Matching and the Allocation of Indivisible Objects via Deferred-Acceptance under Responsive Priorities

by: Bettina Klaus

It is well-known that economics is the science of allocating scarce resources. Often this is done using money as our daily shopping routines confirm. Sometimes, there is more to the allocation mechanism than simple price setting and taking, but even auction mechanisms as encountered on e-bay are now well accepted rationing mechanisms.

However, there are economic problems in which using money and prices to match resources and consumers is not usually done or is considered immoral or even illegal. Examples for this type of problems are the assignments of students to public schools or universities or the assignment of transplant organs to patients who urgently need these transplants. If money cannot or should not be used to determine who gets what, how else can we decide on matching resources to consumers[1]?

Matching Practice: the National Resident Matching Program (NRMP)[2]

A by now famous and classic example of two-sided matching is the National Resident Matching Program (NRMP): a centralized clearing house to assign medical students to so-called intern or residency positions after their M.D. Degree. As with many entry-level jobs, work conditions and salaries are not very flexible and therefore do not play any role in the matching process.

The NRMP was established in 1952 because of persisting problems in the assignment of residents to hospitals. First, between 1900 and 1945, the medical resident market was decentralized and experienced unraveling of appointment dates – in an attempt to compete for the best students, hospitals offered residencies to medical students earlier and earlier up to a point when in 1945 positions were offered two years in advance of graduation. This led to some inefficiency in the matching. Second,when medical schools tried to control unraveling by not releasing information about candidates before a specified date that they all agreed on, the practice of exploding offers (extremely short decision times to accept offers) again destabilized the market (through missed as well as broken agreements). The original NRMP matching algorithm introduced in 1952 as well as the redesigned NRMP matching algorithm introduced in 1998 are based on the so-called deferred acceptance algorithm that I will explain next.

Matching Theory and Deferred Acceptance[3]

In 1962 two game theorists/mathematicians - David Gale and Lloyd Shapley - wrote a now famous paper about “College Admissions and the Stability of Marriage”. The college admissions model exactly captures the situation for medical residents (but the college admissions model was formulated by Gale and Shapley, 1962, independently of the NRMP): a set of students (medical residents) has to be assigned to a set of colleges (hospitals with residency programs). The ingredients of a college admissions

[1] A recent survey on these types of markets is Sönmez and Ünver (2009).

[2] For a detailed discussion of the American medical interns/residents market I refer to Roth (1984,2003).

[3] I strongly recommend Gale and Shapley's (1962) seminal paper – simply beautiful!

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problem are as follows:

- a set of students,
- a set of colleges each with a quota, i.e., the maximal number of students it can admit,
- students strict preferences over colleges (including the option to not go to college),
- and colleges' responsive preferences over sets of students.

Responsiveness means that colleges' preferences over sets of students are based on a strict ranking of the students that the college has (e.g., based on an entrance exam). For simplicity, I assume that all students are acceptable (they pass the minimal entrance requirements for all colleges). Then, a college's preference relation is *responsive* if it would like to add a student if its quota is not exhausted and if it would prefer to exchange one of its students for a better student in case the quota is already met.

A solution to a colleges admissions problem is a *matching* of students to colleges such that each student gets assigned at most one college and such that colleges' quotas are respected. A matching is *stable*, if no student is assigned to a college (s)he does not want to go to (*individual rationality*) and there is *no blocking* by a student-college pair such that the student prefers the college to the current match and the college also would like to add the student (either because the quota is not exhausted or because the college would then not admit a less preferred student). Stability is not only a theoretically appealing property, it plays an important role in entry-level job-matching since an unstable matching is not likely to persist (it is too easy to check if there is a win-win improvement for both sides of the market – you would definitely make a couple of phone calls to check if the employers you would prefer by chance would also prefer you before final hiring decisions are made).

However, given a college assignment or similar problem, do stable matchings always exist? And if they do, how do we find one? Gale and Shapley (1962) provided a simple and fast algorithm, called the *student proposing deferred acceptance algorithm*, to compute a stable matching:

Step 1 for students: each student proposes to her\his favorite college.
Step 1 for colleges: each college tentatively assigns the best students who proposed to it without exceeding its quota (and rejects some students if too many propose).

Step 2 for students: each student currently not tentatively assigned proposes to her/his favorite college among those who have not yet rejected her/him.
Steps 2 for colleges: each college tentatively assigns the best students who proposed to it and who were tentatively assigned without exceeding its quota (and rejects some students if too many propose).

Continue this procedure until all students are either tentatively assigned to a college or have exhausted their list of acceptable colleges. This algorithm produces a stable matching!

There is also a deferred acceptance algorithm where the colleges propose to students, but here I exclusively focus on the student proposing deferred acceptance algorithm, which has the following nice properties.

