A RISK MODEL WITH MULTI-LAYER DIVIDEND STRATEGY

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Abstract. In recent years, various dividend payment strategies for the classical collective risk model have been studied in great detail. In this paper, we consider both the dividend payment intensity and the premium intensity to be step functions depending on the current surplus level. Algorithmic schemes for the determination of explicit expressions for the Gerber-Shiu discounted penalty function and the expected discounted dividend payments are derived. This enables the analytical investigation of dividend payment strategies that, in addition to having a sufficiently large expected value of discounted dividend payments, also take the solvency of the portfolio into account. Since the number of layers is arbitrary, it can also be viewed as an approximation to a continuous surplus-dependent dividend payment strategy. A recursive approach with respect to the number of layers is developed that to a certain extent allows to improve upon computational disadvantages of related calculation techniques that have been proposed for specific cases of this model in the literature. The tractability of the approach is illustrated numerically for a risk model with four layers and an exponential claim size distribution.

Keywords: multiple-threshold ruin model; discounted dividends, discounted penalty function, Laplace transform

1. Introduction

Consider an insurance portfolio with surplus $U(t)$ at time t and initial surplus $U(0) = u$. Let $N(t)$, the number of claims up to time t, be modelled by a Poisson process with intensity $\lambda > 0$. Further, the claim sizes X_i are assumed to be independent and identically distributed random variables with distribution function P and mean $\mu < \infty$. The premium income and the dividend payout are assumed to depend on the surplus level in the following way: Define k layers $0 = b_0 < b_1 < b_2 < \dots b_{k-1} < b_k = \infty$. Whenever the surplus $U(t)$ is in layer i, i.e. $b_{i-1} \le U(t) < b_i$, premium is collected with intensity c_i and dividends are paid out with intensity $0 \le a_i \le c_i$ and hence the surplus process $U(t)$ increases with density $(c_i - a_i)$ until a claim causes a jump to a lower layer or the surplus grows to the next higher layer. If we denote the total claim amount up to time t by $S(t) = \sum_{i=1}^{N(t)} X_i$, the dynamics of the surplus process in the ith layer are thus determined by

$$dU(t) = (c_i - a_i)\, dt - dS(t).$$

Furthermore, the dynamics of the dividend payments at time t is given by

$$dD(t) = a_i\, dt \qquad \text{if } b_{i-1} \le U(t) < b_i$$

(Figure 1 illustrates a sample path of $U(t)$ for the case $c_1 = c_2 = c_3$ and $0 = a_1 < a_2 < a_3 = c_3$). For simplicity, assume that the net profit condition is fulfilled in each layer, i.e.

$$(c_i - a_i) > \lambda\mu, \quad i = 1, \dots, k. \tag{1}$$

The ruin probability for the surplus process $U(t)$ is defined through

$$\psi(u) = \mathbb{P}\Big(\inf_{t>0}\{t : U(t) < 0 | U(0) = u\} < \infty\Big)$$

and the (more general) discounted penalty function, originally introduced in the seminal paper of Gerber & Shiu [12], comprises information on the time of ruin and the distribution of both the deficit at ruin

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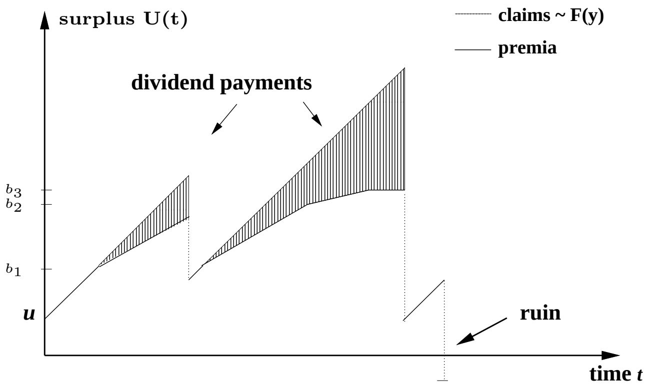

FIGURE 1. A sample path of the risk process $U(t)$

and the surplus before ruin.
Variants and special cases of this risk model have recently been studied extensively in the literature. Among these, in the case $k = 2$, $c_1 = c_2$ and $a_1 = 0$, the so-called threshold model is retrieved for $0 < a_2 < c_1$ and the case of a horizontal dividend barrier model is obtained through the further specification $a_2 = c_1$. Apart from its analytical tractability, one particular motivation to study these two models is the fact that for exponential claim sizes, these dividend strategies can be identified as maximizing the expected discounted dividend payout until ruin in case of bounded and unbounded dividend payment intensity, respectively (see Gerber [10] and Gerber & Shiu [14]). One should, however, keep in mind that this optimality property does not carry over to arbitrary claim size distributions, see for instance Azcue & Muler [7], Albrecher & Thonhauser [4] and Schmidli [18]. At the same time, the solvency of the insurance portfolio also might be of particular concern, and in such a case it is natural to consider dividend payment policies that depend on the current level of the surplus (larger dividend payments for larger surplus level). The layer model considered in this paper can serve as an example for such dividend payment strategies that still enables an analytical treatment. Analytical formulae are of particular interest for tuning the model towards a target in terms of profitability and safety and for risk management purposes in general.

Looking solely at properties of the surplus process (i.e. $a_i = 0$ for $i = 1, \ldots, k$), a surplus-dependent premium rate $c(u)$ already found considerable attention in the literature (see Schock Petersen [19] for an early numerical account and Asmussen & Nielsen [6] for an asymptotic analysis of the resulting risk process based on large deviations). If $c(u)$ is a step function (as in the layer model considered in this paper), an exact expression for the ruin probability was given for two layers (i.e. $k = 2$) in Asmussen [5]. More recently, Lin & Pavlova [16] derived a system of integro-differential equations for the discounted penalty function for two layers. Based on renewal-theoretic techniques, they showed that below the threshold b_1, the discounted penalty function is proportional to the discounted penalty function of the classical (1-layer) model plus a product of an exponential and a compound geometric function. Furthermore, they give explicit results for the ruin probability and the time of ruin when the claims are (mixtures of) exponentials. In Lin & Sendova [17], they are able to extend their analysis to a multi-layer setting and, for exponential claims, derive explicit formulae for the probability of ruin up to a sequence of constants, which has to be calculated numerically as the solution of a system of linear equations. For the same multi-layer model, Zhou [22] proposes a scheme to numerically compute the ruin probability (in his approach the net profit condition (1) is not needed). Furthermore, he uses level crossing techniques to study further ruin-related quantities for $\delta = 0$ and $k = 2$. Laplace transforms of the first passage times across a given level before ruin are investigated in a perturbed risk model in Zhou [20].

The purpose of this paper is to approach the multi-layer model in a generic way, with the aim of unifying and extending some of the results mentioned above. In particular, we intend to contribute to the study of the discounted penalty function and the expected discounted sum of dividend payments up to ruin for this multi-layer setting (where the discount rate is denoted by $\delta \geq 0$).

In Section 2 the well-known differential approach will be used to derive a system of integro-differential equations for the discounted penalty function, which will be investigated by means of Laplace transforms. For exponential claim sizes, we give the exact solution for the probability of ruin in this multi-layer model (this includes an explicit solution of the linear system of equations for the involved constants in the expression, which had been left to a numerical approach in [17]). Furthermore, for exponential claim sizes the Laplace transform of the ruin time in the multi-layer model is rederived in a particularly transparent way. Subsequently, a system of integro-differential equations for the moment-generating function of the discounted dividends is obtained by the infinitesimal approach. The structure of the expected discounted dividends under this multi-layer dividend strategy is then given explicitly for exponential claim sizes, whereas the involved constants are expressed as the solution of a system of linear equations. The approach can in principle be generalized to higher moments of dividend payments and Erlang claim sizes along the same lines.

One computational disadvantage of the methods available in the literature and also the one presented in Section 2 is the fact that for a complete solution (with all involved constants) the obtained recursions have to be evaluated in both directions (that is up and down), typically making the approach infeasible for larger values of k or more involved penalty functions and claim distributions. More concretely, the resulting systems of equations for the coefficients can in most cases not be solved symbolically, so that a representation of the solution in terms of the model parameters is not at hand. In Section 3, we will indicate an alternative approach that improves on this problem to some extent. Rather than going through all the layers recursively at the same time, the explicit solution is calculated successively for an increasing number of layers, always adding one additional layer to the already available solution. In this way a "global" recursion is set up in terms of the number of layers (justifying the name *recursive approach*) and hence the above-mentioned solution of the possibly huge system of linear equations is avoided. The technique in the background is a conditioning on the exit times of the surplus process out of each layer. First, the *time value of upper exit* out of a layer is determined, which is a basic building block for both the discounted penalty and expected dividends. Then recursive formulae are given for these three quantities for general claim sizes. Finally, more explicit results are given for exponential claim sizes.

Throughout the paper, the tractability of the approach is illustrated by an example of a risk model with four layers and exponential claim size, with parameters taken from an example of Zhou [22].

The analysis of this paper can in principle be extended to the case where the claim size distribution also depends on the current layer i (such a model might be appropriate when considering moral hazard behavior of the insurance company; for instance the determination of claim sizes might be stricter (resulting in modified claim distributions) in case the surplus level in the portfolio is low). However, merely for simplification of notation, we here restrict the analysis to the case $P_i(x) \equiv P(x)\,\forall i = 1, \dots, k$.

Notation. The following notation will be used: $\overline{U}(t) := \sup_{0\le s\le t} U(s)$ is the supremum of the surplus up to time t. The subscript $k\,(k \in \mathbb{N})$ will always refer to the corresponding quantity in a model with k layers (for instance, $U_k(t)$ denotes the surplus at time t in a model with k layers and $U_{k-1}(t)$ then is the same quantity in the $(k-1)$-layer model resulting from shifting b_{k-1} to infinity). The subscript 1 correspondingly refers to the classical model with premium density c_1 and dividend rate a_1. Furthermore, let $U_{1,i}(t)$ be the surplus process of the classical (1-layer) model with parameters c_i, a_i (and in general let the subscript $\{\cdot\}_{1,i}$ refer to the corresponding quantity in the classical (1-layer) model with parameters c_i, a_i). Moreover, $\rho_{1,i}$ and $-R_{1,i}$ denote the nonnegative and negative, respectively, solution of the corresponding Lundberg fundamental equation (in ξ)

$$\lambda + \delta - (c_i - a_i)\xi = \lambda \int_0^\infty e^{-\xi t}\,\mathrm{d}P(t), \tag{2}$$

see Gerber & Shiu [12]. Note that $\rho_{1,i}$ is strictly positive when $\delta > 0$. For $a \le b$, define the stopping times $\tau^*(u,a,b) = \inf\{t \ge 0 : U(t) \notin [a,b)\,|U(0) = u\}$,

$$\tau^+(u,a,b) = \begin{cases} \tau^*(u,a,b) & \text{if } U(\tau^*(u,a,b)) = b, \\ \infty & \text{if } U(\tau^*(u,a,b)) < a, \end{cases}$$

and

$$\tau^-(u,a,b) = \begin{cases} \infty & \text{if } U(\tau^*(u,a,b)) = b, \\ \tau^*(u,a,b) & \text{if } U(\tau^*(u,a,b)) < a. \end{cases}$$

Note that the structure of the process implies that upper exits of any interval $[a,b)$ can only happen through (continuous) premium income (hence attaining the value b), whereas lower exits are due to jumps originating from claims. The time of ruin is denoted by $\tau(u) = \tau^-(u,0,\infty)$. Derivatives at a boundary are always interpreted as the appropriate one-sided derivatives.

