ON THE PROBABILITY OF CONJUNCTIONS OF STATIONARY GAUSSIAN PROCESSES

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Abstract: Let $\{X_i(t), t \ge 0\}, 1 \le i \le n$ be independent centered stationary Gaussian processes with unit variance and almost surely continuous sample paths. For given positive constants u, T, define the set of conjunctions $C_{[0,T],u} := \{t \in [0,T] : \min_{1 \le i \le n} X_i(t) \ge u\}$. Motivated by some applications in brain mapping and digital communication systems, we obtain exact asymptotic expansion of $\mathbb{P}\left\{C_{[0,T],u} \neq \phi\right\}$, as $u \to \infty$. Moreover, we establish the Berman sojourn limit theorem for the random process $\{\min_{1 \le i \le n} X_i(t), t \ge 0\}$ and derive the tail asymptotics of the supremum of each order statistics process.

Key Words: Stationary Gaussian processes; Order statistics processes; Conjunction; Extremes; Berman sojourn limit theorem; Generalized Pickands constant.

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1. Introduction & Main Result

Let $X_i(\boldsymbol{t})$ model the value of an image i at location $\boldsymbol{t} \in \mathbb{R}^d$, $1 \le i \le n$. For a given positive threshold u and a given scan set $\mathcal{T} \subset \mathbb{R}^d$, the set of conjunctions $C_{\mathcal{T},u}$ is defined by

$$C_{\mathcal{T},u} := \{\boldsymbol{t} \in \mathcal{T} : \min_{1 \le i \le n} X_i(\boldsymbol{t}) \ge u\}$$

see the seminal contribution [22]. As mentioned in the aforementioned paper, of interest is the calculation of the probability that the set of conjunctions $C_{\mathcal{T},u}$ is not empty, i.e.,

$$p_{\mathcal{T},u} := \mathbb{P}\left\{C_{\mathcal{T},u} \neq \phi\right\} = \mathbb{P}\left\{\sup_{\boldsymbol{t} \in \mathcal{T}} \min_{1 \le i \le n} X_i(\boldsymbol{t}) \ge u\right\}.$$

Typically, in applications such as the analysis of functional magnetic resonance imaging (fMRI) data, X_i's are assumed to be real-valued Gaussian random fields. Approximations of $p_{\mathcal{T},u}$ are discussed for smooth Gaussian random fields in [22, 5, 10]; results for non-Gaussian random fields can be found in [6].

In this paper, we shall consider the case $d = 1, \mathcal{T} := [0, T]$, with $T > 0$, and that X_i's are independent centered

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stationary Gaussian processes with unit variance and correlation functions $r_i(\cdot), 1 \le i \le n$ that satisfy

$$r_i(t) = 1 - C_i \, |t|^{\alpha_i} + o(|t|^{\alpha_i}), \quad t \to 0, \qquad r_i(t) < 1, \quad \forall t \in (0, T] \tag{1}$$

for some positive constants $\alpha_i \in (0, 2]$ and $C_i, 1 \le i \le n$. Further, we assume that X_i's have almost surely continuous sample paths. Since the calculation of $p_{\mathcal{T},u}$ is not possible in general, we shall investigate below the exact asymptotic behavior of $p_{\mathcal{T},u}$ as $u \to \infty$. Although $\{\min_{1\le i\le n} X_i(t), t \ge 0\}$ is not a Gaussian process when $n \ge 2$, as shown in [22], it happens that it is possible to adapt techniques used in the theory of Gaussian processes and random fields to this class of processes. Motivated by a recent paper of Albin and Choi [3] and the extremal theory for stationary Gaussian processes developed mainly by Berman and Albin (see [8, 9, 2, 4]), we shall derive an asymptotic expansion for $p_{\mathcal{T},u}$ as $u \to \infty$, by following the ideas of [3].

