On strategy-proofness and single-peakedness: median-voting over intervals*

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July 2020

Abstract

We study correspondences that choose an interval of alternatives when agents have single-peaked preferences over locations and ordinally extend their preferences over intervals. We extend the main results of Moulin (1980) to our setting and show that the results of Ching (1997) cannot always be similarly extended.

First, *strategy-proofness* and *peaks-onliness* characterize the class of generalized median correspondences (Theorem 1). Second, this result neither holds on the domain of symmetric and single-peaked preferences, nor can in this result *min/max continuity* substitute *peaks-onliness* (see counter-Example 3). Third, *strategy-proofness* and *voter-sovereignty* characterize the class of efficient generalized median correspondences (Theorem 2).

JEL Classification Numbers: C71, D63, D78, H41.

Keywords: correspondences; generalized median correspondences; single-peaked preferences; strategy-proofness.

1 Introduction

We study the problem where an interval of alternatives is chosen from the interval $[0, 1]$ based on the preferences of a finite number of agents. This interval can be considered as the political

*We gratefully acknowledge financial support from the Swiss National Science Foundation (SNFS) for project 100018_156201. This work has also been partly supported by COST Action IC1205 on Computational Social Choice. We would like to thank Sidartha Gordon, Flip Klijn, Jordi Massó, William Thomson, and two anonymous referees and an associate editor for their very valuable feedback. Faculty of Business and Economics, University of Lausanne, Internef, CH-1015 Lausanne, Switzerland; e-mails: `bettina.klaus@unil.ch`; `panos.protopapas@outlook.com`.

spectrum, while the chosen interval can in turn be considered as the legislative constitution or the governmental coalition (in the sense that some "extreme" views are not accounted for by the constitution or are not represented by any member(s) of the governmental coalition). We assume that agents have *single-peaked preferences* defined over all alternatives on $[0, 1]$; that is, an agent's welfare is strictly increasing up to his "peak" (his favorite alternative), and is strictly decreasing thereafter. Other examples for the type of social choice problems we are interested in would be the planning of public parking zones where an agent knows that he will (eventually) find a parking spot in the designated parking zone but he does not know where this will be, or the drafting of an "if-needed" list of candidate locations to build a public facility, e.g., a hospital. The motivation behind our model also resembles that of two-stage voting procedures such as, for example, Black's procedure (e.g. Fishburn, 1977) or the "rule of k names" (e.g. Barberà and Coelho, 2000), or situations where voters select subsets of the outcome space (e.g. Nuñez and Xefteris, 20170). We assume that decisions are made under *ignorance* (Peterson, 2009, p. 40) and that agents, when comparing sets, focus on their best (most favorite) point(s) and their worst (least favorite) point(s) in each set.

We consider voting mechanisms that guarantee that the agents announce their true preferences; in other words, we are interested in voting mechanisms or (choice) *correspondences*, that are *strategy-proof*. Although the classic result of Gibbard and Satterthwaite establishes that on the full domain of preferences -with more than two possible outcomes- *strategy-proofness* and *non-dictatorship* are incompatible (Gibbard, 1973; Satterthwaite, 1975), this is not true for the domain of single-peaked preferences. This compatibility between the two aforementioned properties has been well studied in the context of (choice) *functions* that, based on agents' (single-peaked) preferences, chose one alternative. Specifically, it has been shown that *strategy-proofness* and *peaks-onliness* (the agents only announce their peak) characterize the class of *generalized median rules* (Moulin, 1980). Moreover, when also requiring *(Pareto) efficiency*, the sub-class of efficient *generalized median rules* is characterized (Moulin, 1980). A similar result also holds when the range of the function is closed and not connected (Barberà and Jackson, 1994). For *symmetric single-peaked preferences*, *peaks-onliness* can be substituted by *unanimity* (when a common best alternative exists, it is chosen) (Border and Jordan, 1983); furthermore, it turns out that in these results two of the required properties can be weakened; specifically, *peaks-onliness* and *efficiency* can be substituted by *continuity* (a small change in the announced preferences does not change the outcome a lot) and *voter-sovereignty* (no alternative is a priori excluded from being chosen) respectively (Ching, 1997).[1]

Our main results also make use of either the additional property of *peaks-onliness* or a version of *continuity* adapted for our context (i.e., where an interval of alternatives is chosen). Furthermore, we also study the case where correspondences are *efficient*. Concisely, our results are the following. First, *strategy-proofness* and *peaks-onliness* characterize the class of generalized median correspondences (Theorem 1). Second, this result does not hold on the subdomain of symmetric and single-peaked preferences, nor can in this result *continuity* substitute *peaks-onliness* (see counter-Example 3). Third, *strategy-proofness*, *voter-sovereignty*, and either *peaks-onliness* or *continuity* characterize the class of efficient generalized median correspondences (Theorem 2).

The paper proceeds as follows. We introduce the model and choice correspondences and their properties in Section 2. Section 3 contains the definition of generalized median correspondences and efficient generalized median correspondences and their characterizations. We conclude with a discussion of some model assumptions and by reviewing some related literature (Section 4).

2 The model

Consider a *coalition* (of agents) $N \equiv \{1, \ldots, n\}$ $(n \geq 2)$ and a set of *alternatives* $A \equiv [0, 1]$. We denote generic agents by i and j and generic alternatives by x and y. Each agent i is equipped with *complete*, *transitive*, and *reflexive preferences* R_i over A. As usual, $x\, R_i\, y$ is interpreted as "x is at least as desirable as y", $x\, P_i\, y$ as "x is preferred to y", and $x\, I_i\, y$ as "x is indifferent to y". Moreover, for preferences R_i there exists a number $p(R_i) \in \mathbb{R}$, called the *peak (level) of* i, with the following property: for each pair $x, y \in \mathbb{R}$ such that either $y < x \leq p(R_i)$, or $y > x \geq p(R_i)$, we have $x\, P_i\, y$. We call such preferences *single-peaked* and denote the *domain of single-peaked preferences* by $\mathcal{R}$. Preferences $R_i \in \mathcal{R}$ are *symmetric* if for each pair $x, y \in \mathbb{R}$, $|x - p(R_i)| = |y - p(R_i)|$ implies $x\, I_i\, y$. Let $\mathcal{R}^N$ be the set of profiles $R \equiv (R_i)_{i \in N}$ such that for each $i \in N$, $R_i \in \mathcal{R}$. Given $R \in \mathcal{R}^N$ and $j \in N$, we use R and (R_{-j}, R_j) interchangeably. For each $R \in \mathcal{R}^N$, we denote the *vector of peaks* by $p \equiv (p_i)_{i \in N}$, the *smallest peak* by $\underline{p} \equiv \min\{p_i\}_{i \in N}$, the *largest peak* by $\bar{p} \equiv \max\{p_i\}_{i \in N}$, and the *convex hull* of peaks by $\text{Conv}(p) \equiv [\underline{p}, \bar{p}]$.

[1]Although technically *continuity* is not weaker than *peaks-onliness*, loosely speaking, it imposes fewer restrictions on the result.

Denote the class of closed intervals in A by $\mathcal{A}$, generic sets in $\mathcal{A}$ by X and Y, the *minimum* of X by $\underline{X}$, and the *maximum* of X by $\bar{X}$. For each $R_i \in \mathcal{R}$ denote the set of *best alternative(s)* of i in X by $b_{R_i}(X) \equiv \{x \in X : \text{for each } y \in X,\ x \, R_i \, y\}$ and the set of *worst alternative(s)* of i in X by $w_{R_i}(X) \equiv \{x \in X : \text{for each } y \in X,\ y \, R_i \, x\}$. By single-peakedness, $b_{R_i}(X) \subseteq \{\underline{X}, p_i, \bar{X}\}$ and $|b_{R_i}(X)| = 1$. By single-peakedness, $w_{R_i}(X) \subseteq \{\underline{X}, \bar{X}\}$ and if $w_{R_i}(X) = \{\underline{X}, \bar{X}\}$ (only if $p(R_i) \in (\underline{X}, \bar{X})$), then $\underline{X} \, I_i \, \bar{X}$. With some abuse of notation, we treat sets $b_{R_i}(X)$ and $w_{R_i}(X)$ as if they are points and for each $x \in X$, we write $b_{R_i}(X) \, R_i \, x \, R_i \, w_{R_i}(X)$.

We will consider choice correspondences that assign outcomes in $\mathcal{A}$ under *complete uncertainty* (or *ignorance*) with the interpretation that any agent "*knows the set of possible outcomes ..., but has no information about the probabilities of those outcomes or about their likelihood ranking*" (Bossert et al., 2000, p. 295).[2] We assume that agents when evaluating outcomes focus exclusively on the best and worst points of the outcomes. Various preference extensions with different degrees of optimism or pessimism do so: consider, for example,

- either very optimistic agents who only focus on the best alternative(s) in the outcome set (*max* extension) or (lexicographically) first on the best alternative(s) and then on the worst alternative(s) (*max-min* extension)
- or very pessimistic agents who only focus on the worst alternative(s) in the outcome set (*min* extension) or (lexicographically) first on the worst alternative(s) and then on the best alternative(s) (*min-max* extension),

see Klaus and Protopapas (2020, Appendix A) for a more detailed discussion. All these preference extensions have in common that given $X, Y \in \mathcal{A}$, if an agent prefers his best alternative(s) in X to his best alternative(s) in Y and his worst alternative(s) in X to his worst alternative(s) in Y, then he prefers X to Y. To strike a balance between the opposite assumptions of optimistic versus pessimistic preferences, we use the following *best-worst* extension of preferences over sets (we use the same symbols to denote preferences over points and preferences over sets).

Best-worst extension of preferences to sets. For each $i \in N$ with $R_i \in \mathcal{R}$ and each $X, Y \in \mathcal{A}$, we have

$$X \, R_i \, Y \text{ if and only if } b_{R_i}(X) \, R_i \, b_{R_i}(Y) \text{ and } w_{R_i}(X) \, R_i \, w_{R_i}(Y)$$

[2]For a survey of criteria and methods for ranking subsets of a set of outcomes under complete uncertainty we refer to Barberà et al. (2004, Section 3).

and

$$X \, P_i \, Y \text{ if and only if } X \, R_i \, Y \text{ and } [b_{R_i}(X) \, P_i \, b_{R_i}(Y) \text{ or } w_{R_i}(X) \, P_i \, w_{R_i}(Y)].$$

This extension of preferences is *transitive*; however, it is not *complete* (there exist sets $X, Y \in \mathcal{A}$ such that neither $X R_i Y$ nor $Y R_i X$). In Klaus and Protopapas (2020, Appendix A) we also give a normative foundation of our preference extension based on Bossert et al. (2000, Theorem 1) and illustrate it with the example of public parking allocation.

We use the standard notion of Pareto optimality/efficiency as our efficiency notion.

Efficient sets. Given $R \in \mathcal{R}^N$, set $X \in \mathcal{A}$ is *(Pareto) efficient* if and only if there is no set $Y \in \mathcal{A}$ such that for each $i \in N$, $Y \, R_i \, X$, and for at least one $j \in N$, $Y \, P_j \, X$; we denote the class containing all efficient sets at R by $\mathrm{E}(R)$.

The next characterization of efficient sets follows from Klaus and Protopapas (2020) and it coincides with the well-known characterization of (Pareto) efficient points for choice functions. Note that the original result is a little more complicated since it holds for all compact sets.

