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Analysis of Multiple-Type Housing Markets

Feng Di

UNIL | Université de Lausanne

FACULTÉ DES HAUTES ÉTUDES COMMERCIALES

DÉPARTEMENT D'ÉCONOMIE

Analysis of Multiple-Type Housing Markets

THÈSE DE DOCTORAT

présentée à la

Faculté des Hautes Études Commerciales
de l'Université de Lausanne

pour l'obtention du grade de
Docteur en économie

par

Di FENG

Directrice de thèse
Prof. Bettina Klaus

Jury

Prof. Rafael Lalive, président
Prof. Rustamdjan Hakimov, expert interne
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I M P R I M A T U R

Sans se prononcer sur les opinions de l'auteur, la Faculté des Hautes Etudes Commerciales de l'Université de Lausanne autorise l'impression de la thèse de Monsieur Di FENG, titulaire d'un bachelor en linguistique de l'Université de technologie de Tianjin, ainsi que d'un bachelor en Management et Sciences de l'Information de l'Université internationale Josai, titulaire d'un master en Économie de l'Université nationale de Yokohama, ainsi que d'un master en Économie de l'Université autonome de Barcelone, en vue de l'obtention du grade de docteur en Économie.

La thèse est intitulée :

ANALYSIS OF MULTIPLE-TYPE HOUSING MARKETS

Lausanne, le 7 juin 2023

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and have found it to meet the requirements for a doctoral thesis.
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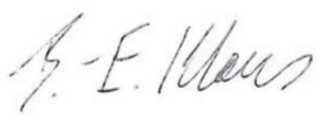

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Analysis of Multiple-Type Housing Markets

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2023

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Contents

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1. Introduction

Assignments of scarce resources are attracting a lot of attention in economic design.[1] Many studies, both theoretical and empirical, focus on indivisible resources, e.g., in auctions (Myerson, 1981; Hortaçsu and McAdams, 2010), school choice (Abdulkadiroğlu and Sönmez, 2003; Kapor et al., 2020), and medical resource allocation (Pathak et al., 2021). Existing literature often investigates such indivisible resource allocation problems with "unit-demand", i.e., each agent only consumes one object. Additionally, when monetary transfers are not allowed, many studies focus on the so-called housing market model (Shapley and Scarf, 1974). This model is an exchange economy in which each agent owns an object (say, a house); each agent has preferences over houses and consumes exactly one house. When preferences are strict, the strict core (defined by a weak blocking notion) is non-empty (Shapley and Scarf, 1974),[2] and can be easily calculated by the so-called top-trading-cycles (TTC) algorithm (due to David Gale). Moreover, TTC also satisfies important incentive properties, *strategy-proofness* (Roth, 1982) and even *group strategy-proofness* (Bird, 1984). Furthermore, TTC is the unique mechanism satisfying *Pareto efficiency*, *individual rationality*, and *strategy-proofness* (Ma, 1994; Svensson, 1999).

Although in some important cases (e.g., kidney exchange, school choice, etc.), unit-demand is an appropriate assumption, in many other cases, agents may wish to receive more than one object. Many studies analyze such situations (Pápai, 2001, 2007; Manjunath and Westkamp, 2021; Biró et al., 2022a,b; Echenique et al., 2022). However, if we relax the unit-demand assumption to allow for "multi-unit demands," i.e., each agent can consume more than one object, most of the positive results obtained under the unit-demand assumption disappear (Klaus and Miyagawa, 2002; Ehlers and Klaus, 2003). In the thesis, we consider an extension of the classical Shapley-Scarf housing markets by allowing multi-unit demands: multiple-type housing markets (Moulin, 1995).[3] In this model, objects are of different types (say, houses, cars, etc.), and agents are "balanced" in the sense that all agents have the same numbers and types of endowments and demands.

The analysis of multiple-type housing markets is relevant for three reasons. First, similar to Shapley-Scarf markets, they are applicable to many real-world problems: a familiar example for most readers would be students' enrollments at many universities, where courses are taught in small groups and multiple sessions (Klaus, 2008). Additionally, various other scenarios exist where individuals may wish to exchange their assigned resources, including term paper topics and dates during a course (Mackin and Xia, 2016), staff, operating rooms, and dates in hospitals to improve surgery schedules (Huh et al., 2013), and shifts in work settings due to personal reasons (Manjunath and Westkamp, 2021). Furthermore, recent technological developments have enabled the allocation of several resource types together, such as cloud computing (Ghodsi et al., 2011, 2012) and 5G network slicing (Peng et al., 2015; Bag et al., 2019; Han et al.,

[1] See the Nobel prize lectures in economic sciences in 2012 and 2020 for examples.

[2] Roth and Postlewaite (1977) show that the strict core is single-valued.

[3] In Echenique et al. (2022), multiple-type housing markets are called categorical economies.

2019). Such situations can be modeled as multiple-type housing markets. Thus, the analysis of multiple-type housing markets may impact the real world. Second, from a theoretical point of view, this is a simple extension of Shapley-Scarf housing markets with multi-unit demands. Therefore, the analysis of multiple-type housing markets, as a first step, is potentially useful for addressing issues for other multi-unit demand models (Pápai, 2007; Anno and Kurino, 2016). Third, for multiple-type housing markets, agents are balanced in the sense that all agents have the same numbers and types of endowments and demands. This balanced structure provides some tractability and hence some hope for positive results.

Despite their importance and generality, there is little research on multiple-type housing markets. One main reason for this is that for multiple-type housing markets, mainly we only have negative results. For instance, even with additively separable preferences, (i) the strict core may be empty, and (ii) no mechanism satisfies *Pareto efficiency*, *individual rationality*, and *strategy-proofness* (Konishi et al., 2001).

In this thesis, we revisit multiple-type housing markets and aim to uncover potential positive results based on the two negative results above. Specifically, in response to the first negative result, we focus on the situation where a strict core allocation always exists and re-examine the classical questions of implementation theory for the strict core. Contrary to classical results (Maskin, 1999; Sönmez, 1999; Takamiya, 2003, 2009), we find that the implementation of the strict core is tighter in our model; loosely speaking, our results suggest that not all strict core allocations are implementable.

Then, we turn to more general domains where a strict core allocation may not exist. In this case, a possibility is to weaken the strict core to other solution concepts. Following Wako (2005), instead of the strict core, we consider another solution concept, the commodity-wise competitive allocation, and its corresponding mechanism, the coordinatewise top trading cycles mechanism (cTTC). By providing a full characterization of cTTC, we prove that cTTC is outstanding as it respects *individual rationality*, achieves some efficiency, e.g., *unanimity*, and keeps the incentive robustness, e.g., *strategy-proofness*.

Finally, to address the second negative result, we explore weaker efficiency properties to determine their compatibility with *individual rationality* and *strategy-proofness*. By providing two characterizations, we prove that our efficiency properties are desirable because they (i) are compatible with *individual rationality* and *strategy-proofness*, and (ii) help us to identify two specific mechanisms, cTTC and the bundle top-trading-cycles mechanism (bTTC).

Overall, in the thesis, we have been successful in uncovering new insights. To the best of our knowledge, we are the first to have obtained characterizations of TTC extensions from studying the multiple-type housing markets, making our findings an important contribution to the field. Not only are our results valuable in the context of the multiple-type housing market model, but they also have potential applications in other studies within the broader field of economic design. Our findings provide a new perspective on the model and open up new avenues for further research. Additionally, the methodology to obtain our results is new: we obtain results for the relatively small and technical domain of lexicographic preferences, then we use these results to obtain further results for large preference domains. Our methodology could be useful for other studies looking for more efficient ways to analyze higher dimensional models. In summary, the positive results we obtained from this study not only contribute to a better understanding of the multiple-type housing market model, but also have implications for other areas of research and can be applied to other fields.

We proceed by providing a more in-depth summary of each chapter. Since all chapters focus on the same model, we formally introduce the model in the technical introduction.

In the first chapter, we focus on a restricted preference domain where strict core allocations always exist. We investigate the strict core implementation problem through various equilibrium concepts. For classical models, such as Shapley-Scarf housing markets and allocation problems with rich preferences, strict core allocations are fully implementable, i.e., all strict core allocations can be obtained in Nash or strict strong Nash equilibrium (Ma, 1994; Sönmez, 1999; Takamiya, 2009). In contrast to this full implementation, we show that for multiple-type housing markets, all strict strong Nash equilibrium outcomes are strict core allocations, but not vice versa, i.e., there are strict core allocations that cannot be implemented in strict strong Nash equilibrium. Moreover, by providing examples, we show the tightness of our results. This chapter was written in collaboration with Bettina Klaus. I played a role in every aspect of this chapter, from proposing the research question and formally defining the concepts, to proving the results and writing the research paper.

In the second chapter, we examine a more general preference domain where strict core allocations may not exist. Furthermore, we consider another solution concept, the commodity-wise competitive allocation, which is always selected by cTTC. We then investigate cTTC and fully characterize it by compelling properties. Specifically, we show that cTTC is the only mechanism satisfying *individual rationality*, *unanimity*, *strategy-proofness*, and *non-bossiness*. This chapter was written in collaboration with Bettina Klaus and Flip Klijn. I played a role in every aspect of this chapter, from proposing the research question and formally defining the concepts, to proving the results and writing the research paper.

The third chapter is single-authored. Since for multiple-type housing markets, *Pareto efficiency* is incompatible with *individual rationality* and *strategy-proofness*, I consider two efficiency properties that are weaker than *Pareto efficiency*: *coordinatewise efficiency* and *pairwise efficiency*. I show that these two properties (i) are both compatible with *individual rationality* and *strategy-proofness* and (ii) help us identify two specific mechanisms. On various domains of preference profiles, together with other well-known properties *individual rationality*, *strategy-proofness*, and *non-bossiness*, *coordinatewise efficiency* and *pairwise efficiency* respectively characterize two mechanisms, cTTC and bTTC.

Finally, we provide some information in the appendices, including the proofs and the independence examples.

2. Technical Introduction

2.1. Multiple-type housing markets

We consider a barter economy formed by n agents and $n \times m$ objects. Let $N = \{1, \ldots, n\}$ be a finite *set of agents*, where $n \geq 2$. A nonempty subset of agents $S \subseteq N$ is a *coalition*. There exist $m \geq 1$ *(distinct) types of objects* and n *(distinct) objects of each type*. We denote the *set of object types* by $T = \{1, ..., m\}$. For each $t \in T$, the set of type-t objects is $O^t = \{o_1^t, ..., o_n^t\}$, and the *set of all objects* is $O = \{o_1^1, o_1^2, \ldots, o_n^1, o_n^2, \ldots, o_n^m\}$, where $|O| = n \times m$. Each agent owns exactly one object of each type. Without loss of generality, let o_i^t be agent i's endowment of type-t. Thus, each agent i's endowment is a list $e_i = (o_i^1, \ldots, o_i^m)$. Moreover, each agent consumes exactly one object of each type, and hence, each agent's *(feasible) consumption set* is $\Pi_{t \in T} O^t$. An element in $\Pi_{t \in T} O^t$ is a *(consumption) bundle*. Note that for $m = 1$, our model is the classical Shapley-Scarf housing market model (Shapley and Scarf, 1974).

An *allocation* x partitions the set of all objects O into n bundles assigned to agents. Formally, $x = \{x_1, \ldots, x_n\}$ is such that for each $t \in T$, $\bigcup_{i \in N} x_i^t = O^t$ and for each pair $i \neq j$, $x_i^t \neq x_j^t$. The *set of all allocations* is denoted by X, and the *endowment allocation* is denoted by $e = \{e_1, \ldots, e_n\}$. Given an allocation $x \in X$, for each agent $i \in N$, we say that x_i is agent i's *allotment* at x and for each type $t \in T$, x_i^t is agent i's *type-t allotment* at x. For simplicity, sometimes we will restate an allocation as a list $x = (x_1, \ldots, x_n) \in (\Pi_{t \in T} O^t)^N$. Given x, let $x^t = (x_1^t, \ldots, x_n^t)$, and $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ be the list of all agents' allotments, except for agent i's allotment; and $x_S = (x_i)_{i \in S}$ to be the list of allotments of the members of coalition S.

Each agent i has *complete*, *antisymmetric*, and *transitive preferences* R_i over all bundles (allotments), i.e., R_i is a linear order over $\Pi_{t \in T} O^t$.[1] For two allotments x_i and y_i, x_i is *weakly better than* y_i if $x_i \, R_i \, y_i$, and x_i is *strictly better than* y_i if $[x_i \, R_i \, y_i$ and not $y_i \, R_i \, x_i]$, denoted $x_i \, P_i \, y_i$. Since preferences over allotments are strict, x_i and y_i are indifferent only if $x_i = y_i$. We denote preferences as ordered lists, e.g., $R_i : x_i, \; y_i, \; z_i$ instead of $x_i \, P_i \, y_i \, P_i \, z_i$. The *set of all preferences* is denoted by $\mathcal{R}$, which we will also refer to as the *strict preference domain*.

