# APPROXIMATION OF SOME MULTIVARIATE RISK MEASURES FOR GAUSSIAN RISKS

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**Abstract:** Gaussian random vectors exhibit the loss of dimension phenomena, which relate to their joint survival tail behaviour. Besides, the fact that the components of such vectors are light-tailed complicates the approximations of various multivariate risk measures significantly. In this contribution we derive precise approximations of marginal mean excess, marginal expected shortfall and multivariate conditional tail expectation of Gaussian random vectors and highlight links with conditional limit theorems. Our study indicates that similar results hold for elliptical and Gaussian like multivariate risks.

**Key Words**: Gaussian random vectors; marginal mean excess; marginal expected shortfall; multivariate conditional tail expectation; conditional limit theorem.
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## 1. Introduction

The recent article [1] investigates two important measures of risk contagion for a given bivariate random vector $(Z_1, Z_2)$, namely the marginal mean excess (MME) and the marginal expected shortfall (MES). Specifically, under the assumption that $\mathbb{E}\{|Z_1|\} < \infty$ the MME is defined for any $p \in (0,1)$ by

$$E(p) = \mathbb{E}\{(Z_1 - VaR_{Z_2}(p))_+ | Z_2 > VaR_{Z_2}(p)\}, \tag{1.1}$$

whereas MES is given as

$$S(p) = \mathbb{E}\{Z_1 | Z_2 > VaR_{Z_2}(p)\}, \tag{1.2}$$

with $VaR_{Z_i}(p)$ the Value-at-Risk at level $p$ of $Z_i$, which is simply the quantile function of $Z_i$ at $p$. In general both $E(p)$ and $S(p)$ cannot be calculated explicitly. Besides, in the risk management practice the main interest is the calculation of these quantities for $p$ being close to 1.

In this paper we shall consider first the approximations of MME and MES for $(Z_1, Z_2)$ being jointly Gaussian with correlation $\rho \in (-1,1)$. Gaussian random vectors are asymptotically independent, i.e., large values occur independently which in our context means that

$$\lim_{p \uparrow 1} \mathbb{P}\{Z_1 > VaR_{Z_1}(p) | Z_2 > VaR_{Z_2}(p)\} = 0.$$

Moreover, Gaussian risks exhibit the dimension reduction phenomenon, i.e., the joint survival probability can be proportional to the marginal survival probability for large values of the threshold, see e.g., [2–4] and the discussion below. Indeed that phenomenon renders the approximations of both MME and MES interesting and challenging.

Under hidden regular variation assumption on $(Z_1, Z_2)$ the recent publications [1, 5] consider approximations of MME and MES under some additional asymptotic conditions. However the Gaussian setup is not covered therein since the marginal distributions are in our setup light-tailed. As discussed recently in [6], see also [7] the light-tailed case is very challenging (even in the one-dimensional setup) and surprisingly very little investigated in the literature.

Given the central role of multivariate Gaussian distributions, and the interesting behaviour of light-tailed risks, our principal goal in this contribution is to derive approximations of MME and MES in the Gaussian setup. We state next the result for the bivariate case.

Throughout in the following $\Phi$ denotes the distribution function (df) of an $N(0,1)$ random variable with inverse $\Phi^{-1}$ and $\varphi$ the probability density function (pdf) of a standard Gaussian random vector $(X_1, X_2)$ with correlation $\rho \in (-1,1)$.

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**Theorem 1.1.** *Let $\boldsymbol{Z}=(Z_1,Z_2)$ be jointly Gaussian with $Z_i$ having $N(\mu_i,\sigma_i^2), i=1,2$ df and correlation $\rho\in(-1,1)$ and set $u_p=\Phi^{-1}(p), \beta=(\mu_2-\mu_1)/\sigma_1, \eta=\beta/\sqrt{1-\rho^2}$.*
*i) If $\sigma_2>\rho\sigma_1$ and $\sigma_1>\rho\sigma_2$, then*

$$E(p) \sim \frac{\sigma_1}{h_1^2 h_2}\sqrt{2\pi}u_p^{-2}e^{\frac{u_p^2}{2}}\varphi(\sigma_2 u_p/\sigma_1+\beta, u_p)\to 0, \quad p\uparrow 1, \tag{1.3}$$

*where*

$$h_1=\frac{\sigma_2-\rho\sigma_1}{\sigma_1(1-\rho^2)}>0, \quad h_2=\frac{\sigma_1-\rho\sigma_2}{\sigma_1(1-\rho^2)}>0.$$

*ii) If $\sigma_2=\rho\sigma_1$, then*

$$\lim_{p\uparrow 1}E(p) = \sigma_1\sqrt{1-\rho^2}\Big(\Phi'(\eta)-\eta[1-\Phi(\eta)]\Big)\in(0,\infty). \tag{1.4}$$

*iii) If $\sigma_2<\rho\sigma_1$, then*

$$E(p) \sim (\rho\sigma_1-\sigma_2)u_p\to\infty, \quad p\uparrow 1. \tag{1.5}$$

*iv) If $\sigma_1\le\rho\sigma_2$, then*

$$E(p) \sim \sigma_1 e^{-\frac{\beta^2}{2}}\Phi(\eta\rho^*)u_p e^{-\beta\frac{\sigma_2}{\sigma_1}u_p}e^{-\frac{\sigma_2^2-\sigma_1^2}{2\sigma_1^2}u_p^2}\to 0, \quad p\uparrow 1, \tag{1.6}$$

*where $\rho^*=\rho$ if $\sigma_2=\rho\sigma_1$ and $\rho^*=\infty$ otherwise.*
*v) As $p\uparrow 1$ we have*

$$S(p)-\mu_1-\sigma_1\rho u_p \to 0. \tag{1.7}$$

The above findings show that $E(p)$ and $S(p)$ have a completely different behaviour as $p$ approaches 1. Both (1.3) and (1.6) prove that $E(p)$ tends super-exponentially fast to 0 as $p\to 1$. A completely different behaviour is observed in (1.4) and (1.5). For the approximation of MES we have only one case as shown in (1.7), since its definition is invariant to $\sigma_2$.
The bivariate setup is however restrictive; it is possible to have in (1.7) a non-zero limit in higher dimensions, see Remark 2.4. Indeed, the two-dimensional setup is easier to deal with and there are no additional notation needed, but it does not show how to derive corresponding results in multivariate setup.
It is worth mentioning that extensions of our results to elliptical random vectors are also possible, but those require more technical efforts and additional assumptions similar to [8][Assumption 4]. Moreover, extensions to the larger class of Gaussian like random vectors treated in [9] can also be obtained, but again further technical treatments are needed and will therefore not be addressed here. Besides, our findings are of certain importance for considering approximations of other risk measures such as multivariate expectiles considered in [10].
Brief outline of the rest of the paper: In the next section we focus on the multivariate setup deriving the approximations of MME, MES and the multivariate conditional tail expectation (MCTE). Section 3 contains all the proofs followed by an Appendix.

## 2. Main Results

In this section we shall be concerned with the multivariate setup deriving first an extension of Theorem 1.1 and then discussing further some related conditional limit results. Given its importance in application we shall consider also the approximation of MCTE. In the last subsection the three dimensional case will be briefly explored.
In our notation below bold lower case symbols are column vectors in $\mathbb{R}^d$. The Hadamard product $r\boldsymbol{x}$ stands for the vector $(rx_1,\dots,rx_d)$ where $r\in\mathbb{R}, \boldsymbol{x}=(x_1,\dots,x_d)^\top\in\mathbb{R}^d$. All other operations with vectors are defined as usual, component-wise. For instance $\boldsymbol{a}\boldsymbol{x}$ is the vector $(a_1x_1,\dots,a_dx_d)^\top$ for any $\boldsymbol{a},\boldsymbol{x}\in\mathbb{R}^d$ and $\boldsymbol{x}\ge\boldsymbol{a}$ means that $x_i\ge a_i, i\le d$.

2.1. **Approximation of MME and MES.** Let in the following $\boldsymbol{Z} = (Z_1, \ldots, Z_d)$ be a $d$-dimensional Gaussian random vector with mean $\boldsymbol{\mu}$. As in the bivariate case we define MME for give level $p \in (0,1)$ by

$$E(p) = \mathbb{E}\{(Z_1 - A_p)_+ | Z_2 > VaR_{Z_2}(p), \ldots, Z_d > VaR_{Z_d}(p)\},$$

with $A_p = \sum_{i=1}^{d-1} a_i VaR_{Z_{i+1}}(p)$ where $a_i$'s are given constants. Writing $\sigma_i^2$ for the variance of $Z_i$ we have thus

$$\begin{aligned} E(p) &= \sigma_1 \mathbb{E}\{(X_1 - (A_p - \mu_1)/\sigma_1)_+ | X_2 > VaR_{X_2}(p), \ldots, X_d > VaR_{X_d}(p)\} \\ &= \sigma_1 \mathbb{E}\{(X_1 - (\sum_{i=1}^{d-1} a_i(\sigma_{i+1} u_p + \mu_{i+1}) - \mu_1)/\sigma_1)_+ | X_2 > u_p, \ldots, X_d > u_p\}, \end{aligned}$$

with $\boldsymbol{X} = (X_1, \ldots, X_d)$ a centered Gaussian random vector with covariance matrix $\Sigma$ equal to the correlation matrix of $\boldsymbol{Z}$ and $u_p = \Phi^{-1}(p)$. For notational simplicity, throughout this paper random vectors are row vectors and therefore we do not use the transpose sign.