2. The differential approach

2.1. **The discounted penalty function.** The discounted penalty function as introduced by Gerber & Shiu [12] is in the k-layer model given by

$$m_k(u) = \mathbb{E}\left[e^{-\delta\tau_k(u)} w\left(U_k(\tau_k(u)-), |U_k(\tau_k(u))|\right) I_{\{\tau_k(u)<\infty\}} \middle| U(0) = u \right]. \tag{3}$$

Within each layer, i.e. for $i = 1,\dots,k-1$ and $b_{i-1} \le u < b_i$, conditioning on the occurrence of a jump within an infinitesimal time interval gives

$$\begin{aligned} m_k(u) = (1-\lambda dt)e^{-\delta dt} m_k(u+(c_i-a_i)dt) + \lambda dt e^{-\delta dt} \Bigg(\int_0^{u+(c_i-a_i)dt} m_k(u+(c_i-a_i)dt - z)\,\mathrm{d}P(z) \\ + \int_{u+(c_i-a_i)dt}^{\infty} w(u+(c_i-a_i)dt, z-u+(c_i-a_i)dt)\,\mathrm{d}P(z) \Bigg) + o(dt). \end{aligned}$$

Taylor expansion and collection of terms of order dt then yields

$$(c_i - a_i)\frac{\partial}{\partial u} m_k(u) - (\lambda+\delta) m_k(u) + \lambda \left(\int_0^u m_k(u-z)\,\mathrm{d}P(z) + \int_u^\infty w(u, z-u)\,\mathrm{d}P(z) \right) = 0, \tag{4}$$

where again $u \in [b_{i-1}, b_i)$, $(i = 1,\dots,k-1)$. We are looking for a continuous solution across all layers, so that we get the boundary conditions

$$\lim_{u\to b_i+} m_k(u) = \lim_{u\to b_i-} m_k(u) \tag{5}$$

and

$$\lim_{u\to\infty} m_k(u) = 0. \tag{6}$$

The above derivation of (4) is a short alternative proof of Theorem 3.1 of Lin & Sendova [17] (although their proof based on the renewal approach has the advantage to directly imply the continuity of the solution across the layers, whereas above one has to ask for it).

Remark 2.1. Note that $m_k(u)$ is not differentiable at the boundaries of the layers. This can be seen directly from rewriting the conditions (5) in the form

$$(c_{i+1} - a_{i+1}) \frac{\partial^+}{\partial u} m_k(u) \Big|_{u=b_i} = (c_i - a_i) \frac{\partial^-}{\partial u} m_k(u) \Big|_{u=b_i}.$$

For specific cases (in particular for Erlang distributed claim sizes), the system of integro-differential equations (4) together with boundary conditions (5) and (6) may be solved by a direct approach. We illustrate the idea in the following example:

Example 2.1. Consider the ruin probability $\psi_k(u)$ (i.e. $w \equiv 1$ and $\delta = 0$), $c_i = c$ for all layers and assume exponential claims ($P(u) = 1 - e^{-\beta u}$). Then for $i = 1,\dots,k-1$, equations (4) take the form

$$(c - a_i)\frac{\partial}{\partial u}\psi_k(u) - \lambda\psi_k(u) + \lambda \left(\int_0^u \psi_k(u-z)\,\mathrm{d}P(z) + 1 - P(u) \right) = 0. \tag{7}$$

Multiplying (7) by the operator $\left(\frac{\partial}{\partial u} + \beta\right)$ gives

$$\left((c-a_i)\frac{\partial^2}{\partial u^2} + ((c-a_i)\beta - \lambda)\frac{\partial}{\partial u} \right) \psi_k(u) = 0, \quad u \in [b_{i-1}, b_i). \tag{8}$$

Using the notation

$$\psi_k(u) = \sum_{i=1}^{k} \psi_k^{(i)}(u) I_{\{b_{i-1} \le u < b_i\}}, \tag{9}$$

where $I_{\mathcal{A}}$ denotes the indicator function of set $\mathcal{A}$ and the function $\psi_k^{(i)}(u)$ is the solution of (8) for $u \in [b_{i-1}, b_i)$ for each i, we obtain

$$\psi_k^{(i)}(u) = A_k^{(i)} + C_k^{(i)} e^{-R_{1,i}u}.$$

Here

$$-R_{1,i} = \frac{\lambda}{c - a_i} - \beta \tag{10}$$

refers to the negative solution of the Lundberg equation (2) for $\delta = 0$.
It remains to establish the constants in the above representation: from $\psi_k(\infty) = 0$, we have $A_k^{(k)} = 0$. From the continuity conditions (5) we immediately have

$$A_k^{(i+1)} + C_k^{(i+1)} e^{-R_{1,i+1}b_i} - A_k^{(i)} - C_k^{(i)} e^{-R_{1,i}b_i} = 0, \quad (i = 1, 2, \ldots, k-1). \tag{11}$$

Using (7) and comparing the coefficients of $e^{-\beta u}$, we get after some manipulations

$$-A_k^{(i+1)} - C_k^{(i+1)} \frac{\beta e^{-R_{1,i+1}b_i}}{\beta - R_{1,i+1}} + A_k^{(i)} + C_k^{(i)} \frac{\beta e^{-R_{1,i}b_i}}{\beta - R_{1,i}} = 0, \quad (i = 1, 2, \ldots, k-1), \tag{12}$$

together with $A_k^{(1)} + C_k^{(1)} \frac{\beta}{\beta - R_{1,1}} = 1$.
Adding (11) and (12) we get

$$C_k^{(i+1)} = \frac{(\beta - R_{1,i+1})R_{1,i}}{(\beta - R_{1,i})R_{1,i+1}} \, e^{(R_{1,i+1} - R_{1,i})b_i} \, C_k^{(i)} = \frac{\lambda - \beta(c - a_i)}{\lambda - \beta(c - a_{i+1})} \, e^{\lambda(\frac{1}{c-a_i} - \frac{1}{c-a_{i+1}})b_i} \, C_k^{(i)}$$

leading to

$$C_k^{(i)} = \frac{(\beta - R_{1,i})R_{1,1}}{(\beta - R_{1,1})R_{1,i}} \, e^{\sum_{j=1}^{i-1}(R_{1,j+1} - R_{1,j})b_j} \, C_k^{(1)} = \frac{\lambda - \beta(c - a_1)}{\lambda - \beta(c - a_i)} \, e^{\lambda \sum_{j=1}^{i-1}(\frac{1}{c-a_j} - \frac{1}{c-a_{j+1}})b_j} \, C_k^{(1)}, \quad (i \ge 1).$$

For $i \in \mathbb{N}$, define

$$L_i := \; R_{1,1} e^{-R_{1,1}b_1} \sum_{j=1}^{i-1} \left(\frac{1}{R_{1,j}} - \frac{1}{R_{1,j+1}} \right) e^{\sum_{l=2}^{j}(b_{l-1} - b_l)R_{1,l}}. \tag{13}$$

Then, again from (11), we have

$$A_k^{(i)} = A_k^{(1)} + C_k^{(1)} \frac{\beta}{\beta - R_{1,1}} L_i = A_k^{(1)} + (1 - A_k^{(1)}) L_i.$$

Hence,

$$A_k^{(1)} = -\frac{L_k}{1 - L_k} \quad \text{and} \quad C_k^{(1)} = \frac{\beta - R_{1,1}}{\beta} \frac{1}{1 - L_k}.$$

Altogether we thus arrive at (9) with

$$\begin{aligned} \psi_k^{(i)}(u) &= \frac{1}{1 - L_k}\Big(L_i - L_k + \frac{\beta - R_{1,i}}{\beta} \frac{R_{1,1}}{R_{1,i}} \, e^{\sum_{j=1}^{i-1}(R_{1,j+1} - R_{1,j})b_j} e^{-R_{1,i}u} \Big) \\ &= \frac{1}{1 - L_k}\Big(L_i - L_k + \frac{R_{1,1}}{R_{1,i}} \, e^{\sum_{j=1}^{i-1}(R_{1,j+1} - R_{1,j})b_j} \psi_{1,i}(u) \Big), \quad (i = 1, \ldots, k), \end{aligned} \tag{14}$$

where according to the notation introduced in Section 1 $\psi_{1,i}(u)$ denotes the ruin probability in the classical (1-layer) model with parameter c_i and a_i. In view of (10) and (13), equation (14) is an explicit formula in terms of the model parameters and hence simplifies the algorithm for the determination of $\psi_k(u)$ in Lin & Sendova [17, Sec.5]. For $k = 2$, formula (14) matches the corresponding expression in

Lin & Pavlova [16]. Also, Zhou [22] (Example 1 in Section 3 of his paper) gives, in the exponential case, numerical results for $\psi_4(u)$ for some values of u for the parameter set

$$\begin{aligned} &\lambda = 1,\, \beta = 1,\, b_1 = 5,\, b_2 = 10,\, b_3 = 15, \\ &a_1 = 0, a_2 = 0.1, a_3 = 0.2, a_4 = 0.3, c_i = 1.4\,(i = 1, \ldots, 4) \end{aligned} \tag{15}$$

(note that here we interpret the decrease of the premium income of [22] as the payout of dividends for constant premiums, which for the ruin process considered in [22] is equivalent).
From (14) one can read off the explicit formula for $\psi_4(u)$ for this case, which (at the corresponding values of u) is in agreement with the numerical values from [22] (all values are rounded to their last digit):

$$\psi_4(u) = \begin{cases} 0.123 + 0.627\, e^{-0.286u} & 0 \le u \le 5 \\ 0.0727 + 0.635\, e^{-0.231u} & 5 \le u \le 10 \\ 0.041 + 0.502\, e^{-0.167u} & 10 \le u \le 15 \\ 0.322\, e^{-0.091u} & 15 \le u \le \infty \end{cases}$$

More generally, Figure 2 depicts the ruin probability as a function of u for $k = 1, \ldots, 4$ (where for instance $k = 2$ means that the third and fourth layer are shifted to infinity). Hence, the figure illustrates the effect of adding additional (upper) layers in the risk model.