There are no consumption externalities: each agent $i \in N$ only cares about his own allotment x_i. Hence, i's preferences over allotments can be extended to his preferences over allocations. So, with some abuse of notation, we use the same notation R_i to denote agent i's preferences over allocations as well. That is, for each agent $i \in N$ and for any two allocations $x, y \in X$, $x \, R_i \, y$ if and only if $x_i \, R_i \, y_i$.[2]

[1]Preferences R_i are *complete* if for any two allotments x_i, y_i, $x_i \, R_i \, y_i$ or $y_i \, R_i \, x_i$; they are *antisymmetric* if $x_i \, R_i \, y_i$ and $y_i \, R_i \, x_i$ imply $x_i = y_i$; and they are *transitive* if for any three allotments x_i, y_i, z_i, $x_i \, R_i \, y_i$ and $y_i \, R_i \, z_i$ imply $x_i \, R_i \, z_i$.

[2]Note that when extending strict preferences over allotments to preferences over allocations without consumption externalities, strictness is lost because any two allocations where an agent gets the same allotment are indifferent to that agent.

A *preference profile* is a list $R = (R_1, \ldots, R_n) \in \mathcal{R}^N$. We use the standard notation $R_{-i} = (R_1, \ldots, R_{i-1}, R_{i+1}, \ldots, R_n)$ to denote the list of all agents' preferences, except for agent i's preferences. For each coalition $S \subseteq N$ we define $R_S = (R_i)_{i \in S}$ and $R_{-S} = (R_i)_{i \in N \setminus S}$ to be the lists of preferences of the members of S and $N \setminus S$, respectively.

In addition to the domain of strict preferences, we consider preference subdomains based on agents' "marginal preferences": assume that for each agent $i \in N$ and for each type $t \in T$, i has complete, antisymmetric, and transitive preferences R_i^t over the set of type-t objects O^t. We refer to R_i^t as *agent i's type-t marginal preferences*, and denote by $\mathcal{R}^t$ the *set of all type-t marginal preferences*. We use the standard notation $R^t = (R_1^t, \ldots, R_n^t)$ to denote the list of all agents' marginal preferences of type-t, and $R^{-t} = (R^1, \ldots, R^{t-1}, R^{t+1}, \ldots, R^m)$ to denote the list of all agents' marginal preferences of all types except for type-t. Then, we can define the following two preference domains.

(Strictly) Separable preferences. Agent i's preferences $R_i \in \mathcal{R}$ are *separable* if for each $t \in T$ there exist type-t marginal preferences $R_i^t \in \mathcal{R}^t$ such that for any two allotments x_i and y_i,

$$\text{if for all } t \in T,\ x_i^t\, R_i^t\, y_i^t, \text{ then } x_i\, R_i\, y_i.$$

$\mathcal{R}_s$ denotes the *domain of separable preferences.*

Before introducing our next preference domain, we introduce some notation. We use a bijective function $\pi_i : T \to T$ to order types according to agent i's "importance", with $\pi_i(1)$ being the most important and $\pi_i(m)$ being the least important object type. For simplicity, sometimes we restate π_i by an ordered list of types, e.g., by $\pi_i = (2, 3, 1)$, we mean that $\pi_i(1) = 2$, $\pi_i(2) = 3$, and $\pi_i(3) = 1$. For each agent $i \in N$ and each allotment $x_i = (x_i^1, \ldots, x_i^m)$, We rearrange x_i with respect to the *object-type importance order* π_i to obtain $x_i^{\pi_i} = (x_i^{\pi_i(1)}, \ldots, x_i^{\pi_i(m)})$.

(Separably) Lexicographic preferences. Agent i's preferences $R_i \in \mathcal{R}$ are *(separably) lexicographical* if they are separable with type-t marginal preferences $(R_i^t)_{t \in T}$ and there exists an object-type importance order $\pi_i : T \to T$ such that for any two allotments x_i and y_i,

$$\begin{aligned}
&\text{if } x_i^{\pi_i(1)} P_i^{\pi_i(1)} y_i^{\pi_i(1)} \text{ or} \\
&\text{if there exists a positive integer } k \leq m-1 \text{ such that} \\
&x_i^{\pi_i(1)} = y_i^{\pi_i(1)},\ \ldots,\ x_i^{\pi_i(k)} = y_i^{\pi_i(k)}, \text{ and } x_i^{\pi_i(k+1)} P_i^{\pi_i(k+1)} y_i^{\pi_i(k+1)}, \\
&\text{then } x_i\, P_i\, y_i.
\end{aligned}$$

$\mathcal{R}_l$ denotes the *domain of lexicographic preferences.*

Remark 1. Representation of lexicographic preferences
Note that any lexicographic preference relation $R_i \in \mathcal{R}_l$ is uniquely determined by agent i's marginal preferences $(R_i^t)_{t \in T}$ and an object-type importance order π_i. For example, consider a situation with two object-types $T = \{H(ouse), C(ar)\}$ and three agents $N = \{1, 2, 3\}$ with each agent i's endowment equal to $o_i = (H_i, C_i)$. Assume that agent i has lexicographic preferences $R_i : (H_1, C_1), (H_1, C_2), (H_1, C_3), (H_2, C_1), (H_2, C_2), (H_2, C_3), (H_3, C_1), (H_3, C_2), (H_3, C_3)$. Then, agent i's type importance order is $\pi_i : H,\ C$, and his marginal preferences are $R_i^H : H_1,\ H_2,\ H_3$, and $R_i^C : C_1,\ C_2,\ C_3$. Hence, agent i's preferences R_i can alternatively be written as $R_i = (R_i^H, R_i^C, \pi_i)$.

For an even compacter description of agent i's lexicographic preferences, we can also rely on the strict ordering of objects that is induced by the object-type importance order together with his marginal preferences:

$$R_i : H_1,\ H_2,\ H_3,\ C_1,\ C_2,\ C_3.$$

Additionally,

$$\mathcal{R}_l \subsetneq \mathcal{R}_s \subsetneq \mathcal{R}.$$

◇

A *(multiple-type housing) market* is a triple (N, e, R); as the set of agents N and the endowment allocation remain fixed throughout, we will simply denote the market (N, e, R) by R. Thus, the strict preference profile domain $\mathcal{R}^N$ also denotes the set of all markets with strict preferences. Similarly, $\mathcal{R}_s^N$ is also the set of all markets with separable preferences and $\mathcal{R}_l^N$ is also the set of all markets with lexicographic preferences.

2.2. Mechanisms and their properties

Note that all following definitions for the domain of strict preference profiles $\mathcal{R}^N$ can be formulated for the domain of separable preference profiles $\mathcal{R}_s^N$ or the domain of lexicographic preference profiles $\mathcal{R}_l^N$.

A *mechanism* is a function $f : \mathcal{R}^N \to X$ that selects for each market R an allocation $f(R) \in X$, and

- for each $i \in N$, $f_i(R)$ denotes agent i's allotment
- for each $i \in N$ and each $t \in T$, $f_i^t(R)$ denotes agent i's type-t allotment. Moreover, $f^t(R)$ denotes the allocation of type-t, i.e., $f^t(R) = (f_1^t(R), \ldots, f_n^t(R))$.

We next introduce and discuss some well-known properties for allocations and mechanisms.

First, we consider a voluntary participation condition for an allocation x to be implementable without causing agents any harm: no agent will be worse off than at his endowment. Let $R \in \mathcal{R}^N$. An allocation $x \in X$ is *individually rational* (at R) if for each agent $i \in N$, $x_i\, R_i\, e_i$.

Individual rationality: For each $R \in \mathcal{R}^N$, $f(R)$ is *individually rational*.

Next, we consider two well-known efficiency criteria. Let $R \in \mathcal{R}^N$. An allocation $y \in X$ is a *Pareto improvement* over allocation $x \in X$ at R if for each agent $i \in N$, $y_i\, R_i\, x_i$, and for at least one agent $j \in N$, $y_j\, P_j\, x_j$. An allocation is *Pareto efficient* at R if no *Pareto improvement*.

Pareto efficiency: For each $R \in \mathcal{R}^N$, $f(R)$ is *Pareto efficient*.

An allocation $x \in X$ is *unanimously best* at R if for each agent $i \in N$ and each allocation $y \in X$, we have $x\, R_i\, y$.[3]

Unanimity: For each $R \in \mathcal{R}^N$, $f(R)$ is *unanimously best* whenever it exists.

If a *unanimously best* allocation exists (at R), then that allocation is the only *Pareto efficient* allocation (at R). Hence, *Pareto efficiency* implies *unanimity*.

[3]Since all preferences are strict, the set of unanimously best allocations is empty or single-valued.

The next two properties, *strategy-proofness* and *group strategy-proofness*, are two of the properties that are most frequently used in the literature on mechanism design. They model that no agent / coalition can benefit from misrepresenting his / their preferences.

Strategy-proofness: For each $R \in \mathcal{R}^N$, each $i \in N$, and each $R'_i \in \mathcal{R}$, $f_i(R_i, R_{-i}) R_i f_i(R'_i, R_{-i})$.

Group strategy-proofness: For each $R \in \mathcal{R}^N$, there are no coalition $S \subseteq N$ and no preference list $R'_S = (R'_i)_{i \in S} \in \mathcal{R}^S$ such that for each $i \in S$, $f_i(R'_S, R_{-S})\ R_i\ f_i(R)$, and for some $j \in S$, $f_j(R'_S, R_{-S})\ P_j\ f_j(R)$.

Next, we consider a well-known property for mechanisms that restricts each agent's influence: no agent can change other agents' allotments without changing his own allotment by changing his reported preference.

Non-bossiness: For each $R \in \mathcal{R}^N$, each $i \in N$, and each $R'_i \in \mathcal{R}$, $f_i(R_i, R_{-i}) = f_i(R'_i, R_{-i})$ implies $f(R_i, R_{-i}) = f(R'_i, R_{-i})$.

By the definition, it is easy to verify that *group strategy-proofness* implies *strategy-proofness* and *non-bossiness*. However, the converse is not true. To be more precise, (i) for strict preferences, *group strategy-proofness* coincides with *strategy-proofness* and *non-bossiness* (Alva, 2017), but (ii) for separable preferences and lexicographic preferences, the former is stronger than the latter. See Lemma 1 and Proposition 1 in Chapter 2 for details.

2.3. TTC extensions

We next focus on the domain of lexicographic preferences $\mathcal{R}_l$ and extend Gale's famous top trading cycles (TTC) algorithm to multiple-type housing markets.

The multiple-type top trading cycles (mTTC) algorithm
Input. A multiple-type housing market problem $R \in \mathcal{R}_l^N$.

Step 1. ***Building step.*** We construct a (directed) graph $G(1)$ with the set of nodes $N \cup O$. For each object $o \in O$, we add a directed edge to its owner and for each agent $i \in N$, we add a directed edge to his most preferred object in O. For each directed edge $(i, o) \in N \times O$ we say that agent i points to object o.

Implementation step. A *trading cycle* is a directed cycle in graph $G(1)$. Given the finite number of nodes, at least one trading cycle exists. We assign to each agent i in a trading cycle the object that he pointed to, and denote the object assigned to him in this step by $a_i(1)$; we denote the corresponding set of objects assigned through trading cycles by $A(1)$. Thus, each agent $i \in N$ receives a (possibly empty) partial allotment $x_i(1) = \{a_i(1)\}$.

Removal step. Let $N(2) := N$ (for $m \geq 2$, no agents are removed at Step 1). We now remove all objects that were assigned through trading cycles from set O and set $U(2) := O \setminus A(1)$ (these are the objects that have not been allocated yet). For each agent $i \in N$, we now derive the set of *feasible continuation objects* $U_i(2)$ by removing all objects in $U(2)$ that are of a type that is already present in agent i's partial allotment $x_i(1)$. Go to the next step **2**.

In general, at Step q (≥ 2) we have the following:

Step *q*. If $U(q)$ (or alternatively $N(q)$) is empty, then stop; otherwise do the following.

Building step. We construct a (directed) graph $G(q)$ with the set of nodes $N(q) \cup U(q)$. For each object $o \in U(q)$, we add a directed edge to its owner and for each agent $i \in N$, we add a directed edge to his most preferred feasible continuation object in $U_i(q)$ (according to R_i).

Implementation step. A *trading cycle* is a directed cycle in graph $G(q)$. Given the finite number of nodes, at least one trading cycle exists. We assign to each agent i in a trading cycle the object that he pointed to, and denote the object assigned to him in this step by $a_i(q)$; we denote the corresponding set of objects assigned through trading cycles by $A(q)$. Up to and including this step, each agent $i \in N(q)$ has received a (possibly empty) partial allotment $x_i(q) = \cup_{p=1}^{q}\{a_i(p)\}$.

