta_n(kq) \le \frac{u_n - \rho^{1/2}(T(n))z}{(1-\rho(T(n)))^{1/2}}, kq \in \cup I_j\right\} \varphi(z)\, dz. \quad (3.27)
\end{aligned}
$$

Since as $n \to \infty$

$$
u_n^{(z)} := \frac{u_n - \rho^{1/2}(T(n))z}{(1-\rho(T(n)))^{1/2}} = u_n + \frac{r - \sqrt{2r}z}{u_n} + o(u_n^{-1})
$$

and

$$
v_n^{(z)} := \frac{u_n + \rho^{1/2}(T(n))z}{(1-\rho(T(n)))^{1/2}} = u_n + \frac{r + \sqrt{2r}z}{u_n} + o(u_n^{-1}).
$$

So, the assumption $\lim_{n\to\infty} T(n)\mu(u_n) = \theta \in (0,\infty)$ implies that

$$
\lim_{n\to\infty} T(n)\mu(u_n^{(z)}) = \theta e^{-r+\sqrt{2r}z}, \quad \lim_{n\to\infty} T(n)\mu(v_n^{(z)}) = \theta e^{-r-\sqrt{2r}z}. \tag{3.28}
$$

Next, by the definition of $\{\eta_n(t), t \ge 0\}$, (3.14), (3.16) and (3.28) we have

$$
\begin{aligned}
\mathbb{P}\left\{-v_n^{(z)} \le \eta_n(kq) \le u_n^{(z)}, kq \in \cup I_j\right\} &= \prod_{j=1}^{[T(n)]} \mathbb{P}\left\{-v_n^{(z)} \le X_n^{(j)}(kq) \le u_n^{(z)}, kq \in I_j\right\} \\
&= \mathbb{P}\left\{-v_n^{(z)} \le X_n(kq) \le u_n^{(z)}, kq \in I_1)\right\}^{[T(n)]} \\
&= \mathbb{P}\left\{-v_n^{(z)} \le X_n(t) \le u_n^{(z)}, t \in I_1\right\}^{[T(n)]} (1+o(1)) \\
&= \mathbb{P}\left\{-v_n^{(z)} \le X_n(t) \le u_n^{(z)}, t \in [0,1]\right\}^{[T(n)]} (1+o(1)) \\
&= \left(1 - \mathbb{P}\left\{\inf_{s\in[0,1]} X_n(s) < -v_n^{(z)}\right\} - \mathbb{P}\left\{\sup_{s\in[0,1]} X_n(t) > u_n^{(z)}\right\}\right. \\
&\quad \left. + \mathbb{P}\left\{\inf_{s\in[0,1]} X_n(s) < -v_n^{(z)}, \sup_{s\in[0,1]} X_n(t) > u_n^{(z)}\right\}\right)^{[T(n)]} (1+o(1)) \quad (3.29)
\end{aligned}
$$

as $n \to \infty$. In the light of Theorem A(i) and Lemma 3.5

$$
\begin{aligned}
\mathbb{P}\left\{-v_n^{(z)} \le \eta_n(kq) \le u_n^{(z)}, kq \in \cup I_j\right\} &= \left(1 - \mu(u_n^{(z)}) - \mu(v_n^{(z)}) + o(\mu(u_n^{(z)}) + \mu(v_n^{(z)}))\right)^{[T(n)]} (1+o(1)) \\
&= \left(1 - \frac{\theta e^{-(r-\sqrt{2r}z)} + \theta e^{-(r+\sqrt{2r}z)}}{T(n)} + o\left(\frac{1}{T(n)}\right)\right)^{[T(n)]} (1+o(1)) \\
&= \exp\left(-\theta e^{-(r-\sqrt{2r}z)} - \theta e^{-(r+\sqrt{2r}z)}\right)(1+o(1))
\end{aligned}
$$

as $n \to \infty$. Combining the last result with (3.17),(3.27) and applying the dominated convergence theorem we have

$$
\lim_{n\to\infty} \mathbb{P}\left\{\max_{kq\in\cup I_j} |X_n(kq)| \le u_n\right\} = \int_{-\infty}^{+\infty} \exp\left(-\theta e^{-(r-\sqrt{2r}z)} - \theta e^{-(r+\sqrt{2r}z)}\right)\varphi(z)\, dz.
$$

Consequently, the proof follows by utilising further (3.13), (3.15) and (3.17).

(2). Case $\theta = \infty$. From the definition of $\mu(\cdot)$, we know that for arbitrarily large $\theta' < \infty$, there exist a real sequence v_n such that $\lim_{n\to\infty} n\mu(v_n) = \theta'$. Clearly, for n sufficient large, $u_n \le v_n$, hence

$$
\mathbb{P}\left\{\sup_{t\in[0,T(n)]} |X_n(t)| \le u_n\right\} \le \mathbb{P}\left\{\sup_{t\in[0,T(n)]} |X_n(t)| \le v_n\right\} \to \Lambda_r(-\log\theta'), \quad n\to\infty.
$$

Since this holds for arbitrarily large $\theta' < \infty$, by letting $\theta' \to \infty$ we see that

$$
\lim_{n\to\infty} \mathbb{P}\left\{\sup_{t\in[0,T(n)]} |X_n(t)| \le u_n\right\} = 0,
$$

which completes the proof. □

For the proof of Theorem 2.2 we need a result which is formulated in the next lemma. By Polya's criterion (see e.g., (3.10) in Durrett 2004) if we assume the convexity of the correlation functions $r_n(t)$ (hence $0<\alpha\le 1$, cf. Theorem 3.1 of Mittal and Ylvisaker (1975)), then there exists a separable standard stationary Gaussian process $Y_n(t), n\in\mathbb{N}$ with correlation function

$$\rho_{n,T(n)}(t)=\frac{r_n(t)-r_n(T(n))}{1-r_n(T(n))},\quad \text{for}\quad t\le T(n).$$