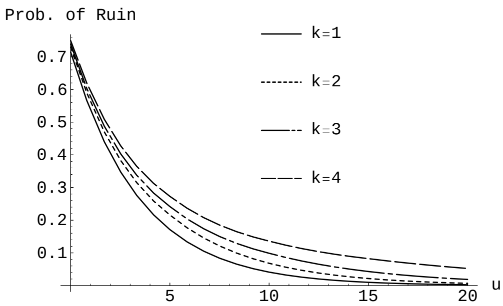

FIGURE 2. Ruin probabilities $\psi_k(u)$ for the parameter set (15) and $k = 1, \ldots, 4$.

Remark 2.2. The main tool in Example 2.1 was the fact that the reformulation of the integro-differential equations as ordinary differential equations allowed to find the solution for each layer separately (that is, instead of using information of the solution from outside the layer (in the integral term), only “local” properties are needed). The remaining task was then to determine the involved coefficients through the continuity assumptions between the solutions in different layers. The latter was achieved through a solution of a system of linear equations (which had a particularly simple structure in the above example). This program can still be carried through when the claim sizes follow an Erlang(n)-distribution, in which case the ordinary differential equations are of order $n + 1$ (but still constant coefficients). Also, more general penalty functions are possible. However, in particular for $\delta > 0$, the solution of the resulting linear system of equations gets highly involved and can then usually only be evaluated numerically.

Another, more general way to approach the system of integro-differential equations (4) with boundary conditions (5) and (6) is to solve it by using Laplace transforms. This can be done in several ways. For the special case of the ruin probability, Kerekesha [15] looked at (4) for arbitrary $u > 0$ by introducing, for each i, a correction term that is zero inside the interval $u \in [b_{i-1}, b_i)$ and corrects for the error outside this interval. Formulating the result in terms of Fourier transforms and exploiting analytical projection properties of the latter, he managed to relate these correction terms to each other and finally express them through truncated Fourier transforms of known quantities (whether these can then be handled depends on the complexity of the considered example). In the following we provide an alternative approach based on the definition of several penalty functions, each “starting” at the height of a layer (this represents an

adaptation of a technique used in Albrecher et al. [2]):
Similarly to (9), use the notation

$$m_k(u) = \sum_{i=1}^{k} m_k^{(i)}(u) I_{\{b_{i-1}\le u<b_i\}},$$

where, for each i, the function $m_k^{(i)}(u)$ is set to 0 for $u \in [0, b_{i-1})$, and for $u \ge b_{i-1}$ is defined by the equation

$$\begin{aligned}(16)\quad & (c_i - a_i)\frac{\partial}{\partial u} m_k^{(i)}(u) - (\lambda+\delta) m_k^{(i)}(u) \\ & + \lambda\left(\int_0^{u-b_{i-1}} m_k^{(i)}(u-z)\,\mathrm{d}P(z) + \int_{u-b_{i-1}}^{u} m_k(u-z)\,\mathrm{d}P(z) + \int_u^\infty w(u,z-u)\,\mathrm{d}P(z)\right) = 0.\end{aligned}$$

For $i = 1,\ldots,k$, define the Laplace transforms $\tilde{m}_k^{(i)}(s) = \int_0^\infty e^{-su} m_k^{(i)}(u)\,\mathrm{d}u$ and $\tilde{p}(s) = \int_0^\infty e^{-su}\,\mathrm{d}P(u)$. Multiplying (16) with e^{-su} and integrating from b_{i-1} to ∞, one then obtains

(17)

$$\tilde{m}_k^{(i)}(s) = \frac{m_k(b_{i-1})(c_i-a_i)e^{-sb_{i-1}} - \lambda\left(\int_{b_{i-1}}^\infty e^{-su}\left(\int_{u-b_{i-1}}^{u} m_k(u-z) + \int_u^\infty w(u,z-u)\right)\mathrm{d}P(z)\,\mathrm{d}u\right)}{(c_i-a_i)s - (\lambda+\delta) + \lambda\,\tilde{p}(s)}.$$

Clearly, this provides an iterative solution algorithm by determining the Laplace transform of the discounted penalty function in each layer, starting with the lowest level. Whenever one can explictly invert this Laplace transform in a certain layer, the solution for the next layer can be determined up to constants. In a final step the boundary conditions can then be used to determine the value of these constants. The following example illustrates such a procedure for a 2-layer model with exponential claim sizes and penalty function $w \equiv 1$, rederiving a result of Lin & Pavlova [16] in this alternative way:

Example 2.2. Consider the time value of ruin in the k-layer model $m_k(u)$ (i.e. $w \equiv 1$ and $\delta \ge 0$). For exponentially distributed claim sizes ($P(u) = 1 - e^{-\beta u}$), we have

$$(18)\quad \tilde{m}_k^{(i)}(s) = \frac{m_k(b_{i-1})(c_i-a_i)e^{-sb_{i-1}} - \lambda\left(\int_{b_{i-1}}^\infty e^{-su}\int_{u-b_{i-1}}^{u} m_k(u-z)\beta e^{-\beta z}\,\mathrm{d}z\,\mathrm{d}u + \frac{e^{-(\beta+s)b_{i-1}}}{\beta+s}\right)}{(c_i-a_i)s - (\lambda+\delta) + \lambda\frac{\beta}{\beta+s}}$$

These Laplace transforms and its inverses may now be calculated up to constants successively, for $i = 1,\ldots,k$. Finally using continuity, we get a system of linear equations to establish the values at the boundaries b_i, $i = 0,\ldots,k-1$.
Let us have a look at the 2-layer model: Equation (18) for $i = 1$ yields

$$\tilde{m}_2^{(1)}(s) = \frac{(s+\beta)\,m_2(0) - \frac{(\beta+\rho_{1,1})(\beta-R_{1,1})}{\beta}}{(s-\rho_{1,1})(s+R_{1,1})}$$

or

$$m_2^{(1)}(u) = \frac{1}{\rho_{1,1}+R_{1,1}}\left((\beta+\rho_{1,1})\left(m_2(0) - \frac{\beta-R_{1,1}}{\beta}\right)e^{\rho_{1,1}u} - (\beta-R_{1,1})\left(m_2(0) - \frac{\beta+\rho_{1,1}}{\beta}\right)e^{-R_{1,1}u}\right)$$

Equation (17) for $i = 2$ gives

$$\tilde{m}_2^{(2)}(s) = e^{-sb_1}\frac{(s+\beta)\,m_2(b_1) - \frac{(\beta+\rho_{1,2})(\beta-R_{1,2})}{(\rho_{1,1}+R_{1,1})}\left(\left(m_2(0)-\frac{\beta-R_{1,1}}{\beta}\right)e^{\rho_{1,1}b_1} - \left(m_2(0)-\frac{\beta+\rho_{1,1}}{\beta}\right)e^{-R_{1,1}b}\right)}{(s-\rho_{1,2})(s+R_{1,2})}.$$

Define $A := \left(\left(m_2(0)-\frac{\beta-R_{1,1}}{\beta}\right)e^{\rho_{1,1}b_1} - \left(m_2(0)-\frac{\beta+\rho_{1,1}}{\beta}\right)e^{-R_{1,1}b}\right)$, then

$$\begin{aligned} m_2^{(2)}(u) = \frac{I_{\{u\ge b_1\}}}{\rho_{1,2}+R_{1,2}}&\left((\beta+\rho_{1,2})\left(m_2(b_1) - \frac{\beta-R_{1,2}}{(\rho_{1,1}+R_{1,1})}A\right)e^{\rho_{1,2}(u-b_1)}\right. \\ &\left. - (\beta-R_{1,2})\left(m_2(b_1) - \frac{\beta+\rho_{1,2}}{(\rho_{1,1}+R_{1,1})}A\right)e^{-R_{1,2}(u-b_1)}\right).\end{aligned}$$

The constant $m_2(0)$ may be calculated by (5) and (6):

$$m_2(0) = \frac{1}{\beta}\frac{(\rho_{1,1}+R_{1,2})(\beta-R_{1,1})e^{\rho_{1,1}b_1}-(R_{1,2}-R_{1,1})(\beta+\rho_{1,1})e^{-R_{1,1}b_1}}{(\rho_{1,1}+R_{1,2})e^{\rho_{1,1}b_1}-(R_{1,2}-R_{1,1})e^{-R_{1,1}b_1}}.$$

Thus, we get

$$m_2(u) = \begin{cases} \frac{1}{\beta}\frac{(R_{1,2}-R_{1,1})e^{-R_{1,1}b_1}(\beta+\rho_{1,1})e^{\rho_{1,1}u}+(\rho_{1,1}+R_{1,2})e^{\rho_{1,1}b_1}(\beta-R_{1,1})e^{-R_{1,1}u}}{(\rho_{1,1}+R_{1,2})e^{\rho_{1,1}b_1}-(R_{1,2}-R_{1,1})e^{-R_{1,1}b_1}}, & u \le b_1 \\ \frac{(\rho_{1,1}+R_{1,1})e^{(\rho_{1,1}-R_{1,1})b_1}}{(\rho_{1,1}+R_{1,2})e^{\rho_{1,1}b_1}-(R_{1,2}-R_{1,1})e^{-R_{1,1}b_1}}\frac{\beta-R_{1,2}}{\beta}e^{-R_{1,2}(u-b_1)}, & u \ge b_1 \end{cases}, \tag{19}$$

which for the case $a_1 = 0$ reproduces the result of Example 7.1 of Lin & Pavlova [16].

In the general case, the corresponding Laplace transforms cannot be explicitly inverted. Another approach that can still lead to explicit results in such cases is outlined in Section 3.