Removal step. We now remove all agents who have received a (complete) allotment and denote the set of remaining agents by $N(q+1)$. Next, we remove all objects that were assigned through trading cycles from set $U(q)$ and set $U(q+1) := U(q) \setminus A(q)$ (these are the objects that have not been allocated yet). For each agent $i \in N(q)$, we now derive the set of *feasible continuation objects* $U_i(q+1)$ by removing all objects in $U(q+1)$ that are of a type that is already present in agent i's partial allotment $x_i(q)$. Go to the next step $\boldsymbol{q+1}$.

Output. The mTTC algorithm terminates when all objects in O are assigned to some agent $i \in N$ (it takes at most $n \cdot m$ steps). Let the final step be Q, so the final allocation is $x(Q) = \{x_1(Q), \ldots, x_n(Q)\}$. We set $x(R) := x(Q)$ as the mTTC allocation at R.

Multiple-type top trading cycles mechanism (mTTC): The *multiple-type top trading cycles mechanism (mTTC)* mTTC (introduced by Sikdar et al., 2017), selects for each market $R \in \mathcal{R}_l^N$ the mTTC allocation $x(R)$, i.e., $\text{mTTC}(R) = x(R)$ and $\text{mTTC}_i(R) = x_i(R)$.

The type-*t* TTC algorithm

Consider a market (N, e, R) such that $R \in \mathcal{R}_l^N$. For each type $t \in T$, let $(N, e^t, R^t) = (N, (o_1^t, \ldots, o_n^t), (R_1^t, \ldots, R_n^t))$ be its *associated type-t submarket*.

We define the top trading cycles (TTC) allocation for each type-t submarket as follows.

Input. A type-t submarket (N, e^t, R^t).

Step 1. Let $N_1 := N$ and $O_1^t := O^t$. We construct a directed graph with the set of nodes $N_1 \cup O_1^t$. For each agent $i \in N_1$, there is an edge from the agent to his most preferred type-t object in O_1^t according to R_i^t. For each edge (i, o) we say that agent i points to type-t object o. For each type-t object $o \in O_1^t$, there is an edge from the object to its owner.

A *trading cycle* is a directed cycle in the graph. Given the finite number of nodes, at least one trading cycle exists. We assign each agent in a trading cycle the type-t object he points to and remove all trading cycle agents and type-t objects. If there are some agents (and hence objects) left, we move to the next step. Otherwise we stop.

Step *k*. Let N_k be the set of agents that remain after Step $k-1$ and O_k^t be the set of type-t objects that remain after Step $k-1$. We construct a directed graph with the set of nodes $N_k \cup O_k^t$. For each agent $i \in N_k$, there is an edge from the agent to his most preferred type-t object in O_k^t according to R_i^t. For each type-t object $o \in O_k^t$, there is an edge from the object to its owner. At least one trading cycle exists and we assign each agent in a trading cycle the type-t object he points to and remove all trading cycle agents and objects. If there are some agents (and hence objects) left, we move to the next step. Otherwise we stop.

Output. The type-t TTC algorithm terminates when each agent in N is assigned an object in O^t, which takes at most n steps. We denote the object in O^t that agent $i \in N$ obtains in the type-t TTC algorithm by $TTC_i^t(e^t, R^t)$ and the final type-t allocation by $TTC^t(e^t, R^t)$.

The cTTC allocation / mechanism: The *coordinatewise top trading cycles (cTTC) allocation*, $cTTC(R)$, is the collection of all type-t TTC allocations, i.e., for each $R \in \mathcal{R}_s^N$,

$$cTTC(R) = \left(\left(TTC_1^1(R^1), \ldots, TTC_1^m(R^m)\right), \ldots, \left(TTC_n^1(R^1), \ldots, TTC_n^m(R^m)\right) \right).$$

The $cTTC$ mechanism (introduced by Wako, 2005) selects for each market its cTTC allocation.

Finally, we consider another TTC extension, which only allows agents to trade their endowment bundles completely.

The bundle top trading cycles (bTTC) algorithm / mechanism The *bundle top trading cycles mechanism (bTTC)* assigns to each market R the unique top-trading allocation that results from the TTC algorithm if agents are only allowed to trade their whole endowments among each other.

Formally, for each market R and $i \in N$, let $R_i|^e$ be the restriction[4] of R_i to endowments $\{e_1, \ldots, e_n\}$ and $R|^e \equiv (R_i|^e)_{i \in N}$ be the restriction profile. We then use the TTC algorithm to compute the bTTC allocation for $R|^e$. Note that the difference with the classical TTC algorithm (for Shapley-Scarf housing markets) is that instead of an object, each agent can only point to a whole endowment.

The $bTTC$ mechanism assigns the bTTC allocation above to each market.

Remark 2. Generalizability
(i) mTTC is only well-defined for lexicographic preferences, (ii) cTTC is well-defined for lexicographic preferences and separable preferences, and (iii) bTTC is well-defined for lexicographic preferences, separable preferences, and strict preferences. ◇

Example 1. Illustration of the three mechanisms
Let $N = \{1, 2, 3\}$. Consider a market $R \in \mathcal{R}_l^N$ with two types: $T = \{H(ouse), C(ar)\}$, $O = \{H_1, H_2, H_3, C_1, C_2, C_3\}$, and each agent i's endowment is (H_i, C_i). The preference profile R is as follows:[5]

$$R_1 : H_2, H_3, \boldsymbol{H_1}, C_3, C_2, \boldsymbol{C_1},$$
$$R_2 : C_1, \boldsymbol{C_2}, C_3, H_3, \boldsymbol{H_2}, H_1,$$
$$R_3 : H_2, H_1, \boldsymbol{H_3}, C_1, \boldsymbol{C_3}, C_2.$$

First, consider mTTC.

Step 1. *Building step.* $G(1) = (N \cup O, E(1))$ with set of directed edges
$E(1) = \{(H_1, 1), (H_2, 2), (H_3, 3), (C_1, 1), (C_2, 2), (C_3, 3), (1, H_2), (2, C_1), (3, H_2)\}$.

Implementation step. The trading cycle $1 \to H_2 \to 2 \to C_1 \to 1$ forms. Then, $a_1(1) = H_2$ and $a_2(1) = C_1$; $x_1(1) = \{H_2\}$, $x_2(1) = \{C_1\}$, and $x_3(1) = \emptyset$; and $A(1) = \{H_2, C_1\}$.

Removal step. $N(2) = N$, $U(2) = O \setminus A(1) = \{H_1, H_3, C_2, C_3\}$, $U_1(2) = \{C_2, C_3\}$, $U_2(2) = \{H_1, H_3\}$, and $U_3(2) = \{H_1, H_3, C_2, C_3\}$.

[4] That is, for each $i \in N$, $R_i|^e$ are preferences over $\{e_1, \ldots, e_n\}$ such that for each $e_j, e_k \in \{e_1, \ldots, e_n\}$, $e_j \; R_i|^e \; e_k$ if and only if $e_j \; R_i \; e_k$.

[5] In all examples we indicate endowments in boldface.

Step 2. *Building step.* $G(2) = (N(2) \cup U(2), E(2))$ with set of directed edges
$E(2) = \{(H_1, 1), (H_3, 3), (C_2, 2), (C_3, 3), (1, C_3), (2, H_3), (3, H_1)\}$.

Implementation step. The trading cycle $1 \to C_3 \to 3 \to H_1 \to 1$ forms. Then, $a_1(2) = C_3$ and $a_3(2) = H_1$; $x_1(2) = \{H_2, C_3\}$, $x_2(2) = \{C_1\}$, and $x_3(2) = \{H_1\}$; and $A(2) = \{H_1, C_3\}$.

Removal step. $N(3) = \{2, 3\}$, $U(3) = U(2) \setminus A(2) = \{H_3, C_2\}$, $U_1(3) = \emptyset$, $U_2(3) = \{H_3\}$, and $U_3(3) = \{C_2\}$.

Step 3. *Building step.* $G(3) = (N(3) \cup U(3), E(3))$ with set of directed edges
$E(3) = \{(H_3, 3), (C_2, 2), (2, H_3), (3, C_2)\}$.

Implementation step. The trading cycle $2 \to H_3 \to 3 \to C_2 \to 2$ forms. Then, $a_2(3) = H_3$ and $a_3(3) = C_2$; $x_1(3) = \{H_2, C_3\}$, $x_2(3) = \{H_3, C_1\}$, and $x_3(3) = \{H_1, C_2\}$; and $A(3) = \{H_3, C_2\}$.

Removal step. $N(4) = \emptyset$ and $U(4) = \emptyset$.

Thus, $\text{mTTC}(R) = ((H_2, C_3), (H_3, C_1), (H_1, C_2))$.

Second, consider $cTTC$.

We have two submarkets: the house submarket and the car submarket. The corresponding preference profiles are as follows.

$$R_1^H : H_2, H_3, \boldsymbol{H_1},$$
$$R_2^H : H_3, \boldsymbol{H_2}, H_1,$$
$$R_3^H : H_2, H_1, \boldsymbol{H_3}.$$

$$R_1^C : C_3, C_2, \boldsymbol{C_1},$$
$$R_2^C : C_1, \boldsymbol{C_2}, C_3,$$
$$R_3^C : C_1, \boldsymbol{C_3}, C_2.$$

It is easy to see that $TTC^H(R^H) = (H_1, H_3, H_2)$ and $TTC^C(R^C) = (C_3, C_2, C_1)$, and hence $cTTC(R) = ((H_1, C_3), (H_3, C_2), (H_2, C_1))$.

Finally, consider $bTTC$.

The corresponding restrictions of R are as follows.

$$R_1|^e : e_2, e_3, \boldsymbol{e_1},$$
$$R_2|^e : e_1, \boldsymbol{e_2}, e_3,$$
$$R_3|^e : e_2, e_1, \boldsymbol{e_3}.$$

Thus, $bTTC(R) = (e_2, e_1, e_3) = ((H_2, C_2), (H_1, C_1), (H_3, C_3))$. ◇

3. CHAPTER 1
Preference Revelation Games and Strict Cores of Multiple-type Housing Markets

3.1. Introduction

In a classical Shapley-Scarf housing market (Shapley and Scarf, 1974), each agent is endowed with an indivisible object, e.g., a house, consumes exactly one house, and ranks all houses in the market. The question then is to (re)allocate houses among the agents without using monetary transfers and by taking agents' preferences and endowments into account.

A common solution concept for Shapley-Scarf housing markets is the strict core solution, which assigns the set of allocations where no group of agents has an incentive (via weak blocking) to deviate by exchanging their endowments within the group. When agents' preferences are strict, the strict core solution exhibits a remarkable number of positive features: it is non-empty (Shapley and Scarf, 1974), always a singleton, and coincides with the unique competitive allocation (Roth and Postlewaite, 1977). In addition, it can be easily calculated by the so-called top-trading-cycles (TTC) algorithm (due to David Gale). Furthermore, as a *strictly core-stable* mechanism, TTC is *strategy-proof* (Roth, 1982), and it is the unique mechanism satisfying *individual rationality*, *Pareto efficiency*, and *strategy-proofness* (Ma, 1994; Svensson, 1999).

Multiple-type housing markets are an extension of Shapley-Scarf housing markets, which are first introduced by Moulin (1995).[1] In multiple-type housing markets, there are multiple types of indivisible objects, each agent is endowed with one object of each type and consumes exactly one object of each type. Multiple-type housing markets are often described with houses and cars as metaphors for indivisible object types. While these and related housing market models appear to be rather stylized, they give valuable insights into many real-world applications such as dynamic resource allocation problems (Monte and Tumennasan, 2015), the assignment of student-presentations (Mackin and Xia, 2016), cloud computing (Ghodsi et al., 2011, 2012), the assignment of medical resources (Huh et al., 2013), and 5G network slicing (Peng et al., 2015; Bag et al., 2019; Han et al., 2019). A more familiar example for most readers would be the situation of students' enrollment at many universities where courses are taught in parallel sessions (Klaus, 2008).

Konishi et al. (2001) are the first to analyze multiple-type housing markets. They demonstrate that when increasing the dimension of the classical Shapley-Scarf housing market model by adding other types of indivisible objects, most of the positive results obtained for the one-dimensional single-type case disappear: even for additively separable preferences, the strict core

[1] There are many other extensions, such as the multi-demand models of Pápai (2001, 2007), Ehlers and Klaus (2003), and Manjunath and Westkamp (2021).

may be empty and no *individually rational*, *Pareto efficient*, and *strategy-proof* mechanism exists. One of the reasons for this is that, in contrast to single-type housing markets, multiple-type housing markets cannot be transformed into well-behaved coalition formation games (Banerjee et al., 2001; Bogomolnaia and Jackson, 2002; Quint and Wako, 2004); e.g., an agent may exchange his house within a trading coalition S_1 but exchange his car with a different trading coalition S_2.

There has been very little work on multiple-type housing markets after Konishi et al. (2001)'s negative results. The following papers considered different solutions for different sub-domains of preferences.