For the formulation of our main result we need to introduce some notation. Let $\{B_{\alpha_i}(t), t \ge 0\}, 1 \le i \le n$ be mutually independent standard fractional Brownian motions with Hurst indexes $\alpha_i/2 \in (0, 1]$, $1 \le i \le n$, respectively, i.e., B_{α_i} is a centered Gaussian process with continuous sample paths and covariance function

$$Cov(B_{\alpha_i}(t), B_{\alpha_i}(s)) = \frac{1}{2}\Big(t^{\alpha_i} + s^{\alpha_i} - |t-s|^{\alpha_i}\Big), \quad t, s > 0, \quad 1 \le i \le n.$$

Next define

$$\mathcal{Z}(t) := \min_{1\le i\le n}\Big(\Big(\sqrt{2}B_{\alpha_i}(C_i^{1/\alpha_i}t) - C_i t^{\alpha_i}\Big)\mathbf{1}(\alpha_i = \alpha_{\min}) + E_i\Big), \quad t \ge 0, \quad \alpha_{\min} := \min_{1\le i\le n} \alpha_i, \tag{2}$$

where $\mathbf{1}(\cdot)$ denotes the indicator function, and E_i's are mutually independent unit exponential random variables being further independent of B_{α_i}'s. Finally, let $\mathcal{H}_{\alpha_1,\dots,\alpha_n}(C_1, \dots, C_n) \in (0, \infty)$ denote *generalized Pickands constant*, determined by

$$\mathcal{H}_{\alpha_1,\dots,\alpha_n}(C_1, \dots, C_n) = \lim_{a\downarrow 0} \frac{1}{a}\mathbb{P}\left\{\max_{k\ge 1} \mathcal{Z}(ak) \le 0\right\}. \tag{3}$$

The following theorem constitutes our principle result.

Theorem 1.1. *Let $\{X_i(t), t \ge 0\}$, $1 \le i \le n$ be mutually independent centered stationary Gaussian processes with unit variance and correlation functions satisfying* (1)*. Then, for any $T > 0$*

$$\mathbb{P}\left\{\sup_{t\in[0,T]} \min_{1\le i\le n} X_i(t) > u\right\} = \mathcal{H}_{\alpha_1,\dots,\alpha_n}(C_1, \dots, C_n)\ T u^{\frac{2}{\alpha_{\min}}} \frac{\exp(-nu^2/2)}{(2\pi)^{n/2}u^n}(1 + o(1)), \quad u \to \infty, \tag{4}$$

where $\mathcal{H}_{\alpha_1,\dots,\alpha_n}(C_1, \dots, C_n) \in (0, \infty)$ is defined in (3)*.*

The organization of the paper. Section 2 presents brief discussions and shows the validity of the Berman sojourn limit theorem for the random process $\{\min_{1\le i\le n} X_i(t), t \ge 0\}$. Additionally, utilizing the fact that minimum is a particular case of the order statistics, in Theorem 2.2 we get a counterpart of Theorem 1.1 for order statistics processes. The case of non-standard stationary Gaussian processes is treated in Theorem 2.3. Section 3 contains all the proofs.

2. Discussions & Extensions

In his seminal contribution [15] J. Pickands III established the exact asymptotic tail behavior of the supremum of the stationary Gaussian process $\{X_1(t), t \in [0,T]\}$ under the condition (1), using a double-sum method. The first crucial step to that result is the celebrated Pickands lemma which states that, for any positive constant S

$$\mathbb{P}\left\{\sup_{t\in[0,u^{-\frac{2}{\alpha_1}}S]} X_1(t) > u\right\} = \mathcal{H}_{\alpha_1}\left[0, C_1^{\frac{1}{\alpha_1}}S\right]\Psi(u)(1+o(1)), \qquad u\to\infty,$$

where $\Psi(\cdot)$ is the survival function of an $N(0,1)$ random variable and

$$\mathcal{H}_{\alpha_1}[0,S] = \mathbb{E}\left\{\exp\left(\sup_{t\in[0,S]}\left(\sqrt{2}B_{\alpha_1}(t) - t^{\alpha_1}\right)\right)\right\} \in (0,\infty).$$

Recall that $\Psi(u) = \exp(-u^2/2)/\sqrt{2\pi u^2}(1+o(1))$ as $u\to\infty$.

An application of Pickands lemma, together with the double-sum method, yields (see, e.g., [15, 16, 17])

$$\mathbb{P}\left\{\sup_{t\in[0,T]} X_1(t) > u\right\} = TC_1^{\frac{1}{\alpha_1}}\mathcal{H}_{\alpha_1}u^{\frac{2}{\alpha_1}}\Psi(u)(1+o(1)), \quad u\to\infty, \tag{5}$$

where $\mathcal{H}_{\alpha_1}\in(0,\infty)$ is the Pickands constant, defined by

$$\mathcal{H}_{\alpha_1} = \lim_{S\to\infty}\frac{1}{S}\mathcal{H}_{\alpha_1}[0,S].$$

We refer to the recent contribution [13], where alternative representations of Pickands constant are derived; see also [12, 14] and the references therein for properties and generalizations of Pickands constant.