Consequently, without loss of generality we shall determine next the asymptotics of

$$E(\boldsymbol{c}, u) = \mathbb{E}\{(X_1 - c_1 u - \mu)_+ | X_2 > c_2 u, \ldots, X_d > c_d u\}$$

as $u \to \infty$ for given $\boldsymbol{c} = (c_1, \ldots, c_d)^\top, \mu$ assuming that $\Sigma$ is a non-singular correlation matrix.

In the two-dimensional setup the aimed approximation can be obtained without discussing a closely related and crucial quadratic optimisation problem. However, in the higher dimensional settings we need to solve the following quadratic programming problem $\Pi_\Sigma(\boldsymbol{c})$: determine the minimum of $\boldsymbol{x}^\top \Sigma^{-1} \boldsymbol{x}$ subject to $\boldsymbol{x} \geq \boldsymbol{c}$ for given $\boldsymbol{c} \in \mathbb{R}^d \setminus (-\infty, 0]^d$ with solution $\tilde{\boldsymbol{c}}$. The reason for discussing $\Pi_\Sigma(\boldsymbol{c})$ is that our investigation is closely related to the asymptotic tail behaviour as $u \to \infty$ of $\mathbb{P}\{\boldsymbol{X} > \boldsymbol{c}u\}$. In view of [2] (see below Lemma 4.2) the aforementioned asymptotic tail behaviour is solely determined by $\Pi_\Sigma(\boldsymbol{c})$.

In view of Lemma 4.1 in Appendix we have that $\tilde{\boldsymbol{c}}$ exists, is unique and there exists a unique index set $I \subset \{1, \ldots, d\}$ with $m \geq 1$ elements such that

$$\tilde{\boldsymbol{c}}_I = \boldsymbol{c}_I, \quad \tilde{\boldsymbol{c}}_{I^c} = \Sigma_{I^c I}(\Sigma_{II})^{-1} \boldsymbol{c}_I \geq \boldsymbol{c}_{I^c}, \quad \tilde{\boldsymbol{c}}^\top \Sigma^{-1} \tilde{\boldsymbol{c}} = \boldsymbol{c}_I^\top (\Sigma_{II})^{-1} \boldsymbol{c}_I > 0, \tag{2.1}$$

where $I^c = \{1, \ldots, d\} \setminus I$; note in passing that $I^c$ can be empty.

Throughout this paper $\Sigma_{IJ}$ is the matrix obtained by $\Sigma$ keeping the rows and columns with indices in $I$ and $J$, respectively and similar notation applies for vectors.

Denote next by $L \subset \{1, \ldots, d\}$ the maximal index set that contains $I$ such that $\tilde{\boldsymbol{c}}_L = \boldsymbol{c}_L$. We have by Lemma 4.1 that

$$\boldsymbol{c}^\top \Sigma^{-1} \boldsymbol{c} = \boldsymbol{c}_L^\top (\Sigma_{LL})^{-1} \boldsymbol{c}_L = \boldsymbol{c}_I^\top (\Sigma_{II})^{-1} \boldsymbol{c}_I$$

and moreover

$$h_i = \boldsymbol{c}_I^\top (\Sigma_{II})^{-1} \boldsymbol{e}_i > 0, \quad \forall i \in I,$$

where $\boldsymbol{e}_i$ is the unit vector in $\mathbb{R}^m$ with all components equal to 0 apart from the $i$th component equal to 1. Denote by $L^c$ the complement of index set $L$ with respect to $\{1, \ldots, d\}$.

For illustration purposes, we discuss briefly the case $d = 2$. Consider therefore $\Sigma$ to be a correlation matrix with off diagonal elements equal $\rho \in (-1, 1)$ and let $\boldsymbol{c} = (1, c)^\top$. If $c \in (\rho, 1)$, then $\tilde{\boldsymbol{c}} = \boldsymbol{c}$ and hence $I = L = \{1, 2\}$ implying that $I^c, L^c$ are empty. The assumption $c = \rho$ yields

$$I = \{1\}, \quad L = \{1, 2\},$$

whereas supposing that $c < \rho$ implies $\tilde{\boldsymbol{c}} = (1, \rho)^\top$ and $I = L = \{1\}$.

Below we write $\boldsymbol{z}_{-1}$ instead of $\boldsymbol{z}_I$ with $I = \{2, \ldots, d\}$ for any $\boldsymbol{z} \in \mathbb{R}^d$. We present next the approximation of $E(\boldsymbol{c}, u)$.

**Theorem 2.1.** *Let $\boldsymbol{c}, \mu$ be two given constants and let $\Sigma$ be the non-singular covariance matrix of the centered Gaussian random vector $\boldsymbol{X}$. Let $I, L$ be the index sets identified by $\Pi_\Sigma(\boldsymbol{c})$, where $\boldsymbol{c} \in \mathbb{R}^d$ has at least one positive component.*

*i) If $1 \in I$, then we have*

$$E(\boldsymbol{c}, u) \sim \frac{1}{\boldsymbol{c}_I^\top (\Sigma_{II})^{-1} \boldsymbol{e}_1} \frac{\mathbb{P}\{X_1 > c_1 u + \mu, \boldsymbol{X}_{-1} > \boldsymbol{c}_{-1} u\}}{u \mathbb{P}\{\boldsymbol{X}_{-1} > \boldsymbol{c}_{-1} u\}} \to 0, \quad u \to \infty. \tag{2.2}$$

*ii) If $1 \in L \setminus I$, then $c_1 = (\Sigma_{I^c I}(\Sigma_{II})^{-1}\boldsymbol{c}_I)_1$ and further*

$$\lim_{u\to\infty} E(\boldsymbol{c},u) \;=\; \mathbb{E}\{(Y-\mu)_+\} \in (0,\infty), \tag{2.3}$$

*where $Y$ has survival function $\overline{G}(x) = \mathbb{P}\{X_1 > x | \boldsymbol{X}_I = \boldsymbol{0}_I\}$ if $L = I \cup \{1\}$ and if $N^* = L \setminus (I \cup \{1\})$ is non-empty*

$$\overline{G}(x) = \frac{\mathbb{P}\{X_1 > x, \boldsymbol{X}_{N^*} > \boldsymbol{0}_{N^*} | \boldsymbol{X}_I = \boldsymbol{0}_I\}}{\mathbb{P}\{\boldsymbol{X}_{N^*} > \boldsymbol{0}_{N^*} | \boldsymbol{X}_I = \boldsymbol{0}_I\}}, \quad x \in \mathbb{R}. \tag{2.4}$$

*iii) If $1 \in L^c$, then as $u \to \infty$*

$$E(\boldsymbol{c},u) \sim u((\Sigma_{I^c I}(\Sigma_{II})^{-1}\boldsymbol{c}_I)_1 - c_1) \to \infty. \tag{2.5}$$

**Remark 2.2.** *i) The tail asymptotics of Gaussian random vectors is well-known, see below Lemma 4.2 for a minor refinement. Hence the exact asymptotic behaviour of $E(\boldsymbol{c},u)$ in (2.2) can be explicitly calculated by approximating both $\mathbb{P}\{X_1 > c_1 u + \mu, \boldsymbol{X}_{-1} > \boldsymbol{c}_{-1}u\}$ and $\mathbb{P}\{\boldsymbol{X}_{-1} > \boldsymbol{c}_{-1}u\}$ as $u \to \infty$.*
*ii) As we demonstrate in the Appendix, $E(\boldsymbol{c},u)$ in (2.2) equals $o(e^{-\varepsilon u^2})$ for some small $\varepsilon > 0$.*

In order to discuss the approximation of MES in this $d$-dimensional setting we define

$$\begin{aligned} S(\boldsymbol{c},u) := \mathbb{E}\{X_1 | \boldsymbol{X}_{-1} > \boldsymbol{c}_{-1}u\} &= c_1 u + \mathbb{E}\{X_1 - c_1 u | \boldsymbol{X}_{-1} > \boldsymbol{c}_{-1}u\} \\ &=: c_1 u + A(\boldsymbol{c},u), \quad \boldsymbol{c} = (c_1,\ldots,c_d)^\top, \end{aligned}$$

where $\boldsymbol{c} \in \mathbb{R}^d \setminus (-\infty,0]^d$. Since we are interested in the approximation of $\mathbb{E}\{X_1 | \boldsymbol{X}_{-1} > \boldsymbol{c}_{-1}u\}$ as $u \to \infty$, the natural question here is if we can determine $c_1$ such that $A(\boldsymbol{c},u)$ is bounded for all large $u$.
In view of [4][Thm 5.1], we know that for particular choices of $\boldsymbol{c}$ the following convergence in distribution

$$(X_1 - c_1 u) | (\boldsymbol{X}_{-1} > \boldsymbol{c}_{-1}u) \stackrel{d}{\to} Y, \quad u \to \infty \tag{2.6}$$

holds with $Y$ being a Gaussian or some truncated Gaussian random variable. The aforementioned result suggests that $\lim_{u\to\infty} A(\boldsymbol{c},u) = \mathbb{E}\{Y\}$ could be valid, which then for the specific choice of $c_1$ implies

$$S(\boldsymbol{c},u) - c_1 u \to \mathbb{E}\{Y\}, \quad u \to \infty. \tag{2.7}$$

Our next result shows that indeed (2.7) holds.