For separable preferences, Konishi et al. (2001) and Wako (2005) suggest an alternative solution to the strict core solution by first using separability to decompose a multiple-type housing market into "coordinate-wise submarkets" and second, determining the strict core in each submarket. Wako (2005) calls the resulting outcome the commodity-wise competitive allocation and shows that it is implementable in (self-enforcing) coalition-proof Nash equilibria. Klaus (2008) calls the mechanism that always selects this unique allocation the coordinate-wise core rule, and shows that it satisfies *individual rationality*, second-best incentive compatible,[2] and *strategy-proofness*.

On the domain of generalized lexicographic preferences, Sikdar et al. (2017, 2019) extend the TTC algorithm and define a new mechanism: the multiple-type top-trading-cycles mechanism (mTTC), and they show that mTTC outputs a strict core allocation; hence, the strict core for generalized lexicographic preferences is non-empty. Strict-core stability implies *individual rationality* and *Pareto efficiency* of mTTC. However, they demonstrate that mTTC is not *strategy-proof* and that the strict core may be multi-valued.

Our contributions

Takamiya (2009) considers the more generalized model of objects allocation introduced by Sönmez (1999), which contains Shapley-Scarf housing markets as a special case (see Appendix A for a description of the generalized indivisible goods allocation model and some further discussions). In particular, Takamiya's results imply that for Shapley-Scarf housing markets and for individually rational and Pareto efficient mechanisms, the set of strict strong Nash equilibrium outcomes of the preference revelation game is the strict core (we state this result as Corollary 1).

Similarly, we examine the relationship between the strict strong Nash equilibrium outcomes of the preference revelation games and the strict core allocations of multiple-type housing markets. Takamiya's (2009) results do not translate into our higher dimensional model. First, multiple-type housing markets may have an empty strict core, even if preferences are separable. Then, a promising subdomain that guarantees the non-emptiness of the strict core is the domain of lexicographic preferences. However, lexicographic preferences do not satisfy the domain richness condition Takamiya (2009) needs for his main result (we discuss this in detail in Appendix A).

We prove that on the domain of lexicographic preferences, the set of all strict strong Nash equilibrium outcomes of the preference revelation game, induced by a *strictly core-stable* mechanism, is a subset of the strict core, but not vice versa, i.e., there are strict core allocations

[2] That is, there exists no other *strategy-proof* mechanism that *Pareto dominates* the coordinate-wise core rule.

that cannot be implemented via strict strong Nash equilibrium (Theorem 1). This result can be extended to a more general set of preference domains that satisfy strict core non-emptiness and a minimal preference domain richness assumption (Theorem 2). Throughout the chapter, we motivate our approach and discuss some comparative statics aspects of our results via various examples.

3.2. Preliminaries

Note that all following definitions for the domain of strict preference profiles $\mathcal{R}^N$ can be formulated for the domain of separable preference profiles $\mathcal{R}_s^N$ or the domain of lexicographic preference profiles $\mathcal{R}_l^N$.

Core

In order to introduce the standard cooperative solutions of the weak and the strict core, we introduce two blocking notions: an allocation $x \in X$ is strictly blocked by coalition $S \subseteq N$ if each member of S is better off after coalition S reallocated their endowments among themselves. Formally, for market $R \in \mathcal{R}^N$, an *allocation $x \in X$ is strictly blocked by coalition $S \subseteq N$* if there exists an allocation $y \in X$ such that:

(1) at allocation y agents in S reallocate their endowments, i.e., for each $i \in S$ and each $t \in T$, $y_i^t \in \{o_j^t\}_{j \in S}$, and

(2) all agents in S are strictly better off, i.e., for each $i \in S, y_i \, P_i \, x_i$.

An *allocation $x \in X$ is weakly blocked by coalition $S \subseteq N$* if condition (2) is replaced by

(2') all agents in S are weakly better off with at least one of them being strictly better off, i.e., for each $i \in S, y_i \, R_i \, x_i$, and for at least one $j \in S$, $y_j \, P_j \, x_j$.

Given the blocking notions above, we can restate *individual rationality* and *Pareto efficiency* as follows. An allocation is *individually rational* if it is not weakly or strictly blocked by any singleton coalition $\{i\}$ and an allocation is *Pareto efficient* if it is not weakly blocked by the set of all agents N.

We now introduce the first type of (possibly empty- or multi-valued) solution to multiple-type housing markets that we will consider: core solutions.

Definition (The strict / weak core and strict / weak core-stability).
Let $R \in \mathcal{R}^N$. An allocation is a *strict core allocation* (at R) if it is not weakly blocked by any coalition; the set of all strict core allocations is the *strict core*.

Similarly, an allocation is a *weak core allocation* (at R) if it is not strictly blocked by any coalition; the set of all weak core allocations is the *weak core*.

Let $\mathrm{SC}(R)$ and $\mathrm{WC}(R)$ denote its strict core and weak core, respectively.

A mechanism f on $\mathcal{R}^N$ is *strictly core-stable* if it selects only strict core allocations. Similarly, a mechanism f on $\mathcal{R}^N$ is *weakly core-stable* if it selects only weak core allocations.

Note that for all $R \in \mathcal{R}^N$, $\text{SC}(R) \subseteq \text{WC}(R)$, and that all strict core allocations satisfy *individual rationality* and *Pareto efficiency*. So, if a mechanism is *strictly core-stable*, then it is *individually rational* and *Pareto efficient* as well. Furthermore, for some $R \in \mathcal{R}^N$, $\text{WC}(R)$ is empty.

Preference revelation games

We now formulate a natural preference revelation game.

Given $R \in \mathcal{R}^N$ and a mechanism $f : \mathcal{R}^N \to X$, the *preference revelation game* induced by f is the *strategic game* $\Gamma_f(R) = (\mathcal{R}^N, f, R)$, where $\mathcal{R}$ is each agent's *strategy space*, f is the *outcome function*, and each agent i evaluates outcomes with R_i.

Definition (Nash / strict strong Nash equilibria).
Let $R \in \mathcal{R}^N$ and consider its corresponding preference revelation game $\Gamma_f(R)$.

A strategy profile $R^* \in \mathcal{R}^N$ is a *Nash equilibrium* of $\Gamma_f(R)$ if for each agent $i \in N$ and each strategy $R'_i \in \mathcal{R}$, $f_i(R^*) = f_i(R^*_i, R^*_{-i})\ R_i\ f_i(R'_i, R^*_{-i})$. We denote the *set of Nash equilibria* by $\text{Nash}(\Gamma_f(R))$ and the *set of Nash equilibrium outcomes* by $f(\text{Nash}(\Gamma_f(R)))$.

A strategy profile $R^* \in \mathcal{R}_l^N$ is a *strict strong Nash equilibrium*[3] of $\Gamma_f(R)$ if for each coalition $S \subseteq N$ and each strategy list $R'_S \in \mathcal{R}^S$,

$$[\text{for each agent } i \in S,\ f_i(R'_S, R^*_{-S})\ R_i\ f_i(R^*_S, R^*_{-S})] \text{ implies}$$

$$[\text{for each agent } i \in S,\ f_i(R'_S, R^*_{-S}) = f_i(R^*_S, R^*_{-S})].$$

We denote the *set of strict strong Nash equilibria* by $\text{sNash}(\Gamma_f(R))$ and the *set of strict strong Nash equilibrium outcomes* by $f(\text{sNash}(\Gamma_f(R)))$. Note that $\text{sNash}(\Gamma_f(R)) \subseteq \text{Nash}(\Gamma_f(R)) \subseteq \mathcal{R}^N$.

Given a preference revelation game $\Gamma_f(R)$, we say that agent i plays a *truth-telling strategy* if he truthfully reports his preferences R_i. If all agents play truth-telling strategies, then $R = (R_i)_{i \in N}$ is a *truth-telling strategy profile* at $\Gamma_f(R)$. Note that if f is *strategy-proof*, then truth-telling is a weakly dominant strategy for each agent and the truth-telling strategy profile is a weakly dominant strategy Nash equilibrium.[4]

3.3. Results

3.3.1. Motivating examples

As mentioned in the introduction, for Shapley-Scarf housing markets with strict preferences, the unique strict core allocation can be obtained by a unique *individually rational*, *Pareto efficient*,

[3] The set of strict strong Nash equilibria is a refinement of the set of strong Nash equilibria: a strategy profile $R^* \in \mathcal{R}^N$ is a *strong Nash equilibrium* of $\Gamma_f(R)$ if for each coalition $S \subseteq N$ and each strategy list $R'_S \in \mathcal{R}^S$, [for each agent $i \in S$, $f_i(R'_S, R^*_{-S})\ R_i\ f_i(R^*_S, R^*_{-S})$] implies [for some agent $j \in S$, $f_j(R'_S, R^*_{-S}) = f_j(R^*_S, R^*_{-S})$]. For a discussion of the existence of strict strong Nash equilibria we refer to Remark 3. Note that in Example 3, our proof shows that the strict core allocation x' cannot be obtained in a strong Nash equilibrium, either.

[4] For $\Gamma_f(R)$ and $i \in N$, a strategy $R^*_i \in \mathcal{R}$ is *weakly dominant* if for each $R' \in \mathcal{R}^N$, $f_i(R^*_i, R'_{-i})\ R_i\ f_i(R')$. A Nash equilibrium R^* is a *weakly dominant strategy Nash equilibrium* if for each $i \in N$, R^*_i is a weakly dominant strategy.

and *strategy-proof* mechanism (Ma, 1994; Svensson, 1999), the top-trading cycles mechanism. Later, Sönmez (1999) considers a generalization of Shapley and Scarf (1974)'s housing markets, *generalized indivisible goods allocation problems* (see Appendix A), and shows that, whenever the preference domain satisfies a certain condition of richness and if there exists a mechanism satisfying *individual rationality*, *Pareto efficiency*, and *strategy-proofness*, then for any problem having a non-empty strict core, the strict core is essentially single-valued[5] and the mechanism chooses a strict core allocation. Takamiya (2003) shows the following converse result: whenever the preference domain satisfies a certain condition of richness and if the strict core solution is essentially single-valued, then any selection from the strict core solution is *strategy-proof*.

However, for multiple-type housing markets, these results do not hold anymore: Konishi et al. (2001) (Sikdar et al., 2017, respectively) show that on the domain of separable preferences (lexicographic preferences, respectively), no mechanism satisfies *individual rationality*, *Pareto efficiency*, and *strategy-proofness*. Note that neither the domain of separable preferences nor the domain of lexicographic preferences satisfies the domain richness condition of Sönmez (1999) (see Appendix A).

The following example shows that on the one hand an individually rational and Pareto efficient mechanism can pick an allocation at which no agent has an incentive to misrepresent his preferences while on the other hand the strict core may be multi-valued (without being essentially single-valued).

Example 2 (**Non-manipulability and a multi-valued strict core**).
Consider $R \in \mathcal{R}_l^N$ with $N = \{1, 2, 3\}$, $T = \{H(ouse), C(ar)\}$, $O = \{H_1, H_2, H_3, C_1, C_2, C_3\}$, each agent i's endowment (H_i, C_i), and

$$\boldsymbol{R_1} : H_2, \boldsymbol{H_1}, H_3, C_3, \boldsymbol{C_1}, C_2,$$

$$\boldsymbol{R_2} : H_3, \boldsymbol{H_2}, H_1, C_1, \boldsymbol{C_2}, C_3,$$

$$\boldsymbol{R_3} : H_2, \boldsymbol{H_3}, H_1, C_1, \boldsymbol{C_3}, C_2.$$

Applying mTTC to R, at Step 1, the trading cycle $2 \to H_3 \to 3 \to H_2 \to 2$ forms; the trading cycle at Step 2 is $1 \to H_1 \to 1$; the trading cycle at Step 3 is $1 \to C_3 \to 3 \to C_1 \to 1$; and at Step 4, we have $2 \to C_2 \to 2$. The final outcome is the strict core allocation $x = ((H_1, C_3), (H_3, C_2), (H_2, C_1))$.

Note that at R, no agent has an incentive to misrepresent his preferences: agent 3 has no incentive to misreport his preferences because he receives his best allotment. Agent 1 cannot obtain his best house H_2 by misreporting his preferences (it is traded in Step 1 between agents 2 and 3). Given that, he receives the best possible allotment and has no incentive to misreport his preferences. Finally, agent 2 already obtains his best house and if he tries to obtain his best car by misreporting his preferences, he cannot obtain his best house; thus, he has no incentive to misreport his preferences. Finally, the strict core is not unique: $((H_1, C_3), (H_3, C_1), (H_2, C_2))$ is also a strict core allocation. $\diamond$

Recall that for multiple-type housing markets with lexicographic preferences, no mechanism satisfies *individual rationality*, *Pareto efficiency*, and *strategy-proofness* (Sikdar et al., 2017). Hence, *strict core stability* and *strategy-proofness* are also not compatible. Thus, in our context,

[5]The strict core is essentially single-valued if for each agent, any two strict core allocations are equivalent.

strategy-proofness, or truth-telling being a weakly dominant strategy Nash equilibrium in the corresponding preference revelation game, is a very strong requirement. Therefore, we next consider implementation through a different equilibrium concept: strict strong Nash equilibrium.