- it finds a stable matching (Gale and Shapley, 1962);
- it finds the best/optimal stable matching for all students (Gale and Shapley, 1962).
- it is a weakly dominant strategy for all students to state their true preferences (Roth, 1985). In other words, a student cannot obtain a better match by lying about her/his preferences.

School Choice and Deferred Acceptance[4]

Some cities in the US operate so-called school choice programs in which students submit preferences about different district schools and based on "priorities" students are matched with schools. The school choice problem is very similar to the college admissions problem with the difference that schools' preferences over sets of students are replaced by priorities, i.e., rankings of students that might reflect objective priority criteria determined by the school district. In New York City, priorities at schools are determined by exam scores and in Boston aspects such as walking distance and siblings in the same school are taken into account. Both school systems exhibited signs of market failure and were redesigned using the student proposing deferred acceptance algorithm as main building block (see Abdulkadiroğlu, Pathak, and Roth, 2005 a,b).

Allocating Indivisible Objects using Deferred-Acceptance

So far we have argued that deferred acceptance has nice theoretical properties for college admissions and school choice (note that preferences of colleges or priorities of schools are given). Furthermore, deferred acceptance emerged "naturally" as the mechanism of the NRMP clearinghouse and it was implemented in Boston and New York City's school choice programs. Finally, I will demonstrate that deferred acceptance also has very nice and robust properties for allocation models without preferences or fixed priorities.

In Ehlers and Klaus (2009) we study the allocation of indivisible objects to a set of agents. The assignment of students to schools could be considered an example as well as the assignment of dormitory rooms to students. We assume that agents (e.g., students) in these situations have

[4] A recent survey on school choice is Klijn (2008).

strict preferences over the (object) types (e.g., dormitory rooms of a certain type or in a certain building) and that (object) types might come with a capacity constraint (the maximal number of dormitory rooms of the same type). We have various results for this general class of allocation problems, but here – due to space limitations - I will only present one of our results for the class of problems where exactly one object of each type is available.

Our approach is to first define desirable properties that an allocation rule should satisfy and then see which class of rules satisfies all these properties.

We consider situations where resources may change, i.e., it could be that additional objects are available. When the change of the environment is exogenous, it would be unfair if the agents who were not responsible for this change were treated unequally. We apply this idea of solidarity and require that if additional resources become available, then all agents (weakly) gain. This requirement is called *resource-monotonicity*.

Next, we impose the mild efficiency requirement of weak *non-wastefulness*[5] as well as the very basic and intuitive properties of *individual rationality*[6] and *unavailable object invariance*[7].

We also impose the invariance property *truncation invariance*[8]. Our last property is the well-known strategic robustness condition of strategy-proofness (which we mentioned before when stating that it is a weakly dominant strategy for students to state their true preferences when the student proposing deferred acceptance algorithm is used).

Assuming that an allocation rule satisfies all six elementary and intuitive properties described, we construct a priority structure of objects over agents. In other words, if an allocation rule satisfies all properties mentioned above, then any object is just like a college (with quota one due to our assumption that we have one object of each type) and has a priority ranking over agents. Thus, objects can be endowed with responsive preferences over agents and we can apply the agent proposing deferred acceptance algorithm to obtain an assignment or matching of objects to agents. We call a rule that is based on the agents-proposing deferred-acceptance algorithm with responsive priorities a *responsive DA-rule*. Not only can we construct a priority structure of objects over agents based on our properties, but the only allocation rule satisfying the properties is the DA-rule based on the constructed priority structure.

To summarize, we characterize the class of responsive DA-rules by a set of basic and intuitive properties, namely, unavailable object invariance, individual rationality, weak non-wastefulness, resource-monotonicity, truncation invariance, and strategy-proofness (Ehlers and Klaus, 2009, Theorem 1). For further characterizations along this line (and technical details of the model, the properties etc.) we refer to Ehlers and Klaus (2009).

The main conclusion of Ehlers and Klaus (2009) is, that deferred acceptance also plays an important role for allocation problems that might not come in the form of a college admissions problem (where colleges have responsive preferences) or a school choice problem (where schools have fixed priorities over students). When allocating objects to agents, it might be necessary to assign pseudo (responsive) preferences or priorities to objects in order to guarantee the desirable properties mentioned above.

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[5] No agent who does not receive any object would prefer to obtain an object that is not assigned.

[6] Each agent weakly prefers an object she\he is assigned to not receiving any object.

[7] The rule only depends on the set of available objects and not on objects that are not currently available.

[8] If an agent reduces his set of acceptable objects without changing the preference between any two real objects and his assigned object remains acceptable under the new preference, then the rule should choose the same allocation for the initial profile and the new one.

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