Proposition 1 (Klaus and Protopapas (2020))**.** *At $R \in \mathcal{R}^N$, a closed interval is efficient if and only if it is a subset of the convex hull of peaks.*

A *(choice) correspondence* F assigns to each $R \in \mathcal{R}^N$ a set $F(R) \in \mathcal{A}$, i.e., $F\colon \mathcal{R}^N \to \mathcal{A}$. Given $R \in \mathcal{R}^N$, we denote the *minimum* of $F(R)$ by $\underline{F}(R)$ and the *maximum* of $F(R)$ by $\bar{F}(R)$. We denote the family of correspondences by $\mathcal{F}$. Moreover, if $F \in \mathcal{F}$ assigns to each $R \in \mathcal{R}^N$ an interval consisting of a single point we will refer to it as a *function* and use notation $f \in \mathcal{f}$, i.e., $f\colon \mathcal{R}^N \to A$.

The first two properties we consider are related; the first is *(Pareto) efficiency* for correspondences while the second, being weaker than the first, requires no alternative in A to be a priori excluded from being selected.

Efficiency. For each $R \in \mathcal{R}^N$, $F(R) \in \mathrm{E}(R)$.

Voter-sovereignty. For each $x \in A$, there exists $R \in \mathcal{R}^N$ such that $F(R) = \{x\}$.

The next property, which is central in our results, requires no agent to gain by deviating; it also implies comparability between the chosen sets before and after an agent's deviation.

Note that it is a strong extension of strategy-proofness to correspondences in line with the one introduced by Duggan and Schwartz (2000) (see Barberà, 2011, for a comprehensive survey on strategy-proof social choice correspondences).

Strategy-proofness. For each $i \in N$, each $R \in \mathcal{R}^N$, and each $R'_i \in \mathcal{R}$, $F(R)\, R_i\, F(R_{-i}, R'_i)$.

The next property expresses a considerable reduction or simplification of the information used by a correspondence by requiring the chosen set to depend only on the vector of peaks.

Peaks-onliness. For each pair $R, R' \in \mathcal{R}^N$ such that $p = p'$, $F(R) = F(R')$.

So far we have introduced typical "economic" properties. Our last two properties are a bit more technical (although commonly considered in various economic contexts). First, we adapt continuity to our context; loosely speaking, it requires that when the announced preferences of an agent change "a little", the minimum and maximum alternatives chosen do not change "a lot". Before describing it formally, we first define the three following notions. First, the *indifference relation* $r_{R_i} \colon [0,1] \to [0,1]$, given preferences $R_i \in \mathcal{R}$, loosely speaking maps each alternative x to an alternative y that i finds indifferent to x, according to R_i, i.e., for each $x \in [0, p_i]$, $r_{R_i}(x) = y$ if $y \in [p_i, 1]$ exists such that $y\, I_i\, x$, or $r_{R_i}(x) = 1$ otherwise; while for each $x \in [p_i, 1]$, $r_{R_i}(x) = y$ if $y \in [0, r_i]$ exists such that $y\, I_i\, x$, or $r_{R_i}(x) = 0$ otherwise. Second, the *distance* between a pair $R_i, R'_i \in \mathcal{R}$ is measured by $\mathrm{d}(R_i, R'_i) \equiv \max_{x\in[0,1]} |r_{R_i}(x) - r_{R'_i}(x)|$. Finally, a sequence $\{R_i^k\}_{k\in\mathbb{N}^+}$ in $\mathcal{R}$ *converges* to R_i, if $k \to \infty$ implies that distance $\mathrm{d}(R_i, R_i^k) \to 0$. We denote this convergence by $R_i^k \to R_i$.

Min/max continuity. For each $R \in \mathcal{R}^N$, each $i \in N$, and each $\{R_i^k\}_{k\in\mathbb{N}^+}$ in $\mathcal{R}$,

$$\text{if } R_i^k \to R_i, \text{ then } \begin{cases} \underline{F}(R_{-i}, R_i^k) \to \underline{F}(R), \text{ and} \\ \bar{F}(R_{-i}, R_i^k) \to \bar{F}(R). \end{cases}$$

Min/max continuity for functions is equivalent to the regular *continuity* property for functions and Protopapas (2018, Appendix A) shows that it is equivalent to *upper-hemi continuity* and *lower-hemi continuity* for correspondences.

A choice correspondence satisfies *uncompromisingness* (Border and Jordan, 1983) if whenever an agent's preferences change such that his peaks, before and after this change, both lie on the same side of the minimum (maximum) point chosen, then the minimum (maximum) point chosen does not change.

Uncompromisingness. For each $i \in N$ and each pair $R, R' \in \mathcal{R}^N$ such that $R'_{-i} = R_{-i}$, **(i)** if $[p_i < \underline{F}(R)$ and $p'_i \leq \underline{F}(R)]$ or $[p_i > \underline{F}(R)$ and $p'_i \geq \underline{F}(R)]$, then $\underline{F}(R) = \underline{F}(R')$ and **(ii)** if $[p_i < \bar{F}(R)$ and $p'_i \leq \bar{F}(R)]$ or $[p_i > \bar{F}(R)$ and $p'_i \geq \bar{F}(R)]$, then $\bar{F}(R) = \bar{F}(R')$.

3 Generalized median rules and correspondences

Before defining the classes of functions and correspondences that our results revolve around, the following definition is necessary: for each odd and positive integer k, and each vector $T \in \mathbb{R}^k$, label the coordinates of T such that $t_1 \leq \cdots \leq t_k$; we define the *median* (coordinate) of T by $\text{med}(T) \equiv t_{\frac{k+1}{2}}$.

The class of functions we consider was introduced and characterized by *strategy-proofness* and *peaks-onliness* (Moulin, 1980, Proposition 3). It was later shown that *peaks-onliness* can be substituted with the "weaker" property of *continuity* (Ching, 1997, Theorem). In order to provide an intuition in understanding this class, we present an example inspired by the one provided in Arribillaga and Massó (2016, p. 564).

Example 1. Let $N = \{1, 2\}$ and $\alpha = (\alpha_\emptyset, \alpha_{\{1\}}, \alpha_{\{2\}}, \alpha_N)$ such that $\alpha_N \leq \alpha_{\{1\}} \leq \alpha_{\{2\}} \leq \alpha_\emptyset$. Define $f^\alpha \in \not{f}$ as follows. For each $R \in \mathcal{R}^N$, if $p_1 \leq p_2$, choose $\tilde{\alpha}_p = (\alpha_\emptyset, \alpha_{\{1\}}, \alpha_N)$ and set $f^\alpha(R) = \text{med}(\tilde{\alpha}_p, p)$, and if $p_1 > p_2$, choose $\tilde{\alpha}_p = (\alpha_\emptyset, \alpha_{\{2\}}, \alpha_N)$ and set $f^\alpha(R) = \text{med}(\tilde{\alpha}_p, p)$. The range of f^α equals $[\alpha_N, \alpha_\emptyset]$. Note that if $\alpha_{\{1\}} \neq \alpha_{\{2\}}$, then the agents have asymmetric power in influencing the chosen alternative; since $\alpha_{\{1\}} \leq \alpha_{\{2\}}$, agent 1 has a greater power than agent 2 in influencing the chosen alternative: agent 1 can make sure that the chosen alternative is not larger than $\alpha_{\{1\}}$ and not smaller than p_1 (by announcing $p_1 \leq \alpha_{\{1\}}$), or that it is not larger than p_1 and not smaller than $\alpha_{\{1\}}$ (by announcing $p_1 \geq \alpha_{\{1\}}$). In addition, he is a dictator on the interval $[\alpha_{\{1\}}, \alpha_{\{2\}}]$.

Next, agent 2 only has the power to influence the chosen alternative if agent 1 "allows" him to do so. That is, if $\alpha_N \leq p_1 \leq \alpha_{\{1\}}$, then agent 2 can choose an alternative in $[p_1, \alpha_{\{1\}}]$, and if $p_1 \leq \alpha_N \leq \alpha_{\{1\}}$, then agent 2 can choose an alternative in $[\alpha_N, \alpha_{\{1\}}]$. Similarly, if $\alpha_{\{2\}} \leq p_1 \leq \alpha_\emptyset$, then agent 2 can choose an alternative in $[\alpha_{\{2\}}, p_1]$, and if $\alpha_{\{2\}} \leq \alpha_\emptyset \leq p_1$, then agent 2 can choose an alternative in $[\alpha_{\{2\}}, \alpha_\emptyset]$. □

The general n-agent case is defined next. We use the terminology of *generalized median rules* (see Border and Jordan, 1983). Moulin (1980) was the first to introduce this class of rules using a "minmax representation." Ching (1997) refers to *augmented median voter* *schemes* and explains "Moulin (1980) first characterized the class of solutions satisfying strategy-proofness and peak only as the class of minmax solutions. We show how to relate a minmax solution and an augmented median-voter solution in terms of their parameters."

Generalized median rules. Let $\alpha \in A^{2^n}$ be such that $\alpha \equiv (\alpha_M)_{M \subseteq N}$, where for each pair of sets $L \subseteq M \subseteq N$, $\alpha_L \geq \alpha_M$. Also, for each $R \in \mathcal{R}^N$, let bijection $\pi\colon N \to N$ be such that $p_{\pi(1)} \leq \cdots \leq p_{\pi(n)}$ and construct vector $\tilde{\alpha}_p = (\alpha_\emptyset, \alpha_{\{\pi(1)\}}, \alpha_{\{\pi(1),\pi(2)\}}, \ldots, \alpha_N)$. We denote the *generalized median rule* associated with vector α by f_G^α, where for each $R \in \mathcal{R}^N$, $f_G^\alpha(R) \equiv \text{med}(p, \tilde{\alpha}_v)$. We denote the *class of generalized median rules* by $\mathscr{f}_G$.

Clearly, if all agents announce different peaks, a unique ordering of them by their announced peak exists. Moreover, for the case where some agents announce the same peak and hence such a unique ordering does not exist, the chosen alternative does not depend on the particular ordering chosen; as shown in Ching (1997, Remark 1) (see also Protopapas, 2018, Lemma 1).

The following class of correspondences extends the spirit of generalized median rules to correspondences.

Generalized median correspondences. Let $\alpha, \beta \in A^{2^n}$ be such that $\alpha \equiv (\alpha_M)_{M \subseteq N}$ and $\beta \equiv (\beta_M)_{M \subseteq N}$, with $\alpha \leq \beta$, where for each pair of sets $L \subseteq M \subseteq N$, $\alpha_L \geq \alpha_M$ and $\beta_L \geq \beta_M$. Also, for each $R \in \mathcal{R}^N$, let bijection $\pi\colon N \to N$ be such that $p_{\pi(1)} \leq \cdots \leq p_{\pi(n)}$ and construct vectors $\tilde{\alpha}_p = (\alpha_\emptyset, \alpha_{\{\pi(1)\}}, \alpha_{\{\pi(1),\pi(2)\}}, \ldots, \alpha_N)$ and $\tilde{\beta}_p = (\beta_\emptyset, \beta_{\{\pi(1)\}}, \beta_{\{\pi(1),\pi(2)\}}, \ldots, \beta_N)$. We denote the *generalized median correspondence* associated with vectors α and β by $F_G^{\alpha,\beta}$, where for each $R \in \mathcal{R}^N$, $F_G^{\alpha,\beta}(R) \equiv [\text{med}(p, \tilde{\alpha}_p), \text{med}(p, \tilde{\beta}_p)]$. We denote the *class of generalized median correspondences* by $\mathcal{F}_G$.