For generalized indivisible goods allocation problems, Takamiya (2009) studies the relationship between coalitional equilibria and the strict core. Takamiya's main result implies that for Shapley-Scarf housing markets and for a preference revelation game induced by an *individually rational* and *Pareto efficient* mechanism f, the set of strict strong Nash equilibrium outcomes equals the strict core.

Corollary 1 (**Takamiya, 2009**). *For each Shapley-Scarf housing market $R \in \mathcal{R}^N$ and each individually rational and Pareto efficient mechanism f, we have*

$$f(\text{sNash}(\Gamma_f(R))) = \text{SC}(R).$$

The following example shows that Corollary 1 does not extend to multiple-type housing markets with lexicographic preferences.

Example 3 (**Corollary 1 does not extend to $\boldsymbol{R_l^N}$**).
Consider $R \in \mathcal{R}_l^N$, $N = \{1, 2, 3\}$, $T = \{H(ouse), C(ar)\}$, $O = \{H_1, H_2, H_3, C_1, C_2, C_3\}$, each agent i's endowment (H_i, C_i), and

$$\boldsymbol{R_1} : H_2, \boldsymbol{H_1}, H_3, C_3, C_2, \boldsymbol{C_1},$$

$$\boldsymbol{R_2} : H_1, \boldsymbol{H_2}, H_3, C_1, \boldsymbol{C_2}, C_3,$$

$$\boldsymbol{R_3} : H_1, \boldsymbol{H_3}, H_2, C_1, \boldsymbol{C_3}, C_2.$$

Applying mTTC to R, at Step 1, the trading cycle $1 \to H_2 \to 2 \to H_1 \to 1$ forms; the trading cycle at Step 2 is $3 \to H_3 \to 3$; the trading cycle at Step 3 is $1 \to C_3 \to 3 \to C_1 \to 1$; and at Step 4 we have $2 \to C_2 \to 2$. The final outcome is the strict core allocation $x = ((H_2, C_3), (H_1, C_2), (H_3, C_1))$.

There is another strict core allocation $x' = ((H_2, C_2), (H_1, C_1), (H_3, C_3))$. We show that $mTTC(\text{sNash}(\Gamma_{mTTC}(R))) = \{x\} \subsetneq \{x, x'\} = SC(R)$.

First, we prove that truth-telling, i.e., reporting preference profile R, is a strict strong Nash equilibrium. Suppose that truth-telling is not a strict strong Nash equilibrium. Then, a profitable deviation from R and x exists, i.e., there exist $S \subseteq N$ and $R'_S \in \mathcal{R}_l^S$ such that for each agent $i \in S$, $mTTC_i(R'_S, R_{-S}) \; R_i \; x_i$ and for some agent $j \in S$, $mTTC_j(R'_S, R_{-S}) \; P_j \; x_j$. We now show that no such $S \subseteq N$ exists.

Note that at x, agent 1 receives his best allotment and thus coalition $\{1\}$ has no profitable deviation from R. Furthermore, if agent 1 takes part in a profitable deviation, then he must still receive (H_2, C_3).

For coalitions $\{2\}$ or $\{1, 2\}$, agent 2 can only be better off by receiving (H_1, C_1). However, he can only receive C_1 at Step 1 of mTTC by misreporting his preferences as $R'_2 : C_1, \cdots$. However, with R'_2, at Step 2, agent 3 would receive H_1 and agent 2 would not be better off.

For coalitions $\{3\}$ or $\{1, 3\}$, agent 3 can only be better off by receiving (H_1, C_1). However, then H_3 has to be assigned to agent 1 or agent 2, which would violate *individual rationality* (recall that agent 1 only participates in a deviating coalition if he still receives H_2 and agent 2 then would receive H_3).

Next, consider coalition $\{2, 3\}$, which has a conflict of interest. Agent 2 can only be better off by receiving (H_1, C_1), which leaves agent 3 with an allotment that is worse than $x_3 = (H_3, C_1)$, and agent 3 can only be better off by receiving (H_1, C_1), which leaves agent 2 with an allotment that is worse than $x_2 = (H_1, C_2)$.

Finally, the grand coalition $\{1, 2, 3\}$ cannot profitably deviate because mTTC is *Pareto efficient*. Hence, no profitable deviation from R and x exists.

Second, we prove that x' is not a strict strong Nash equilibrium outcome. Assume that there is a strict strong Nash equilibrium $R' = (R'_1, R'_2, R'_3)$ such that $mTTC(R') = x'$. We show that there is a profitable deviation for coalition $\{1, 3\}$, i.e., there exists $R'' = (R''_1, R'_2, R''_3)$ such that for each agent $i \in \{1, 3\}$, $mTTC_i(R'')\ R_i\ x'_i$ and for some agent $j \in \{1, 3\}$, $mTTC_j(R'')\ P_j\ x'_j$.

There are two cases depending on agent 2's object-type importance order at R'_2.

Case 1. Agent 2 misreported his importance order at R'_2, i.e., $\pi'_2 : C, H$.

Recall that $mTTC_2(R') = (H_1, C_1)$. Hence, by *individual rationality* of $mTTC$, at R', agent 2 ranked C_1 above C_2 and

$$\boldsymbol{R'_2} : C_3, C_1, \boldsymbol{C_2}, \cdots, \quad \text{or}$$
$$\boldsymbol{R'_2} : C_1, \boldsymbol{C_2}, C_3, \cdots, \quad \text{or}$$
$$\boldsymbol{R'_2} : C_1, C_3, \boldsymbol{C_2}, \cdots.$$

Next, consider strategy profile $R'' = (R''_1, R'_2, R_3)$ obtained by agents 1 and 3 deviating from R' such that

$$\boldsymbol{R''_1} : C_3, \boldsymbol{C_1}, C_2, H_2, \boldsymbol{H_1}, H_3.$$

Applying mTTC to R'', at Step 1, the trading cycle $1 \to C_3 \to 3 \to H_1 \to 1$ forms; and at Step 2 we have $2 \to C_1 \to 1 \to H_2 \to 2$. The final outcome is $mTTC(R'') = y = ((H_2, C_3), (H_3, C_1), (H_1, C_2))$. Since $y_1\ P_1\ x'_1$ and $y_3\ P_3\ x'_3$, coalition $\{1, 3\}$ has an incentive to deviate from R' to R'', which implies that R' is not a strict strong Nash equilibrium; a contradiction.

Case 2. Agent 2 truthfully reported his importance order at R'_2, i.e., $\pi'_2 : H, C$.

Recall that $mTTC_2(R') = (H_1, C_1)$. Hence, by *individual rationality* of $mTTC$, at R', agent 2 ranked H_1 above H_2 and

$$\boldsymbol{R'_2} : H_3, H_1, \boldsymbol{H_2}, \cdots, \quad \text{or}$$
$$\boldsymbol{R'_2} : H_1, \boldsymbol{H_2}, H_3, \cdots, \quad \text{or}$$
$$\boldsymbol{R'_2} : H_1, H_3, \boldsymbol{H_2}, \cdots \quad .$$

Next, consider strategy profile $R'' = (R_1, R'_2, R''_3)$ obtained by agents 1 and 3 deviating from R' such that

$$\boldsymbol{R''_3} : \boldsymbol{H_3}, H_1, H_2, C_1, \boldsymbol{C_3}, C_2.$$

Applying mTTC to R'', at Step 1, the trading cycle $3 \to H_3 \to 3$ forms; and we also have $1 \to H_2 \to 2 \to H_1 \to 1$ (this cycle, depending on R'_2, occurs at Step 1 or Step 2). Subsequently, we have the trading cycle $1 \to C_3 \to 3 \to C_1 \to 1$. The final outcome is $mTTC(R'') = x = ((H_2, C_3), (H_1, C_2), (H_3, C_1))$. Since $x_1\ P_1\ x'_1$ and $x_3\ P_3\ x'_3$, coalition $\{1, 3\}$ has an incentive to deviate from R' to R'', which implies that R' is not a strict strong Nash equilibrium; a contradiction. ◇

Based on Corollary 1 and Example 3 one could now conjecture that for each multiple-type housing market $R \in \mathcal{R}_l^N$ and each *individually rational* and *Pareto efficient* mechanism f, we have $f(\text{sNash}(\Gamma_f(R))) \subseteq \text{SC}(R)$. That conjecture is almost correct; however, we need to strengthen *individual rationality* and *Pareto efficiency* to *strict core-stability* (see Example 5).

3.3.2. Main results

We show that for lexicographic preferences, if a mechanism is *strictly core-stable*, then any strict strong Nash equilibrium of the corresponding preference revelation game will induce a strict core allocation. However, for some multiple-type housing markets with lexicographic preferences, there exist strict core allocations that cannot be implemented via strict strong Nash equilibrium.

Theorem 1. *Let f be a strictly core-stable mechanism on $\mathcal{R}_l^N$.*
Then, for each $R \in \mathcal{R}_l^N$ and the corresponding preference revelation game $\Gamma_f(R) = (\mathcal{R}_l^N, f, R)$, the set of strict strong Nash equilibrium outcomes is a subset of the strict core, that is,

$$f(\mathrm{sNash}(\Gamma_f(R))) \subseteq \mathrm{SC}(R).$$

Furthermore, there exist $R \in \mathcal{R}_l^N$ such that $f(\mathrm{sNash}(\Gamma_f(R))) \subsetneq \mathrm{SC}(R)$.

We would like to emphasize that *strict core-stability* of f is key for this result. Clearly, if for some preference profiles, the strict core is empty, then a *strictly core-stable* mechanism f cannot exist. Thus, in this first result, we restrict the preference profile domain to $\mathcal{R}_l^N$ with the intent to generalize Theorem 1 later on.

Proof. Let f be a *strictly core-stable* mechanism on $\mathcal{R}_l^N$.

First, let $R \in \mathcal{R}_l^N$ and assume by contradiction, that $f(\mathrm{sNash}(\Gamma_f(R))) \not\subseteq \mathrm{SC}(R)$. Let $R' \in \mathcal{R}_l^N$ be such that $R' \in \mathrm{sNash}(\Gamma_f(R))$ and $f(R') = x \notin \mathrm{SC}(R)$. Hence, x can be weakly blocked by a coalition S and there exists an allocation y such that (1) for each $i \in S$ and each $t \in T$, $y_i^t \in \{o_j^t\}_{j \in S}$, and (2') for each $i \in S$, $y_i \, R_i \, x_i$, and for some $j \in S$, $y_j \, P_j \, x_j$.

Now we consider the profile $(\hat{R}_S, R'_{-S}) \in \mathcal{R}_l^N$ such that each agent $i \in S$ ranks allotment y_i as his best allotment; for each $i \in S$, it then holds that $\hat{R}_i : y_i, \ \cdots$, i.e., each agent i, for each object type t, ranks y_i^t as best type-t object. We want to show that coalition S has an incentive to deviate from R'_S to $\hat{R}_S$. To this end, we first prove the following claim.

Claim 1. For each $i \in S$, we have $f_i(\hat{R}_S, R'_{-S}) = y_i$.

Let $z = f(\hat{R}_S, R'_{-S})$. Suppose that for some agent $j \in S$, $z_j \neq y_j$. We show that z is not a strict core allocation at $(\hat{R}_S, R'_{-S})$, i.e., $z \notin \mathrm{SC}(\hat{R}_S, R'_{-S})$.

At $(\hat{R}_S, R'_{-S})$, for each agent $i \in S$, $y_i \, \hat{R}_i \, z_i$ because y_i is his best allotment. Since $z_j \neq y_j$, $y_j \, \hat{P}_j \, z_j$. Therefore, at $(\hat{R}_S, R'_{-S})$, allocation z can also be weakly blocked by coalition S via allocation y. Thus, $f(\hat{R}_S, R'_{-S}) \notin \mathrm{SC}(\hat{R}_S, R'_{-S})$, which contradicts that f is *strictly core-stable*. □

Strictly speaking, by Claim 1, we now only know that $f(\hat{R}_S, R'_{-S}) = y'$ such that $y'_S = y_S$. However, since allotments to agents in $N \setminus S$ play no role in our proof, it is without loss of generality to assume that $y' = y$. Hence, when coalition S deviates from R'_S to $\hat{R}_S$, by Claim 1 and without loss of generality, $f(\hat{R}_S, R'_{-S}) = y$. Thus, since $f(R')$ is weakly blocked by S via y, for each $i \in S$, $f_i(\hat{R}_S, R'_{-S}) \, R_i \, f_i(R')$ and for $j \in S$, $f_j(\hat{R}_S, R'_{-S}) \, P_j \, f_j(R')$; contradicting that R' is a strict strong Nash equilibrium.