2.2. **Moment-generating function for the discounted dividends.** Let now $\delta > 0$ and define

$$M_k(u,y) = \mathbb{E}\left[\exp\left(y\int_0^{\tau_k(u)} e^{-\delta t}\,\mathrm{d}D_k(t)\right)\middle| U_0 = u\right]$$

as the moment-generating function of the discounted dividend payments up to ruin. Then, using the differential approach again, one can condition on the occurrence of a claim in the next infinitesimal time step, use Taylor expansion and collect terms of order dt to arrive at a partial integro-differential equation (PIDE) for $M_k(u,y)$ in the usual way. Let $W_{k,m}(u,b)$ denote the mth moment of the discounted dividend payments until ruin in the model with k layers. As in Gerber & Shiu [13], the representation

$$M_k(u,y) = 1 + \sum_{m=1}^{\infty}\frac{y^m}{m!}W_{k,m}(u)$$

in the PIDE for $M_k(u,y)$ then gives integro-differential equations for $W_{k,m}(u,b)$:

$$\left((c_i-a_i)\frac{\partial}{\partial u}-\lambda-\delta m\right)W_{k,m}(u)+\lambda\int_0^u W_{k,m}(u-z)\,\mathrm{d}P(z)+m\,a_iW_{k,m-1}(u)=0, \quad u\in[b_{i-1},b_i) \tag{20}$$

for $m\in\mathbb{N}$ and $i=1,\dots,k$, with boundary conditions

$$\lim_{u\to\infty}W_{k,m}(u) = \left(\frac{a_k}{\delta}\right)^m \tag{21}$$

and asking for continuity of the solution across the layers implies

$$\lim_{u\to b_i^+}W_{k,m}(u) = \lim_{u\to b_i^-}W_{k,m}(u). \tag{22}$$

Remark 2.3. In fact the required continuity across layers of the solution above is again automatically implied by a renewal type proof of (20), cf. Section 2.1. For the special case of two layers with $a_1 = 0$, $a_2 = c_1 = c_2$ (i.e. the horizontal barrier strategy), Dickson & Waters [9] used a direct conditioning technique to derive equations for $W_{2,m}(u)$; however, in the general setting the above approach via the moment-generating function is more convenient.

Just like in Section 2.1, equations (20) can be reformulated as ordinary differential equations with constant coefficients in the case of Erlang-distributed claim sizes, allowing to find the solution for each layer separately. Finally the continuity conditions can be used to determine the involved constants through a system of linear equations. The idea is demonstrated in the following example:

Example 2.3. Consider the setting of Example 2.1 with exponential claims and define

$$W_{k,1}(u) = \sum_{i=1}^{k}W_{k,1}^{(i)}(u)I_{\{b_{i-1}\le u<b_i\}}.$$

Multiplying (20) by the operator $\left(\frac{\partial}{\partial u}+\beta\right)$, we get for $i=1,\dots,k$

$$a_i\beta+\left(\frac{\partial}{\partial u}+\beta\right)\left((c_i-a_i)\frac{\partial}{\partial u}-\lambda-\delta\right)W_{k,1}^{(i)}(u)+\lambda\beta^n W_{k,1}^{(i)}(u)=0,\quad u\in[b_{i-1},b_i).$$

Thus, the solutions take the form

$$W_{k,1}^{(i)}(u)=\frac{a_i}{\delta}+A_{k,1}^{(i)}e^{-R_{1,i}u}+B_{k,1}^{(i)}e^{\rho_{1,i}u},\quad u\in[b_{i-1},b_i), \tag{23}$$

for some real constants $A_{k,1}^{(i)}, B_{k,1}^{(i)}$. From (21), we get

$$A_1^{(k)}=0.$$

The continuity conditions give, for $i=1,2,\dots,k-1$,

$$A_{k,1}^{(i)}e^{-R_{1,i}b_i}+B_{k,1}^{(i)}e^{\rho_{1,i}b_i}-A_{k,1}^{(i+1)}e^{-R_{1,i+1}b_i}-B_{k,1}^{(i+1)}e^{\rho_{1,i+1}b_i}=\frac{a_{i+1}}{\delta}-\frac{a_i}{\delta}.$$

Moreover, comparing the coefficients of $e^{-\beta u}$ of these solutions in (20) gives, after some algebra,

$$A_{k,1}^{(i)}\frac{e^{-R_{1,i}b_i}\beta}{\beta-R_{1,i}}+B_{k,1}^{(i)}\frac{e^{\rho_{1,i}b_i}\beta}{\beta+\rho_{1,i}}-A_{k,1}^{(i+1)}\frac{e^{-R_{1,i+1}b_i}\beta}{\beta-R_{1,i+1}}-B_{k,1}^{(i+1)}\frac{e^{\rho_{1,i+1}b_i}\beta}{\beta+\rho_{1,i+1}}=\frac{a_{i+1}}{\delta}-\frac{a_i}{\delta},\quad (i=1,2,\dots,k-1)$$

and

$$A_{k,1}^{(1)}\frac{\beta}{\beta-R_{1,1}}+B_{k,1}^{(1)}\frac{\beta}{\beta+\rho_{1,1}}=\frac{a_1}{\delta}.$$

Hence, we have a system of $2k$ linear equations for the $2k$ coefficients $A_{k,1}^{(i)}, B_{k,1}^{(i)}$ $(i=1,\dots,k)$, leading to the solution (23) on each layer. This solution algorithm generalizes formula (6.14) of Gerber & Shiu [14] obtained for $k=2$.

For larger values of k it usually becomes cumbersome to calculate these closed-form solutions as a function of the model parameters (note that the symmetry of the matrix exploited in the explicit solution for the ruin probability in Example 2.1 is not present in the case of dividend payments). On the other hand, numerical solutions can always be obtained using this approach. For an illustration, see Section 3.3.

3. A recursive approach

As already mentioned in the introduction, a computational disadvantage of the above methods to derive explicit formulae is the fact that usually the obtained recursions among the different layers have to be solved with unknown constants that can only be evaluated numerically a posteriori which makes the method rather infeasible for larger k or more involved quantities under study. In this section we will pursue another approach based on level crossings which can lead to explicit solutions also in more general situations. More concretely, we will set up a recursion with respect to the numbers of layers, meaning that the complete solution of the (classical) 1-layer model is used to obtain the complete solution of the 2-layer model and so on. When feasible, this approach avoids the above problem of numerical evaluation of constants.

3.1. Preliminaries.

Recollect first a few concepts developed for the 1-layer model that will be needed later on.

- The expected discounted payment of 1 at the time of recovery $\tau_0=\inf\{t>\tau\geq 0 : U(t)=0\}$ is given by

$$\Psi_\delta(u):=\mathbb{E}\left[e^{-\delta\tau_0}I_{\{\tau<\infty\}}|U(0)=u\right]=\mathbb{E}\left[e^{-\delta\tau+\rho U(\tau)}I_{\{\tau<\infty\}}|U(0)=u\right]. \tag{24}$$

  (cf. Gerber & Shiu [12, Eqn. (6.4),(6.10)]). Clearly, $\Psi_\delta(u)$ is a discounted penalty function with $w(x,y)=e^{-\rho y}$, where ρ is itself a function of δ ($\Psi_0(u)$ is just the ruin probability). For exponential claim sizes, $\Psi_\delta(u)$ has a simple form (see Gerber & Shiu [12, Eqn. (6.37)]).
- Let $B(u,0,b)$ denote the Laplace transform of τ^+ (which can be interpreted as the expected present value of a payment of 1 at the time when the surplus reaches the level b for the first time provided that ruin has not occurred yet). From Gerber & Shiu [12, Eqn. (6.25)] we have that

$$B(u,0,b):=\mathbb{E}\left[e^{-\delta\tau^+(u,0,b)}\middle|U(0)=u\right]=\frac{e^{\rho u}-\Psi_\delta(u)}{e^{\rho b}-\Psi_\delta(b)}.$$

- Let $f(x,y,t|u)$ denote the joint probability density of $U(\tau_-(u)), |U(\tau(u))|$ and $\tau(u)$. The joint density of the first two quantities is given by

$$f(x,y|u) = \int_0^\infty e^{-\delta t} f(x,y,t|u)\,dt = \begin{cases} \dfrac{\lambda}{c-a} e^{-\rho x} p(x+y), & u = 0, \\ f(x,y|0) \dfrac{e^{\rho x}\Psi_\delta(u-x) - \Psi_\delta(u)}{1-\Psi_\delta(0)}, & 0 < x \le u, \\ f(x,y|0) \dfrac{e^{\rho u} - \Psi_\delta(u)}{1-\Psi_\delta(0)}, & x > u, \end{cases}$$

  see Gerber & Shiu [12, Eqn.(6.34)-(6.36)] (for the renewal model see Dickson & Drekic [8] and for a more general semi-Markovian framework, see Albrecher & Boxma [1]).

Gerber, Lin & Yang [11] recently used the quantity $B(u,0,b)$ to derive a dividend-penalty identity for upwards skip-free strong Markov models. This nice identity allows to construct constant dividend barriers that maximize the expected difference between the present value of the accumulated dividends and the discounted penalty at ruin.

The following observation is a crucial step in our derivations (see Zhou [22] for the case $\delta = 0$).

Lemma 3.1. *For $s_1 \le u < s_2$,*

$$\begin{aligned} (25)\quad & \mathbb{E}\left[e^{-\delta\tau^-(u,s_1,s_2)} H\left(U(\tau^-(u,s_1,s_2)-), |U(\tau^-(u,s_1,s_2))|\right)\right] \\ &= \mathbb{E}\left[e^{-\delta\tau^*(u,s_1,\infty)} H\left(U(\tau^*(u,s_1,\infty)-), |U(\tau^*(u,s_1,\infty))|\right)\right] \\ &- \mathbb{E}\left[e^{-\delta\tau^+(u,s_1,s_2)}\right] \mathbb{E}\left[e^{-\delta\tau^*(s_2,s_1,\infty)} H\left(U(\tau^*(s_2,s_1,\infty)-), |U(\tau^*(s_2,s_1,\infty))|\right)\right], \end{aligned}$$

where H is any function such that the expectations exist.

Proof. The result follows from

$$\begin{aligned} & \mathbb{E}\left[e^{-\delta\tau^-(u,s_1,s_2)} H\left(U(\tau^-(u,s_1,s_2)-), |U(\tau^-(u,s_1,s_2))|\right)\right] \\ &= \mathbb{E}\left[e^{-\delta\tau^*(u,s_1,s_2)} H\left(U(\tau^*(u,s_1,s_2)-), |U(\tau^*(u,s_1,s_2))|\right) I_{\{\overline{U}(\tau^*(u,s_1,s_2))<s_2\}}\right] \end{aligned}$$

and splitting $I_{\{\overline{U}<a\}} = 1 - I_{\{\overline{U}\ge a\}}$, using the strong Markov property at time $\tau^+(u,s_1,s_2)$ in the last term. ☐

3.2. **Time value of "upper exit".** Let $B_k(u,b) := B_k(u,0,b) = \mathbb{E}[e^{-\delta\tau_k^+(u,0,b)}]$ denote the Laplace transform of the stopping time $\tau_k^+(u,0,b)$ in the risk model with k layers. Recall that the notation $B_{k-1}(u,b)$ then corresponds to $B_k(u,b)$ with b_{k-1} shifted to infinity.