Remark 1. By definition of $\mathcal{F}_G$ and $\mathscr{f}_G$, a generalized median correspondence $F_G^{\alpha,\beta}$ can be decomposed into two generalized median rules f_G^α and f_G^β, i.e., for each $R \in \mathcal{R}^N$, $F_G^{\alpha,\beta}(R) \equiv [\text{med}(p, \tilde{\alpha}_p), \text{med}(p, \tilde{\beta}_p)] = [f_G^\alpha(R), f_G^\beta(R)]$. □

Given $F_G^{\alpha,\beta} \in \mathcal{F}_G$, if for each $R \in \mathcal{R}^N$, $F_G^{\alpha,\beta}(R) \in \text{E}(R)$, we say that $F_G^{\alpha,\beta}$ is an *efficient generalized median correspondence* and denote the *class of efficient generalized median correspondences* by $\mathcal{F}_{EG}$. We obtain the following characterization.

Proposition 2. *A generalized median correspondence $F_G^{\alpha,\beta}$ is an efficient generalized median correspondence if and only if α, β are such that $\alpha_\emptyset = \beta_\emptyset = 1$ and $\alpha_N = \beta_N = 0$.*

Proof. Let $F_G^{\alpha,\beta} \in \mathcal{F}_G$. First, assuming that $F_G^{\alpha,\beta} \in \mathcal{F}_{EG}$ such that α, β are not as described above, results in a contradiction as follows. If $\alpha_N \neq 0$ or $\beta_N \neq 0$, choose $R \in \mathcal{R}^N$ such that $p = (0, \ldots, 0)$. By Proposition 1, $\mathrm{E}(R) = \{0\}$ and by the definition of $\mathcal{F}_G$, $F_G^{\alpha,\beta}(R) = [\alpha_N, \beta_N]$. Hence, $F_G^{\alpha,\beta}(R) \notin \mathrm{E}(R)$. Similarly, if $\alpha_\emptyset \neq 1$ or $\beta_\emptyset \neq 1$, choose $R \in \mathcal{R}^N$ such that $p = (1, \ldots, 1)$, $\mathrm{E}(R) = \{1\}$, $F_G^{\alpha,\beta}(R) = [\alpha_\emptyset, \beta_\emptyset]$, and $F_G^{\alpha,\beta}(R) \notin \mathrm{E}(R)$.

Second, if $\alpha_N = \beta_N = 0$ and $\alpha_\emptyset = \beta_\emptyset = 1$, then for each $R \in \mathcal{R}^N$, $\mathrm{med}(p, \tilde{\alpha}_p) \in \mathrm{Conv}(p)$ and $\mathrm{med}(p, \tilde{\beta}_p) \in \mathrm{Conv}(p)$. Hence, $F_G^{\alpha,\beta}(R) \subseteq \mathrm{Conv}(p)$, and thus, by Proposition 1, $F_G^{\alpha,\beta}(R) \in \mathrm{E}(R)$. □

Generalized median correspondences are *strategy-proof*, similar to the results on functions by Moulin (1980). However, in contrast to Moulin's results these correspondences are not *group strategy-proof*.[3] The following example illustrates this.

Example 2 (**Group strategy-proofness counter-example**)**.**
Let $N = \{1, 2, 3\}$ and define $F' \in \mathcal{F}$ such that for all $R \in \mathcal{R}^N$, $F'(R) = [\mathrm{med}(0, 0, 0, p_1, p_2, p_2, 1), \mathrm{med}(0, 0, p_1, p_2, p_3, 1, 1)]$, i.e., $\underline{F}'$ selects the smallest peak and $\bar{F}'$ the second smallest peak. Note that $F' = F_G^{\alpha,\beta} \in \mathcal{F}_{EM}$ with $\alpha_\emptyset = \beta_\emptyset = 1$, $\alpha_{\{i\}} = 0$, $\beta_{\{i\}} = 1$, $\alpha_{\{i,j\}} = \beta_{\{i,j\}} = 1$, and $\alpha_N = \beta_N = 0$.

Now, consider symmetric $R, R' \in \mathcal{R}^N$ such that $p_1 = p'_1 = 0$, $p_2 = 0.5$, $p'_2 = 0.6$, and $p_3 = p'_3 = 1$. Then, $F'(R) = [0, 0.5]$ and $F'(R') = [0, 0.6]$. Hence, agent 2 is indifferent when changing preferences from R_2 to R'_2 while agent 3 is strictly better off after this deviation; a contradiction to *group strategy-proofness*. □

We now present our first main result, which generalizes Moulin (1980, Proposition 3).[4]

Theorem 1. *The following three statements for a correspondence $F \in \mathcal{F}$ are equivalent.*

(i) *F satisfies strategy-proofness and peaks-onliness.*

(ii) *F satisfies uncompromisingness.*

(iii) *F is a generalized median correspondence.*

[3] No group of agents can deviate such that all members of the group are weakly better off and at least one member of the group is strictly better off.

[4] Note that this result only holds on the full domain of single-peaked preferences that we consider here but not on the subdomain of symmetric single-peaked preferences. We explain this aspect of our result after discussing the logical independence of characterizing properties and in Remark 2.

We prove Theorem 1 in Appendix A. The properties in the above characterization are independent: first, correspondence F^* proposed in the following example satisfies *strategy-proofness* but neither *peaks-onliness* nor *uncompromisingness*; second, $\tilde{F}(R) = \{\frac{p+\bar{p}}{2}\}$ satisfies *peaks-onliness* but neither *strategy-proofness* nor *uncompromisingness*.

Ching (1997, Theorem) provided an alternative characterization to Moulin (1980, Proposition 3) by replacing *peaks-onliness* with *continuity*. Next, we show that this result does not extend to correspondences. We illustrate this with a correspondence satisfying *strategy-proofness* and *min/max continuity* but violating *peaks-onliness* and *uncompromisingness*. Moreover, the example also demonstrates that the equivalence of (i) and (ii) in Theorem 1 does not hold on the subdomain of symmetric single-peaked preferences (see also Remark 2).

Example 3 (**Counter-example corresponding to Ching (1997, Theorem)**).
Let $|N| \geq 1$ and define $r^*_R \equiv \max\{r_{R_i}(0)\}_{i \in N}$, that is, at profile R, among the indifferent announced alternatives to 0 of each agent $i \in N$, r^*_R is the largest one. Next, define $F^* \in \mathcal{F}$ as follows. For each $R \in \mathcal{R}^N$, $F^*(R) = [0, r^*_R]$. By definition, it follows that F^* satisfies *min/max continuity*. Note that F^* satisfies neither *peaks-onliness*, nor *voter-sovereignty*, nor *efficiency*.

To show that F^* satisfies *strategy-proofness* let $R \in \mathcal{R}^N$ and $R'_i \in \mathcal{R}$ such that $R'_i \neq R_i$.
Case 1 ($r_{R_i}(0) = r^*_R$). By single-peakedness, $b_{R_i}(F^*(R)) = \{p_i\}$, implying i's best point does not improve by deviating at R, and $0 \in w_{R_i}(F^*(R))$. By the definition of F^*, $0 \in F^*(R_{-i}, R'_i)$, hence i's worst point(s) does not improve by deviating at R. Therefore, $F^*(R) R_i F^*(R_{-i}, R'_i)$.
Case 2 ($r_{R_i}(0) < r^*_R$). By single-peakedness, $b_{R_i}(F^*(R)) = \{p_i\}$, implying i's best point does not improve by deviating at R, and $w_{R_i}(F^*(R)) = \{r^*_R\}$. By the definition of F^*, $r^*_R \in F^*(R_{-i}, R'_i)$, hence i's worst point does not improve by deviating at R. Therefore, $F^*(R)\; R_i\; F^*(R_{-i}, R'_i)$.

To show that F^* does not satisfy *uncompromisingness* let $N = \{1, 2, 3\}$ and consider symmetric profiles $R, R' \in \mathcal{R}^N$ such that $p_1 = 0.2$, $p'_1 = 0.3$, and $p_2 = p'_2 = p_3 = p'_3 = 0$. Hence, $r^*_R = r_{R_1}(0) = 0.4$ and $r^*_{R'} = r_{R'_1}(0) = 0.6$. Therefore, $F(R) = [0, 0.4]$ and $F(R') = [0, 0.6]$ and F^* does not satisfy *uncompromisingness*. □

We conclude this section by presenting the "efficient version" of Theorem 1. Notice that now *strategy-proofness* and *voter sovereignty* imply *peaks-onliness* and *min/max continuity*; this generalizes a result by Ching (1997, Proposition 2).

Theorem 2. *The following three statements for a correspondence $F \in \mathcal{F}$ are equivalent.*

(i) *F satisfies strategy-proofness and voter-sovereignty.*

(ii) *F satisfies uncompromisingness and voter-sovereignty.*

(iii) *F is an efficient generalized median correspondence.*

We prove Theorem 2 in Appendix B. The properties in the above characterization are independent: first, a constant choice correspondence that always chooses a fixed set satisfies *strategy-proofness* and *uncompromisingness*, but not *voter-sovereignty*. Second, $\tilde{F}(R) = \{\frac{p+\bar{p}}{2}\}$ satisfies *voter-sovereignty* but neither *strategy-proofness* nor *uncompromisingness*.

4 Conclusion

We have presented two characterization results when agents have single-peaked preferences over locations and ordinally extend their preferences over intervals. First, *strategy-proofness* and *peaks-onliness* characterize the class of generalized median correspondences (Theorem 1). Second, *strategy-proofness* and *voter-sovereignty* characterize the class of efficient generalized median correspondences (Theorem 2). Furthermore, in both characterizations, *strategy-proofness* (and *peaks-onliness*) can be replaced by *uncompromisingness*. We next discuss two extensions of these results.

Remark 2 (Results for symmetric single-peaked preferences). On the subdomain of symmetric single-peaked preferences, *peaks-onliness* is vacuously satisfied. Then, Example 3 illustrates that Theorem 1 does not hold on the subdomain of symmetric single-peaked preferences since correspondence F^* satisfies *strategy-proofness* but neither satisfies *uncompromisingness*, nor is it a generalized median correspondence. In contrast, Theorem 2 does hold on the subdomain of symmetric single-peaked preferences (Protopapas, 2018, Theorem 5). □

Remark 3 (Results with *anonymity*). Moulin (1980, Proposition 3) also characterized the set of choice functions satisfying *strategy-proofness*, *peaks-onliness*, and *anonymity*[5] (introduced as *strategy-proof and anonymous voting schemes*). The extension to our model is as follows: let vectors $a, b \in A^{n+1}$ be such that $a \equiv (a_1, \ldots, a_{n+1})$ and $b \equiv (b_1, \ldots, b_{n+1})$,

[5]The names of the agents do not affect the chosen alternative.

with $a \leq b$, $a_1 \leq \cdots \leq a_{n+1}$, and $b_1 \leq \cdots \leq b_{n+1}$. We define the *median correspondence* associated with vectors a and b for each $R \in \mathcal{R}^N$ by $F_M^{a,b}(R) \equiv [\text{med}(p, a), \text{med}(p, b)]$.