Example 3 exhibits a profile $R \in \mathcal{R}_l^N$ such that $f(\mathrm{sNash}(\Gamma_f(R))) \subsetneq \mathrm{SC}(R)$ (recall that in Example 3 there is a unique strict strong Nash equilibrium outcome while multiple strict core allocations exist). □

Remark 3. Existence of strict strong Nash equilibria, an open problem
The existence of (strict) strong Nash equilibria is proven for specific classes of games, such as social choice / voting (Dutta and Sen, 1991), congestion games (Holzman and Law-Yone, 1997), cost sharing games (Epstein et al., 2009), and continuously convex games (Nessah and Tian, 2014). However, in general, (strict) strong Nash equilibria need not exist.[6]

> **Question:** Let f be a strictly core-stable mechanism on $\mathcal{R}_l^N$. For each problem $R \in \mathcal{R}_l^N$, do we have $f(\text{sNash}(\Gamma_f(R))) \neq \emptyset$?

For Shapley-Scarf housing markets and the TTC mechanism, truth-telling is a strict strong Nash equilibrium. Thus, for higher-dimensional multiple-type housing markets, one could conjecture that, mTTC allocations can always be implemented in strict strong Nash equilibrium. The following example shows that the mTTC allocation cannot always be implemented truthfully in strict strong Nash equilibrium.

Consider $R \in \mathcal{R}_l^N$ with $N = \{1,2,3\}$, $T = \{H(ouse), C(ar)\}$, $O = \{H_1, H_2, H_3, C_1, C_2, C_3\}$, each agent i's endowment (H_i, C_i), and

$$R_1 : H_2, \boldsymbol{H_1}, H_3, C_3, \boldsymbol{C_1}, C_2,$$

$$R_2 : H_3, H_1, \boldsymbol{H_2}, \boldsymbol{C_2}, C_1, C_3,$$

$$R_3 : H_1, H_2, \boldsymbol{H_3}, \boldsymbol{C_3}, C_1, C_2.$$

Applying mTTC to R, at Step 1, the trading cycle $1 \to H_2 \to 2 \to H_3 \to 3 \to H_1 \to 1$ forms; the trading cycles at Step 2 are $2 \to C_2 \to 2$ and $3 \to C_3 \to 3$; and at Step 3 we have $1 \to C_1 \to 1$. The final outcome is the strict core allocation $x = ((H_2, C_1), (H_3, C_2), (H_1, C_3))$.

However, the truth-telling profile R is not a strict strong Nash equilibrium: agent 1 has an incentive to misreport the following preferences

$$R_1' : C_3, \boldsymbol{C_1}, C_2, H_2, \boldsymbol{H_1}, H_3.$$

For profile $R' = (R_1', R_2, R_3)$, the mTTC allocation is $x' = ((H_2, C_3), (H_3, C_2), (H_1, C_1))$. Since $x_1' = (H_2, C_3)\ P_1\ (H_2, C_1) = x_1$, R is not a strict strong Nash equilibrium.

The above example illustrates that an implementation of the mTTC allocation in strict strong Nash equilibrium might require some agents to (possibly mutually) change their object type sequences. We neither found a systematic way for agents to change their object type sequences to show existence of strict strong Nash equilibria, nor did we manage to construct a counter example. ◇

A more general result

Note that the proof of Theorem 1 did not use many properties of the lexicographic preference domain. It turns out that our result can easily be extended to other preference domains. Consider a subdomain of preferences $\hat{\mathcal{R}} \subseteq \mathcal{R}$ that satisfies the following two assumptions.

Assumption 1 (**Strict core existence and minimal preference domain richness**)**.** Preference domain $\hat{\mathcal{R}} \subseteq \mathcal{R}$ satisfies

[6] Hoefer and Skopalik (2013) point out the following technical difficulty of finding strong Nash equilibria: "a strong Nash equilibrium must be the optimal solution of multiple non–convex optimization problems."

(a) *strict core existence* if for each problem $R \in \hat{\mathcal{R}}^N$, $\mathrm{SC}(R) \neq \emptyset$; and

(b) *minimal preference domain richness* if for each allocation $x \in X$, each agent i can position x_i as his best allotment; i.e., for each $x \in X$, there exists a profile $\hat{R} \in \hat{\mathcal{R}}^N$ such that for each $i \in N$, $\hat{R}_i : x_i, \cdots$.

Assumption 1 is simple and reasonable. Assumption 1 (a) allows us to focus on the solution of the strict core and for that the strict core should always be non-empty. Assumption 1 (b) is a very weak preference domain richness condition that is different from the one used by Sönmez (1999, Assumption B) and weaker than the one imposed by Takamiya (2009, Condition A). We discuss the preference domain richness conditions of Sönmez (1999) and Takamiya (2009) in Appendix A.

Remark 4. Preference domains satisfying Assumption 1
The domains of weak and strict preferences for Shapley-Scarf housing markets and the lexicographic preference domain for multiple-type housing markets all satisfy Assumption 1. There are various larger lexicographic domains, e.g., those of Monte and Tumennasan (2015) and Sikdar et al. (2017, generalized lexicographical preferences), that satisfy Assumption 1. Hence, our Theorem 1 applies to these settings as well (see the following Theorem 2). $\diamond$

We now show that Theorem 1 can be extended to any preference domain $\hat{\mathcal{R}} \subseteq \mathcal{R}$ satisfying Assumption 1.

Theorem 2. *Let $\hat{\mathcal{R}}$ satisfy Assumption 1 and let f be a strictly core-stable mechanism on $\hat{\mathcal{R}}^N$. Then, for each problem $R \in \hat{\mathcal{R}}^N$ and the corresponding preference revelation game $\Gamma_f(R) = (\hat{\mathcal{R}}^N, f, R)$, the set of strict strong Nash equilibrium outcomes is a subset of the strict core, that is, $f(\mathrm{sNash}(\Gamma_f(R))) \subseteq \mathrm{SC}(R)$.*

Furthermore, there exist $R \in \hat{\mathcal{R}}^N$ such that $f(\mathrm{sNash}(\Gamma_f(R))) \subsetneq \mathrm{SC}(R)$.

***Proof*.** The proof is the same as that of Theorem 1 since in that proof the only properties of the preference domain that were (implicitly) used were strict core existence and minimal domain richness. $\square$

The role of assumptions in Theorems 1 and 2

In Theorems 1 and 2, we make three sufficient assumptions: (a) strict core existence, (b) minimal preference domain richness, and (c) *strict core-stability* of f. We now show that if assumptions (a) and (c) do not hold, then our result(s) need not be true. We do not discuss the role of minimal preference domain richness for our result since we believe that, once ranking certain allotments first is not possible for the agents, one starts to discuss very unstructured preference domains.

(a) For some $R \in \mathcal{R}_s^N$ it is possible that $\mathrm{SC}(R) = \emptyset$ and $\mathrm{sNash}(\Gamma_f(R)) \neq \emptyset$. See Example 4 below.

(c) If f is *individually rational* and *Pareto efficient* but not *strictly core-stable* (even if f is defined on $\mathcal{R}_l^N$), then the allocation induced by f may not be a strict core allocation. See Example 5 below.

Example 4 (Strict core existence is important for our result to hold).
Consider Example 2.2 in Konishi et al. (2001), i.e., consider $R \in \mathcal{R}_s^N$ with $N = \{1, 2, 3\}$, $T = \{H(ouse), C(ar)\}$, $O = \{H_1, H_2, H_3, C_1, C_2, C_3\}$, each agent i's endowment (H_i, C_i), and

$\boldsymbol{R_1}$: $(H_1, C_3), (H_3, C_3), (H_1, C_2), \boldsymbol{(H_1, C_1)}, \cdots$,

$\boldsymbol{R_2}$: $(H_2, C_3), (H_2, C_1), (H_3, C_3), (H_3, C_1), \boldsymbol{(H_2, C_2)}, \cdots$,

$\boldsymbol{R_3}$: $(H_2, C_1), (H_2, C_2), (H_3, C_1), (H_1, C_1), (H_3, C_2), (H_1, C_2), (H_2, C_3), \boldsymbol{(H_3, C_3)}, (H_1, C_3)$.

Note that $\mathrm{SC}(R) = \emptyset$ (see Konishi et al., 2001, Example 2.2).

Now, consider a mechanism f that chooses a strict core allocation whenever the strict core is nonempty and otherwise it determines an allocation by serial dictatorship based on the sequence of agents $1 \triangleright 3 \triangleright 2$: agent 1 moves first and chooses his most preferred allotment; then agent 3 moves and, considering only the remaining objects, chooses his most preferred allotment; finally agent 2 receives the remaining objects. By definition, f is *strictly core-stable* whenever this is possible.

At the preference revelation game $\Gamma_f(R)$, by truth-telling agents 1 and 3 get their best allotments, and hence, agent 2's strategy does not influence the outcome. Thus, the truth-telling strategy profile is a strict strong Nash equilibrium, i.e., $R \in \mathrm{sNash}(\Gamma_f(R))$, and its outcome is $f(R) = ((H_1, C_3), (H_3, C_2), (H_2, C_1))$. Since the strict core is empty, we thus have $f(\mathrm{sNash}(\Gamma_f(R))) \not\subseteq \mathrm{SC}(R)$. ◇

Example 5 (Strict core-stability of $\boldsymbol{f}$ is important for our result to hold).
We introduce the so-called *trading cycles and chains (TCC) algorithm / mechanism* on $\mathcal{R}_l^N$. The first step of the TCC algorithm is the same as that of the mTTC algorithm: let each agent point to his most preferred object and clear all trading cycles. Furthermore, "deactivate" all agents who have received an object from *another agent*; a deactivated agent cannot point to an object in the next steps of the algorithm.

The main difference between TCC and mTTC now arises: only agents who did not receive any object from another agent yet are active and can proceed to the next step and point to their most preferred object among those objects that have not been assigned yet. In the following steps, if there is a trading cycle, then we assign objects to the agents in the trading cycle.

However, if there is no trading cycle (this can happen due to the fact that some agents are inactive), then we clear a so-called trading chain where the last object belongs to a de-activated agent - we select a trading chain according to a tie-breaking rule, e.g., if two agents i and j point to the same object, then we break the tie in favor of agent $j > i$. Agents along the chain receive the object they point to (except the last agent in the chain who was inactive). Again, deactivate all agents who have received an object from another agent.

Continue clearing first trading cycles then trading chains until all agents are deactivated. Then, we activate all agents and repeat the TCC algorithm to allocate the remaining objects.

Since at each step of the TCC algorithm, each agent who receives an object receives his most preferred object among all of his feasible continuation objects, the TCC mechanism is *individually rational* and *Pareto efficient*. The following example shows that the TCC mechanism is not *strictly core-stable*.

Consider $R \in \mathcal{R}_l^N$ with $N = \{1, 2, 3\}$, $T = \{H(ouse), C(ar)\}$, $O = \{H_1, H_2, H_3, C_1, C_2, C_3\}$, each agent i's endowment (H_i, C_i), and

$$\boldsymbol{R_1} : C_2, \boldsymbol{C_1}, C_3, H_2, \boldsymbol{H_1}, H_3,$$

$$\boldsymbol{R_2} : C_1, \boldsymbol{C_2}, C_3, H_1, H_3, \boldsymbol{H_2},$$
$$\boldsymbol{R_3} : H_1, H_2, \boldsymbol{H_3}, C_2, C_1, \boldsymbol{C_3}.$$

Applying mTTC to R yields the unique strict core allocation $x = ((H_2, C_2), (H_1, C_1), (H_3, C_3))$.

The TCC algorithm at R proceeds as follows. At Step 1 trading cycle $1 \to C_2 \to 2 \to C_1 \to 1$ forms, is executed, and agents 1 and 2 are deactivated. At Step 2, agent 3 points to his most preferred object H_1 and since this forms a trading chain, agent 3 receives H_1 and is deactivated.

Then, all agents are activated again. At Step 3, trading cycle $3 \to C_3 \to 3$ forms, is executed, and hence agent 3 is removed since he cannot receive more objects. Finally, agent 1 points to H_2 and agent 2 points to H_3, and since this forms a trading chain, agent 1 and 2 receive H_2 and H_3, respectively. The final allocation is $y = ((H_2, C_2), (H_3, C_1), (H_1, C_3))$. Allocation y is *individually rational* and *Pareto efficient*, but it is weakly blocked by coalition $\{1, 2\}$ via allocation x.

Let f^{TCC} be the TCC mechanism. We prove that truth-telling, i.e., reporting preference profile R, is a strict strong Nash equilibrium. Suppose that truth-telling is not a strict strong Nash equilibrium. Then, there exist $S \subseteq N$ and $R'_S \in \mathcal{R}_l^S$ such that for each agent $i \in S$, $f_i^{\text{TCC}}(R'_S, R_{-S})\ R_i\ y_i$ and for some agent $j \in S$, $f_j^{\text{TCC}}(R'_S, R_{-S})\ P_j\ y_j$.

Note that at y, agent 1 receives his best allotment and thus, if agent 1 takes part in a profitable deviation, then he must still receive (H_2, C_2). This implies that

(a) there is no profitable deviation for coalition $\{1\}$.