Lemma 3.2. *For $k \in \mathbb{N}$ and $\delta > 0$, we have:*

(i)

$$B_k(u,b) = 1 \quad \text{if} \quad u \ge b,$$
$$B_k(u,b) = 0 \quad \text{if} \quad u < 0.$$

(ii) *For $0 \le u < b_{k-1}$:*

$$B_k(u,b) = \begin{cases} B_{k-1}(u,b) & \text{if} \quad b \le b_{k-1}, \\ B_{k-1}(u,b_{k-1}) B_k(b_{k-1},b) & \text{if} \quad b \ge b_{k-1} \end{cases}.$$

(iii) *For $b_{k-1} \le u \le b$:*

$$\begin{aligned} B_k(u,b) &= B_{1,k}(u-b_{k-1}, b-b_{k-1}) + \mathbb{E}\left[e^{-\delta\tau_{1,k}(u-b_{k-1})} B_k\left(b_{k-1} - |U_{1,k}(\tau_{1,k}(u-b_{k-1}))|, b\right)\right] \\ &- B_{1,k}(u-b_{k-1}, b-b_{k-1}) \mathbb{E}\left[e^{-\delta\tau_{1,k}(b-b_{k-1})} B_k\left(b_{k-1} - |U_{1,k}(\tau_{1,k}(b-b_{k-1}))|, b\right)\right]. \end{aligned}$$

Proof. (i) is true by definition. In case (ii), as the processes $U_k(t)$ and $U_{k-1}(t)$ coincide before the first exit of the interval $[0, b_{k-1})$ we have $\tau_k^+(u,0,b) = \tau_{k-1}^+(u,0,b)$ and hence $B_k(u,b) = B_{k-1}(u,b)$ for $0 < b \le b_{k-1}$. For $b \ge b_{k-1}$, the strong Markov property of the surplus process at the stopping time $\tau_{k-1}^+(u,0,b_{k-1})$ gives

$$\mathbb{E}\left[e^{-\delta\tau_k^+(u,0,b)}\right] = \mathbb{E}\left[e^{-\delta(\tau_{k-1}^+(u,0,b_{k-1})+\tau_k^+(b_{k-1},0,b))}\right] = B_{k-1}(u,b_{k-1})B_k(b_{k-1},b),$$

establishing (ii).
Let now $b_{k-1} \le u < b$. Then

$$\begin{aligned}
\mathbb{E}\left[e^{-\delta\tau_k^+(u,0,b)}\right] &= \mathbb{E}\left[e^{-\delta(\tau_k^*(u,b_{k-1},b)+\tau_k^+(U_k(\tau_k^*(u,b_{k-1},b)),0,b))}\right] \\
&= \mathbb{E}\left[e^{-\delta(\tau_k^*(u,b_{k-1},b))}I_{\{U_k(\tau_k^*(u,b_{k-1},b))=b\}}\right] \\
&\quad + \mathbb{E}\left[e^{-\delta(\tau_k^*(u,b_{k-1},b)+\tau_k^+(U_k(\tau_k^*(u,b_{k-1},b)),0,b))}I_{\{\overline{U_k}(\tau_k^*(u,b_{k-1},b))<b\}}\right] \\
&= B_{1,k}(u-b_{k-1},b-b_{k-1}) + \mathbb{E}\left[e^{-\delta\tau_{1,k}(u-b_{k-1})}B_k(b_{k-1}-|U_{1,k}(\tau_{1,k}(u-b_{k-1}))|,b)\right] \\
&\quad - B_{1,k}(u-b_{k-1},b-b_{k-1})\mathbb{E}\left[e^{-\delta\tau_{1,k}(b-b_{k-1})}B_k(b_{k-1}-|U_{1,k}(\tau_{1,k}(b-b_{k-1}))|,b)\right],
\end{aligned}$$

where the last term corrects for those trajectories that arrive at b before dropping below b_{k-1} (since $\tau_k^-(u,b_{k-1},b)$ is replaced by the ruin time $\tau_{1,k}(u-b_{k-1})$). □

From Lemma 3.2 (ii) at $u=0$, we get

$$B_k(0,b) = B_{k-1}(0,b_{k-1})\,B_k(b_{k-1},b). \tag{26}$$

On the other hand, we have from Lemma 3.2 (iii) at $u = b_{k-1}$ that

$$\begin{aligned}
B_k(b_{k-1},b) &= B_{1,k}(0,b-b_{k-1}) + \mathbb{E}\left[e^{-\delta\tau_{1,k}(0)}B_k\left(b_{k-1}-|U_{1,k}(\tau_{1,k}(0))|,b\right)\right] \\
&\quad - B_{1,k}(0,b-b_{k-1})\mathbb{E}\left[e^{-\delta\tau_{1,k}(b-b_{k-1})}B_k\left(b_{k-1}-|U_{1,k}(\tau_{1,k}(b-b_{k-1}))|,b\right)\right].
\end{aligned}$$

Using Lemma 3.2 (ii) inside the above expectations, we obtain

$$B_k(b_{k-1},b) = B_{1,k}(0,b-b_{k-1}) + H_k(0)\frac{B_k(0,b)}{B_{k-1}(0,b)} - B_{1,k}(0,b-b_{k-1})H_k(b-b_{k-1})\frac{B_k(0,b)}{B_{k-1}(0,b)}, \tag{27}$$

where

$$H_k(v) = \mathbb{E}\left[e^{-\delta\tau_{1,k}(v)}B_{k-1}(b_{k-1}-|U_{1,k}(\tau_{1,k}(v))|,b_{k-1})\right], \quad v \ge 0.$$

This finally gives

$$B_k(0,b) = \frac{B_{1,k}(0,b-b_{k-1})B_{k-1}(0,b_{k-1})}{1-H_k(0)+B_{1,k}(0,b-b_{k-1})H_k(b-b_{k-1})}, \tag{28}$$

providing a formula for $B_k(0,b)$ that solely depends on quantities from lower layers. Note that in the numerator we have the expression for a path from 0 to b_{k-1} and then going on to b, and the numerator gives the correction term for the possibility to fall again below b_{k-1} in between. This immediately gives now a formula for $B_k(u,b)$ in terms of quantities from lower layers:

Proposition 3.1. *For $k = 2,3,\ldots,$ we have the following formulae:*

(i) *For $0 \le u < b_{k-1} < b$:*

$$B_k(u,b) = \frac{B_{1,k}(0,b-b_{k-1})B_{k-1}(u,b_{k-1})}{1-H_k(0)+B_{1,k}(0,b-b_{k-1})H_k(b-b_{k-1})}.$$

(ii) *For $b_{k-1} \le u < b$:*

$$B_k(u,b) = \frac{B_{1,k}(u-b_{k-1},b-b_{k-1})\Big(1-H_k(0)\Big)+H_k(u-b_{k-1})B_{1,k}(0,b-b_{k-1})}{1-H_k(0)+B_{1,k}(0,b-b_{k-1})H_k(b-b_{k-1})}.$$

(iii) *For $b < b_{k-1}$: $B_k(u,b) = B_{k-1}(u,b)$.*

Proof: Assertion (i) follows from Lemma 3.2(ii), (26) and (28). Assertion (ii) follows from substitution of assertion (i) inside the expectations in Lemma 3.2(iii). Finally, assertion (iii) is just Lemma 3.2(ii). □

We now work out an explicit expression for $B_k(u,b)$ for exponential claim sizes. To that end, introduce the following notation:

$$\begin{aligned}
Z_i(u) &= (\beta+\rho_{1,i})e^{\rho_{1,i}u} - (\beta - R_{1,i})e^{-R_{1,i}u},\\
\overline{Z}_i(u) &= e^{\rho_{1,i}u} - e^{-R_{1,i}u},\\
\widehat{Z}_i(u) &= e^{\rho_{1,i}u}/(\beta+\rho_{1,i}) - e^{-R_{1,i}u}/(\beta - R_{1,i}).
\end{aligned}$$

Proposition 3.2. *Let $k \in \mathbb{N}$ be the number of layers, $P(u) = 1 - e^{-\beta u}$ and $c_i = c$ for $i = 1, \dots, k$. Consider the following recursions:*

$$\begin{aligned}
N_1(u) &= Z_1(u), &\quad N_i(u) &= N_{i-1}(b_{i-1})Z_i(u-b_{i-1}) - (\beta - R_{1,i})(\beta+\rho_{1,i})M_i\overline{Z}_i(u-b_{i-1}) \quad \text{for } i \geq 2\\
M_2 &= \overline{Z}_1(b_1), &\quad M_{i+1} &= N_{i-1}(b_{i-1})\overline{Z}_i(b_i-b_{i-1}) - (\beta - R_{1,i})(\beta+\rho_{1,i})M_i\widehat{Z}_i(b_i-b_{i-1}) \quad \text{for } i \geq 2.
\end{aligned}$$

Let $b \geq b_{k-1}$. Then, for $i = 1, \dots, k$ and $b_{i-1} \leq u < b_i$ and $u < b$, we have

$$B_k(u,b) = \frac{N_i(u)}{N_k(b)} \prod_{j=i+1}^{k} (\rho_{1,j} + R_{1,j}), \tag{29}$$

where the convention $\prod_{j=k+1}^{k} \cdot = 1$ is used.

From the previous considerations it is clear that this determines the expression $B_k(u,b)$ for all the other combinations of u and b, too.