Based on Theorem 1 and Moulin (1980, Proposition 3) (*anonymity* also applies to the functions $\underline{F}$ and $\bar{F}$), we obtain the following characterization: a choice correspondence satisfies *strategy-proofness*, *peaks-onliness*, and *anonymity* if and only if it is a median correspondence (Protopapas, 2018, Theorem 2). By adding *voter sovereignty* and using Theorem 2, an "efficient version" of this characterization where $a_1 = b_1 = 0$ and $a_{n+1} = b_{n+1} = 1$ is obtained (Protopapas, 2018, Theorem 4). Finally, on the domain of symmetric single-peaked preferences, the class of efficient median correspondences is characterized by *strategy-proofness* (*uncompromisingness*), *voter-sovereignty*, and *anonymity* (Protopapas, 2018, Theorem 6). □

Related Literature

When choosing a single alternative from a finite set (of alternatives), *strategy-proofness* and *voter-sovereignty* characterize, on the domain of strict preferences, a class of functions similar to the class of efficient generalized median rules (Barberà et al., 1993). Moreover, the admissible preferences of all agents being *top-connected*[6] characterizes the maximal domain in which (i) every *strategy-proof* and *unanimous* function is a generalized median rule, and (ii) every generalized median rule is *strategy-proof* (Achuthankutty and Roy, 2018).

When departing from the setting where agents have single-peaked preferences and only one alternative is chosen, a few more results should be mentioned. First, in the case of *probabilistic functions*,[7] where the agents' single-peaked preferences are ordinarily extended over probability distributions via first-order stochastic dominance, similar results to Moulin's results (1980) were achieved (Ehlers et al., 2002). Next, if agents have *single-dipped preferences*,[8] *strategy-proofness* and *unanimity* characterize the class of *collections of 0-decisive sets with a tie-breaker*[9] (Manjunath, 2014). Klaus and Storcken (2002) consider location

[6]For every agent and every pair of "neighboring" alternatives (a, b), there exist admissible preferences such that a is the most favorite alternative and b is the second most favorite alternative.

[7]Given the agents' preferences, a probability distribution over all alternatives is chosen.

[8]An agent's welfare is strictly decreasing up to his "dip" (his least favorite alternative), and is strictly increasing thereafter.

[9]Each such function chooses either the minimum or the maximum alternative. Loosely speaking, if all agents are indifferent between the two alternatives the choice depends on the preference profile (over all other alternatives). Otherwise, the choice depends on the number of agents preferring the minimum over the maximum alternative, their identities, and their preferences.

problems in Euclidean space when agents have separable quadratic single-peaked preferences but in contrast to Border and Jordan (1983), they consider choice correspondences and characterize the class of "coordinatewise median voting schemes" as an extension of the median without using additional and fixed coordinates (so-called phantom voters in Border and Jordan, 1983). Klaus and Storcken (2002) also use the best-worst extension of preferences to sets we use here. In a predecessor paper, Klaus and Protopapas (2020), for the same model as in this paper, considered so-called solidarity properties and show that *efficiency* and *replacement-dominance*[10] characterize the class of target point functions while *efficiency* and *population-monotonicity*[11] characterize the larger class of target set correspondences.

Finally, we would like to discuss two results when preferences are single-peaked and two alternatives can be chosen, with the agents comparing different pairs of alternatives using the *max-extension*, i.e., when comparing two pairs of alternatives $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$, an agent first locates in each pair the alternative he ranks higher, say x^* and y^*. If he prefers x^* to y^*, then he also prefers X to Y. If he is indifferent between x^* and y^*, then he is also indifferent between X and Y.

(Heo, 2013, Theorem 1) *Strategy-proofness*, *continuity*, *anonymity*, and *users-only*[12] characterize the class of *double median functions*[13].

(Miyagawa, 2001, Theorem 1) If $|N| > 3$, then *efficiency* and *replacement-dominance* characterize the class of rules comprised of the *left-peaks function* and the *right-peaks function*[14].

The class of generalized median correspondences and the class of median correspondences discussed in Remark 3 share the property of decomposability into two functions with the double median functions characterized in Heo (2013) and the left-peaks (right-peaks) functions characterized in Miyagawa (2001). However, there are some notable differences between our and their results.

[10] *Replacement-dominance*: if the preferences of an agent change, then the other agents, whose preferences remained unchanged, should all be made at least as well off as they were initially, or they should all be made at most as well off.

[11] *Population-monotonicity*: if additional agents join a population, then the agents who were initially present should all be made at least as well off as they were initially, or they should all be made at most as well off.

[12] For each pair of chosen alternatives (a, b), the choice of a depends only on agents preferring a over b.

[13] A double median function can be decomposed into two median rules, where for each preference profile each one selects one alternative.

[14] The left (right) peaks function chooses the two unique left-most (right-most) peaks.

First, by using the max-extension of preferences in our setting, as is the case in the two aforementioned papers, the only *efficient* correspondence would be the one always choosing the interval of the peaks, that is, for all $R \in \mathcal{R}$, $F(R) = [\underline{p}, \bar{p}]$. This follows from the fact that agents only care about their best alternative in a set and that in our setting we do not a priori limit the number of alternatives that may be chosen at a given profile.

Second, the class of double median functions of Heo (2013) seems to be equivalent to the class of median correspondences (see Remark 3). Specifically, the pair of alternatives (x_1, x_2) chosen by a double median function are essentially the minimum and maximum alternatives of the interval chosen by a median correspondence. However, Heo's characterization result makes use of *users-only*, a property that partitions each coalition of agents into two sub-coalitions; one preferring x_1 over x_2, and everyone else, with only the first partition (second partition) influencing the choice of alternative x_1 (x_2). In our setting, for each $F^{a,b} \in \mathcal{F}_M$, the choice of both vectors a and b depends on the preferences of all agents.

Third, although the left-peaks function of Miyagawa (2001) seems to be a special case of a median correspondence (see Remark 3), this is not the case; the left-peaks function always chooses the two *distinct* left-most peaks, and moreover, Miyagawa's setting requires that at least two distinct peaks exist in each profile. In our setting, the median correspondence that looks "similar" to the left-peaks function is $F^{a,b} \in \mathcal{F}$ where $a = (0, \ldots, 0)$ and $b = (0, \ldots, 0, 1)$. Although this correspondence seems to choose the two left-most peaks, when two or more agents share the minimum peak, it only chooses the minimum peak. Furthermore, in Klaus and Protopapas (2020) the same properties as in Miyagawa (2001) (namely, *efficiency* and *replacement-dominance*) are considered, for (almost) the same setting as in this paper. There, it is shown that each correspondence satisfying said properties is essentially a function, reconfirming a characterization of Vohra (1999) for *target-point functions*.[15]

[15]Each target point function is determined by its target point: if the target point is *efficient*, it is chosen; if it is not *efficient*, the closest *efficient* point is chosen.

Appendix

A Proof of Theorem 1

Before starting the proof of Theorem 1 we show that following an agent's preference deviation, there are restrictions on the chosen set.

Lemma 1. *For each $F \in \mathcal{F}$ satisfying strategy-proofness and peaks-onliness, each $i \in N$, and each $R, R' \in \mathcal{R}^N$ such that $R_{-i} = R'_{-i}$, the following hold.*

(a) *If $p_i < \bar{F}(R)$, then $\bar{F}(R) \leq \bar{F}(R')$, and if in addition $p_i < \underline{F}(R)$, then $\underline{F}(R) \leq \underline{F}(R')$.*

(b) *If $p_i > \underline{F}(R)$, then $\underline{F}(R) \geq \underline{F}(R')$, and if in addition $p_i > \bar{F}(R)$, then $\bar{F}(R) \geq \bar{F}(R')$.*

Proof. (a) Let $F \in \mathcal{F}$ satisfy *strategy-proofness* and *peaks-onliness*. Let $R, R' \in \mathcal{R}^N$ and $i \in N$ be such that $R_{-i} = R'_{-i}$ and $p_i < \bar{F}(R)$. By *peaks-onliness* it is without loss of generality to assume that R_i is such that $0 \; P_i \; \bar{F}(R)$. Hence, $w_{R_i}(F(R)) = \bar{F}(R)$. Then, by *strategy-proofness*, $\bar{F}(R) \; R_i \; w_{R_i}(F(R'))$. Hence, since for all $y \in [0, \bar{F}(R))$, $y \; P_i \; \bar{F}(R)$, we have $w_{R_i}(F(R')) \geq \bar{F}(R)$ and $\bar{F}(R') \geq \bar{F}(R)$.

If in addition $p_i < \underline{F}(R)$, then by *peaks-onliness* it is without loss of generality to additionally assume that R_i is such that $0 \; P_i \; \underline{F}(R)$. Hence, $b_{R_i}(F(R)) = \underline{F}(R)$. Then, by *strategy-proofness*, $\underline{F}(R) \; R_i \; b_{R_i}(F(R'))$. Hence, since for all $y \in [0, \underline{F}(R))$, $y \; P_i \; \underline{F}(R)$, we have $b_{R_i}(F(R')) \geq \underline{F}(R)$ and $\underline{F}(R') \geq \underline{F}(R)$.

The proof of statement (b) is based on symmetric arguments. □

Proof of Theorem 1: (i) ⇒ (ii). Let $F \in \mathcal{F}$ satisfy *strategy-proofness* and *peaks-onliness*. We show that F satisfies *uncompromisingness*. Let $R, R' \in \mathcal{R}^N$ and $i \in N$ be such that $R_{-i} = R'_{-i}$. Since by *peaks-onliness* $p_i = p'_i$ implies $F(R) = F(R')$, let $p_i < p'_i$. There are four (partially overlapping) cases.

Case 1.1 ($p_i < p'_i \leq \underline{F}(R)$). By Lemma 1(a), $\underline{F}(R) \leq \underline{F}(R')$. Moreover, assuming $\underline{F}(R) < \underline{F}(R')$ results in a contradiction since then $p'_i < \underline{F}(R')$ and by Lemma 1(a), $\underline{F}(R') \leq \underline{F}(R)$. Therefore, $\underline{F}(R) = \underline{F}(R')$.

Case 1.2 ($\underline{F}(R) < p_i < p'_i$). By Lemma 1(b), $\underline{F}(R') \leq \underline{F}(R)$. Hence, $\underline{F}(R') < p'_i$ and by Lemma 1(b), $\underline{F}(R) \leq \underline{F}(R')$. Therefore, $\underline{F}(R) = \underline{F}(R')$.

Case 2.1 ($p_i < p'_i \leq \bar{F}(R)$). By Lemma 1(a), $\bar{F}(R) \leq \bar{F}(R')$. Moreover, assuming $\bar{F}(R) < \bar{F}(R')$ results in a contradiction since then $p'_i < \bar{F}(R')$ and by Lemma 1(a), $\bar{F}(R') \leq \bar{F}(R)$. Therefore, $\bar{F}(R) = \bar{F}(R')$.

Case 2.2 ($\bar{F}(R) < p_i < p'_i$). By Lemma 1(b), $\bar{F}(R') \leq \bar{F}(R)$. Hence, $\bar{F}(R') < p'_i$ and by Lemma 1(b), $\bar{F}(R) \leq \bar{F}(R')$. Therefore, $\bar{F}(R) = \bar{F}(R')$. □

Before continuing the proof of Theorem 1 we prove the following result.