(b) We next show that there is no profitable deviation for coalitions $\{2\}$ and $\{1, 2\}$. Note that agent 2 can only be better off by receiving (H_1, C_1). Consider (R'_1, R'_2) (possibly $R'_1 = R_1$) and note that for this to be a profitable deviation, agents 1 and 2 still need to trade in the first step of the TCC algorithm, which means that they are not active in the second step where agent 3 now can choose an object.

If agents 1 and 2 traded cars C_1 and C_2 or car C_1 and house H_2, then agent 3 will choose H_1 in the second step. Thus, agent 2 cannot receive (H_1, C_1) and hence would not be better off.

If agents 1 and 2 traded house H_1 and car C_2, then agent 3 will choose H_2 in the second step. Thus, the third step is a normal trading step in which agent 1 will point to the only remaining house H_3, agent 3 points at car C_1, and hence the final allocation is $((H_3, C_2), (H_1, C_3), (H_2, C_1))$. Thus, agent 2 would not be better off.

If agents 1 and 2 traded houses H_1 and H_2, then both of them participated in the preference deviation. Then, agent 3 will choose H_3. However, since agent 3 still did not receive any object from another agent yet, he remains the only active agent and will choose car C_2 in the third step. Now, the final allocation is either $((H_2, C_1), (H_1, C_3), (H_3, C_2))$ or $((H_2, C_3), (H_1, C_1), (H_3, C_2))$. Thus, agent 1 or agent 2 is worse off.

(c) We next show that there is no profitable deviation for coalitions $\{3\}$ and $\{1, 3\}$. Note that agent 3 can only be better off by receiving (H_1, C_1) or (H_1, C_2). Consider (R'_1, R'_3) (possibly $R'_1 = R_1$) and note that agents 1 and 2 still trade in the first step of the TCC algorithm (either because agents 1 and 2 report true preferences, or because agent 1 participates in the deviation and hence must still receive (H_2, C_2)).

If agent 1 did not participate in the deviation, then agents 1 and 2 trade cars C_1 and C_2 in the first step of the TCC algorithm. But then, agent 3 cannot be bettor off since C_1 and C_2 are already assigned.

Assume that agent 1 participated in the deviation. Then, agent 1 must still receive (H_2, C_2). Hence, since agent 2 reports preference truthfully, agent 2 will receive car C_1 in the first step.

Thus, agent 3 can only be better off by receiving (H_1, C_2), which is not compatible with agent 1 receiving (H_2, C_2). Hence, agent 3 cannot be better off.

(d) We next show that there is no profitable deviation for coalition $\{2, 3\}$. Note that agent 2 can only be better off by receiving (H_1, C_1), but this would make agent 3 worse off. Thus, agent 2 would have to receive $y_2 = (H_3, C_1)$ at a profitable deviation and in order for agent 3 to be better off, he would have to receive (H_1, C_2). This implies that agent 1 receives (H_2, C_3), which violates *individual rationality*.

(e) Finally, the grand coalition $\{1, 2, 3\}$ cannot profitably deviate because the TCC mechanism is *Pareto efficient*. ◇

3.4. Conclusion

We consider multiple-type housing markets when agents have lexicographic preferences $\mathcal{R}_l$; or alternatively, preferences are drawn from a preference domain $\hat{\mathcal{R}}$ that guarantees strict core existence and that satisfies a minimal preference domain richness condition (see Assumption 1). We show that if a mechanism is *strictly core-stable*, then any strict strong Nash equilibrium outcome of its corresponding preference revelation game is a strict core allocation (Theorems 1 and 2). The converse statement is not true, i.e., there exist markets with strict core allocations that cannot be implemented in strict strong Nash equilibrium (Example 3). We also demonstrate the necessity of two crucial assumptions (strict core non-emptiness and *strict core-stability* of mechanisms) in our results (Examples 4 and 5).

Comparing our results to Takamiya's result for Shapley-Scarf housing markets, Corollary 1 (see Appendix A for the generalized individual goods allocation model considered in Takamiya, 2009), our results (Theorems 1 and 2) have two differences with his main result.

First, we show that not all strict core allocations may be implementable through strict strong Nash equilibria of the preference revelation game while Takamiya (2009) shows full implementation for Shapley-Scarf housing markets. The main reason for our partial implementation versus his full implementation result is that our preference domains are less rich than the ones he considers. Neither separable nor lexicographic preferences satisfy Takamiya's preference domain richness condition (Takamiya, 2009, Condition A, see Appendix A). For example, for multiple-type housing market problems with lexicographic preferences, no agent can protect an allotment by positioning it as his first best and his endowment as his second best allotment (an argument that is crucial in Takamiya's proof).

Second, we require *strict core-stability* for our mechanisms while Takamiya (2009) only requires *individual rationality* and *Pareto efficiency*. In Takamiya's model, each agent only demands one object. Thus, each agent will trade within only one coalition. Therefore, once the induced allocation (the equilibrium outcome) is *individually rational* and *Pareto efficient*, no coalition can block it.[7] However, the same is not true for multiple-type housing markets because each agent may trade different objects with different coalitions. That is, a multiple-type housing market cannot be easily transformed into a coalition formation game.

[7] See Takamiya's (2009) proof of Theorem (B) for details.

4. CHAPTER 2 A Characterization of the Coordinate-Wise Top-Trading-Cycles Mechanism for Multiple-Type Housing Markets

4.1. Introduction

In many applied matching problems, indivisible goods that are in unit demand have to be assigned without monetary transfers. One of the most prominent such problems is modeled by classical Shapley-Scarf housing markets (Shapley and Scarf, 1974). Shapley and Scarf (1974) consider an exchange economy in which each agent owns an indivisible object (say, a house); each agent has preferences over houses and consumes exactly one house. The objective of the market designer then is to reallocate houses among agents. When preferences are strict, Shapley and Scarf (1974) show that the strict core (defined by a weak blocking notion) has remarkable features: it is non-empty, and can be easily calculated by the so-called top-trading-cycles (TTC) algorithm (due to David Gale). Moreover, TTC satisfies important incentive properties, *strategy-proofness* (Roth, 1982) as well as the stronger property of *group strategy-proofness* (Bird, 1984). Furthermore, it is known that TTC is the unique mechanism satisfying *Pareto efficiency*, *individual rationality*, and *strategy-proofness* (Ma, 1994; Svensson, 1999).

However, more general problems of exchanging indivisible objects that are in multi-unit demand are known to be very difficult. In this chapter, we consider an extension of the classical Shapley-Scarf housing markets by allowing multi-unit demand: multiple-type housing markets, to use the language of Moulin (1995). In this model, objects are of different types (say, houses, cars, etc.) and agents initially own and exactly wish to consume one object of each type. A familiar example for most readers would be the situation of students' enrollment at many universities where courses are taught in small groups and in multiple sessions (Klaus, 2008). Furthermore, for term paper presentations during a course, students may want to exchange their assigned topics and dates (Mackin and Xia, 2016); hospitals may want to improve their surgery schedule for surgeons by swapping surgery staff, operating rooms, and dates (Huh et al., 2013); and in cloud computing (Ghodsi et al., 2011, 2012) and 5G network slicing (Peng et al., 2015; Bag et al., 2019; Han et al., 2019), there may be several types of resources that agents require, including CPU, memory, and storage.

This model is firstly studied by Konishi et al. (2001). Their results are mainly negative: they show that even if we further restrict preferences to be strict and additively separable, the strict core may still be empty. Moreover, no mechanism satisfies *Pareto efficiency*, *individual* *rationality*, and *strategy-proofness*.

Despite their negative results, for (strictly) separable preferences, Wako (2005) suggests an alternative solution concept to the strict core by first decomposing a multiple-type housing market into coordinatewise submarkets and second, determining the strict core in each submarket. Wako (2005) calls this unique outcome the commoditywise competitive allocation and shows that it is implementable in (self-enforcing) coalition-proof Nash equilibria but not in strong Nash equilibria.[1]

Based on Wako's result, we investigate the mechanism that always selects the commoditywise competitive allocation; since this allocation can be obtained by using the TTC algorithm for each object type, we refer to it as the *coordinatewise TTC mechanism (cTTC)*. Although cTTC is not *Pareto efficient*, it does have many desirable properties: it is *individually rational*, *strategy-proof*, and *second-best incentive compatible*, i.e., it is *not Pareto dominated* by any other *strategy-proof* mechanism (Klaus, 2008). In view of these positive results, one may wonder whether cTTC can be characterized by weakening *Pareto efficiency* and strengthening *strategy-proofness*.

For Shapley-Scarf housing markets with strict preferences, a characterization along these lines is provided by Takamiya (2001): he shows that TTC is the only mechanism satisfying *unanimity*, *individual rationality*, and *group strategy-proofness*.[2] Based on Takamiya's result, one could now conjecture that this characterization of TTC for Shapley-Scarf housing markets can be carried over to cTTC for multiple-type housing markets. That conjecture is almost true; however, we need to weaken *group strategy-proofness* to *strategy-proofness* and *non-bossiness*.[3] In other words, inspired by Takamiya's result for Shapley-Scarf housing markets, we show that, remarkably, cTTC is the only mechanism satisfying *unanimity* (or *ontoness*), *individual rationality*, *strategy-proofness*, and *non-bossiness* (see Theorems 3 and 4 for lexicographic and separable preferences, respectively). We obtain corresponding results when replacing [*strategy-proofness* and *non-bossiness*] with *effective group (or pairwise) strategy-proofness* (Corollaries 2 and 3).

Our characterizations of cTTC constitute the first characterizations of an extension of the prominent TTC to multiple-type housing markets. Furthermore, our results suggest that when preferences are separable, cTTC is outstanding; first, because some efficiency in the form of *unanimity* is preserved (even if full *Pareto efficiency* cannot be reached), and second, because of its incentive robustness in the form of *strategy-proofness*, *non-bossiness*, and *effective group (pairwise) strategy-proofness* (even if full *group strategy-proofness* cannot be reached). Moreover, we also provide several impossibility results (Theorem 5 and Corollary 4) for strict (but otherwise unrestricted) preferences:

- there is no mechanism satisfying *unanimity*, *individual rationality*, and *strategy-proofness* (Theorem 5);
- there is no mechanism satisfying *ontoness*, *individual rationality*, *strategy-proofness*, and *non-bossiness* (Corollary 4).

[1]However, (1) the commoditywise competitive allocation may be *Pareto inefficient*; and (2) the mechanism that always selects this allocation is *not group strategy-proof* (see Wako, 2005, Section 6, for details).

[2]In fact, Takamiya's characterization is based on *ontoness*, a weakening of *unanimity*. However, in the presence of *group strategy-proofness*, *ontoness* coincides with *unanimity*.

[3]When preferences are strict but otherwise unrestricted, the combination of *strategy-proofness* and *non-bossiness* is equivalent to *group strategy-proofness*. Example 6 shows that this is not true for separable preferences.

The chapter proceeds as follows. In the following section, Section 4.2, we introduce some properties of mechanisms, and some relevant results. We state our results in Section 4.3. In Subsection 4.3.1, we first show that for lexicographic preferences, a mechanism is *unanimous* (or *onto*), *individually rational*, *strategy-proof*, and *non-bossy* if and only if it is cTTC (Theorem 3). In Subsection 4.3.2, using Theorem 3, we obtain a corresponding characterization for separable preferences (Theorem 4). We would like to emphasize that the proof strategy to use lexicographic preferences as a "stepping stone" to obtain a corresponding result for separable preferences is, to the best of our knowledge, new. In Subsections 4.3.1 and 4.3.2 we obtain corresponding results when replacing [*strategy-proofness* and *non-bossiness*] with *effective group (or pairwise) strategy-proofness* (Corollaries 2 and 3). In Subsection 4.3.3, we finally show several impossibility results (Theorem 5 and Corollary 4). Section 4.4 concludes with a discussion of our results and how they relate to the literature.

4.2. Preliminaries

Note that all following definitions and results for the domain of strict preference profiles $\mathcal{R}^N$ can be formulated for the domain of separable preference profiles $\mathcal{R}_s^N$ or the domain of lexicographic preference profiles $\mathcal{R}_l^N$.

First, we introduce a weaker condition than *unanimity* that guarantees that no allocation is a priori excluded.

Definition (Ontoness).
A mechanism on $\mathcal{R}^N$ is *onto* if each allocation is selected to some markets. In other words, a mechanism is *onto* if it is an onto function.

It is immediate that *unanimity* implies *ontoness* (see also Lemma 2).