**Theorem 2.3.** *Let $\boldsymbol{b} = \boldsymbol{c}_{-1}$ have at least one positive component and let $\mathcal{I}, \mathcal{L}$ be the index sets corresponding to $\Pi_B(\boldsymbol{b})$ with unique solution $\tilde{\boldsymbol{b}}$, where $B$ is the covariance matrix of $\boldsymbol{X}_{-1}$. Suppose for simplicity that $\mathcal{I} = \{k,\ldots,d-1\}$. Then (2.7) holds with*

$$c_1 = \Sigma_{1,I}(\Sigma_{II})^{-1}\boldsymbol{c}_I, \quad I = \{k+1,\ldots,d\}.$$

*Moreover, for the above choice of $c_1$ (2.6) is satisfied with $Y$ having survival function $\overline{G}(x) = \mathbb{P}\{X_1 > x | \boldsymbol{X}_I = \boldsymbol{0}\}$ if $\mathcal{L} = \mathcal{I}$. In case that $\mathcal{N} = \mathcal{L} \setminus \mathcal{I}$ is non-empty, then $\overline{G}$ is given from (2.4) with $N^* = \mathcal{N} + 1$.*

**Remark 2.4.** *In the two dimensional setup $\boldsymbol{b}$ has only one element and thus $\mathcal{I} = \mathcal{L}$. Hence the limiting random variable $Y$ has $N(0, 1-\rho^2)$ distribution and therefore $\mathbb{E}\{Y\} = 0$ confirming (1.7). If $\mathcal{I} \neq \mathcal{L}$, then in general $\mathbb{E}\{Y\}$ does not equal 0.*

2.2. **Approximation of MCTE.** Another interesting risk measure is the multivariate conditional tail expectation (abbreviated here as MCTE), which for elliptically symmetric random vectors can be calculated explicitly, see [11, 12]. For a given random vector $\boldsymbol{X} = (X_1,\ldots,X_d)$ with integrable components and given $\boldsymbol{c} \in \mathbb{R}^d$ it is defined by

$$M(\boldsymbol{c},u) = \mathbb{E}\{X_1 | \boldsymbol{X} > \boldsymbol{c}u\}$$

for $u > 0$ and $\boldsymbol{c}$ with at least one positive component.
Note in passing that for any $\boldsymbol{c}, u$ and taking for simplicity $\mu = 0$ we have (hereafter where $\mathbb{I}(\cdot)$ denotes the indicator function)

$$E(\boldsymbol{c},u) \;=\; \frac{\mathbb{E}\{(X_1 - c_1 u)_+ \mathbb{I}(\boldsymbol{X}_{-1} > \boldsymbol{c}_{-1}u)\}}{\mathbb{P}\{\boldsymbol{X}_{-1} > \boldsymbol{c}_{-1}u\}}$$

$$
\begin{aligned}
&= \frac{\mathbb{E}\{(X_1 - c_1 u)\mathbb{I}(X_1 > c_1 u)\mathbb{I}(\boldsymbol{X}_{-1} > \boldsymbol{c}_{-1} u)\}}{\mathbb{P}\{\boldsymbol{X}_{-1} > \boldsymbol{c}_{-1} u\}} \\
&= \frac{\mathbb{P}\{\boldsymbol{X} > \boldsymbol{c} u\}}{\mathbb{P}\{\boldsymbol{X}_{-1} > \boldsymbol{c}_{-1} u\}} \frac{\mathbb{E}\{(X_1 - c_1 u)\mathbb{I}(\boldsymbol{X} > \boldsymbol{c} u)\}}{\mathbb{P}\{\boldsymbol{X} > \boldsymbol{c} u\}} \\
&= \frac{\mathbb{P}\{\boldsymbol{X} > \boldsymbol{c} u\}}{\mathbb{P}\{\boldsymbol{X}_{-1} > \boldsymbol{c}_{-1} u\}} \mathbb{E}\{(X_1 - c_1 u)|\boldsymbol{X} > \boldsymbol{c} u)\} \\
&=: r(u)[M(\boldsymbol{c}, u) - c_1 u],
\end{aligned}
$$

where we assumed that $\mathbb{P}\{\boldsymbol{X} > \boldsymbol{c}u\} > 0$. In view of Lemma 4.2, under the assumption *iii)* in Theorem 2.1 it follows that $\lim_{u\to\infty} r(u) = 1$. Consequently, Theorem 2.1 implies

$$
M(\boldsymbol{c}, u) \sim \Sigma_{1,I}(\Sigma_{II})^{-1}\boldsymbol{c}_I u, \quad u \to \infty. \tag{2.8}
$$

Under the assumption *ii)* in Theorem 2.1 since by Lemma 4.2 we have $\lim_{u\to\infty} r(u) = C \in (0,\infty)$, then again Theorem 2.1 yields that for some $C_1 > 0$ that can be calculated explicitly

$$
\lim_{u\to\infty} [M(\boldsymbol{c}, u) - c_1 u] = C_1, \quad u \to \infty. \tag{2.9}
$$

Finally, under the assumptions of Theorem 2.1, *i)* we have that

$$
\lim_{u\to\infty} u[M(\boldsymbol{c}, u) - c_1 u] = \frac{1}{\boldsymbol{c}_I^\top (\Sigma_{II})^{-1}\boldsymbol{e}_1} > 0, \quad u \to \infty. \tag{2.10}
$$

An intuition for the above approximations comes from the conditional limit theorem derived in [4][Thm 5.1]. For instance if $1 \in I$ being the index set related to $\Pi_\Sigma(\boldsymbol{c})$ for some general $\boldsymbol{c}$ with at least one positive component, we have the convergence in distribution

$$
u(X_1 - c_1 u)\big|(\boldsymbol{X} > \boldsymbol{c}u) \overset{d}{\to} \mathcal{E}, \quad u \to \infty,
$$

where $\mathcal{E}$ is an exponential random variable with mean $1/\boldsymbol{c}_I^\top (\Sigma_{II})^{-1}\boldsymbol{e}_1$.
The following result is new and gives a minor refinement of (2.8).

**Theorem 2.5.** *Under the assumptions of Theorem 2.1 iii) we have* $\tilde{c}_1 = \Sigma_{1,I}(\Sigma_{II})^{-1}\boldsymbol{c}_I > c_1$

$$
(X_1 - \tilde{c}_1 u)|\boldsymbol{X} > \boldsymbol{c}u \overset{d}{\to} Y, \quad u \to \infty, \tag{2.11}
$$

*where* $Y$ *has survival function* $\overline{G}$ *given in Theorem 2.1 with* $N^* = L \setminus I$. *Moreover as* $u \to \infty$

$$
M(\boldsymbol{c}, u) - \tilde{c}_1 u \to \mathbb{E}\{Y\}. \tag{2.12}
$$

**Remark 2.6.** *If* $L = I$, *then* $\mathbb{E}\{Y\} = 0$ *since* $Y$ *with survival function* $\overline{G}$ *defined above is a centered Gaussian random variable.*

2.3. **Trivariate Case.** In order to apply our results we need to determine the index sets $I$ and $L$ related to the quadratic programming problem $\Pi_\Sigma(\boldsymbol{c})$. The index set $I$ has $m \le d$ elements and it is possible that $m = 1$ for given $\boldsymbol{c}$ with at least one positive component. If $X_1$ is independent of $\boldsymbol{X}_{-1}$, then it follows easily that $m \ge 2$ and $1 \in I$, whereas for the case $d = 2$ and $c_1 = c_2$ we have $m = 2$ and $I = L$. In general, $m = d$ if and only if the so-called Savage condition (see [13, 14])

$$
\Sigma^{-1}\boldsymbol{c} > \boldsymbol{0} = (0, \ldots, 0)^\top \in \mathbb{R}^d
$$

holds, which can be easily checked for given $\boldsymbol{c}$ and $\Sigma$. If the Savage condition does not hold, then $m < d$ but the exact value of $m$ cannot be known without the knowledge of $\Sigma$ and $\boldsymbol{c}$. In the following we discuss in details the trivariate case $\boldsymbol{c} = (1,1,1)^\top$ and $\Sigma$ is a non-singular correlation matrix with entries $\sigma_{ij}, i, j \le 3$.
First note that the Savage condition is equivalent with

$$
1 + 2\sigma - \sigma_{12} - \sigma_{13} - \sigma_{23} > 0, \quad \sigma = \min(\sigma_{12}, \sigma_{13}, \sigma_{23}), \tag{2.13}
$$

which is equivalent with $m=3$ as mentioned above. Consequently, assuming (2.13), by statement $i)$ in Theorem 2.1

$$E(\boldsymbol{c},u) \sim \frac{\sqrt{1-\sigma_{23}}}{(1+\sigma_{23})^{3/2}} \frac{1}{\sqrt{2\pi det(\Sigma)}(\boldsymbol{c}^\top \Sigma^{-1}\boldsymbol{e}_1)^2 \prod_{i=2}^3 \boldsymbol{c}^\top \Sigma^{-1}\boldsymbol{e}_i} \frac{1}{u^2} e^{-\frac{u^2}{2}[(\boldsymbol{c}+\mu\boldsymbol{e}_1)^\top \Sigma^{-1}(\boldsymbol{c}+\mu \boldsymbol{e}_1)/2-1/(1+\sigma_{23})]}$$

as $u\to\infty$, where $\boldsymbol{e}_i$'s are unit vectors in $\mathbb{R}^d$ with 1 in the $i$th coordinate and all other coordinates equal 0.
Suppose next that (2.13) does not hold, i.e.,

$$1+2\sigma-\sigma_{12}-\sigma_{13}-\sigma_{23}\le 0$$

and $m=2$ since $m=1$ is impossible in the two dimensional setup when the coordinates of $\boldsymbol{c}$ are equal and positive. If (2.14) is satisfied with equality, then $L=\{1,2,3\}$. Assuming that $\sigma_{12}\le \min(\sigma_{13},\sigma_{23})$ implies $I=\{1,2\}$ and thus $1\in I$ and the asymptotics of $E(\boldsymbol{c},u)$ follows again from statement $i)$ in Theorem 2.1. The case $\sigma=\sigma_{13}$ is similar and therefore we assume next that $\sigma=\sigma_{23}$, which implies that $I=\{2,3\}$ and thus $1\not\in I$ and $1\in L$, provided that $1+\sigma_{23}=\sigma_{12}+\sigma_{13}$. For this case, by (2.3)

$$\lim_{u\to\infty} E(\boldsymbol{c},u)=\mathbb{E}\{(X_1-\mu)_+|X_2=0,X_3=0\}. \tag{2.14}$$

Finally, if $\sigma_{12}+\sigma_{13}-\sigma_{23}-1>0$, then $I=L$ and $1\in L^c$. Hence by statement $iii)$ in Theorem 2.1

$$E(\boldsymbol{c},u)\sim \frac{\sigma_{12}+\sigma_{13}-\sigma_{23}-1}{1+\sigma_{23}}u$$

and from (2.12)

$$M(\boldsymbol{c},u)-\frac{\sigma_{12}+\sigma_{13}}{1+\sigma_{23}}u\to 0$$

as $u\to\infty$.