Lemma 2. *If $F \in \mathcal{F}$ satisfies uncompromisingness, then it satisfies peak-monotonicity, i.e., for each $i \in N$ and each $R, R' \in \mathcal{R}^N$ such that $R'_{-i} = R_{-i}$, if $p_i \leq p'_i$, then $\underline{F}(R) \leq \underline{F}(R')$ and $\bar{F}(R) \leq \bar{F}(R')$.*

Note that *peak-monotonicity* implies *peaks-onliness*.

Proof. Let $F \in \mathcal{F}$ satisfy *uncompromisingness*. Let $i \in N$ and $R, R' \in \mathcal{R}^N$ be such that $R'_{-i} = R_{-i}$ and $p_i \leq p'_i$. There are three cases.

Case 1 ($p_i < \underline{F}(R)$). We first consider $\underline{F}(R)$. If $p'_i \leq \underline{F}(R)$, then by *uncompromisingness*, $\underline{F}(R) = \underline{F}(R')$. Let $R^1_i \in \mathcal{R}$ be such that $p^1_i = \underline{F}(R)$. Then, by *uncompromisingness*, $\underline{F}(R_{-i}, R^1_i) = \underline{F}(R)$. If $p'_i > \underline{F}(R)$, then assuming $\underline{F}(R') < \underline{F}(R)$ leads to a contradiction as follows. Beginning from R', change i's preference to R^1_i. Since $\underline{F}(R') < \underline{F}(R) = p^1_i < p'_i$, by *uncompromisingness*, $\underline{F}(R'_{-i}, R^1_i) = \underline{F}(R')$. However, since $R'_{-i} = R_{-i}$, we also have $\underline{F}(R'_{-i}, R^1_i) = \underline{F}(R_{-i}, R^1_i) = \underline{F}(R)$, contradicting $\underline{F}(R') < \underline{F}(R)$. Therefore, we have $\underline{F}(R') \geq \underline{F}(R)$.

Next, we consider $\bar{F}(R)$. Note that $p_i < \bar{F}(R)$. Then, the arguments to show that $\bar{F}(R') \geq \bar{F}(R)$ are the same as above with $\bar{F}$ in the role of $\underline{F}$.

Case 2 ($\underline{F}(R) \leq p_i < \bar{F}(R)$). First, $\bar{F}(R') \geq \bar{F}(R)$ follows by Case 1 (the argument there was based on $p_i < \bar{F}(R)$). Next, if $\underline{F}(R') < \underline{F}(R)$, then $\underline{F}(R') < p_i \leq p'_i$ and *uncompromisingness* imply $\underline{F}(R') = \underline{F}(R)$, a contradiction. Hence, $\underline{F}(R') \geq \underline{F}(R)$.

Case 3 ($\bar{F}(R) \leq p_i$). If $\bar{F}(R') < \bar{F}(R)$, then $\bar{F}(R') < \bar{F}(R) \leq p_i \leq p'_i$ and *uncompromisingness* imply $\bar{F}(R') = \bar{F}(R)$, a contradiction. Hence, $\bar{F}(R) \leq \bar{F}(R')$.

Next, we consider $\underline{F}(R)$. Note that $\underline{F}(R) \leq p_i$. Then, the arguments to show that $\underline{F}(R') \geq \underline{F}(R)$ are the same as above with $\underline{F}$ in the role of $\bar{F}$. □

Proof of Theorem 1: (ii) ⇒ (iii). Let $F \in \mathcal{F}$ satisfy *uncompromisingness*. By Lemma 2, F satisfies *peak-monotonicity* and hence *peaks-onliness*. For each $i \in N$, let $R^{\min}_i, R^{\max}_i \in \mathcal{R}$ be such that $p^{\min}_i = 0$ and $p^{\max}_i = 1$. We proceed in three steps.

Step 1. Let $i \in N$, $R \in \mathcal{R}^N$, and consider profiles $R^{\min} := (R_{-i}, R_i^{\min})$ and $R^{\max} := (R_{-i}, R_i^{\max})$. Since $R_{-i} = R_{-i}^{\min} = R_{-i}^{\max}$ and $p_i^{\min} \leq p_i \leq p_i^{\max}$, by Lemma 2,

$$\underline{F}(R^{\min}) \leq \underline{F}(R) \leq \underline{F}(R^{\max}).$$

We show that

$$\underline{F}(R) = \text{med}(\underline{F}(R_{-i}, R_i^{\min}), p_i, \underline{F}(R_{-i}, R_i^{\max})).$$

Case 1 ($p_i < \underline{F}(R^{\min}) \leq \underline{F}(R)$). Then, $\text{med}(\underline{F}(R^{\min}), p_i, \underline{F}(R^{\max})) = \underline{F}(R^{\min})$. Since $0 = p_i^{\min} \leq p_i < \underline{F}(R)$, *uncompromisingness* implies $\underline{F}(R^{\min}) = \underline{F}(R)$. Therefore, $\underline{F}(R) = \underline{F}(R^{\min}) = \text{med}(\underline{F}(R^{\min}), p_i, \underline{F}(R^{\max}))$.

Case 2 ($p_i > \underline{F}(R^{\max}) \geq \underline{F}(R)$). Then, $\text{med}(\underline{F}(R^{\min}), p_i, \underline{F}(R^{\max})) = \underline{F}(R^{\max})$. Since $1 = p_i^{\max} \geq p_i > \underline{F}(R)$, *uncompromisingness* implies $\underline{F}(R^{\max}) = \underline{F}(R)$. Therefore, $\underline{F}(R) = \underline{F}(R^{\max}) = \text{med}(\underline{F}(R^{\min}), p_i, \underline{F}(R^{\max}))$.

Case 3 ($\underline{F}(R^{\min}) \leq p_i \leq \underline{F}(R^{\max})$). Then, $\text{med}(\underline{F}(R^{\min}), p_i, \underline{F}(R^{\max})) = p_i$. Assuming $p_i < \underline{F}(R)$ and thus $\underline{F}(R^{\min}) < \underline{F}(R)$ results in a contradiction as follows. Since $0 = p_i^{\min} \leq p_i < \underline{F}(R)$, *uncompromisingness* implies $\underline{F}(R^{\min}) = \underline{F}(R)$. Similarly, assuming $\underline{F}(R) < p_i$ and thus $\underline{F}(R) < \underline{F}(R^{\max})$ results in a contradiction as follows. Since $\underline{F}(R) < p_i \leq p_i^{\max}$, *uncompromisingness* implies $\underline{F}(R) = \underline{F}(R^{\max})$. Therefore, $\underline{F}(R) = p_i = \text{med}(\underline{F}(R^{\min}), p_i, \underline{F}(R^{\max}))$.

The arguments to show that

$$\bar{F}(R) = \text{med}(\bar{F}(R_{-i}, R_i^{\min}), p_i, \bar{F}(R_{-i}, R_i^{\max}))$$

are the same as above with $\bar{F}$ in the role of $\underline{F}$.

Step 2. We construct two vectors α and β. We will use a different letter (U instead of R) to label the following profiles.[16] For each $M \subseteq N$, let $U^M \in \mathcal{R}^N$ be such that all agents in M announce 0 as their peak and all other agents announce 1 as their peak, i.e.,

$$u^M = (\underbrace{0, \ldots, 0}_{i \in M}, \underbrace{1, \ldots, 1}_{i \in N \setminus M}).$$

Next, let vectors $\alpha = (\alpha_M)_{M \subseteq N}$ and $\beta = (\beta_M)_{M \subseteq N}$ be such that

$$\alpha_M = \underline{F}(U^M) \text{ and } \beta_M = \bar{F}(U^M) \quad \text{(note that } \alpha_M \leq \beta_M\text{)}.$$

[16]This is done to distinguish different types of profiles in Step 3 of the proof.

Let $L, M \subseteq N$ be such that $L \subsetneq M$. Then, for each $i \in M \setminus L$, $U_i^L \neq U_i^M$ (because $u_i^L = 1 > 0 = u_i^M$), and for each $i \notin M \setminus L$, $U_i^L = U_i^M$. Beginning from profile U^L, (sequentially) change the preferences of agents $i \in M \setminus L$ to U_i^M. Since for each $i \in M \setminus L$, $u_i^L > u_i^M$, by (sequentially) applying *peak-monotonicity*, we conclude that $\alpha_L \geq \alpha_M$ and $\beta_L \geq \beta_M$.

Step 3. We show that F is a generalized median correspondence associated with vectors α and β that were constructed in Step 2.

Let $R \in \mathcal{R}^N$. Without loss of generality, index the agents in N such that $p_1 \leq p_2 \leq \cdots \leq p_n$. Recall vectors α, β and profiles U^M ($M \subseteq N$) defined in Step 2. Let vectors $\tilde{\alpha}_p, \tilde{\beta}_p \in A^{n+1}$ be such that

$$\tilde{\alpha}_p = (\alpha_\emptyset, \alpha_{\{1\}}, \alpha_{\{1,2\}}, \ldots, \alpha_N) \text{ and } \tilde{\beta}_p = (\beta_\emptyset, \beta_{\{1\}}, \beta_{\{1,2\}}, \ldots, \beta_N).$$

Since the coordinates of $\tilde{\alpha}_p$ and $\tilde{\beta}_p$ are such that $0 \leq \alpha_N \leq \cdots \leq \alpha_\emptyset \leq 1$ and $0 \leq \beta_N \leq \cdots \leq \beta_\emptyset \leq 1$, and $u^\emptyset = (1, \ldots, 1)$, we have $\underline{F}(U^\emptyset) = \text{med}(u^\emptyset, \tilde{\alpha}_p) = \alpha_\emptyset$ and $\bar{F}(U^\emptyset) = \text{med}(u^\emptyset, \tilde{\beta}_p) = \beta_\emptyset$. Moreover, for each $i \in \{1, \ldots, n\}$,

$$u^{\{1,\ldots,i\}} = (\underbrace{0, \ldots, 0}_{j \in \{1,\ldots,i\}}, \underbrace{1, \ldots, 1}_{j \in \{i+1,\ldots,n\}})$$

implies

$$\underline{F}(U^{\{1,\ldots,i\}}) = \text{med}(u^{\{1,\ldots,i\}}, \tilde{\alpha}_p) = \alpha_{\{1,\ldots,i\}} \text{ and } \bar{F}(U^{\{1,\ldots,i\}}) = \text{med}(u^{\{1,\ldots,i\}}, \tilde{\beta}_p) = \beta_{\{1,\ldots,i\}}.$$

Next, for each $i \in \{1, \ldots, n\}$, let $R^i \in \mathcal{R}^N$ be such that

$$R^i = (R_1, \ldots, R_i, R_{i+1}^{\max}, \ldots, R_n^{\max}) \quad \text{(note that } R^n = R).$$

Note that at profiles R^i the order of agents' peaks does not change, i.e., for each $i \in \{1, \ldots, n\}$, we have $p_1^i \leq p_2^i \leq \cdots \leq p_n^i$. Hence, for each $i \in \{1, \ldots, n\}$, $\tilde{\alpha}_{p^i} = \tilde{\alpha}_p$ and $\tilde{\beta}_{p^i} = \tilde{\beta}_p$; in the sequel we will therefore use $\tilde{\alpha}_p$ and $\tilde{\beta}_p$ instead of $\tilde{\alpha}_{p^i}$ and $\tilde{\beta}_{p^i}$.

We show that $F(R) = F_G^{\alpha,\beta}(R) = [\text{med}(p, \tilde{\alpha}_p), \text{med}(p, \tilde{\beta}_p)]$ by induction.