The constant $\mathcal{H}_{\alpha_1,\ldots,\alpha_n}(C_1,\ldots,C_n)$, appearing in (3), is more complicated than $\mathcal{H}_{\alpha_1}$. A simple lower bound for $\mathcal{H}_{\alpha_1,\ldots,\alpha_n}(C_1,\ldots,C_n)$ can be found as follows:

$$\begin{aligned}
\mathcal{H}_{\alpha_1,\ldots,\alpha_n}(C_1,\ldots,C_n) &\geq \max_{1\le i\le n:\alpha_i=\alpha_{\min}}\lim_{a\downarrow 0}\frac{1}{a}\mathbb{P}\left\{\max_{k\ge1}\left(\sqrt{2}B_{\alpha_i}(C_i^{1/\alpha_i}ak) - C_i(ak)^{\alpha_i}\right) + E_i \le 0\right\}\\
&\geq \max_{1\le i\le n:\alpha_i=\alpha_{\min}} C_i^{1/\alpha_i}\lim_{a\downarrow0}\frac{1}{C_i^{1/\alpha_i}a}\mathbb{P}\left\{\max_{k\ge1}(\sqrt{2}B_{\alpha_i}(C_i^{1/\alpha_i}ak) - (C_i^{1/\alpha_i}ak)^{\alpha_i}) + E_i \le 0\right\}\\
&\geq \max_{1\le i\le n:\alpha_i=\alpha_{\min}} C_i^{1/\alpha_{\min}}\mathcal{H}_{\alpha_{\min}} > 0, \qquad (6)
\end{aligned}$$

where in the last step we used the alternative expression of the Pickands constant given in [3].

Theorem 1.1 can also be proved using the double-sum method, extending thus the Pickands lemma and Pickands theorem to include the non-Gaussian process $\{\min_{1\le i\le n} X_i(t), t\ge0\}$; due to heavy technical details the proof will be displayed in a forthcoming article.

Finally, we remark that in view of the recent contributions [21, 20] it is possible to derive the exact asymptotics of $p_{\mathcal{T},u}$ considering $X_i(\boldsymbol{t}), \boldsymbol{t} \in \mathbb{R}^d$ stationary isotropic Gaussian random fields.

We continue below with four results, the first one establishes a Berman sojourn limit theorem, the second one deals with order statistics processes of X_i's, the third one focuses on a time-changed model, and the last one concerns a generalization of Theorem 1.1 to non-standard stationary Gaussian X_i's.

2.1. **A Berman sojourn limit theorem.** Let, for $t \ge 0$,

$$L_t(u) = \int_0^t \mathbf{1}(\min_{1\le i\le n} X_i(s) > u)\, ds \tag{7}$$

be the sojourn time of the process $\{\min_{1\le i\le n} X_i(t), t \ge 0\}$ above a level $u > 0$ on the time interval $[0, t]$. The next result is the Berman sojourn limit theorem for the process $\{\min_{1\le i\le n} X_i(t), t \ge 0\}$.

Theorem 2.1. *Let $\{X_i(t), t \ge 0\}, 1 \le i \le n$ be independent centered stationary Gaussian processes with unit variance and correlation functions that satisfy* (1)*, and let $L_t(u)$ be defined as in* (7) *for any positive constants t, u. Then we have, for all $t > 0$ small enough, that*

$$\lim_{u\to\infty} \int_x^\infty \frac{\mathbb{P}\left\{u^{\frac{2}{\alpha_{\min}}} L_t(u) > y\right\}}{u^{\frac{2}{\alpha_{\min}}} \mathbb{E}\left\{L_t(u)\right\}}\, dy = B(x)$$

holds at all continuity points $x > 0$ of $B(x) = \mathbb{P}\left\{\int_0^\infty \mathbf{1}(\mathcal{Z}(s) > 0)\, ds > x\right\}$.