## 3. Proofs

Proof of Theorem 1.1 Let $(X_1,X_2)$ be jointly Gaussian with mean vector zero, correlation $\rho\in(-1,1)$ and set

$$u:=u_p=VaR_{X_2}(p),\quad \beta=\frac{\mu_2-\mu_1}{\sigma_1},\quad c=\frac{\sigma_2}{\sigma_1}.$$

For any $u>0$ we have

$$\begin{aligned}
E(p) &= \mathbb{E}\{(\sigma_1X_1+\mu_1-\sigma_2u-\mu_2)_+|X_2>u\}\\
&= \sigma_1\mathbb{E}\Big\{\Big(X_1-\frac{\sigma_2}{\sigma_1}u-\frac{\mu_2-\mu_1}{\sigma_1}\Big)_+\Big|X_2>u\Big\}\\
&= \frac{\sigma_1}{\mathbb{P}\{X_1>u\}}\mathbb{E}\{(X_1-cu-\beta)\mathbb{I}(X_1>cu+\beta,X_2>u\}\\
&=: \frac{\sigma_1}{\mathbb{P}\{X_1>u\}}\theta_u\in(0,\infty).
\end{aligned}$$

Let below $\varphi$ denote the pdf of $(X_1,X_2)$.
$i)$ First note that in this case $c\in(\rho,1]$. Let $h_1^*,h_2^*$ be defined by

$$h_1^*=\frac{c-\rho}{1-\rho^2}>0,\quad h_2^*=\frac{1-c\rho}{1-\rho^2}>0. \tag{3.1}$$

Using the transformation

$$s=cu+\beta+x/u,\quad t=u+y/u$$

for any $u>0$, we have further

$$\begin{aligned}
\theta_u=\int_{cu+\beta}^\infty\int_u^\infty (s-cu-\beta)\varphi(s,t)dsdt\\
&= u^{-3}\int_0^\infty\int_0^\infty x\varphi(cu+\beta+x/u,u+y/u)dxdy\\
&=: u^{-3}\varphi(cu+\beta,u)\int_0^\infty\int_0^\infty x\exp\left(-h_1^*x-h_2^*y\right)\psi_u(x,y)dxdy.
\end{aligned}$$

After some calculations for any $x, y$ positive we obtain

$$\lim_{u\to\infty} \psi_u(x,y) = 1 \tag{3.2}$$

and further for all $\varepsilon > 0$ sufficiently small and all $u$ large $\psi_u(x,y) \le e^{\varepsilon(x+y)}$. Consequently, since $h_1^*, h_2^*$ are positive, applying the dominated convergence theorem we obtain

$$\begin{aligned}\theta_u &\sim u^{-3}\varphi(cu+\beta,u)\int_0^\infty\int_0^\infty x\exp\left(-h_1^*x-h_2^*y\right)dxdy\\ &= \frac{1}{(h_1^*)^2h_2^*}u^{-3}\varphi(cu+\beta,u), \quad u\to\infty,\end{aligned}$$

hence the claim follows.

*ii)* If $c=\rho$ the above transformation cannot be used since then $h_1^*=0$ and the limiting integral is not finite. We use another transformation, namely

$$s=\rho u+\beta+x, \quad t=u+y/u$$

for any $u>0$. Consequently, we have

$$\begin{aligned}\theta_u &= u^{-1}\int_0^\infty\int_0^\infty x\varphi(\rho u+\beta+x,u+y/u)dxdy\\ &=: u^{-1}\varphi(\rho u,u)\int_0^\infty\int_0^\infty xe^{-\frac{(x+\beta)^2}{2(1-\rho^2)}-y}\psi_u(x,y)dxdy.\end{aligned}$$

By the definition of $\varphi$

$$u^{-1}\varphi(\rho u,u)\sim\frac{1}{2\pi\sqrt{1-\rho^2}}u^{-1}e^{-u^2/2}, \quad \mathbb{P}\{X_1>u\}\sim u^{-1}e^{-u^2/2}/\sqrt{2\pi}, \quad u\to\infty, \tag{3.3}$$

where the second approximation is a direct consequence of the well-known Mill's ratio asymptotics. Clearly (3.2) holds and the domination of the integrand follows easily. Hence by the dominated convergence theorem as $u\to\infty$

$$\begin{aligned}\theta_u &\sim u^{-1}\varphi(\rho u,u)\int_0^\infty\int_0^\infty xe^{-\frac{(x+\beta)^2}{2(1-\rho^2)}-y}dxdy\\ &\sim \frac{1}{\sqrt{2\pi(1-\rho^2)}}\int_\beta^\infty(x-\beta)e^{-\frac{x^2}{2(1-\rho^2)}}dx\mathbb{P}\{X_1>u\}\\ &= \mathbb{E}\{(\sqrt{1-\rho^2}X_1-\beta)_+\}\mathbb{P}\{X_1>u\}.\end{aligned}$$

Since for any $a>0, b\in\mathbb{R}$

$$\mathbb{E}\{(aX_1-b)_+\}=a\Phi'(b/a)-b[1-\Phi(b/a)] \tag{3.4}$$

the claim follows.

*iii)* If $c<\rho$, then

$$\begin{aligned}\mathbb{P}\{X_1>cu+\beta,X_2>u\} &= u^{-1}\int_0^\infty\mathbb{P}\{\sqrt{1-\rho^2}X_1>(c-\rho)u-\rho x/u+\beta\}\frac{1}{\sqrt{2\pi}}e^{-(u+x/u)^2/2}\,dx\\ &\sim \mathbb{P}\{X_1>u\}, \quad u\to\infty.\end{aligned}$$

Next, using the same transformation as for the case $c=\rho$ gives letting $u\to\infty$

$$\begin{aligned}\theta_u &= \int_{cu+\beta}^\infty\int_u^\infty(x+(\rho-c)u-\rho u-\beta)\varphi(x,y)dxdy\\ &= (\rho-c)u\mathbb{P}\{X_1>cu+\beta,X_2>u\}+\int_{cu+\beta}^\infty\int_u^\infty(x-\rho u-\beta)\varphi(x,y)dxdy\\ &\sim (\rho-c)u\mathbb{P}\{X_1>u\}+u^{-1}\int_{(c-\rho)u}^\infty\int_0^\infty x\varphi(\rho u+\beta+x,u+y/u)dxdy\\ &= (\rho-c)u\mathbb{P}\{X_1>u\}+u^{-1}\varphi(\rho u,u)\int_{(c-\rho)u}^\infty\int_0^\infty xe^{-\frac{(x+\beta)^2}{2(1-\rho^2)}-y}\psi_u(x,y)dxdy.\end{aligned}$$

As above, by (3.2) and $\lim_{u\to\infty}(c-\rho)u=-\infty$

$$\lim_{u\to\infty}\int_{(c-\rho)u}^{\infty}\int_0^{\infty} xe^{-\frac{(x+\beta)^2}{2(1-\rho^2)}-y}\psi_u(x,y)dxdy=\int_{\mathbb{R}}\int_0^{\infty} xe^{-\frac{(x+\beta)^2}{2(1-\rho^2)}-y}dxdy=0.$$

Utilising further (3.3) we obtain

$$\theta_u\sim(\rho-c)u\mathbb{P}\{X_1>u\},\quad u\to\infty$$

establishing the claim.
*iv*) Since $c\ge 1/\rho$, then $h_2^*$ defined in (3.1) is non-positive. Hence we need to use another transform, namely

$$s=cu+\beta+x/u,\quad t=c\rho u+y$$

for any $u>0$. Consequently, for any $u>0$

$$\begin{aligned}\theta_u &= u^{-2}\int_0^{\infty}\int_{(1-c\rho)u}^{\infty} x\varphi(cu+\beta+x/u,c\rho u+y)dxdy\\ &=: (cu)^{-2}\varphi(cu,c\rho u)e^{-\beta cu}\int_0^{\infty}\int_{(1-c\rho)u}^{\infty} xe^{-x}e^{-\frac{y^2-2\rho\beta y+\beta^2}{2(1-\rho^2)}}\psi_u(x,y)dxdy,\end{aligned}$$

where $\psi_u(x,y)\to 1$ as $u\to\infty$. The domination of the integrad follows easily, hence applying the dominated convergence theorem and (3.3), for $c=1/\rho$

$$\begin{aligned}\theta_u &\sim (cu)^{-2}\varphi(cu,c\rho u)e^{-\frac{\beta^2}{2}-\beta cu}\int_0^{\infty}\int_0^{\infty} xe^{-x}e^{-\frac{(y-\rho\beta)^2}{2(1-\rho^2)}}dxdy\\ &= (cu)^{-2}\varphi(cu,c\rho u)e^{-\frac{\beta^2}{2}-\beta cu}\sqrt{2\pi(1-\rho^2)}[1-\Phi(-\rho\beta/\sqrt{1-\rho^2})]\\ &\sim (cu)^{-1}e^{-\frac{\beta^2}{2}-\beta cu}\Phi(\rho\beta/\sqrt{1-\rho^2})[1-\Phi(cu)]\end{aligned}$$

as $u\to\infty$. If $c>1/\rho$, then

$$\begin{aligned}\theta_u &\sim (cu)^{-2}\varphi(cu,c\rho u)e^{-\frac{\beta^2}{2}-\beta cu}\int_{\mathbb{R}} e^{-\frac{(y-\rho\beta)^2}{2(1-\rho^2)}}dy\\ &\sim (cu)^{-1}e^{-\frac{\beta^2}{2}-\beta cu}[1-\Phi(cu)],\quad u\to\infty,\end{aligned}$$

hence the claim follows.
*v*) First note that for any $p\in(0,1)$ and $u:=u_p=VaR_{Z_2}(p)$

$$S(p)=\mu_1+\mathbb{E}\{(Z_1-\mu_1)|Z_2>VaR_{Z_2}(p)\}=\mu_1+\sigma_1\rho u+\sigma_1\mathbb{E}\{X_1-\rho u|X_2>u\}.$$