Induction basis. We show that $F(R^1) = F_G^{\alpha,\beta}(R^1)$.

Consider profile $R^1 = (R_1, R_2^{\max}, \ldots, R_n^{\max})$. Recall profiles $U^{\{1\}} = (R_1^{\min}, R_2^{\max}, \ldots, R_n^{\max})$ and $U^\emptyset = (R_1^{\max}, \ldots, R_n^{\max})$. Hence, $U^{\{1\}} = (R_{-1}^1, R_1^{\min})$ and $U^\emptyset = (R_{-1}^1, R_1^{\max})$. By Step 1,

$$\underline{F}(R^1) = \text{med}(\underline{F}(U^{\{1\}}), p_1, \underline{F}(U^\emptyset)) \text{ and } \bar{F}(R^1) = \text{med}(\bar{F}(U^{\{1\}}), p_1, \bar{F}(U^\emptyset)).$$

Hence,

$$\underline{F}(R^1) = \text{med}(\alpha_{\{1\}}, p_1, \alpha_{\{\emptyset\}}) \text{ and } \bar{F}(R^1) = \text{med}(\beta_{\{1\}}, p_1, \beta_{\{\emptyset\}}).$$

Moreover, since

$$\underbrace{\alpha_N \leq \cdots \leq \alpha_{\{1\}}}_{n \text{ terms}} \leq \underbrace{\alpha_\emptyset \leq p_2^1 = \cdots = p_n^1}_{n \text{ terms}} = 1 \text{ and } \underbrace{\beta_N \leq \cdots \leq \beta_{\{1\}}}_{n \text{ terms}} \leq \underbrace{\beta_\emptyset \leq p_2^1 = \cdots = p_n^1}_{n \text{ terms}} = 1,$$

we have that

$$\underline{F}(R^1) = \text{med}(p^1, \tilde{\alpha}_p) \text{ and } \bar{F}(R^1) = \text{med}(p^1, \tilde{\beta}_p).$$

Therefore, $F(R^1) = F_G^{\alpha,\beta}(R^1)$.

Induction hypothesis. For $i \in \{2, \ldots, n\}$, $F(R^{i-1}) = F_G^{\alpha,\beta}(R^{i-1})$.

Induction step. We show that $F(R^i) = F_G^{\alpha,\beta}(R^i)$. More specifically, we show $\underline{F}(R^i) = \underline{F}_G^{\alpha,\beta}(R^i)$. The proof that $\bar{F}(R^i) = \bar{F}_G^{\alpha,\beta}(R^i)$ is obtained by using the same arguments with $\bar{F}$ in the role of $\underline{F}$ and $\tilde{\beta}_p$ in the role of $\tilde{\alpha}_p$. Recall that

$$R^{i-1} = (R_1, \ldots, R_{i-1}, R_i^{\max}, \ldots, R_n^{\max}) \text{ and } R^i = (R_{-i}^{i-1}, R_i).$$

There are three cases.

Case 1 ($p_i > \underline{F}(R^i)$). Since $R_{-i}^{i-1} = R_{-i}^i$ and $\underline{F}(R^i) < p_i = p_i^i \leq p_i^{i-1} = p_i^{\max} = 1$, by *uncompromisingness*, $\underline{F}(R^i) = \underline{F}(R^{i-1}) = \text{med}(p^{i-1}, \tilde{\alpha}_p)$. Thus, $R_{-i}^{i-1} = R_{-i}^i$ and $\text{med}(p^{i-1}, \tilde{\alpha}_p) < p_i = p_i^i \leq p_i^{i-1} = 1$ implies $\text{med}(p^{i-1}, \tilde{\alpha}_p) = \text{med}(p^i, \tilde{\alpha}_p)$. Hence, $\underline{F}(R^i) = \text{med}(p^i, \tilde{\alpha}_p) = \underline{F}_G^{\alpha,\beta}(R^i)$.

Case 2 ($p_i < \underline{F}(R^i)$). Recall that $U^{\{1,\ldots,i\}} = (R_1^{\min}, \ldots, R_i^{\min}, R_{i+1}^{\max}, \ldots, R_n^{\max})$. Since $p_1 \leq \cdots \leq p_n$ and $p_i = p_i^i < \underline{F}(R^i)$, for each $j \in \{1, \ldots, i\}$, $0 = p_j^{\min} \leq p_j = p_j^i < \underline{F}(R^i)$. Hence, by *uncompromisingness*, $\underline{F}(R_{-j}^i, R_j^{\min}) = \underline{F}(R^i)$. Beginning from profile R^i, (sequentially) change the preferences of agents $j \in \{1, \ldots, i\}$ to $R_j^{\min}$. Then, by (sequentially applying) *uncompromisingness*, $\underline{F}(R^i) = \underline{F}(U^{\{1,\ldots,i\}})$. We have shown at the beginning of Step 3 that $\underline{F}(U^{\{1,\ldots,i\}}) = \alpha_{\{1,\ldots,i\}}$. Thus, $\underline{F}(R^i) = \alpha_{\{1,\ldots,i\}}$. Then, $p_1^i \leq \ldots \leq p_i^i < \alpha_{\{1,\ldots,i\}}$ and $p_{i+1}^i = \ldots = p_n^i = 1 \geq \alpha_{\{1,\ldots,i\}}$. This implies $\text{med}(p^i, \tilde{\alpha}_p) = \alpha_{\{1,\ldots,i\}}$. Hence, $\underline{F}(R^i) = \text{med}(p^i, \tilde{\alpha}_p) = \underline{F}_G^{\alpha,\beta}(R^i)$.

Case 3 ($p_i = \underline{F}(R^i)$). Since $R_{-i}^{i-1} = R_{-i}^i$ and $p_i = p_i^i \leq p_i^{i-1} = p_i^{\max} = 1$, by *peak-monotonicity*, $\underline{F}(R^i) \leq \underline{F}(R^{i-1})$. Thus, $p_1 \leq \cdots \leq p_n$ and $p_i = p_i^i = \underline{F}(R^i)$ imply $p_{i-1}^i = p_{i-1}^{i-1} \leq \underline{F}(R^{i-1})$. There are two sub-cases.

Case 3.1 ($p^i_{i-1} = \underline{F}(R^{i-1})$). Thus, $p^i_{i-1} = p^i_i = p_i = \underline{F}(R^i) = \underline{F}(R^{i-1})$. Hence, $\underline{F}(R^{i-1}) = \underline{F}^{\alpha,\beta}_G(R^{i-1})$ implies $\text{med}(p^{i-1}, \tilde{\alpha}_p) = p_i = p^i_i \leq p^{i-1}_i = 1$. This implies $\text{med}(p^i, \tilde{\alpha}_p) = \text{med}(p^{i-1}, \tilde{\alpha}_p)$. Hence, $\underline{F}(R^i) = \text{med}(p^i, \tilde{\alpha}_p) = \underline{F}^{\alpha,\beta}_G(R^i)$.

Case 3.2 ($p^i_{i-1} < \underline{F}(R^{i-1})$). Recall that $U^{\{1,\dots,i-1\}} = (R^{\min}_1, \dots, R^{\min}_{i-1}, R^{\max}_i, \dots, R^{\max}_n)$ with $\underline{F}(U^{\{1,\dots,i-1\}}) = \alpha_{\{1,\dots,i-1\}}$. By the same arguments as at the beginning of Case 2, $p^i_{i-1} < \underline{F}(R^{i-1})$ implies that $\underline{F}(R^{i-1}) = \underline{F}(U^{\{1,\dots,i-1\}}) = \alpha_{\{1,\dots,i-1\}}$. Since $p_i = \underline{F}(R^i) \leq \underline{F}(R^{i-1})$, it follows that $p_i \leq \alpha_{\{1,\dots,i-1\}}$.

Recall that $U^{\{1\dots,i\}} = (R^{\min}_1, \dots, R^{\min}_i, R^{\max}_{i+1}, \dots, R^{\max}_n)$ with $\underline{F}(U^{\{1,\dots,i\}}) = \alpha_{\{1,\dots,i\}}$. Beginning from profile $U^{\{1,\dots,i\}}$, (sequentially) change the preferences of all agents $j \in \{1, \dots, i\}$ to R_j and obtain profile $R^i = (R_1, \dots, R_i, R^{\max}_{i+1}, \dots, R^{\max}_n)$. Since for each $j \in \{1, \dots, i\}$, $p^i_j \geq p^{\min}_j = 0$, by (sequentially) applying *peak-monotonicity*, we obtain $\underline{F}(R^i) \geq \underline{F}(U^{\{1,\dots,i\}}) = \alpha_{\{1,\dots,i\}}$. Hence, $p_i \geq \alpha_{\{1,\dots,i\}}$ and it follows that $\alpha_{\{1,\dots,i\}} \leq p_i \leq \alpha_{\{1,\dots,i-1\}}$.

Thus, since $\alpha_N \leq \dots \leq \alpha_\emptyset$, vector $\tilde{\alpha}_p$ contains at least $n+1-i$ coordinates not larger than p_i (i.e., coordinates $\alpha_{\{1,\dots,i\}}, \dots, \alpha_N$) and at least i coordinates not smaller than p_i (i.e., coordinates $\alpha_\emptyset, \dots, \alpha_{\{1,\dots,i-1\}}$). In addition, since $p_1 \leq \dots \leq p_n$, at least i agents announce peaks not larger than p_i (i.e., agents $1, \dots, i$) and $n-i+1$ agents announce peaks not smaller than p_i (i.e., agents $i, \dots, n$). This implies $\text{med}(p^i, \tilde{\alpha}_p) = p_i$. Hence, $\underline{F}(R^i) = p_i = \text{med}(p^i, \tilde{\alpha}_p) = \underline{F}^{\alpha,\beta}_G(R^i)$. □

Proof of Theorem 1: (iii) ⇒ (i). Let $F^{\alpha,\beta}_G \in \mathcal{F}_G$. By definition, $F^{\alpha,\beta}_G$ satisfies *peaks-onliness*. By Moulin (1980, Proposition 3), $\underline{F}^{\alpha,\beta}_G$ and $\bar{F}^{\alpha,\beta}_G$ are *strategy-proof* and by Border and Jordan (1983, Proposition 1), $\underline{F}^{\alpha,\beta}_G$ and $\bar{F}^{\alpha,\beta}_G$ satisfy *uncompromisingness*. Then, by Lemma 2, $\underline{F}^{\alpha,\beta}_G$ and $\bar{F}^{\alpha,\beta}_G$ satisfy *peak-monotonicity*. To show that $F^{\alpha,\beta}_G$ satisfies *strategy-proofness* let $i \in N$, $R, R' \in \mathcal{R}^N$ such that $R_{-i} = R'_{-i}$.

Case 1 ($p_i \leq \underline{F}^{\alpha,\beta}_G(R)$). First, if $p_i > p'_i$, by *uncompromisingness*, $\underline{F}^{\alpha,\beta}_G(R) = \underline{F}^{\alpha,\beta}_G(R')$ and $\bar{F}^{\alpha,\beta}_G(R) = \bar{F}^{\alpha,\beta}_G(R')$. Hence, $F^{\alpha,\beta}_G(R) = F^{\alpha,\beta}_G(R')$.