Proof. In a first step, we show by induction that for all $k \geq 1$ the auxiliary quantity M_k fulfills

$$e^{\beta b_k} M_{k+1} = \sum_{i=1}^{k} \int_{b_{i-1}}^{b_i} N_i(y)\, e^{\beta y}\, \mathrm{d}y \prod_{j=i+1}^{k} (\rho_{1,j} + R_{1,j}). \tag{30}$$

Trivially, we have $e^{\beta b_1} M_2 = e^{\beta b_1}(e^{\rho_{1,1}b_1} - e^{-R_{1,1}b_1}) = \int_0^{b_1} Z_1(y)e^{\beta y}\,\mathrm{d}y$, which is (30) for $k = 1$. Assume then that (30) holds for $k-1$, then we have for k

$$\begin{aligned}
e^{\beta b_k} M_{k+1} &= e^{\beta b_k}\Big(\overline{Z}_k(b_k - b_{k-1})N_{k-1}(b_{k-1}) - (\beta - R_{1,k})(\beta+\rho_{1,k})M_k\widehat{Z}_k(b_k-b_{k-1})\Big)\\
&= e^{\beta b_k}\overline{Z}_k(b_k - b_{k-1})N_{k-1}(b_{k-1}) - (\rho_{1,k}+R_{1,k})e^{\beta b_{k-1}}M_k\\
&\quad - e^{\beta b_k}(\beta - R_{1,k})(\beta+\rho_{1,k})M_k\widehat{Z}_k(b_k-b_{k-1}) + (\rho_{1,k}+R_{1,k})e^{\beta b_{k-1}}M_k\\
&= \int_{b_{k-1}}^{b_k} N_k(y)\, e^{\beta y}\,\mathrm{d}y + (\rho_{1,k}+R_{1,k})e^{\beta b_{k-1}}M_k\\
&= \sum_{i=1}^{k} \int_{b_{i-1}}^{b_i} N_i(y)\, e^{\beta y}\,\mathrm{d}y \prod_{j=i+1}^{k}(\rho_{1,j}+R_{1,j}),
\end{aligned}$$

where the induction hypothesis was used in the last line. Now we can prove (29) by another induction: First, we know from Gerber & Shiu [12] that

$$B_1(u,b) = \frac{N_1(u)}{N_1(b)} = \frac{Z_1(u)}{Z_1(b)},$$

which coincides with (29) for $k = 1$. Assume then that (29) holds for k. We start by calculating $H_{k+1}(0)$. Observe that $H_i(v) = H_i(0)e^{-R_{1,i}v}$ (see Gerber & Shiu [12, Eqn.(5.42)]) and furthermore

$$H_i(0) = (\beta - R_{1,i})\int_0^{b_{i-1}} B_{i-1}(b_{i-1}-y, b_{i-1})e^{-\beta y}\,\mathrm{d}y,$$

for all $i \in \mathbb{N}$. Thus by the validity of (29) and (30) for k,

$$H_{k+1}(0) = (\beta - R_{1,k+1})e^{-\beta b_k} \int_0^{b_k} B_k(y, b_k)e^{\beta y}\, \mathrm{d}y = \frac{(\beta - R_{1,k+1})M_{k+1}}{N_k(b_k)}.$$

The denominator in Proposition 3.1 (i) and (ii) for $k+1$ layers is now given by

$$\begin{aligned} &1 - H_{k+1}(0) + B_{1,k+1}(0, b - b_k)H_{k+1}(b - b_k) \\ &= \frac{(N_k(b_k) - (\beta - R_{1,k+1})M_{k+1})Z_{k+1}(b - b_k) + (\rho_{1,k+1} + R_{1,k+1})(\beta - R_{1,k+1})M_{k+1}e^{-R_{1,k+1}(b-b_k)}}{N_k(b_k)Z_{k+1}(b - b_k)} \\ &= \frac{N_{k+1}(b)}{N_k(b_k)Z_{k+1}(b - b_k)}, \end{aligned}$$

since

$$Z_{k+1}(b - b_k) - (\rho_{1,k+1} + R_{1,k+1})e^{-R_{1,k+1}(b-b_k)} = (\beta + \rho_{1,k+1})\overline{Z}_{k+1}(b - b_k).$$

Now, Proposition 3.1 (ii) and a little algebra gives for $u \geq b_k$:

$$B_{k+1}(u, b) = \frac{N_k(b_k)Z_{k+1}(u - b_k) - (\beta - R_{1,k+1})(\beta + \rho_{1,k+1})M_k\overline{Z}_{k+1}(u - b_k)}{N_{k+1}(b)} = \frac{N_{k+1}(u)}{N_{k+1}(b)}.$$

Finally, Proposition 3.1 (i) and (29) for $i = 1, \ldots, k$ gives for $b_{i-1} \leq u < b_i$:

$$B_{k+1}(u, b) = \frac{\frac{N_i(u)(\rho_{1,k+1}+R_{1,k+1})\prod_{j=i+1}^{k}(\rho_{1,j}+R_{1,j})}{N_k(b_k)Z_{k+1}(b-b_k)}}{\frac{N_{k+1}(b)}{N_k(b_k)Z_{k+1}(b-b_k)}} = \frac{N_i(u)\prod_{j=i+1}^{k+1}(\rho_{1,j} + R_{1,j})}{N_{k+1}(b)}.$$

$\square$

Example 3.1. For $k = 2$ and exponential claim amounts we obtain the explicit expression $B_2(u, b) = A_1/A_2$ with

$$A_1 = (\rho_{1,2} + R_{1,2})\left((\beta + \rho_{1,1})e^{\rho_{1,1}u} - (\beta - R_{1,1})e^{-R_{1,1}u}\right),$$

for $0 \leq u < b_1$,

$$\begin{aligned} A_1 = &\left((\beta + \rho_{1,1})e^{\rho_{1,1}b_1} - (\beta - R_{1,1})e^{-R_{1,1}b_1}\right)\left((\beta + \rho_{1,2})e^{\rho_{1,2}(u-b_1)} - (\beta - R_{1,2})e^{-R_{1,2}(u-b_1)}\right) \\ &- (\beta - R_{1,2})(\beta + \rho_{1,2})(e^{\rho_{1,1}b_1} - e^{-R_{1,1}b_1})(e^{\rho_{1,2}(u-b_1)} - e^{-R_{1,2}(u-b_1)}), \end{aligned}$$

for $u \geq b_1$, and

$$\begin{aligned} A_2 = &\left((\beta + \rho_{1,1})e^{\rho_{1,1}b_1} - (\beta - R_{1,1})e^{-R_{1,1}b_1}\right)\left((\beta + \rho_{1,2})e^{\rho_{1,2}(b-b_1)} - (\beta - R_{1,2})e^{-R_{1,2}(b-b_1)}\right) \\ &- (\beta - R_{1,2})(\beta + \rho_{1,2})(e^{\rho_{1,1}b_1} - e^{-R_{1,1}b_1})(e^{\rho_{1,2}(b-b_1)} - e^{-R_{1,2}(b-b_1)}). \end{aligned}$$

For illustrative purposes, we give the corresponding explicit formula for the 4-layer model with parameters (15) and $\delta = 0.01$ (with $b = 20$):

$$B_4(u, 20) = \begin{cases} 0 & u < 0 \\ 0.483\, e^{0.024u} - 0.329\, e^{-0.302u} & 0 \leq u \leq 5 \\ 0.475\, e^{0.030u} - 0.290\, e^{-0.253u} & 5 \leq u \leq 10 \\ 0.413\, e^{0.0417u} - 0.0498\, e^{-0.2u} & 10 \leq u \leq 15 \\ 0.537\, e^{-0.145u} + 0.276\, e^{0.063u} & 15 \leq u \leq 20 \\ 1 & u \geq 20 \end{cases}$$

Figure 3 depicts $B_k(u, 20)$ as a function of u for $k = 1, \ldots, 4$ for this parameter setting (where, again, for each k the higher layers are shifted to infinity).

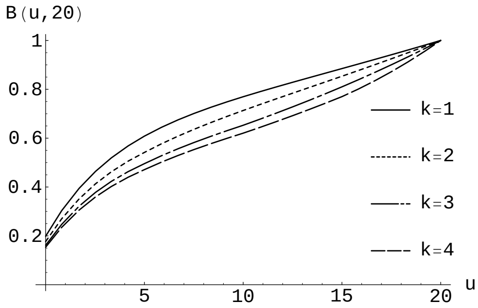

FIGURE 3. $B_k(u,20)$ for the parameter set (15) and $k=1,\dots,4$.

3.3. **The expected discounted dividends.** We now show how to use the quantities of the previous section for the calculation of the expected discounted dividends $W_{k,1}$ in the k-layer model. In the sequel $W_{(1,k),1}$ denotes the expected discounted dividends in a 1-layer model with parameters from the k-th layer. Let $0 \le u < b_{k-1}$ and condition on the event of either reaching b_{k-1} or getting ruined:

$$W_{k,1}(u) = \mathbb{E}\left[\int_0^{\tau_k^*(u,0,b_{k-1})} e^{-\delta t}\,\mathrm{d}D_k(t)\right] + \mathbb{E}\left[\int_{\tau_k^*(u,0,b_{k-1})}^{\tau_k(u)} e^{-\delta t}\,\mathrm{d}D_k(t)\right] := I_1 + I_2 \tag{31}$$

The second term only gives a non-zero contribution in case $\tau_k^*(u,0,b_{k-1}) = \tau_k^+(u,0,b_{k-1})$, which (using $\tau_k^+(u,0,b_{k-1}) = \tau_{k-1}^+(u,0,b_{k-1})$) leads to

$$I_2 = \mathbb{E}\left[e^{-\delta\tau_{k-1}^+(u,0,b_{k-1})}\right] W_{k,1}(b_{k-1}) = B_{k-1}(u,b_{k-1})W_{k,1}(b_{k-1}).$$

In the first term in (31) we replace $\tau_k^*(u,0,b_{k-1}) = \tau_{k-1}^*(u,0,b_{k-1})$ by the ruin time $\tau_{k-1}(u)$ and correct, for all trajectories of the process that exceed b_{k-1}, for the dividend contribution after this event to obtain

$$\begin{aligned} I_1 &= \mathbb{E}\left[\int_0^{\tau_{k-1}^*(u,0,b_{k-1})} e^{-\delta t}\,\mathrm{d}D_{k-1}(t)\right] \\ &= \mathbb{E}\left[\int_0^{\tau_{k-1}(u)} e^{-\delta t}\,\mathrm{d}D_{k-1}(t)\right] - \mathbb{E}\left[e^{-\delta\tau_{k-1}^+(u,0,b_{k-1})}\right]\mathbb{E}\left[\int_{\tau_{k-1}^+(u,0,b_{k-1})}^{\tau_{k-1}(b_{k-1})} e^{-\delta t}\,\mathrm{d}D_{k-1}(t)\right] \\ &= W_{k-1,1}(u) - B_{k-1}(u,b_{k-1})W_{k-1,1}(b_{k-1}). \end{aligned}$$

Hence

$$W_{k,1}(u) = W_{k-1,1}(u) + B_{k-1}(u,b_{k-1})\Big(W_{k,1}(b_{k-1}) - W_{k-1,1}(b_{k-1})\Big).$$

Letting $u=0$ in the previous equation, we arrive at

$$W_{k,1}(u) = W_{k-1,1}(u) + \frac{B_{k-1}(u,b_{k-1})}{B_{k-1}(0,b_{k-1})}\Big(W_{k,1}(0) - W_{k-1,1}(0)\Big). \tag{32}$$