Let

$$M_{T(n)}(Y)=\max_{0\le t\le T(n)} Y_n(t),\quad M_{T(n)}(-Y)=\max_{0\le t\le T(n)} -Y_n(t).$$

Lemma 3.6. *Let $Y_n(t)$ be defined as above. Under the conditions of Theorem 2.2* for any $\varepsilon>0$

$$\lim_{n\to\infty}\mathbb{P}\left\{|M_{T(n)}(Y)-b_{T(n)}|>\varepsilon r_n^{1/2}(T(n))\right\} \;=\; 0 \tag{3.30}$$

and

$$\lim_{n\to\infty}\mathbb{P}\left\{|M_{T(n)}(-Y)-b_{T(n)}|>\varepsilon r_n^{1/2}(T(n))\right\} \;=\; 0 \tag{3.31}$$

are valid.

Proof. Since the proofs are similar, we only give the proof of (3.30). By the assumptions

$$\rho_{n,T(n)}(t)=\frac{r_n(t)-r_n(T(n))}{1-r_n(T(n))}=1-c_n(T(n))|t|^\alpha+\epsilon_n(t)|t|^\alpha$$

as $t\to 0$, where $c_n(T(n))=\frac{c_n}{1-r_n(T(n))}\to 1$, as $n\to\infty$, and $\epsilon_n(t)=\frac{\varepsilon_n(t)}{1-r_n(T(n))}\to 0$ as $t\to 0$, uniformly in n. Furthermore, for any $\varepsilon>0$, there exists $\gamma>0$ such that $\sup\{|\rho_{n,T(n)}(t)|, T\ge|t|\ge\varepsilon, n\in\mathbb{N}\}<\gamma<1$. Utilising the stationarity of $\{Y_n(t), 0\le t\le T(n)\}$, Theorem A (i) and the definition of $b_{T(n)}$, we have

$$\begin{aligned}
\mathbb{P}\left\{M_{T(n)}(Y)-b_{T(n)}>\varepsilon r_n^{1/2}(T(n))\right\} &\le ([T(n)]+1)\mathbb{P}\left\{\max_{0\le t\le 1} Y_n(t)>\varepsilon r_n^{1/2}(T(n))+b_{T(n)}\right\}\\
&\le C_6([T(n)]+1)(\varepsilon r_n^{1/2}(T(n))+b_{T(n)})^{\frac{2}{\alpha}-1}e^{-\frac{1}{2}(r_n^{1/2}(T(n))+b_{T(n)})^2}\\
&\le C_6([T(n)]+1)(\log T(n))^{\frac{2-\alpha}{2\alpha}}e^{-\frac{1}{2}(2\log T(n)+\frac{2-\alpha}{\alpha}\log\log T(n)+2(r_n(T(n))\log T(n))^{1/2})}\\
&\le C_6e^{-(r_n(T(n))\log T(n))^{1/2}}.
\end{aligned}$$

Assumption (B1) and the fact that $\lim_{n\to\infty} r_n(T(n))\log T(n)=\infty$ imply

$$\lim_{n\to\infty}\mathbb{P}\left\{M_{T(n)}(Y)-b_{T(n)}>\varepsilon r_n^{1/2}(T(n))\right\}=0.$$

Next, repeating the proof of equation (3.9) in Mital and Ylvisaker (1975), we have

$$\lim_{n\to\infty}\mathbb{P}\left\{M_{T(n)}(Y)-b_{T(n)}<-\varepsilon r_n^{1/2}(T(n))\right\}=0,$$

hence (3.30) holds, and thus the claim follows. □

Proof of Theorem 2.2. Represent $X_n(t)$ as

$$X_n(t)=(1-r_n(T(n)))^{1/2}Y_n(t)+r_n^{1/2}(T(n))\mathcal{W},$$

where $\mathcal{W}$ is a standard Gaussian random variable independent of the process $\{Y_n(t), t\ge 0\}$. Using Lemma 3.6 and setting $a(n):=\sqrt{\frac{1-r_n(T(n))}{r_n(T(n))}}$ we obtain

$$\mathbb{P}\left\{r_n^{-1/2}(T(n))\left(\sup_{t\in[0,T(n)]}|X_n(t)|-(1-r_n(T(n)))^{1/2}b_{T(n)}\right)\le x\right\}$$

$$
\begin{aligned}
&= \mathbb{P}\left\{ \sup_{t\in[0,T(n)]} |X_n(t)| \le r_n^{1/2}(T(n))[a(n)b_{T(n)}+x] \right\} \\
&= \mathbb{P}\left\{ -x \le a(n)(Y_n(t)+b_{T(n)}) + \mathcal{W}, a(n)(Y_n(t)-b_{T(n)}) + \mathcal{W} \le x, t\in[0,T(n)] \right\} \\
&= \mathbb{P}\left\{ a(n)(-Y_n(t)-b_{T(n)}) - \mathcal{W} \le x, a(n)(Y_n(t)-b_{T(n)}) + \mathcal{W} \le x, t\in[0,T(n)] \right\} \\
&= \mathbb{P}\left\{ a(n)(M_{T(n)}(-Y)-b_{T(n)}) - \mathcal{W} \le x, a(n)(M_{T(n)}(Y)-b_{T(n)}) + \mathcal{W} \le x \right\} \\
&\to \mathbb{P}\left\{ -\mathcal{W} \le x, \mathcal{W} \le x \right\}, \quad n\to\infty,
\end{aligned}
$$

and hence the claim follows. □

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