Second, if $p_i \leq p'_i$, by *peak-monotonicity*, $\underline{F}^{\alpha,\beta}_G(R) \leq \underline{F}^{\alpha,\beta}_G(R')$ and $\bar{F}^{\alpha,\beta}_G(R) \leq \bar{F}^{\alpha,\beta}_G(R')$, which implies $F^{\alpha,\beta}_G(R) \; R_i \; F^{\alpha,\beta}_G(R')$.

Case 2 ($p_i \geq \bar{F}^{\alpha,\beta}_G(R)$). Symmetric to Case 1.

Case 3 ($\underline{F}^{\alpha,\beta}_G(R)) < p_i < \bar{F}^{\alpha,\beta}_G(R)$). Then, $b_{R_i}(F^{\alpha,\beta}_G(R)) = p_i \; R_i \; b_{R_i}(F^{\alpha,\beta}_G(R'))$. Next, $w_{R_i}(F^{\alpha,\beta}_G(R)) \in \{\underline{F}^{\alpha,\beta}_G(R), \bar{F}^{\alpha,\beta}_G(R)\}$. By *strategy-proofness* of $\underline{F}^{\alpha,\beta}_G$ and $\bar{F}^{\alpha,\beta}_G$, $\underline{F}^{\alpha,\beta}_G(R) \; R_i \; \underline{F}^{\alpha,\beta}_G(R')$ and $\bar{F}^{\alpha,\beta}_G(R) \; R_i \; \bar{F}^{\alpha,\beta}_G(R')$, which implies $w_{R_i}(F^{\alpha,\beta}_G(R)) \; R_i \; w_{R_i}(F^{\alpha,\beta}_G(R'))$. Hence, $F^{\alpha,\beta}_G(R) \; R_i \; F^{\alpha,\beta}_G(R')$. □

B Proof of Theorem 2

Before proving Theorem 2, we prove some intermediate results.

Lemma 3. *If $F \in \mathcal{F}$ satisfies strategy-proofness and voter-sovereignty, then it satisfies unanimity, i.e., for each $R \in \mathcal{R}^N$ such that $p = (x, \ldots, x)$, $F(R) = \{x\}$.*

Proof. Let $F \in \mathcal{F}$ satisfy *strategy-proofness* and *voter-sovereignty*. We show that F satisfies *unanimity*. Let $a \in A$ and $R^a \in \mathcal{R}^N$ be such that $p^a = (a, \ldots, a)$. By *voter-sovereignty*, there exists $R^0 \in \mathcal{R}^N$ such that $F(R^0) = \{a\}$. First, consider profile $R^1 := (R^a_1, R^0_{-1})$. By *strategy-proofness*, $F(R^1)\, R^a_1\, F(R)$. Since $F(R^0) = \{a\} = \{p^a_1\}$, we then have $F(R^1) = \{a\}$. Next, for each $i \in \{2, \ldots, n\}$ (sequentially) consider profile $R^i := (R^a_i, R^{i-1}_{-i})$. By *strategy-proofness* we again obtain $F(R^i) = F(R^{i-1}) = \{a\}$. Therefore, since $R^n = R^a$, $F(R^a) = \{a\}$. □

Lemma 4. *If $F \in \mathcal{F}$ satisfies strategy-proofness and voter-sovereignty, then F satisfies efficiency.*

Proof. Let $F \in \mathcal{F}$ satisfy *strategy-proofness* and *voter-sovereignty*. By Lemma 3, F satisfies *unanimity*. We show that F satisfies *efficiency*. The proof proceeds in two steps.

Step 1. Let $R \in \mathcal{R}^N$ such that $p_1 = p_2 \leq \ldots \leq p_n$ and $F(R) \in \mathrm{E}(R)$. By Proposition 1, $\underline{p} = p_1 \leq \underline{F}(R) \leq \bar{F}(R) \leq p_n = \bar{p}$. Consider $R'_1 \in \mathcal{R}$ such that $p'_1 \geq p_n$ and let $R' := (R'_1, R_{-1})$. If $p'_1 = p_1$, then $p_1 = \ldots = p_n$ and by *unanimity*, $F(R') = \{p_1\} \in \mathrm{E}(R')$. Assume $p'_1 > p_1$. We show that $F(R') \in \mathrm{E}(R)$.

If $\underline{F}(R') < p_1 = \underline{p}'$, then $\underline{F}(R') < \underline{F}(R) \leq \bar{F}(R) \leq p'_1$. Hence, $w_{R'_1}(F(R)) = \underline{F}(R)\; P'_1\; \underline{F}(R')$. Thus, $w_{R'_1}(F(R))\; P'_1\; w_{R'_1}(F(R'))$, contradicting *strategy-proofness*. Therefore, $\underline{F}(R') \geq \underline{p}'$.

Recall that $p_1 = p_2 < p'_1$. If $\bar{F}(R') > p'_1 = \bar{p}'$, then $p_2 \leq \underline{F}(R') \leq \bar{F}(R')$ and $w_{R_2}(F(R')) = \bar{F}(R')$. Consider $\hat{R}_2 \in \mathcal{R}$ such that $\hat{p}_2 = p'_1$ and $p_2\; \hat{I}_2\; \bar{F}(R')$. Note that $w_{\hat{R}_2}(F(R'))\; \hat{I}_2\; \bar{F}(R')$. Let $\hat{R} := (\hat{R}_2, R'_{-2})$. By *strategy-proofness*, $w_{\hat{R}_2}(F(\hat{R}))\; \hat{R}_2\; w_{\hat{R}_2}(F(R'))$. Hence, $w_{\hat{R}_2}(F(\hat{R}))\; \hat{I}_2\; \bar{F}(R')\; \hat{R}_2\; p_2$. Thus, $w_{\hat{R}_2}(F(\hat{R})) \in [p_2, \bar{F}(\hat{R})]$. Then, $w_{R_2}(F(\hat{R})) = \bar{F}(\hat{R})$. If $\bar{F}(\hat{R}) \leq \hat{p}_2$, then $w_{R_2}(F(\hat{R}))\; P_2\; w_{R_2}(F(R'))$, contradicting *strategy-proofness*. Hence, $\bar{F}(\hat{R}) > \hat{p}_2 = p'_1$.

Set $\hat{R}^2 \equiv \hat{R}$. Thus, $\bar{F}(\hat{R}^2) > \hat{p}^2_2 = p'_1$. We now (sequentially) consider for each $j \in \{3, \ldots, k\}$ with $p_j < p'_1$ preferences $\hat{R}_j \in \mathcal{R}$ such that $\hat{p}_j = p'_1$ and $p_j\; \hat{I}_j\; \bar{F}(\hat{R}^{j-1})$. Let $\hat{R}^j := (\hat{R}_j, \hat{R}^{j-1}_{-j})$. Then, by the same arguments as above for $\hat{R}^j$ instead of $\hat{R}^2$, we obtain that $\bar{F}(\hat{R}^j) > p'_1$. In particular, for the final profile $\hat{R}^k$, $\bar{F}(\hat{R}^k) > p'_1$. However, profile

$\hat{R}^k$ is such that for all agents $i \in N$, $p(\hat{R}^k_i) = p'_1$. Thus, by *unanimity*, $F(\hat{R}^k) = \{p'_1\}$, a contradiction. Hence, $\bar{F}(R') \leq p'_1 = \bar{p}'$.

Therefore, $\underline{p}' \leq \underline{F}(R') \leq \bar{F}(R') \leq \bar{p}'$ and $F(R') \in \mathrm{E}(R')$.

Step 2. Let $R \in \mathcal{R}^N$ such that $p_1 \leq \cdots \leq p_n$. Let $R^1 \in \mathcal{R}^N$ be such that $R^1 := (R_1, \ldots, R_1)$. By *unanimity*, $F(R^1) = \{p_1\}$. Hence, by Proposition 1, $F(R^1) \in \mathrm{E}(R^1)$. Next, consider profile $R^2 := (R_2, R^1_{-2})$ where $p_2 \geq p^1_2 = p_1$, $\underline{p}^2 = \underline{p}^1$, and $\bar{p}^2 \geq \bar{p}^1$. Since $F(R^1) \in \mathrm{E}(R^1)$, by Step 1 (with agent 2 in the role of agent 1) we conclude $F(R^2) \in \mathrm{E}(R^2)$. We (sequentially) consider for each $i = \{3, \ldots, n\}$ profile $R^i := (R_i, R^{i-1}_{-i})$. Since $F(R^{i-1}) \in \mathrm{E}(R^{i-1})$, by Step 1 (with agent i in the role of agent 1) we conclude $F(R^i) \in \mathrm{E}(R^i)$. Therefore, since $R^n \equiv R$, $F(R) \in \mathrm{E}(R)$. □

Lemma 5. *If $F \in \mathcal{F}$ satisfies strategy-proofness and voter-sovereignty, then for each $R, R' \in \mathcal{R}^N$ and each $i \in N$ such that $R_{-i} = R'_{-i}$ the following holds:*

(a) *if $p_i < \underline{F}(R)$ and $p'_i \leq \underline{F}(R)$, then $F(R) = F(R')$ and*

(b) *if $p_i > \bar{F}(R)$ and $p'_i \geq \bar{F}(R)$, then $F(R) = F(R')$.*

Proof. Let $F \in \mathcal{F}$ satisfy *strategy-proofness* and *voter-sovereignty*. By Lemma 4, F satisfies *efficiency*. Let $R, R' \in \mathcal{R}^N$ and $i \in N$ be such that $R_{-i} = R'_{-i}$. Assume that $p_1 \leq \cdots \leq p_n$.

(a) Let $p_i < \underline{F}(R)$ and $p'_i \leq \underline{F}(R)$. We have that $b_{R_i}(F(R)) = \underline{F}(R)$ and $w_{R_i}(F(R)) = \bar{F}(R)$. If $\underline{F}(R') > \underline{F}(R)$, then $b_{R'_i}(F(R)) = \underline{F}(R)\ P'_i\ \underline{F}(R') = b_{R'_i}(F(R'))$, contradicting *strategy-proofness*. Hence, $\underline{F}(R') \leq \underline{F}(R)$. If $\underline{F}(R') \in [p_i, \underline{F}(R))$, then $b_{R_i}(F(R')) = \underline{F}(R')\, P_i\, \underline{F}(R) = b_{R_i}(F(R))$, contradicting *strategy-proofness*. Hence, $\underline{F}(R') = \underline{F}(R)$ or $\underline{F}(R') < p_i$.

Assuming $\underline{F}(R') < p_i$ leads to a contradiction as follows. If $\bar{F}(R') \geq p_i$, then $b_{R_i}(F(R')) = p_i\ P_i\ \underline{F}(R) = b_{R_i}(F(R))$, contradicting *strategy-proofness*. Hence, $\bar{F}(R') < p_i$. By *efficiency*,

$$p_1 \leq \underline{F}(R') \leq \bar{F}(R') < p_i < \underline{F}(R) \leq \bar{F}(R).$$

Hence, $1 \neq i$. Consider $\hat{R}_1 \in \mathcal{R}$ such that $\hat{p}_1 = p_i$ and $\bar{F}(R')\ \hat{P}_1\ \underline{F}(R)$ and let $R^1 := (\hat{R}_1, R_i, R_{-1,i})$ and $\tilde{R}^1 = (\hat{R}_1, R'_i, R_{-1,i})$. Note that $\tilde{R}^1 = (\hat{R}_1, R'_{-1}) = (\hat{R}_i, R^1_{-i})$. By *efficiency*, $p_1 \leq \underline{F}(R^1)$. Then, $p_1 < \hat{p}_1 < \underline{F}(R) \leq \bar{F}(R)$ and *strategy-proofness* imply $F(R^1) = F(R)$.