Above the upper threshold, $u \ge b_{k-1}$, we have

$$\begin{aligned} W_{k,1}(u) &= W_{(1,k),1}(u-b_{k-1}) + \mathbb{E}\left[e^{-\delta\tau_{1,k}(u-b_{k-1})}W_{k,1}\left(b_{k-1} - \Big|U_{1,k}\Big(\tau_{1,k}(u-b_{k-1})\Big)\Big|\right)\right] \\ &= W_{(1,k),1}(u-b_{k-1}) + \mathbb{E}\left[e^{-\delta\tau_{1,k}(u-b_{k-1})}W_{k-1,1}\left(b_{k-1} - \Big|U_{1,k}\Big(\tau_{1,k}(u-b_{k-1})\Big)\Big|\right)\right] \\ &\quad + \mathbb{E}\left[e^{-\delta\tau_{1,k}(u-b_{k-1})}B_{k-1}(b_{k-1} - \Big|U_{1,k}\Big(\tau_{1,k}(u-b_{k-1})\Big)\Big|, b_{k-1})\right]\frac{W_{k,1}(0) - W_{k-1,1}(0)}{B_{k-1}(0,b_{k-1})}, \end{aligned} \tag{33}$$

where (32) was used in the last step. Now, evaluating (33) and (32) at $u = b_{k-1}$, we get by continuity

$$\frac{W_{k,1}(0) - W_{k-1,1}(0)}{B_{k-1}(0,b_{k-1})} = \frac{\mathbb{E}\left[e^{-\delta\tau_{1,k}(0)}W_{k-1,1}\left(b_{k-1} - |U_{1,k}(\tau_{1,k}(0))|\right)\right] + W_{(1,k),1}(0) - W_{k-1,1}(b_{k-1})}{1 - H_k(0)}$$

and subsequently

$$W_{k,1}(u) = \begin{cases} W_{k-1,1}(u) + \frac{B_{k-1}(u,b_{k-1})}{1-H_k(0)} \left(\mathbb{E}\left[e^{-\delta\tau_{1,k}(0)} W_{k-1,1}\left(b_{k-1} - |U_{1,k}(\tau_{1,k}(0))|\right)\right] + W_{(1,k),1}(0) - W_{k-1,1}(b_{k-1})\right), \\ \qquad\qquad \text{for } \; 0 \le u < b_{k-1} \\ W_{(1,k),1}(u-b_{k-1}) + \mathbb{E}\left[e^{-\delta\tau_{1,k}(u-b_{k-1})} W_{k-1,1}\left(b_{k-1} - |U_{1,k}(\tau_{1,k}(u-b_{k-1}))|\right)\right] \\ \quad + \frac{H_k(u-b_{k-1})}{1-H_k(0)} \left(\mathbb{E}\left[e^{-\delta\tau_{1,k}(0)} W_{k-1,1}\left(b_{k-1} - |U_{1,k}(\tau_{1,k}(0))|\right)\right] + W_{(1,k),1}(0) - W_{k-1,1}(b_{k-1})\right), \\ \qquad\qquad \text{for } \; u \ge b_{k-1} \end{cases}.$$

The latter formula gives, for all values of u, $W_{k,1}(u)$ explicitly as a function of $W_{k-1,1}(u)$ only, i.e. we can recursively determine the solution "bottom-up" (whereas in the differential approach of Section 2 it was necessary to simultaneously go through all the layers to obtain the solution through a system of linear equations). Hence this approach leads to analytic solution formulae whenever $B_j(u,b)$ $(j = 1, \ldots, k)$ is analytically available. From Proposition 3.2 we know that this is for instance the case for exponential claim sizes:

Example 3.2. For Exp(β) claim sizes, one has

$$W_{1,1}(u) = \frac{a_1}{\delta}\left(1 - \mathbb{E}\left[e^{-\delta\tau_1(u)}\right]\right) = \frac{a_1}{\delta}\left(1 - \frac{\beta - R_{1,1}}{\beta} e^{-R_{1,1}u}\right)$$

(where the formula for the Laplace transform of τ_1 can for instance be found in Gerber & Shiu [12]). Moreover, together with Gerber & Shiu [12, Eqn.(5.42)],

$$\mathbb{E}\left[e^{-\delta\tau_{1,2}(u-b_1)} W_{1,1}\left(b_1 - |U_{1,2}(\tau_{1,2}(u-b_1))|\right)\right] = \frac{a_1}{\delta}\frac{\beta - R_{1,2}}{\beta} e^{-R_{1,2}(u-b_1)}\left(1 - e^{-R_{1,1}b_1}\right)$$

(here one uses the fact that $\tau_{1,2}$ and $U_{1,2}(\tau_{1,2})$ are independent for exponential claims). In the 2-layer case this leads to

$$W_{2,1}(u) = \begin{cases} \frac{a_1}{\delta}\left(1 - \frac{\beta - R_{1,1}}{\beta} e^{-R_{1,1}u}\right) + \frac{(\beta+\rho_{1,1})e^{\rho_{1,1}u} - (\beta - R_{1,1})e^{-R_{1,1}u}}{(R_{1,2}+\rho_{1,1})e^{\rho_{1,1}b_1} - (R_{1,2}-R_{1,1})e^{-R_{1,1}b_1}} \left(\frac{a_2-a_1}{\delta}\frac{R_{1,2}}{\beta} + \frac{a_1}{\delta}\frac{R_{1,2}-R_{1,1}}{\beta} e^{-R_{1,1}b_1}\right) \\ \qquad\qquad \text{for } \; 0 \le u < b_1 \\ \frac{a_2}{\delta} - \frac{1}{\delta}\frac{(a_2-a_1)\left(\rho_{1,1}e^{\rho_{1,1}b_1} + R_{1,1}e^{-R_{1,1}b_1}\right) + a_1(\rho_{1,1}+R_{1,1})e^{(\rho_{1,1}-R_{1,1})b_1}}{(R_{1,2}+\rho_{1,1})e^{\rho_{1,1}b_1} - (R_{1,2}-R_{1,1})e^{-R_{1,1}b_1}} \frac{(\beta - R_{1,2})}{\beta} e^{-R_{1,2}(u-b_1)} \\ \qquad\qquad \text{for } \; u \ge b_1 \end{cases},$$

which for the special case $a_1 = 0$ coincides with formula (6.15) of Gerber & Shiu [14].
Coming back to the example of the 4-layer model with parameters (15) and $\delta = 0.01$, we obtain for this case

$$W_{2,1}(u) = \begin{cases} e^{-0.302u}\left(-15.841 + 23.238\, e^{0.326u}\right) & 0 \le u < 5 \\ 10 - 10.483\, e^{-0.505u} - 8.316\, e^{-0.253u} + 13.609\, e^{0.031u} & 5 \le u < 10 \\ 20 - 195.233\, e^{-0.669u} - 0.638\, e^{-0.2u} + 5.299\, e^{0.0417u} & 10 \le u < 15 \\ 30 - 10885.564\, e^{-0.751u} & u \ge 15 \end{cases}$$

Figure 4 depicts $W_{k,1}(u)$ as a function of u for $k = 2, \ldots, 4$ (again, for each k the higher layers are shifted to infinity).

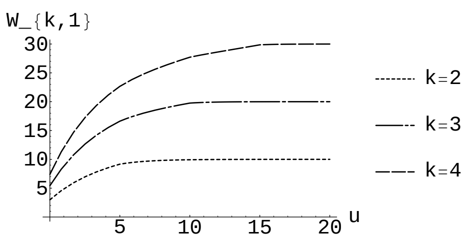

FIGURE 4. $W_{k,1}(u)$ for the parameter set (15) and $k = 2, 3, 4$.

3.4. **The discounted penalty function.** In this section we consider the recursive approach for the discounted penalty function defined in (3). For $0 \le u \le b_{k-1}$, the same reasoning as in Section 3.3 yields

$$\begin{aligned} m_k(u) &= \mathbb{E}\left[e^{-\delta\tau_k^+(u,0,b_{k-1})}\right] m_k(b_{k-1}) + m_{k-1}(u) - B_{k-1}(u,b_{k-1})m_{k-1}(b_{k-1}) \\ &= m_{k-1}(u) + B_{k-1}(u,b_{k-1})\Big(m_k(b_{k-1}) - m_{k-1}(b_{k-1})\Big). \end{aligned}$$

Evaluating this formula at $u = 0$, one obtains

$$m_k(u) = m_{k-1}(u) + \frac{B_{k-1}(u,b_{k-1})}{B_{k-1}(0,b_{k-1})}\Big(m_k(0) - m_{k-1}(0)\Big), \qquad 0 \le u \le b_{k-1}. \tag{34}$$

For $u \ge b_{k-1}$ we distinguish whether ruin occurs directly from the kth layer or some lower layer is reached first, i.e.

$$\begin{aligned} m_k(u) &= \mathbb{E}\left[e^{-\delta\tau_{1,k}(u-b_{k-1})} m_k(b_{k-1} - |U_{1,k}(\tau_{1,k}(u-b_{k-1}))|) I_{\{|U_{1,k}(\tau_{1,k}(u-b_{k-1}))|\le b_{k-1}\}}\right] \\ &+ \mathbb{E}\left[e^{-\delta\tau_{1,k}(u-b_{k-1})} w(b_{k-1} + U_{1,k}(\tau_{1,k}(u-b_{k-1})-), b_{k-1} - |U_{1,k}(\tau_{1,k}(u-b_{k-1}))|) I_{\{|U_{1,k}(\tau_{1,k}(u-b_{k-1}))|> b_{k-1}\}}\right]. \end{aligned}$$

or, using (34),

$$\begin{aligned} m_k(u) &= \mathbb{E}\left[e^{-\delta\tau_{1,k}(u-b_{k-1})} m_{k-1}(b_{k-1} - |U_{1,k}(\tau_{1,k}(u-b_{k-1}))|) I_{\{|U_{1,k}(\tau_{1,k}(u-b_{k-1}))|\le b_{k-1}\}}\right] \\ &+ \mathbb{E}\left[e^{-\delta\tau_{1,k}(u-b_{k-1})} B_{k-1}(b_{k-1} - |U_{1,k}(\tau_{1,k}(u-b_{k-1}))|, b_{k-1}) I_{\{|U_{1,k}(\tau_{1,k}(u-b_{k-1}))|\le b_{k-1}\}}\right] \frac{m_k(0) - m_{k-1}(0)}{B_{k-1}(0,b_{k-1})} \\ &+ \mathbb{E}\left[e^{-\delta\tau_{1,k}(u-b_{k-1})} w(b_{k-1} + U_{1,k}(\tau_{1,k}(u-b_{k-1})-), b_{k-1} - |U_{1,k}(\tau_{1,k}(u-b_{k-1}))|) I_{\{|U_{1,k}(\tau_{1,k}(u-b_{k-1}))|> b_{k-1}\}}\right]. \end{aligned}$$

Note the indicator function in the second line above can be omitted, since if the condition in the indicator is not fulfilled, then the whole line is zero due to Lemma 3.2(i). Now, by the continuity of $m_k(u)$ at $u = b_{k-1}$ one can equate the above expression and (34) at that point to obtain

$$\frac{m_k(0) - m_{k-1}(0)}{B_{k-1}(0,b_{k-1})} = \frac{A_k(0) - m_{k-1}(b_{k-1})}{1 - H_k(0)}$$

with

$$\begin{aligned} A_k(v) := \mathbb{E}\Big[e^{-\delta\tau_{1,k}(v)} \big(& m_{k-1}(b_{k-1} - |U_{1,k}(\tau_{1,k}(v))|) I_{\{|U_{1,k}(\tau_{1,k}(v))|\le b_{k-1}\}} + \\ & w(b_{k-1} + U_{1,k}(\tau_{1,k}(v)-), b_{k-1} - |U_{1,k}(\tau_{1,k}(v))|) I_{\{|U_{1,k}(\tau_{1,k}(v))|> b_{k-1}\}}\big)\Big]. \end{aligned}$$

Finally, we arrive at

$$m_k(u) = \begin{cases} m_{k-1}(u) + \frac{A_k(0) - m_{k-1}(b_{k-1})}{1 - H_k(0)} B_{k-1}(u,b_{k-1}), & \text{for } 0 \le u \le b_{k-1} \\ A_k(u - b_{k-1}) + \frac{A_k(0) - m_{k-1}(b_{k-1})}{1 - H_k(0)}\; H_k(u - b_{k-1}), & \text{for } \; u \ge b_{k-1}. \end{cases} \tag{35}$$

So, again it is possible to express the k-layer solution solely through quantities from the $(k-1)$-layer case.