2.2. **Asymptotics of supremum of order statistics processes.** Let $\{X_{i:n}(t), t \ge 0\}, 1 \le i \le n$ be the order statistics processes of $\{X_i(t), t \ge 0\}, 1 \le i \le n$, i.e., we define

$$X_{1:n}(t) := \max_{1\le i\le n} X_i(t) \ge X_{2:n}(t) \ge \ldots \ge X_{n:n}(t) = \min_{1\le i\le n} X_i(t), \quad t \ge 0.$$

Our next result concerns the exact tail asymptotics of the supremum of the order statistics processes. We refer, e.g., to [19] for motivation of study the exit probabilities of the order statistics processes in electrical engineering.

For clearness of the presentation, we assume further that $\alpha_1 = \ldots = \alpha_n =: \alpha$ and $C_1 = \ldots = C_n = 1$. Furthermore, define

$$\mathcal{H}_{\alpha,j} = \lim_{a\downarrow 0} \frac{1}{a} \mathbb{P}\left\{\max_{k\ge 1} \mathcal{Z}_j(ak) \le 0\right\}, \quad 1 \le j \le n,$$

where

$$\mathcal{Z}_j(t) := \min_{1\le i\le j} \left(\sqrt{2} B_\alpha^{(i)}(t) - t^\alpha + E_i\right), \quad t \ge 0,$$

with E_i's being independent unit exponential random variables which are further independent of mutually independent fractional Brownian motions $B_\alpha^{(i)}$'s.

Theorem 2.2. *Let $\{X_i(t), t \ge 0\}, 1 \le i \le n$ be independent centered stationary Gaussian processes with unit variance and correlation functions that satisfy* (1) *with $\alpha_1 = \ldots = \alpha_n =: \alpha$ and $C_1 = \ldots = C_n = 1$. Then, for any $T > 0$*

$$\mathbb{P}\left\{\sup_{t\in[0,T]} X_{j:n}(t) > u\right\} = \mathcal{H}_{\alpha,j}\ T\frac{n!}{(n-j)!j!}u^{\frac{2}{\alpha}}(\Psi(u))^j(1+o(1)), \quad 1 \le j \le n$$

as $u \to \infty$.

2.3. **Conjunction of time-changed processes.** The technique of Albin and Choi [3] which we applied in the proof of Theorem 1.1, can be utilized also for some other interesting extensions. To illustrate it, we investigate the tail asymptotics of supremum of process $Y_{\boldsymbol{\Theta}}(t) = \min_{1\le i\le n} X_i^*(t), t \ge 0$ on a finite-time interval, say $[0,T]$, where $X_i^*(t) = X_i(\Theta_i t), 1 \le i \le n$ are time-changed centered stationary Gaussian processes with Θ_i's non-degenerate non-negative bounded random variables being independent of X_i's; see [7, 11] for recent results on the extremes of time-changed Gaussian processes. Indeed, it follows easily that the result of Lemma 3.1 (see Section 3) holds with limit process

$$\mathcal{Z}_{\boldsymbol{\Theta}}(t) = \min_{1\le i\le n}\Big(\Big(\sqrt{2}B_{\alpha_i}(C_i^{1/\alpha_i}\Theta_i t) - C_i(\Theta_i t)^{\alpha_i}\Big)\mathbf{1}(\alpha_i = \alpha_{\min}) + E_i\Big), \quad t \ge 0,$$

where B_{α_i}'s and E_i's are given as before which are further independent of Θ_i's. Thus, we have by a similar proof as Theorem 1.1 that

$$\mathbb{P}\left\{\sup_{t\in[0,T]} Y_{\boldsymbol{\Theta}}(t) > u\right\} = \mathcal{H}^*_{\alpha_1,\ldots,\alpha_n}(C_1,\ldots,C_n)\ Tu^{\frac{2}{\alpha_{\min}}}(\Psi(u))^n(1+o(1)), \quad u \to \infty,$$

where

$$\mathcal{H}^*_{\alpha_1,\ldots,\alpha_n}(C_1,\ldots,C_n) = \lim_{a\downarrow 0}\frac{1}{a}\mathbb{P}\left\{\max_{k\ge 1}\mathcal{Z}_{\boldsymbol{\Theta}}(ak) \le 0\right\} \in (0,\infty).$$

2.4. **Non-standard stationary Gaussian processes.** Let $\widetilde{X_i}(t) = X_i(t)/b_i, t \ge 0$ for some $b_i > 0, 1 \le i \le n$ with X_i's being given as in Theorem 1.1. Clearly, $\widetilde{X_i}$'s are again centered stationary Gaussian processes. We have the following result considering the supremum of $\min_{1\le i\le n}\widetilde{X_i}(t), t \in [0,T]$.