As above we have

$$\begin{aligned}\mathbb{E}\{X_1-\rho u|X_2>u\} &= \frac{1}{\mathbb{P}\{X_1>u\}}\int_{x\in\mathbb{R},y>u}(x-\rho u)\varphi(x,y)dxdy\\ &= \frac{\varphi(\rho u,u)}{u\mathbb{P}\{X_1>u\}}\int_{x\in\mathbb{R},y>0} x\varphi(\rho u+x,u+y/u)/\varphi(\rho u,u)dxdy\\ &\sim \frac{1}{\sqrt{2\pi(1-\rho^2)}}\int_{x\in\mathbb{R},y>0} x\varphi(\rho u+x,u+y/u)/\varphi(\rho u,u)dxdy\\ &\sim \frac{1}{\sqrt{2\pi(1-\rho^2)}}\int_{x\in\mathbb{R},y>0} xe^{-\frac{x^2}{2(1-\rho^2)}-y}dxdy\\ &= 0\end{aligned}$$

as $u\to\infty$, establishing thus the claim. □

Proof of Theorem 2.1 The proof is driven by the tail asymptotics of Gaussian random vectors derived in [2]. As therein the index set $I$ is also crucial for the derivation of the asymptotics of $E(\boldsymbol{c},u)$, since the tail asymptotics of $\mathbb{P}\{\boldsymbol{X}>\boldsymbol{c}u\}$ is up to a pre-factor the same as that of $\mathbb{P}\{\boldsymbol{X}_I>\boldsymbol{c}_I u\}$ as $u\to\infty$. The components with indices in the set $L\setminus I$ influence the asymptotics only by the pre-factor, whereas the components with indices in the set $K:=L^c$ are not important. For these reasons we have three different cases which shall be dealt with separately.

Set next for any $u > 0$

$$E^*(u) = \mathbb{P}\{\boldsymbol{X}_{-1} > \boldsymbol{c}_{-1}u\}E(\boldsymbol{c}, u)$$

and write $\varphi$ for the pdf of $\boldsymbol{X}$.

*i)* When $1 \in I$, then $\tilde{c}_1 = c_1$. Hence for any $u$ positive

$$\begin{aligned} E^*(u) &= \int_{\boldsymbol{s} > \boldsymbol{c}u + \mu \boldsymbol{e}_1^*} (s_1 - c_1 u - \mu)_+ \varphi(\boldsymbol{s}) d\boldsymbol{s} \\ &= \frac{1}{u^{m+1}} \int_{\boldsymbol{x} > \overline{\boldsymbol{u}}(\boldsymbol{c}u - \tilde{\boldsymbol{c}}u)} x_1 \varphi(\tilde{\boldsymbol{c}}u + \boldsymbol{x}/\overline{\boldsymbol{u}} + \mu \boldsymbol{e}_1^*) d\boldsymbol{x}, \end{aligned}$$

where $\overline{\boldsymbol{u}}$ has all components with indices in $I$ equal to $u$ and otherwise equal to 1 and $\boldsymbol{e}_1^*$ has all components equal to 0 apart from the first component equal to 1. Recall that $m$ stands for the number of the elements of the index set $I$ which cannot be empty. Using further (4.4) (set next $J = I^c = \{1, \ldots, d\} \setminus I$ and assume for simplicity that $J$ is not empty) we have

$$\begin{aligned} &(\tilde{\boldsymbol{c}}u + \boldsymbol{x}/\overline{\boldsymbol{u}} + \mu \boldsymbol{e}_1^*)^\top \Sigma^{-1} (\tilde{\boldsymbol{c}}u + \boldsymbol{x}/\overline{\boldsymbol{u}} + \mu \boldsymbol{e}_1^*) \\ &= (\tilde{\boldsymbol{c}}u + \mu \boldsymbol{e}_1^*)^\top \Sigma^{-1} (\tilde{\boldsymbol{c}}u + \mu \boldsymbol{e}_1^*) + 2u \tilde{\boldsymbol{c}}^\top \Sigma^{-1} \boldsymbol{x}/\overline{\boldsymbol{u}} + 2\mu (\boldsymbol{e}_1^*)^\top \Sigma^{-1} \boldsymbol{x}/\overline{\boldsymbol{u}} + (\boldsymbol{x}/\overline{\boldsymbol{u}})^\top \Sigma^{-1} \boldsymbol{x}/\overline{\boldsymbol{u}}. \end{aligned} \tag{3.5}$$

By the properties of $\tilde{\boldsymbol{c}}$ (see equation (4.4) in Lemma 4.1) for any $u \neq 0, \boldsymbol{x} \in \mathbb{R}^d$

$$u \tilde{\boldsymbol{c}}^\top \Sigma^{-1} \boldsymbol{x}/\overline{\boldsymbol{u}} = \tilde{\boldsymbol{c}}_I^\top (\Sigma_{II})^{-1} \boldsymbol{x}_I.$$

Hence since $1 \in I$ implies $(\boldsymbol{e}_1^*)^\top \Sigma^{-1} \boldsymbol{x}/\overline{\boldsymbol{u}} = O(1/u)$ as $u \to \infty$, then by (3.5)

$$\varphi(\tilde{\boldsymbol{c}}u + \boldsymbol{x}/\tilde{\boldsymbol{u}} + \mu \boldsymbol{e}_1^*) = \varphi(\tilde{\boldsymbol{c}}u + \mu \boldsymbol{e}_1^*) \psi_u(\boldsymbol{x}) e^{-\boldsymbol{c}_I (\Sigma_{II})^{-1} \boldsymbol{x}_I - \boldsymbol{x}_J^\top (\Sigma^{-1})_{JJ} \boldsymbol{x}_J/2},$$

where $\lim_{u \to \infty} \psi_u(\boldsymbol{x}) = 1$ for any $\boldsymbol{x} \in \mathbb{R}^d$. Using the fact that $\Sigma^{-1}$ is positive definite and $\boldsymbol{c}_I^\top (\Sigma_{II})^{-1} > \boldsymbol{0}_I$ for any $\boldsymbol{x} \in \mathbb{R}^d$ with $\boldsymbol{x}_I > \boldsymbol{0}_I$ we obtain that

$$2\boldsymbol{c}_I^\top (\Sigma_{II})^{-1} \boldsymbol{x}_I + 2\mu (\boldsymbol{e}_1^*)^\top \Sigma^{-1} \boldsymbol{x}/\overline{\boldsymbol{u}} + (\boldsymbol{x}/\overline{\boldsymbol{u}})^\top \Sigma^{-1} \boldsymbol{x}/\overline{\boldsymbol{u}} \le C(\boldsymbol{1}_I^\top \boldsymbol{x}_I + \boldsymbol{x}_J^\top \boldsymbol{x}_J) \tag{3.6}$$

holds for all large $u$ and some positive constant $C$. Using thus the dominated convergence theorem (recall $\tilde{c}_i > c_i$ for any $i \in K = L^c$) we obtain

$$\begin{aligned} &E^*(u) \\ &= \frac{1}{u^{m+1}} \varphi(\tilde{\boldsymbol{c}}u + \mu \boldsymbol{e}_1^*) \int_{\boldsymbol{x}_L > \boldsymbol{0}_L, x_i > u(c_i - \tilde{c}_i), i \in K} x_1 \psi_u(\boldsymbol{x}) e^{-\boldsymbol{c}_I (\Sigma_{II})^{-1} \boldsymbol{x}_I - \boldsymbol{x}_J^\top (\Sigma^{-1})_{JJ} \boldsymbol{x}_J/2} d\boldsymbol{x} \\ &\sim \frac{1}{u^{m+1}} \varphi(\tilde{\boldsymbol{c}}u + \mu \boldsymbol{e}_1^*) \int_{\boldsymbol{x}_L > \boldsymbol{0}_L, x_i \in \mathbb{R}, i \in K} x_1 e^{-\boldsymbol{c}_I (\Sigma_{II})^{-1} \boldsymbol{x}_I - \boldsymbol{x}_J^\top (\Sigma^{-1})_{JJ} \boldsymbol{x}_J/2} d\boldsymbol{x} \\ &= \frac{1}{h_1 u} \frac{1}{u^m} \varphi(\tilde{\boldsymbol{c}}u + \mu \boldsymbol{e}_1^*) \frac{1}{\prod_{i \in I} h_i} \int_{x_i > 0, i \in L \setminus I, x_i \in \mathbb{R}, i \in K} e^{-\boldsymbol{x}_J^\top (\Sigma^{-1})_{JJ} \boldsymbol{x}_J/2} d\boldsymbol{x}_J, \end{aligned}$$

where $h_i = \boldsymbol{c}_I^\top (\Sigma_{II})^{-1} \boldsymbol{e}_i > 0$ with $\boldsymbol{e}_i$ the $i$th unit vector in $\mathbb{R}^m$ with $m$ the number of elements of the index set $I$. Since $1 \in I$, applying (4.6) in Lemma 4.2 yields

$$E^*(u) \sim (uh_1)^{-1} \mathbb{P}\{\boldsymbol{X} > \boldsymbol{c}u + \mu \boldsymbol{e}_1^*\}, \quad u \to \infty,$$

hence the claim follows by the definition of $E^*(u)$.