Now, starting from $\tilde{R}^1 = (\hat{R}_1, R'_{-1})$, *strategy-proofness* implies that $b_{\hat{R}_1}(F(\tilde{R}^1))\ \hat{R}_1\ b_{\hat{R}_1}(F(R')) = \bar{F}(R')$. Since $\bar{F}(R')\ \hat{P}_1\ \underline{F}(R) = \underline{F}(R^1)$, we have $b_{\hat{R}_1}(F(\tilde{R}^1))\ \hat{P}_1\ \underline{F}(R^1) = b_{\hat{R}_1}(F(R^1))$. Hence, $b_{\hat{R}_1}(F(\tilde{R}^1)) \in [\bar{F}(R'), \underline{F}(R^1))$. Recall that $\tilde{R}^1 = (\hat{R}_i, R^1_{-i})$. Hence, if

$b_{\hat{R}_1}(F(\tilde{R}^1)) \in [p_i, \underline{F}(R^1))$, then $b_{R_i}(F(\tilde{R}^1))\ P_i\ b_{R_i}(F(R^1)) = \underline{F}(R^1)$, contradicting *strategy-proofness*. Thus, $b_{\hat{R}_1}(F(\tilde{R}^1)) \in [\bar{F}(\hat{R}), p_i)$ and $\bar{F}(\tilde{R}^1) < p_i < \underline{F}(R)$.

We now (sequentially) consider for each $j \in \{2, \ldots, k\}$ with $p_j \leq \bar{F}(\tilde{R}^{j-1}) < p_i < \underline{F}(R^{j-1})$ preferences $\hat{R}_j \in \mathcal{R}$ such that $\hat{p}_j = p_i$ and $\bar{F}(\tilde{R}^{j-1})\ \hat{P}_j\ \underline{F}(R^{j-1})$. Let $R^j := (\hat{R}_j, R^{j-1}_{-j})$ and $\tilde{R}^j := (\hat{R}_j, \tilde{R}^{j-1}_{-j})$. Then, for each pair of profiles R^j and $\tilde{R}^j$, by the same arguments as above, we obtain that $\bar{F}(\tilde{R}^j) < p_i = \hat{p}_j < \underline{F}(R^j)$. However, final profile $\tilde{R}^k$ is such that $\bar{F}(\tilde{R}^k) < p_i = \underline{\tilde{p}}^k$, contradicting *efficiency*.

We can now conclude that $\underline{F}(R') = \underline{F}(R)$. Then, $p'_i \leq \underline{F}(R') = \underline{F}(R) \leq \bar{F}(R)$ and *strategy-proofness* imply $\bar{F}(R') = \bar{F}(R)$. Hence, $F(R') = F(R)$.

(b) Let $p_i > \bar{F}(R)$ and $p'_i \geq \bar{F}(R)$. By symmetric arguments to the ones presented in Part (a) for $\bar{F}$ instead of $\underline{F}$ it follows that $F(R) = F(R')$. □

Lemma 6. *If $F \in \mathcal{F}$ satisfies strategy-proofness and voter-sovereignty, then F satisfies peaks-onliness.*

Proof. Let $F \in \mathcal{F}$ satisfy *strategy-proofness* and *voter-sovereignty*. By Lemma 4, F satisfies *efficiency*. Let $R, R' \in \mathcal{R}^N$ and $i \in N$ be such that $R_{-i} = R'_{-i}$. Assume that $p_1 \leq \cdots \leq p_n$ and that $p_i = p'_i$. We prove that $F(R) = F(R')$. There are three cases.

Case 1 ($p_i < \underline{F}(R)$). Hence, $p'_i < \underline{F}(R)$ and by Lemma 5 (a), $F(R) = F(R')$.

Case 2 ($p_i > \bar{F}(R)$). Hence, $p'_i > \bar{F}(R)$ and by Lemma 5 (b), $F(R) = F(R')$.

Case 3 ($\underline{F}(R)) \leq p_i \leq \bar{F}(R)$). If $\underline{F}(R) = p_i = \bar{F}(R)$, then $b_{R'_i}(F(R)) = w_{R'_i}(F(R)) = p_i = p'_i$ and by *strategy-proofness*, $b_{R'_i}(F(R')) = w_{R'_i}(F(R')) = p'_i$. Hence, $F(R) = F(R')$.

Without loss of generality, assume that $\underline{F}(R) < p_i \leq \bar{F}(R)$ (the case $\underline{F}(R) \leq p_i < \bar{F}(R)$ is proven symmetrically). Then, $b_{R'_i}(F(R)) = p_i = p'_i$ and by *strategy-proofness*, $b_{R'_i}(F(R')) = p'_i$. Hence, $\underline{F}(R') \leq p'_i \leq \bar{F}(R')$. We prove that $\underline{F}(R) = \underline{F}(R')$ by contradiction. Assume that $\underline{F}(R) \neq \underline{F}(R')$. More specifically, assume that $\underline{F}(R) < \underline{F}(R')$ (the case $\underline{F}(R') < \underline{F}(R)$ is then proven by exchanging the roles of R and R'). If $\bar{F}(R') < \bar{F}(R)$, then $\underline{F}(R) < \underline{F}(R') \leq \bar{F}(R') \leq \bar{F}(R)$ and $w_{R_i}(F(R'))\ P_i\ w_{R_i}(F(R))$, contradicting *strategy-proofness*. Hence, $\bar{F}(R) \leq \bar{F}(R')$. By Lemma 5, it is without loss of generality to assume that $p_1 = \underline{F}(R)$ and $p_n = \bar{F}(R')$. Hence,

$$p_1 = \underline{F}(R) < \underline{F}(R') \leq p_i = p'_i \leq \bar{F}(R) \leq \bar{F}(R') = p_n.$$

If $p_i = p_n = \bar{p}$, then by *efficiency*, $\underline{F}(R) < \underline{F}(R') \leq p_i = p'_i = \bar{F}(R) = \bar{F}(R')$. Thus, $w_{R_i}(F(R')) = \underline{F}(R') P_i \underline{F}(R) = w_{R_i}(F(R))$, contradicting *strategy-proofness*. Hence, $p_i < p_n$.

Consider $\hat{R}_n \in \mathcal{R}$ such that $\hat{p}_n = p_i$ and $\bar{F}(R') \, \hat{P}_n \, \underline{F}(R)$. Then, $w_{\hat{R}_n}(F(R')) \, \hat{P}_n \, \underline{F}(R)$. Let $R^n := (R_i, \hat{R}_n, R_{-n,i})$ and $\tilde{R}^n := (R'_i, \hat{R}_n, R_{-n,i})$. Note that $R^n = (\hat{R}_n, R_{-n})$ and $\tilde{R}^n = (\hat{R}_n, R'_{-n})$.

Since $b_{\hat{R}_n}(F(R)) = \hat{p}_n$, by *strategy-proofness*, $b_{\hat{R}_n}(F(R^n)) = \hat{p}_n$ and $\underline{F}(R^n) \leq \hat{p}_n \leq \bar{F}(R^n)$. Since $R^n = (\hat{R}_n, R_{-n})$ and $w_{R_n}(F(R)) = \underline{F}(R)$, by *strategy-proofness*, $\underline{F}(R^n) \leq \underline{F}(R) = p_1$. Then, by *efficiency*, $\underline{F}(R^n) = \underline{F}(R)$.

Since $\tilde{R}^n = (\hat{R}_n, R'_{-n})$ and $b_{\hat{R}_n}(F(R')) = \hat{p}_n$, by *strategy-proofness*, $b_{\hat{R}_n}(F(\tilde{R}^n)) = \hat{p}_n$ and $\underline{F}(\tilde{R}^n) \leq \hat{p}_n \leq \bar{F}(\tilde{R}^n)$. By *efficiency*, $p_1 = \underline{F}(R) \leq \underline{F}(\tilde{R}^n)$. Recall that $w_{\hat{R}_n}(F(R')) \, \hat{P}_n \, \underline{F}(R)$. Hence, if $\underline{F}(R) = \underline{F}(\tilde{R}^n)$, then $w_{\hat{R}_n}(F(R')) \, \hat{P}_n \, w_{\hat{R}_n}(F(\tilde{R}^n))$, contradicting *strategy-proofness*. Hence, $p_1 = \underline{F}(R) < \underline{F}(\tilde{R}^n)$. Similarly as before, by Lemma 5, we can assume that

$$p_1 = \underline{F}(R^n) < \underline{F}(\tilde{R}^n) \leq p_i = p'_i \leq \bar{F}(R^n) \leq \bar{F}(\tilde{R}^n) = p_{n-1}.$$

We now (sequentially) consider for each $j \in \{n-1, n-2, \ldots, k\}$ with $p_i < p_j$ preferences $\hat{R}_j \in \mathcal{R}$ such that $\hat{p}_j = p_i$ and $\bar{F}(\tilde{R}^{j+1}) \, \hat{P}_j \, \underline{F}(R^{j+1})$. Let $R^j = (\hat{R}_j, R^{j+1}_{-j})$ and $\tilde{R}^j = (\hat{R}_j, \tilde{R}^{j+1}_{-j})$. Then, for each pair of profiles R^j and $\tilde{R}^j$, by the same arguments as above, we obtain that

$$p_1 = \underline{F}(R^j) < \underline{F}(\tilde{R}^j) \leq p_i = p'_i \leq \bar{F}(R^j) \leq \bar{F}(\tilde{R}^j) = p_{j-1}.$$

However, the final profiles R^k and $\tilde{R}^k$ are such that $R^k_{-i} = \tilde{R}^k_i$ and

$$p_1 = \underline{F}(R^k) < \underline{F}(\tilde{R}^k) \leq p_i = p'_i = \bar{F}(R^k) = \bar{F}(\tilde{R}^k).$$

Thus, $w_{R_i}(F(\tilde{R}^k)) = \underline{F}(\tilde{R}^k) \, P_i \, \underline{F}(R^k) = w_{R_i}(F(R^k))$, contradicting *strategy-proofness*. We can now conclude that $\underline{F}(R') = \underline{F}(R)$. Assuming $\bar{F}(R') \neq \bar{F}(R)$ leads to a contradiction in a symmetric fashion. Hence, $F(R') = F(R)$. □

Proof of Theorem 2. We start with statement (i) and F satisfying *strategy-proofness* and *voter sovereignty*. Then, by Lemma 6, F satisfies *peaks-onliness*. Hence, by Theorem 1 ((i) ⇒ (iii)), $F \in \mathcal{F}_G$. Since F satisfies *peaks-onliness*, by Lemma 4, F satisfies *efficiency*. Hence, $F \in \mathcal{F}_{EG}$ and statement (i) implies statement (iii).

Second, we show that statement (iii) implies statement (ii). By Theorem 1 ((iii) ⇒ (ii)), each $F \in \mathcal{F}_G$ satisfies *uncompromisingness*. Since F satisfies *efficiency* it also satisfies the weaker property of *voter-sovereignty*. Hence, statement (iii) implies statement (ii).

Finally, by Theorem 1 ((ii) ⇒ (i)), *uncompromisingness* implies *strategy-proofness* and *peaks-onliness*. Hence, statement (ii) implies statement (i). □

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