Theorem 2.3. *Under the assumptions of Theorem 1.1, we have, for any $T > 0$,*

$$\mathbb{P}\left\{\sup_{t\in[0,T]}\min_{1\le i\le n}\widetilde{X_i}(t) > u\right\} = \widetilde{\mathcal{H}}_{\alpha_1,\ldots,\alpha_n}(C_1,\ldots,C_n)\ Tu^{\frac{2}{\alpha_{\min}}}\prod_{i=1}^n\Psi(b_i u)(1+o(1)), \quad u \to \infty, \tag{8}$$

where

$$\widetilde{\mathcal{H}}_{\alpha_1,\ldots,\alpha_n}(C_1,\ldots,C_n) = \lim_{a\downarrow 0}\frac{1}{a}\mathbb{P}\left\{\max_{k\ge 1}\widetilde{\mathcal{Z}}(ak) \le 0\right\},$$

with

$$\widetilde{\mathcal{Z}}(t) := \min_{1\le i\le n}\left(\Big(\sqrt{2}b_i^{-1}B_{\alpha_i}(C_i^{1/\alpha_i}t) - C_i t^{\alpha_i}\Big)\mathbf{1}(\alpha_i = \alpha_{\min}) + b_i^{-2}E_i\right), \quad t \ge 0,$$

and B_{α_i}'s and E_i's being given as in Section 1.

3. Proofs

The idea of the proof of Theorem 1.1 is based on the technique developed by Albin and Choi [3]. We begin with several lemmas for the minimum process $Y(t) := X_{n:n}(t) = \min_{1\le i\le n} X_i(t)$, which altogether will be used to show the proof of Theorem 1.1. Then we present the proofs of Theorem 2.1 and Theorem 2.2. Hereafter we shall use the notation and the assumptions of Introduction and Section 2. For notational simplicity we shall set below

$$q(u) = u^{-2/\alpha_{\min}}, \quad u > 0$$

and shall use the standard notation $\lfloor\cdot\rfloor$ for the the ceiling function, i.e., $\lfloor x\rfloor$ is the largest integer that is smaller than $x \in \mathbb{R}$.

Lemma 3.1. *For any grid of points $0 \le t_0 < t_1 < \cdots < t_d < \infty$, $d \in \mathbb{N}$, we have the joint convergence in distribution*

$$\Big(nu(Y(q(u)t_1) - u), \ldots, nu(Y(q(u)t_d) - u)\Big)\Big|(Y(0) > u) \quad \overset{d}{\to} \quad n\Big(\mathcal{Z}(t_1), \ldots, \mathcal{Z}(t_d)\Big)$$

as $u \to \infty$, where the process $\mathcal{Z}$ is defined as in (2)*.*

Proof of Lemma 3.1: First note that $Y(0)$ has distribution function $G(\cdot)$ in the Gumbel max-domain of attraction with positive scaling function $w(u) = nu$ i.e.,

$$\lim_{u\to\infty} \frac{1 - G(u + x/w(u))}{1 - G(u)} = \exp(-x), \quad x \in \mathbb{R}.$$

See [14, 18] for more details on the Gumbel max-domain of attraction. Moreover, it follows from Lemma 2 in [3] that, for any $1 \le i \le n$, the following joint convergence in distribution

$$\Big(X_{iu}(t_1), \ldots, X_{iu}(t_d)\Big)\Big|(X_i(0) > u) \overset{d}{\to} \Big(\sqrt{2}B_{\alpha_i}(C_i^{1/\alpha_i} t_1) - C_i t_1^{\alpha_i} + E_i, \ldots, \sqrt{2}B_{\alpha_i}(C_i^{1/\alpha_i} t_d) - C_i t_d^{\alpha_i} + E_i\Big)$$

holds as $u \to \infty$, where $X_{iu}(t) = u(X_i(u^{-2/\alpha_i}t) - u), t \ge 0, u > 0$. Then the claim follows by the independence of X_i's, B_{α_i}'s and E_i's. □