*ii)* In view of Lemma 4.2, the asymptotics of $\mathbb{P}\{X_1 > c_1 u + x, \boldsymbol{X}_{-1} > \boldsymbol{c}_{-1}u\}$ and that of $\mathbb{P}\{\boldsymbol{X}_{-1} > \boldsymbol{c}_{-1}u\}$ as $u \to \infty$ are up to the pre-factor the same. It follows easily that $Y_u := (X_1 - c_1 u) | \boldsymbol{X}_{-1} > \boldsymbol{c}_{-1}u$ converges in distribution as $u \to \infty$ to a random variable $Y$ which has survival function $\mathbb{P}\{X_1 > x | \boldsymbol{X}_I = \boldsymbol{0}_I\}$ if $L \setminus I = \{1\}$ and when $N^* = L \setminus (I \cup \{1\})$ is non-empty, then $Y$ has survival function

$$\frac{\mathbb{P}\{X_1 > x, \boldsymbol{X}_{N^*} > \boldsymbol{0}_{N^*} | \boldsymbol{X}_I = \boldsymbol{0}_I\}}{\mathbb{P}\{\boldsymbol{X}_{N^*} > \boldsymbol{0}_{N^*} | \boldsymbol{X}_I = \boldsymbol{0}_I\}}, \quad x \in \mathbb{R}.$$

In case that $(Y_u-\mu)_+, u>0$ is uniformly integrable, then

$$\lim_{u\to\infty} E(\boldsymbol{u},c)=\mathbb{E}\{(Y-\mu)_+\}.$$

We show next the above convergence directly, which in turn implies the uniform integrability mentioned above. Since $1\in L\setminus I$ we still have that $\tilde{c}_1=c_1$ and as above

$$\begin{aligned}E^*(u) &= \int_{\boldsymbol{s}>\boldsymbol{c}u+\mu \boldsymbol{e}_1^*}(s_1-c_1u-\mu)_+\varphi(\boldsymbol{s})d\boldsymbol{s}\\ &= \frac{1}{u^m}\int_{\boldsymbol{x}>\overline{\boldsymbol{u}}(\boldsymbol{c}u-\tilde{\boldsymbol{c}}u)} x_1\varphi(\tilde{\boldsymbol{c}}u+\boldsymbol{x}/\overline{\boldsymbol{u}}+\mu \boldsymbol{e}_1^*)d\boldsymbol{x}.\end{aligned}$$

Next, since $1\not\in I$ i.e., $1\in J:=I^c$ by (3.5)

$$\begin{aligned}&(\tilde{\boldsymbol{c}}u+\boldsymbol{x}/\overline{\boldsymbol{u}}+\mu \boldsymbol{e}_1^*)^\top\Sigma^{-1}(\tilde{\boldsymbol{c}}u+\boldsymbol{x}/\overline{\boldsymbol{u}}+\mu \boldsymbol{e}_1^*)\\ &\quad= (\tilde{\boldsymbol{c}}u+\mu \boldsymbol{e}_1^*)^\top\Sigma^{-1}(\tilde{\boldsymbol{c}}u+\mu \boldsymbol{e}_1^*)+2\boldsymbol{c}_I^\top(\Sigma_{II})^{-1}\boldsymbol{x}_I+2\mu(\Sigma^{-1})_{1,J}\boldsymbol{x}_J+\boldsymbol{x}_J^\top(\Sigma^{-1})_{JJ}\boldsymbol{x}_J+O(u^{-1})\end{aligned}$$

as $u\to\infty$. Consequently, in view of (3.6), we can apply the dominated convergence theorem to obtain (set $N=L\setminus I$, write $k$ for the number of elements of the index set $K=L^c=\{1,\ldots,d\}\setminus L$ and recall that $\tilde{c}_i>c_i, i\in K$)

$$\begin{aligned}E^*(u) &\sim \frac{1}{u^m}\varphi(\tilde{\boldsymbol{c}}u+\mu \boldsymbol{e}_1^*)\int_{\boldsymbol{x}_L>\boldsymbol{0}_L, x_i>u(c_i-\tilde{c}_i), i\in K} x_1 e^{-\boldsymbol{c}_I(\Sigma_{II})^{-1}\boldsymbol{x}_I-\boldsymbol{x}_J^\top(\Sigma^{-1})_{JJ}\boldsymbol{x}_J/2-\mu(\Sigma^{-1})_{1,J}\boldsymbol{x}_J}d\boldsymbol{x}\\ &= \frac{1}{u^m}\varphi(\tilde{\boldsymbol{c}}u+\mu \boldsymbol{e}_1^*)\frac{1}{\prod_{i\in I}h_i}\int_{\boldsymbol{x}_N>\boldsymbol{0}_N,\boldsymbol{x}_K\in\mathbb{R}^k} x_1 e^{-\boldsymbol{x}_J^\top(\Sigma^{-1})_{JJ}\boldsymbol{x}_J/2-\mu(\Sigma^{-1})_{1,J}\boldsymbol{x}_J}d\boldsymbol{x}_J\end{aligned}$$

as $u\to\infty$. With similar calculations

$$\mathbb{P}\{\boldsymbol{X}>\boldsymbol{c}u+\mu \boldsymbol{e}_1^*\} \sim \frac{1}{u^m}\varphi(\tilde{\boldsymbol{c}}u+\mu \boldsymbol{e}_1^*)\frac{1}{\prod_{i\in I}h_i}\int_{\boldsymbol{x}_N>\boldsymbol{0}_N,\boldsymbol{x}_K\in\mathbb{R}^k} e^{-\boldsymbol{x}_J^\top(\Sigma^{-1})_{JJ}\boldsymbol{x}_J/2-\mu(\Sigma^{-1})_{1,J}\boldsymbol{x}_J}d\boldsymbol{x}_J$$

as $u\to\infty$. Since $1\in J$, by Lemma 4.2

$$\lim_{u\to\infty}\frac{\mathbb{P}\{\boldsymbol{X}>\boldsymbol{c}u+\mu \boldsymbol{e}_1^*\}}{\mathbb{P}\{\boldsymbol{X}_{-1}>\boldsymbol{c}_{-1}u\}}=C_1$$

for some $C_1>0$ which can be calculated explicitly, hence the claim follows.
$iii)$ When $1\in L^c$, then $\tilde{c}_1>c_1$ implying

$$\begin{aligned}E^*(u) &= \int_{\boldsymbol{s}>\boldsymbol{c}u+\mu \boldsymbol{e}_1^*}(s_1-\tilde{c}_1u+(\tilde{c}_1-c_1)u-\mu)\varphi(\boldsymbol{s})d\boldsymbol{s}\\ &= (\tilde{c}_1-c_1)u\mathbb{P}\{\boldsymbol{X}>\boldsymbol{c}u+\mu \boldsymbol{e}_1^*\}+\int_{\boldsymbol{s}>\boldsymbol{c}u+\mu \boldsymbol{e}_1^*}(s_1-\tilde{c}_1u-\mu)\varphi(\boldsymbol{s})d\boldsymbol{s}.\end{aligned}$$

It follows easily that

$$E^*(u) \sim (\tilde{c}_1-c_1)u\mathbb{P}\{\boldsymbol{X}>\boldsymbol{c}u+\mu \boldsymbol{e}_1^*\},\quad u\to\infty$$

and further by Lemma 4.2

$$\mathbb{P}\{\boldsymbol{X}>\boldsymbol{c}u+\mu \boldsymbol{e}_1^*\}\sim\mathbb{P}\{\boldsymbol{X}_{-1}>\boldsymbol{c}_{-1}u\},\quad u\to\infty,$$

hence the proof is complete. □

Proof of Theorem 2.3 We show first the conditional convergence in (2.6). Let $\tilde{\boldsymbol{b}}$ be the solution of the quadratic programming problem $\Pi_B(\boldsymbol{b})$ with corresponding index set $\mathcal{I}=\{k,\ldots,d-1\}$ and let $\mathcal{L}$ be the set of indices such that $\tilde{b}_i=b_i$. Recall that $\boldsymbol{b}=\boldsymbol{c}_{-1}$ is a $(d-1)$-dimensional vector. Let $I=\{k+1,\ldots,d\}$ and set $c_1=\Sigma_{1,I}(\Sigma_{II})^{-1}\boldsymbol{c}_I$. By the definition of $\mathcal{I}$ and $I$ we have that $(\Sigma_{II})^{-1}\boldsymbol{c}_I>\boldsymbol{0}_I$ and $\tilde{\boldsymbol{c}}_{J^*}=\Sigma_{J^*I}(\Sigma_{II})^{-1}\boldsymbol{c}_I$ where $J^*=\{2,\ldots,k\}$ being empty if $k=1$. Note that we agree that when index sets are empty, the defined relationships should be ignored. Let $\tilde{\boldsymbol{c}}$ be such that $\tilde{c}_1=c_1=\Sigma_{1,I}(\Sigma_{II})^{-1}\boldsymbol{c}_I$ and $\tilde{\boldsymbol{c}}_{-1}=\tilde{\boldsymbol{b}}$. Setting $J=\{1\}\cup J^*$ we have that $\tilde{\boldsymbol{c}}_J=\Sigma_{JI}(\Sigma_{II})^{-1}\boldsymbol{c}_I$. Consequently, since $(\Sigma_{II})^{-1}\boldsymbol{c}_I>\boldsymbol{0}_I$ and $I\cup J=\{1,\ldots,k\}$ by the converse statement in Lemma 4.1 we have that $\tilde{\boldsymbol{c}}$ is the unique solution of