Example 3.3. For Exp(β)-claim sizes and $w \equiv 1$ (i.e. considering the time value of ruin), the 1-layer case was studied in Gerber & Shiu [12] giving

$$m_1(u) = \mathbb{E}\left[e^{-\delta\tau_1(u)}\right] = \frac{\beta - R_{1,1}}{\beta} e^{-R_{1,1}u}.$$

With some effort it is possible to calculate (cf. notation of Proposition 3.2

$$\frac{m_2(0) - m_1(0)}{B_1(0,b_1)} B_1(u,b_1) = \frac{1}{\beta}\frac{(R_{1,1} - R_{1,2})e^{-R_{1,1}b_1}}{N_1(b_1) - (\beta - R_{1,2})M_2} Z_1(u)$$

Thus, (35) yields:

$$m_2(u) = \begin{cases} \frac{\beta - R_{1,1}}{\beta} e^{-R_{1,1}u} + \frac{1}{\beta}\frac{(R_{1,1} - R_{1,2})e^{-R_{1,1}b_1}}{Z_1(b_1) - (\beta - R_{1,2})\overline{Z}_1(b_1)} Z_1(u) & \text{for } 0 \le u < b_1, \\ \frac{Z_1(b_1) - (\beta - R_{1,1})\overline{Z}_1(b_1)}{Z_1(b_1) - (\beta - R_{1,2})\overline{Z}_1(b_1)} e^{(R_{1,2} - R_{1,1})b_1} m_{1,2}(u), & \text{for } u \ge b_1, \end{cases}$$

which coincides with formula (19).
In the following step, the advantage of the recursive method becomes particularly transparent: For $b_{i-1} \leq u < b_i, (i = 1, 2)$, we get:

$$\frac{m_3(0) - m_2(0)}{B_2(0, b_2)} B_2(u, b_2) = \frac{(N_1(b_1) - (\beta - R_{1,1})M_2)}{(N_1(b_1) - (\beta - R_{1,2})M_2)} \frac{(R_{1,2} - R_{1,3})e^{-R_{1,2}(b_2-b_1)}}{(N_2(b_2) - (\beta - R_{1,3})M_3)} \frac{N_i(u)}{\beta} \prod_{j=i+1}^{2} (\rho_{1,j} + R_{1,j}),$$

and thus, for $u \geq b_2$, we have

$$m_3(u) = \frac{\prod_{j=1}^{2} (N_j(b_j) - (\beta - R_{1,j})M_{j+1})}{\prod_{j=1}^{2} (N_j(b_j) - (\beta - R_{1,j+1})M_{j+1})} e^{\sum_{j=1}^{2}(R_{1,j+1}-R_{1,j})b_j} m_{1,3}(u).$$

For notational convenience, define $N_0(u) = 1$ and $M_1 = 0$, then for $k \geq 3$, we can derive

$$\begin{aligned} &\frac{m_k(0) - m_{k-1}(0)}{B_{k-1}(0, b_{k-1})} B_{k-1}(u, b_{k-1}) \\ &= \frac{\prod_{j=1}^{k-2}(N_j(b_j) - (\beta - R_{1,j})M_{j+1})}{\prod_{j=1}^{k-1}(N_j(b_j) - (\beta - R_{1,j+1})M_j)} (R_{1,k-1} - R_{1,k})e^{-R_{1,k-1}(b_{k-1}-b_{k-2})} \frac{N_i(u)}{\beta} \prod_{j=i+1}^{k-1} (\rho_{1,j} + R_{1,j}). \end{aligned}$$

Using

$$N_j(b_j) - (\beta - R_{1,j})M_{j+1} = (R_{1,j} + \rho_{1,j})e^{\rho_{1,j}(b_j-b_{j-1})}(N_{j-1}(b_{j-1}) - (\beta - R_{1,j})M_j),$$

we get for $b_{i-1} \leq u < b_i, (i = 1, \ldots, k-1)$:

$$\begin{aligned} Y_k^{(i)} &:= \frac{m_k(0) - m_{k-1}(0)}{B_{k-1}(0, b_{k-1})} \frac{B_{k-1}(u, b_{k-1})}{N_i(u)} \\ &= e^{\sum_{j=1}^{k-2}(R_{1,j+1}-R_{1,j})b_j} \frac{\prod_{j=1}^{k-2}(R_{1,j} + \rho_{1,j})e^{\rho_{1,j}(b_j-b_{j-1})}}{(N_{k-1}(b_{k-1}) - (\beta - R_{1,k})M_k)} \frac{(R_{1,k-1} - R_{1,k})e^{-R_{1,k-1}(b_{k-1}-b_{k-2})}}{(N_{k-2}(b_{k-2}) - (\beta - R_{1,k-1})M_{k-1})} \frac{1}{\beta} \prod_{j=i+1}^{k-1} (\rho_{1,j} + R_{1,j}). \end{aligned}$$

Finally, with the notation of Section 2.1, for $i = 1, \ldots, k$, we arrive at

$$m_k^{(i)}(u) = \frac{\prod_{j=1}^{i-1}(R_{1,j} + \rho_{1,j})e^{\rho_{1,j}(b_j-b_{j-1})}}{(N_{i-1}(b_{i-1}) - (\beta - R_{1,i})M_i)} e^{\sum_{j=1}^{i-1}(R_{1,j+1}-R_{1,j})b_j} m_{1,i}(u) + \left(\sum_{j=i+1}^{k} Y_j^{(i)} \right) N_i(u).$$

Observe that this provides an explicit formula for $m_k(u)$ in terms of the model parameters even for $\delta > 0$ (the quantities N, M are only abbreviations for available functions of the model parameters), whereas in the differential approach of Section 2 the case $\delta > 0$ led to a system of equations that did not give rise to an explicit solution.

Finally, we again illustrate the tractability of the approach by giving the time value of ruin for the 4-layer case with exponential claims and parameter setting (15) and $\delta = 0.01$:

$$\mathbb{E}(e^{-0.01\,\tau_4(u)}) = \begin{cases} 0.662\,e^{-0.302u} + 0.0528\,e^{0.024u} & 0 \leq u < 5 \\ 0.654\,e^{-0.253u} + 0.0183\,e^{0.030u} & 5 \leq u < 10 \\ 0.508\,e^{-0.2u} + 0.005\,e^{0.042\text{ u}} & 10 \leq u < 15 \\ 0.307\,e^{-0.145u} & 15 \leq u < \infty. \end{cases}$$

Figure 5 depicts $\mathbb{E}(e^{-0.01\,\tau_4(u)})$ as a function of u for $k = 1, \ldots, 4$ (for each k the higher layers are shifted to infinity). The formula for arbitrary δ can also be obtained with the aid of a symbolic computation program such as Mathematica, but for larger values of k this can lead to very lengthy expressions. Taking derivatives of this general expression with respect to δ and evaluating the result at $\delta = 0$ gives the moments of the time of ruin, given that ruin occurs. Here we give the corresponding result for the first moment in the 3-layer model (i.e. b_3 is shifted to infinity):

$$\mathbb{E}(\tau_3(u)|\tau_3(u) < \infty) = \begin{cases} -\frac{600.062-32.897u}{7.676+e^{0.286u}} - 2.5u + 73.594 & 0 \leq u \leq 5 \\ -3.343u + \frac{118.856u-2463.073}{20.122+e^{0.231u}} + 121.277 & 5 \leq u < 10 \\ 4.167u + 3.952 & 10 \leq u < \infty \end{cases}$$

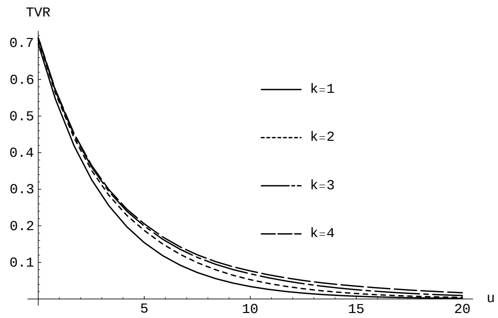

FIGURE 5. $\mathbb{E}(e^{-0.01\,\tau_4(u)})$ for the parameter set (15) and $k = 1, \ldots, 4$.

Figure 6 finally depicts $\mathbb{E}(\tau_k(u)|\tau_k(u) < \infty)$ as a function of u for $k = 1, 2, 3$.

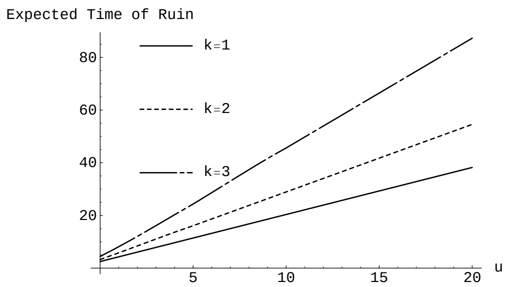

FIGURE 6. $\mathbb{E}(\tau_k(u)|\tau_k(u) < \infty)$ for the parameter set (15) and $k = 1, 2, 3$.

Acknowledgement. We would like to thank Prof. Sheldon Lin and two anonymous referees for helpful comments to improve the presentation of the manuscript.

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