Lemma 3.2. *For any $a > 0$ we have*

$$\lim_{N\to\infty} \lim_{u\to\infty} \frac{1}{N\mathbb{P}\{Y(0) > u\}} \mathbb{P}\left\{\max_{k\in\{0,\ldots,N\}} Y(aq(u)k) > u\right\} \quad = \quad \mathbb{P}\left\{\bigcap_{l=1}^{\infty}\{\mathcal{Z}(al) \le 0\}\right\}.$$

Proof of Lemma 3.2: In view of Lemma 3.1, the proof follows with the same arguments as that of Lemma 3 in [3]. □

Lemma 3.3. *Let $\overline{\alpha} := \alpha_{\min}/4$. We have*

$$\lim_{a\downarrow 0}\limsup_{u\to\infty}\frac{q(u)}{\mathbb{P}\left\{Y(0)>u\right\}}\mathbb{P}\left\{\sup_{t\in[0,T]}Y(t)>u+\frac{a^{\overline{\alpha}}}{u},\max_{k\in\{0,\ldots,\lfloor T/(aq(u))\rfloor\}}Y(aq(u)k)\le u\right\}=0.$$

Proof of Lemma 3.3: The proof follows by similar arguments as that of Lemma 4 in [3]. Since the proof of Lemma 4 in [3] only requires the stationarity and the continuity of the process involved, we obtain, for all large u and small $a>0$, that

$$\begin{aligned}&\mathbb{P}\left\{\sup_{t\in[0,T]}Y(t)>u+\frac{a^{\overline{\alpha}}}{u},\max_{k\in\{0,\ldots,\lfloor Ta/q(u)\rfloor\}}Y(aq(u)k)\le u\right\}\\ &\le \frac{2T}{aq(u)}\sum_{j=1}^{\infty}2^j\mathbb{P}\left\{Y(aq(u)2^{-j})>u_j,Y(0)\le u_{j-1}\right\},\end{aligned}$$

where $u_j := u+a^{\overline{\alpha}}(1-2^{-j\overline{\alpha}})/u>u$ for any $j\ge 1$. Further, for all u large and $a>0$ small and any $1\le i\le n, j\ge 1$, the following inequality

$$r_i(aq(u)2^{-j})X_i(aq(u)2^{-j})-X_i(0)\quad\ge\quad a^{\overline{\alpha}}(2^{\overline{\alpha}}-1)2^{-j\overline{\alpha}-1}/u=:c_{ju}$$

is implied by the event $\{X_i(aq(u)2^{-j})>u_j, X_i(0)\le u_{j-1}\}$. Thus, in view of the fact that $r_i(aq(u)2^{-j})X_i(aq(u)2^{-j})-X_i(0)$ is independent of $X_i(aq(u)2^{-j})$, we conclude that

$$\begin{aligned}&\mathbb{P}\left\{\sup_{t\in[0,T]}Y(t)>u+\frac{a^{\overline{\alpha}}}{u},\max_{k\in\{0,\ldots,\lfloor Ta/q(u)\rfloor\}}Y(aq(u)k)\le u\right\}\\
&\le \frac{2T}{aq(u)}\sum_{j=1}^{\infty}2^j\mathbb{P}\left\{X_1(aq(u)2^{-j})>u_j,\ldots,X_n(aq(u)2^{-j})>u_j,\bigcup_{i=1}^n\{X_i(0)\le u_{j-1}\}\right\}\\
&\le \frac{2T}{aq(u)}\sum_{j=1}^{\infty}2^j\mathbb{P}\left\{X_1(aq(u)2^{-j})>u_j,\ldots,X_n(aq(u)2^{-j})>u_j,\bigcup_{i=1}^n\{r_i(aq(u)2^{-j})X_i(aq(u)2^{-j})-X_i(0)\ge c_{ju}\}\right\}\\
&\le \frac{2T}{aq(u)}\sum_{j=1}^{\infty}2^j(\Psi(u))^n\mathbb{P}\left\{\bigcup_{i=1}^n\{r_i(aq(u)2^{-j})X_i(aq(u)2^{-j})-X_i(0)\ge c_{ju}\}\right\}\\
&\le \frac{2T}{aq(u)}\sum_{j=1}^{\infty}2^j(\Psi(u))^n\sum_{i=1}^n\Psi\left(\frac{c_{ju}}{\sqrt{1-r_i(aq(u)2^{-j})^2}}\right)\end{aligned}$$