$\Pi_\Sigma(\boldsymbol{c})$. From the aforementioned proposition $I$ is the index set that determines the unique solution $\tilde{\boldsymbol{c}}$. In order to show (2.6) we need to determine the asymptotics as $u \to \infty$ of

$$\mathbb{P}\{\boldsymbol{X} > \boldsymbol{c}u + x\boldsymbol{e}_1^*\}/\mathbb{P}\{\boldsymbol{X}_{-1} > \boldsymbol{c}_{-1}u\}$$

for any $x \in \mathbb{R}$. If $\mathcal{L} = \mathcal{I}$, then $L = \{1\} \cup I$ (since $\tilde{c}_1 = c_1$) and thus by Lemma 4.2 we have that (2.6) holds with $Y$ having the same distribution as $X_1|\boldsymbol{X}_I = \boldsymbol{0}_I$. The case that $\mathcal{N} = \mathcal{L} \setminus \mathcal{I}$ is not empty follows from [4][Corr 5.2]. Indeed, the tail asymptotics of the denominator and the nominator are the same up to some positive constant since the $I$ index sets of the corresponding quadratic programming problems are the same. The ratio of those constants is (set $N^* = \mathcal{N} + 1$)

$$\frac{\mathbb{P}\{X_1 > x, \boldsymbol{X}_{N^*} > \boldsymbol{0}_{N^*} | \boldsymbol{X}_I = \boldsymbol{0}_I\}}{\mathbb{P}\{\boldsymbol{X}_{N^*} > \boldsymbol{0}_{N^*} | \boldsymbol{X}_I = \boldsymbol{0}_I\}}$$

and thus (2.6) holds. The proof of (2.7) follows by calculating the asymptotics of

$$\mathbb{E}\{(X_1 - c_1 u)\mathbb{I}(\boldsymbol{X}_{-1} > \boldsymbol{c}_{-1}u)\},$$

which is established similarly to the proof of statement $ii)$ in Theorem 2.1 and therefore we omit the details. □

Proof of Theorem 2.5 Let $I, L$ denote the unique index sets defined from the solution of the quadratic programming problem $\Pi_\Sigma(\boldsymbol{c})$. Suppose first that $N = L \setminus I$ is not empty. By the assumptions $1 \notin N \cup I$. Let $\tilde{\boldsymbol{a}}$ be the unique solution of $\Pi_\Sigma(\boldsymbol{a}), \boldsymbol{a} = (\tilde{c}_1, c_2, \ldots, c_d)^\top$. The corresponding index set $I$ (write this as $I_{\boldsymbol{a}}$) includes $I$ since $1 \notin I$. But we cannot have $1 \in I_{\boldsymbol{a}}$, i.e., $(\Sigma_{I_{\boldsymbol{a}} I_{\boldsymbol{a}}})^{-1}\boldsymbol{a}_{I_{\boldsymbol{a}}} > \boldsymbol{0}_{I_{\boldsymbol{a}}}$ since this contradicts the definition of $a_1 = \tilde{c}_1 > c_1$. Consequently, 1 belongs to the index set $L_{\boldsymbol{a}}$ of all indices $i \le d$ such that $\tilde{a}_i = a_i$. Next, for any $x \in \mathbb{R}$ using Lemma 4.2 and Lemma 4.1 we have

$$\lim_{u\to\infty} \frac{\mathbb{P}\{X_1 > \tilde{c}_1 u + x, \boldsymbol{X}_{-1} > \boldsymbol{c}_{-1}u\}}{\mathbb{P}\{\boldsymbol{X} > \boldsymbol{c}u\}} = \frac{\mathbb{P}\{X_1 > x, \boldsymbol{X}_N > \boldsymbol{0}_N | \boldsymbol{X}_I = \boldsymbol{0}_I\}}{\mathbb{P}\{\boldsymbol{X}_N > \boldsymbol{0}_N | \boldsymbol{X}_I = \boldsymbol{0}_I\}} =: \overline{G}(x), \quad x \in \mathbb{R},$$

where for the asymptotics of the denominator we used the fact that $1 \in L^c$, i.e., $\tilde{c}_1 > c_1$. If $I = L$, then $\overline{G}(x) = \mathbb{P}\{X_1 > x | \boldsymbol{X}_I = \boldsymbol{0}_I\}$. Consequently, $Y$ has the claimed survival function $\overline{G}$. The second claim follows easily and therefore we omit the proof. □

## 4. Appendix

**Lemma 4.1.** *Let $\Sigma$ be a $d\times d$ positive definite matrix and let $\boldsymbol{b} \in \mathbb{R}^d \setminus (-\infty, 0]^d$. The quadratic programming problem $\Pi_\Sigma(\boldsymbol{b})$: minimise $\boldsymbol{x}^\top \Sigma^{-1} \boldsymbol{x}$ under $\boldsymbol{x} \ge \boldsymbol{b}$ has a unique solution $\tilde{\boldsymbol{b}}$ and there exists a unique non-empty index set $I \subseteq \{1, \ldots, d\}$ with $m \le d$ elements such that*

$$\tilde{\boldsymbol{b}}_I = \boldsymbol{b}_I, \quad (\Sigma_{II})^{-1}\boldsymbol{b}_I > \boldsymbol{0}_I \tag{4.1}$$

*and if $I^c := \{1, \ldots, d\} \setminus I \neq \emptyset$, then*

$$\tilde{\boldsymbol{b}}_{I^c} = \Sigma_{I^c I}(\Sigma_{II})^{-1}\boldsymbol{b}_I \ge \boldsymbol{b}_{I^c}, \tag{4.2}$$

$$\min_{\boldsymbol{x} \ge \boldsymbol{b}} \boldsymbol{x}^\top \Sigma^{-1} \boldsymbol{x} = \tilde{\boldsymbol{b}}^\top \Sigma^{-1} \tilde{\boldsymbol{b}} = \boldsymbol{b}_I^\top (\Sigma_{II})^{-1}\boldsymbol{b}_I > 0 \tag{4.3}$$

$$\boldsymbol{x}^\top \Sigma^{-1} \tilde{\boldsymbol{b}} = \boldsymbol{x}_F^\top (\Sigma_{FF})^{-1}\boldsymbol{b}_F, \quad \forall \boldsymbol{x} \in \mathbb{R}^d \tag{4.4}$$

*for any index set $F$ of $\{1, \ldots, d\}$ containing $I$ and if $\boldsymbol{b} = (b, \ldots, b)^\top, b \in (0, \infty)$, then $2 \le |I| \le d$. Conversely, if for some non-empty index set $I \subset \{1, \ldots, d\}$ we have*

$$(\Sigma_{II})^{-1}\boldsymbol{b}_I > \boldsymbol{0}_I, \quad \Sigma_{I^c I}(\Sigma_{II})^{-1}\boldsymbol{b}_I \ge \boldsymbol{b}_{I^c},$$

*then $\tilde{\boldsymbol{b}}$ with $\tilde{\boldsymbol{b}}_{I^c} = \Sigma_{I^c I}(\Sigma_{II})^{-1}\boldsymbol{b}_I, \tilde{\boldsymbol{b}}_I = \boldsymbol{b}_I$ is the solution of $\Pi_\Sigma(\boldsymbol{b})$.*

Proof of Lemma 4.1 The claims in (4.1)-(4.3) are formulated in [15]. Since by (4.2) we have $(\Sigma^{-1}\tilde{\boldsymbol{b}})_M = \boldsymbol{0}_M$ for any $M \subset I^c$ (assuming $I^c$ is not empty) exactly as in proof of [16][Lem 4.1] we have for any $\boldsymbol{x} \in \mathbb{R}^d$ and $F = \{1, \ldots, d\} \setminus M$

$$(\boldsymbol{x} + \tilde{\boldsymbol{b}})^\top \Sigma^{-1} (\boldsymbol{x} + \tilde{\boldsymbol{b}}) = \boldsymbol{x}^\top \Sigma^{-1} \boldsymbol{x} + 2\boldsymbol{x}_F^\top (\Sigma_{FF})^{-1} \tilde{\boldsymbol{b}}_F + \tilde{\boldsymbol{b}}_F^\top (\Sigma_{FF})^{-1} \tilde{\boldsymbol{b}}_F, \tag{4.5}$$

which implies that $\boldsymbol{x}^\top \Sigma^{-1} \tilde{\boldsymbol{b}} = \boldsymbol{x}_F^\top (\Sigma_{FF})^{-1} \tilde{\boldsymbol{b}}_F$ and thus (4.4) holds.

If for some non-empty index set $I$ we have $(\Sigma_{II})^{-1}\boldsymbol{b}_I > \boldsymbol{0}_I$, then $\boldsymbol{b}_I = argmin_{\boldsymbol{x}_I \ge \boldsymbol{b}_I} \boldsymbol{x}_I^\top (\Sigma_{II})^{-1}\boldsymbol{x}_I$. Since for any two non-overlapping index set $A, B, A\cup B = \{1,\dots,d\}$ (using Schur compliments)

$$\boldsymbol{x}^\top \Sigma^{-1}\boldsymbol{x} = \boldsymbol{x}_A^\top(\Sigma_{AA})^{-1}\boldsymbol{x}_A + (\boldsymbol{x}_B - \Sigma_{BA}(\Sigma_{AA})^{-1}\boldsymbol{x}_A)^\top(\Sigma^{-1})_{BB}(\boldsymbol{x}_B - \Sigma_{BA}(\Sigma_{AA})^{-1}\boldsymbol{x}_A), \quad \boldsymbol{x}\in\mathbb{R}^d$$

and $(\Sigma^{-1})_{BB}$ is positive definite, it follows easily that $\tilde{\boldsymbol{b}}$ with $\tilde{\boldsymbol{b}}_I = \boldsymbol{b}_I$ and $\tilde{\boldsymbol{b}}_{I^c} = \Sigma_{I^cI}(\Sigma_{II})^{-1}\boldsymbol{b}_I$ is the unique solution of $\Pi_\Sigma(\boldsymbol{b})$, hence the claim is complete. □

The next result follows from [4][Thm 3.3] since Gaussian random vectors are particular instances of elliptically symmetric ones where the radius has distribution function in the Gumbel max-domain of attraction with scaling function $w(u) = u$. We present however a short proof.

**Lemma 4.2.** *Let $\boldsymbol{c}\in\mathbb{R}^d$ have at least one positive component and let $\boldsymbol{X}$ be a centered d-dimensional Gaussian random vector with non-singular covariance matrix $\Sigma$. Denote by $I, L$ the index sets related to $\Pi_\Sigma(\boldsymbol{c})$ and let further $\boldsymbol{x}(u), u > 0$ be a d-dimensional vector such that $\lim_{u\to\infty} u^{-1}\boldsymbol{x}(u) = \boldsymbol{0}$.*
*As $u\to\infty$ we have*

$$\mathbb{P}\{\boldsymbol{X}_I > (\boldsymbol{c}u + \boldsymbol{x}(u))_I\} \sim \frac{1}{\prod_{i\in I} \boldsymbol{c}_I^\top(\Sigma_{II})^{-1}\boldsymbol{e}_i} u^{-m}\varphi_{\boldsymbol{X}_I}((\boldsymbol{c}u + \boldsymbol{x}(u))_I), \quad u\to\infty, \tag{4.6}$$

*where $m$ is the number of elements of $I$ and $\varphi_{\boldsymbol{X}_I}$ is the pdf of $\boldsymbol{X}_I$. Moreover, with $N = L\setminus I$*

$$\begin{aligned}\lim_{u\to\infty}\frac{\mathbb{P}\{\boldsymbol{X} > \boldsymbol{c}u + \boldsymbol{x}(u)\}}{\mathbb{P}\{\boldsymbol{X}_I > (\boldsymbol{c}u + \boldsymbol{x}(u))_I\}} &= \lim_{u\to\infty}\frac{\mathbb{P}\{\boldsymbol{X}_L > (\boldsymbol{c}u + \boldsymbol{x}(u))_L\}}{\mathbb{P}\{\boldsymbol{X}_I > (\boldsymbol{c}u + \boldsymbol{x}(u))_I\}}\\ &= \mathbb{P}\{\boldsymbol{X}_N > \boldsymbol{x}_N \mid \boldsymbol{X}_I = \boldsymbol{x}_I\},\end{aligned} \tag{4.7}$$

*provided that $\lim_{u\to\infty}(\boldsymbol{x}(u))_{I\cup N} = \boldsymbol{x}_{I\cup N}$ (set $\mathbb{P}\{\boldsymbol{X}_N > \boldsymbol{x}_N \mid \boldsymbol{X}_I = \boldsymbol{x}_I\}$ to 1 if $N$ is empty).*

**Remark 4.3.** *In the particular case $\boldsymbol{x}(u) = \boldsymbol{x}/u, \boldsymbol{x}\in\mathbb{R}^d$ from (4.6) we obtain*

$$\mathbb{P}\{\boldsymbol{X}_I > (\boldsymbol{c}u + \boldsymbol{x}/u)_I\} \sim \mathbb{P}\{\boldsymbol{X}_I > \boldsymbol{c}_I u\} e^{-\boldsymbol{x}_I^\top(\Sigma_{II})^{-1}\boldsymbol{c}_I}, \quad u\to\infty.$$

Proof of Lemma 4.2 Assume for simplicity that $I = \{1,\dots,d\}$. In view of Lemma 4.1 $\Sigma^{-1}\boldsymbol{c} > \boldsymbol{0}$ and this is the crucial condition for the proof. Note further that $\Pi_\Sigma(\boldsymbol{c})$ has unique solution $\boldsymbol{c}$. Hence for any $u\in\mathbb{R}$ we have (set $\boldsymbol{a}(u) = \boldsymbol{c}u + \boldsymbol{x}(u)$)

$$(\boldsymbol{a}(u) + \boldsymbol{x}/u)^\top\Sigma^{-1}(\boldsymbol{a}(u) + \boldsymbol{x}/u) = (\boldsymbol{a}(u))^\top\Sigma^{-1}\boldsymbol{a}(u) + 2\boldsymbol{x}^\top\Sigma^{-1}\boldsymbol{a}(u)/u + \boldsymbol{x}^\top\Sigma^{-1}\boldsymbol{x}/u^2.$$

The term $\boldsymbol{x}^\top\Sigma^{-1}\boldsymbol{x}/u^2$ is important for showing an integrable upper bound for the integrand below, and the finiteness of the integral follows from $\Sigma^{-1}\boldsymbol{c} > \boldsymbol{0}$. More precisely, with $\varphi$ the pdf of $\boldsymbol{X}$ we have

$$\begin{aligned}\mathbb{P}\{\boldsymbol{X} > \boldsymbol{a}(u)\} &= \int_{\boldsymbol{x}>\boldsymbol{a}(u)}\varphi(\boldsymbol{x})d\boldsymbol{x}\\ &= \frac{1}{u^d}\varphi(\boldsymbol{a}(u))\int_{\boldsymbol{y}>\boldsymbol{0}} e^{-\boldsymbol{y}^\top\Sigma^{-1}\boldsymbol{a}(u)/u + \boldsymbol{y}^\top\Sigma^{-1}\boldsymbol{y}/u^2}d\boldsymbol{y} \sim \frac{1}{u^d}\varphi(\boldsymbol{a}(u))\int_{\boldsymbol{y}>\boldsymbol{0}} e^{-\boldsymbol{y}^\top\Sigma^{-1}\boldsymbol{c}}d\boldsymbol{y}\end{aligned}$$

since we assume that $\boldsymbol{x}(u)/u\to\boldsymbol{0}$ as $u\to\infty$.
Next suppose that $I$ has $m < d$ elements and let $J = I^c = \{1,\dots,d\}\setminus I$. We have

$$\mathbb{P}\{\boldsymbol{X} > \boldsymbol{a}(u)\} = \frac{1}{u^m}\int_{\boldsymbol{y}_I>\boldsymbol{0}_I, \boldsymbol{y}_J>(\boldsymbol{c}u-\tilde{\boldsymbol{c}}u)_J}\varphi(\tilde{\boldsymbol{c}}u + \boldsymbol{x}(u) + \boldsymbol{y}/\overline{\boldsymbol{u}})d\boldsymbol{y},$$

where $\overline{\boldsymbol{u}}_I = u\boldsymbol{1}_I$ and $\overline{\boldsymbol{u}}_J = \boldsymbol{1}_J$, hence the proof follows easily using further (4.5). It follows easy that the components of $\boldsymbol{X}$ with indices not in $L$ do not contribute, so we assume without loss of generality that $L$ has $d$ elements. In that case $(\boldsymbol{c}u - \tilde{\boldsymbol{c}}u)_J = \boldsymbol{0}_J$ and the proof follows after some straightforward calculations.
□

To this end we prove that $E(\boldsymbol{c}, u)$ in (2.2) equals $o(e^{-\varepsilon u^2})$ for some small $\varepsilon > 0$. We have that

$$E(\boldsymbol{c}, u) = o(R(u)), \quad R(u) = \mathbb{P}\{X_1 > c_1 u + \mu, \boldsymbol{X}_{-1} > \boldsymbol{c}_{-1}u\}/\mathbb{P}\{\boldsymbol{X}_{-1} > \boldsymbol{c}_{-1}u\}$$

as $u\to\infty$ and $1\in I$ where the index set $I$ determines the solution of $\Pi_\Sigma(\boldsymbol{c})$. The claim now follows if we show that $\lim_{u\to\infty} R(u) = 0$. Indeed this is the case, since in view of Lemma 4.2 the other possibility is that $\lim_{u\to\infty} R(u) = C > 0$. This means that the attained minimum of the quadratic programming problem $\Pi_\Sigma(\boldsymbol{c})$ is $\boldsymbol{c}_I^\top(\Sigma_{II})^{-1}\boldsymbol{c}_I$ being equal to the attained minimum of $\Pi_B(\boldsymbol{b})$ where $B$ is obtained from

$\Sigma$ by deleting the first row and column and $\boldsymbol{b} = \boldsymbol{c}_{-1}$. Since $1 \in I$ there are two different index sets that determine the minimum of the quadratic programming problem $\Pi_\Sigma(\boldsymbol{c})$ which is a contradiction.

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