for all u large and $a>0$ small, hence the claim follows. ☐

Proof of Theorem 1.1: The proof follows by a similar idea as used in the proof of Theorem 1 in [3]. First note that, for any $k>0$

$$\mathbb{P}\left\{Y(0)>u,Y(aq(u)k)>u\right\}=\prod_{i=1}^n\mathbb{P}\left\{X_i(0)>u,X_i(aq(u)k)>u\right\}\le\prod_{i=1}^n\mathbb{P}\left\{X_i(0)+X_i(aq(u)k)>2u\right\}.$$

Therefore, similar arguments as in the proof of Lemma 1 therein imply

$$\lim_{u\to\infty}\frac{q(u)}{\mathbb{P}\left\{Y(0)>u\right\}}\mathbb{P}\left\{\sup_{k\in\{0,\ldots,\lfloor T/(aq(u))\rfloor\}}Y(aq(u)k)>u\right\}\quad=\quad\frac{T}{a}\mathbb{P}\left\{\bigcap_{l=1}^{\infty}\{\mathcal{Z}(al)\le 0\}\right\}$$

for any $a > 0$. Moreover, the finiteness of the generalized Pickands constant $\mathcal{H}_{\alpha_1,\ldots,\alpha_n}(C_1,\ldots,C_n)$ and the asymptotic equation (4) can be established as in [3], using the results of Lemmas 3.1-3.3. In fact, $\mathcal{H}_{\alpha_1,\ldots,\alpha_n}(C_1,\ldots,C_n) > 0$ follows directly from (6). This completes the proof. □

Proof of Theorem 2.1: The claim follows by checking the Assumptions 3.I and 3.II in Theorem 3.1 in [8]. Assumption 3.I can be established with the aid of Lemma 3.1, where we have (with the notation as in [8]) $w(u) = nu$, $v(u) = u^{2/\alpha_{\min}}$ and $Z(t) = \mathcal{Z}(t)$. Furthermore, it follows that

$$\lim_{d\to\infty} \limsup_{u\to\infty} v(u) \int_{d/v(u)}^{t} \mathbb{P}\left\{Y(s) > u | Y(0) > u\right\} \, ds \quad \le \quad \lim_{d\to\infty} \limsup_{u\to\infty} v(u) \int_{d/v(u)}^{t} \mathbb{P}\left\{X_i(s) > u | X_i(0) > u\right\} \, ds,$$

where X_i is some of X_i's such that $\alpha_i = \alpha_{\min}$. Therefore, Assumption 3.II can be verified as in Section 7 therein, and thus the proof is complete. □

Proof of Theorem 2.2: Initially we establish the proof for the case $j = 1$. Introduce a new random process Z defined by

$$Z(t) = X_i(t-(i-1)T), \quad t \in [(i-1)T, iT), \quad 1 \le i \le n.$$

For any $u \ge 0$ we have

$$\mathbb{P}\left\{\sup_{t\in[0,T]} X_{1:n}(t) > u\right\} \quad = \quad \mathbb{P}\left\{\sup_{t\in[0,nT]} Z(t) > u\right\}.$$

By the Bonferroni inequality and Pickands theorem (see Eq. (5))

$$\begin{aligned} \mathbb{P}\left\{\sup_{t\in[0,nT]} Z(t) > u\right\} \quad &\le \quad \sum_{i=1}^{n} \mathbb{P}\left\{\sup_{t\in[0,T]} X_i(t) > u\right\} \\ &= \quad Tn\mathcal{H}_\alpha u^{\frac{2}{\alpha}} \Psi(u)(1+o(1)), \quad u\to\infty \end{aligned} \tag{9}$$

and further

$$\mathbb{P}\left\{\sup_{t\in[0,nT]} Z(t) > u\right\} \quad \ge \quad \sum_{i=1}^{n} \mathbb{P}\left\{\sup_{t\in[0,T]} X_i(t) > u\right\} - \Sigma_1(u), \tag{10}$$

with

$$\Sigma_1(u) = \sum_{1\le i<j\le n} \mathbb{P}\left\{\sup_{t\in[0,T]} X_i(t) > u\right\} \mathbb{P}\left\{\sup_{t\in[0,T]} X_j(t) > u\right\}.$$

Moreover, in view of (5)

$$\Sigma_1(u) = o\left(u^{\frac{2}{\alpha}} \Psi(u)\right), \quad u\to\infty. \tag{11}$$

Consequently, the claim for the case $j = 1$ follows from (9)-(11). Next, we give only the proof of the case $j = n-1$ since the other cases follow by similar arguments. For notational simplicity denote

$$A_i(t,u) = \{X_1(t) > u, \ldots, X_{i-1}(t) > u, X_i(t) \le u, X_{i+1}(t) > u, \ldots, X_n(t) > u\}, \quad 1 \le i \le n,$$

$$B(t,u) = \{X_1(t) > u, X_2(t) > u, \ldots, X_n(t) > u\}.$$

For any $u>0$ we have

$$\begin{aligned}
\mathbb{P}\left\{\sup_{t\in[0,T]} X_{n-1:n}(t)>u\right\} &\le \mathbb{P}\left\{\exists_{t\in[0,T]} \cup_{i=1}^n A_i(t,u)\cup B(t,u)\right\}\\
&\le \mathbb{P}\left\{\exists_{t\in[0,T]} B(t,u)\right\}+\sum_{i=1}^n \mathbb{P}\left\{\exists_{t\in[0,T]} A_i(t,u)\right\}\\
&\le \mathbb{P}\left\{\sup_{t\in[0,T]}\min_{1\le i\le n} X_i(t)>u\right\}+\sum_{i=1}^n \mathbb{P}\left\{\sup_{t\in[0,T]}\min_{1\le j\le n, j\ne i} X_j(t)>u\right\}. \qquad (12)
\end{aligned}$$

Further, for any $u>0$

$$\begin{aligned}
\mathbb{P}\left\{\sup_{t\in[0,T]} X_{n-1:n}(t)>u\right\} &\ge \mathbb{P}\left\{\exists_{t\in[0,T]} \cup_{i=1}^n A_i(t,u)\right\}\\
&= \mathbb{P}\left\{\cup_{i=1}^n \{\exists_{t\in[0,T]} A_i(t,u)\}\right\}\\
&\ge \sum_{i=1}^n \mathbb{P}\left\{\sup_{t\in[0,T]}\min_{1\le j\le n, j\ne i} X_j(t)>u\right\}\mathbb{P}\left\{\sup_{t\in[0,T]} X_i(t)\le u\right\}-\Sigma_2(u), \qquad (13)
\end{aligned}$$

where

$$\Sigma_2(u)=\sum_{1\le i<j\le n}\mathbb{P}\left\{\exists_{t\in[0,T]} A_i(t,u), \exists_{s\in[0,T]} A_j(s,u)\right\}.$$

By the independence of $X_i's$, we conclude that

$$\Sigma_2(u)\le n^2\prod_{i=1}^n \mathbb{P}\left\{\sup_{t\in[0,T]} X_i(t)>u\right\}. \qquad (14)$$

Consequently, the claim follows from (12)-(14) and an application of Theorem 1.1. $\square$

Proof of Theorem 2.3: Denote $\widetilde{Y}(t)=\min_{1\le i\le n}\widetilde{X_i}(t), t\ge 0$, and let $w(u)=\sum_{i=1}^n b_i^2 u$. As in the proof of Lemma 3.1 for any grid of points $0\le t_0<t_1<\dots<t_d<\infty$

$$\Big(w(u)(\widetilde{Y}(q(u)t_1)-u),\dots,w(u)(\widetilde{Y}(q(u)t_d)-u)\Big)\Big|(\widetilde{Y}(0)>u) \quad \overset{d}{\to} \quad \sum_{i=1}^n b_i^2\Big(\widetilde{\mathcal{Z}}(t_1),\dots,\widetilde{\mathcal{Z}}(t_d)\Big)$$

holds as $u\to\infty$. Results analogous to Lemma 3.2 and Lemma 3.3 for $\widetilde{Y}$ can be derived with similar arguments as in the case of Y. Consequently, the proof is established by repeating the arguments in the proof of Theorem 1.1. $\square$

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