Next, we introduce a strategic robustness property that is stronger than *strategy-proofness* and weaker than *group strategy-proofness*. Serizawa (2006) introduces and analyzes effective pairwise strategy-proofness for various economic models: a mechanism that is effectively pairwise strategy-proof excludes unilateral as well as "self-enforcing" pairwise manipulations. Recently, Biró et al. (2022a) extend Serizawa's self-enforcing notion of pairwise strategy-proofness to robustness against coalitional deviations of arbitrary sizes (assuming "minimality of the self-enforcing manipulations").[4]

Definition (Effective group (pairwise) strategy-proofness).
A coalition of agents $S \subseteq N$ can manipulate mechanism f in a self-enforcing manner if there exist some $R \in \mathcal{R}^N$ and some $R'_S \in \mathcal{R}^S$ such that

- coalition S can manipulate mechanism f at R via R'_S:
  for each $i \in S$, $f_i(R'_S, R_{-S})\ R_i\ f_i(R)$ and for some $j \in S$, $f_j(R'_S, R_{-S})\ P_j\ f_j(R)$ and
- *coalition S is self-enforcing*: for each $i \in S$, $f_i(R'_S, R_{-S})\ R_i\ f_i(R_i, R'_{S\setminus\{i\}}, R_{-S})$.

If a coalition of agents S can manipulate mechanism f at R via R'_S, then S is a *minimal manipulating coalition* at R via R'_S if there is no $S' \subsetneq S$ such that S' can manipulate mechanism f at R via $R'_{S'}$. A mechanism on $\mathcal{R}^N$ is *effectively group strategy-proof* if no minimal manipulating

[4] Biró et al. (2022a) refer to their property as *self-enforcing group strategy-proofness.*

coalition of agents can manipulate f in a self-enforcing manner; it is *effectively pairwise strategy-proof* if it is *strategy-proof* and no pair of agents can manipulate f in a self-enforcing manner.

Alva (2017, Proposition 1) shows that *strategy-proofness* and *non-bossiness* are equivalent to *effective pairwise strategy-proofness*, and Biró et al. (2022a, Proposition 11) show that *strategy-proofness* and *non-bossiness* are equivalent to *effective group strategy-proofness*. Thus, these studies provide an intuition of why the invariance property *non-bossiness* can be considered to be an incentive property as well. Both results apply to our model as well.

Lemma 1 (Alva, 2017; Biró et al., 2022a).
A mechanism on $\mathcal{R}^N$ is strategy-proof and non-bossy if and only if it is effectively group (or pairwise) strategy-proof.

We already mentioned that *unanimity* implies *ontoness*. We next show that, in the presence of *strategy-proofness* and *non-bossiness*, *ontoness* implies *unanimity*.

Lemma 2.

(a) *If a mechanism on $\mathcal{R}^N$ is unanimous, then it is onto.*

(b) *If a mechanism on $\mathcal{R}^N$ is strategy-proof, non-bossy, and onto, then it is unanimous.*

***Proof*.** **(a)** Let f on $\mathcal{R}^N$ be *unanimous*. Fix any allocation $x \in X$. Let $R \in \mathcal{R}^N$ be a preference profile such that x is *unanimously* best under R. Then, by *unanimity* of f, $f(R) = x$. Hence, f is an *onto* function.

(b) Let f on $\mathcal{R}^N$ be *strategy-proof*, *non-bossy*, and *onto*. Let $x \in X$ and $R \in \mathcal{R}^N$ be a preference profile such that x is unanimously best under R. By *ontoness* of f, there exists a preference profile $R' \in \mathcal{R}$ such that $f(R') = x$. Let $i \in N$ and $y = f(R_i, R'_{-i})$. By *strategy-proofness* of f, we have $y_i \; R_i \; x_i$. Since x_i is agent i's most preferred allotment, we have $y_i = x_i$. Then, by *non-bossiness* of f, we have $f(R_i, R'_{-i}) = y = x = f(R')$. By applying this argument repeatedly for all agents in $N \backslash \{i\}$, we find that $f(R) = x = f(R')$. So, f is *unanimous*. □

Shapley-Scarf housing market results

As mentioned before, for $m = 1$ our model equals the classical Shapley-Scarf housing market model (Shapley and Scarf, 1974) and cTTC reduces to the standard TTC mechanism. The Shapley-Scarf housing market (with strict preferences) results that are pertinent for our analysis of multiple-type housing markets are the following.

Result 1 (Bird, 1984).
For Shapley-Scarf housing markets, TTC is *group strategy-proof*.

Note that *group strategy-proofness* implies *strategy-proofness* and *non-bossiness*. Thus, Result 1 also implies that TTC is *non-bossy* (Miyagawa, 2002, explicitly shows this). Also note that when preferences are strict and unrestricted, the combination of *strategy-proofness* and *non-bossiness* coincides with *group strategy-proofness*. Recently, Alva (2017) identifies preference domain properties such that this equivalence holds.

Result 2 (Pápai, 2000; Takamiya, 2001; Alva, 2017).
For Shapley-Scarf housing markets, a mechanism on $\mathcal{R}^N$ is *strategy-proof* and *non-bossy* if and only if it is *group strategy-proof.*

Result 3 (Ma, 1994; Svensson, 1999).
For Shapley-Scarf housing markets, only TTC is *Pareto efficient*, *individually rational*, and *strategy-proof.*

Result 4 (Takamiya, 2001).
For Shapley-Scarf housing markets, only TTC is *onto*, *individually rational*, *strategy-proof*, and *non-bossy*.

Extension of existing Shapley-Scarf housing market results to multiple-type housing markets

The results in the previous subsection imply that for Shapley-Scarf housing markets (with strict preferences), TTC satisfies

- *Pareto efficiency* and hence *unanimity* and *ontoness*;
- *individual rationality*; and
- *group strategy-proofness* and hence *strategy-proofness* and *non-bossiness.*

cTTC inherits most of these properties, except for *Pareto efficiency* and *group strategy-proofness.* Hence, TTC Results 1, 2, and 3 do not extend to cTTC when more than one object type is allocated.

Proposition 1. *On the domain of (i) lexicographic preferences and (ii) separable preferences,*

- *cTTC satisfies* unanimity, ontoness, individual rationality, strategy-proofness, non-bossiness, *and* effective group (pairwise) strategy-proofness.
- *cTTC satisfies neither* Pareto efficiency *nor* group strategy-proofness.

Proof. It is straightforward to check that cTTC on $\mathcal{R}_s^N$ is *individually rational* and *unanimous* (and hence *onto*).

We next show that cTTC on $\mathcal{R}_s^N$ inherits *strategy-proofness* from TTC. Let $R \in \mathcal{R}_s^N$, $i \in N$, and $\hat{R}_i \in \mathcal{R}_s$ with marginal preferences $(\hat{R}_i^1, \ldots, \hat{R}_i^m)$. By the definition and *strategy-proofness* of TTC, for each $t \in T$, $cTTC_i^t(R) = TTC_i^t(R^t)\ R_i^t\ TTC_i^t(\hat{R}_i^t, R_{-i}^t) = cTTC_i^t(\hat{R}_i, R_{-i})$. Then, by the separability of preferences, we have $cTTC_i(R) R_i cTTC_i(\hat{R}_i, R_{-i})$ and cTTC is *strategy-proof.*

Finally, to show that cTTC on $\mathcal{R}_s^N$ is *non-bossy*, let $R \in \mathcal{R}_s^N$, $i \in N$, and $\hat{R}_i \in \mathcal{R}_s$, with marginal preferences $(\hat{R}_i^1, \ldots, \hat{R}_i^m)$, be such that $cTTC_i(R) = cTTC_i(\hat{R}_i, R_{-i})$. Thus, for each $t \in T$, $cTTC_i^t(R) = cTTC_i^t(\hat{R}_i, R_{-i})$. Moreover, by definition of cTTC, we have for each $t \in T$, $cTTC_i^t(R) = TTC_i(R^t)$ and $cTTC_i^t(\hat{R}_i, R_{-i}) = TTC_i(\hat{R}_i^t, R_{-i}^t)$. Thus, for each $t \in T$, $TTC_i(R^t) = TTC_i(\hat{R}_i^t, R_{-i}^t)$, and since TTC is *non-bossy*, we have that for each $t \in T$, $TTC(R^t) = TTC(\hat{R}_i^t, R_{-i}^t)$. Then, for each $t \in T$, $cTTC^t(R) = cTTC^t(\hat{R}_i, R_{-i})$. Thus, $cTTC(R) = cTTC(\hat{R}_i, R_{-i})$ and cTTC is *non-bossy.*

Since cTTC on $\mathcal{R}_s^N$ is *strategy-proof* and *non-bossy*, by Lemma 1, it is also *effectively group (pairwise) strategy-proof.*

Example 6 below shows that cTTC on $\mathcal{R}_s^N$ is neither *Pareto efficient* nor *group strategy-proof.* □

Example 6 (***cTTC*** **is neither** ***Pareto efficient*** **nor** ***group strategy-proof*****).**
Consider the market with $N = \{1,2\}$, $T = \{H(ouse), C(ar)\}$, $O = \{H_1, H_2, C_1, C_2\}$, and where each agent i's endowment is (H_i, C_i). The preference profile $R \in \mathcal{R}_l^N$ is as follows:

$$\boldsymbol{R_1} : H_2, \boldsymbol{H_1}, \boldsymbol{C_1}, C_2,$$

$$\boldsymbol{R_2} : C_1, \boldsymbol{C_2}, \boldsymbol{H_2}, H_1.$$

Thus, agent 1, who primarily cares about houses, would like to trade houses but not cars, and agent 2, who primarily cares about cars, would like to trade cars but not houses. One easily verifies that $cTTC(R) = \big((H_1, C_1), (H_2, C_2)\big)$, the no-trade allocation. However, note that since preferences are lexicographic, both agents would be strictly better off if they traded cars and houses. Thus, allocation $\big((H_2, C_2), (H_1, C_1)\big)$ Pareto dominates $cTTC(R)$. Hence, $cTTC$ is not *Pareto efficient.* Furthermore, assume that both agents (mis)report their preferences as follows:

$$\boldsymbol{R_1'} : H_2, \boldsymbol{H_1}, C_2, \boldsymbol{C_1},$$

$$\boldsymbol{R_2'} : C_1, \boldsymbol{C_2}, H_1, \boldsymbol{H_2}.$$

Then, $cTTC(R') = \big((H_2, C_2), (H_1, C_1)\big)$, making both agents better off compared to $cTTC(R)$. Hence, $cTTC$ is not *group strategy-proof.* Finally, note that

$$cTTC_1(R_1, R_2') = (H_2, C_1)\ P_1\ (H_2, C_2) = cTTC_1(R')$$

and

$$cTTC_2(R_1', R_2) = (H_2, C_1)\ P_2\ (H_1, C_1) = cTTC_2(R'),$$

and hence R' is not a manipulation in a self-enforcing manner; cTTC does not violate *effective group (pairwise) strategy-proofness.* ◇

While Example 6 shows that cTTC is not *Pareto efficient*, Klaus (2008) shows that it is *second-best incentive compatible*, i.e., there exists no other *strategy-proof* mechanism that *Pareto dominates* cTTC. At the end of her paper, Klaus (2008) presents a mechanism for classical housing markets that is different from the TTC mechanism and satisfies *individual rationality*, *second-best incentive compatibility*, and *strategy-proofness*. This mechanism can be extended to multiple-type housing markets by applying it coordinatewise; thus, cTTC is not the unique mechanism that satisfies these properties.

Example 6 also shows that cTTC does not satisfy the three properties that are used in Result 3. Is there another mechanism that does satisfy the three properties? The following result gives an answer in the negative: there is no mechanism that satisfies *Pareto efficiency*, *individual rationality*, and *strategy-proofness*, neither on the domain of separable preference profiles nor on the domain of lexicographic preference profiles.

Result 5 (**Impossible trinity**).

(a) For multiple-type housing markets with separable preferences, no mechanism is *Pareto efficient*, *individually rational*, and *strategy-proof* (Konishi et al., 2001, Proposition 4.1).

(b) For multiple-type housing markets with lexicographic preferences, no mechanism is *Pareto efficient*, *individually rational*, and *strategy-proof* (Sikdar et al., 2017, Theorem 2).

Result 5 implies that there is no other mechanism that does better than cTTC by satisfying the three properties on either the domain of separable preference profiles or the domain of lexicographic preference profiles. However, cTTC on $\mathcal{R}_s^N$ ($\mathcal{R}_l^N$, respectively) does satisfy all the properties used in Result 4. In the next section we answer the question if Takamiya's characterization of TTC for Shapley-Scarf housing markets can be extended to characterize cTTC for multiple-type housing markets.

Finally, Proposition 1 also demonstrates that the equivalence of *strategy-proofness* and *non-bossiness* with *group strategy-proofness* (Result 2) does not extend to multiple-type housing markets with separable or lexicographic preferences (because *strategy-proofness* and *non-bossiness* do not imply *group strategy-proofness*).

4.3. Characterizing cTTC

From now on, we focus on the multiple-type extension of the Shapley-Scarf housing market model as introduced by Moulin (1995) with more than 1 agent and more than 1 type, i.e., $|N| = n > 1$ and $|T| = m > 1$.[5]

4.3.1. Characterizing cTTC for lexicographic preferences

We first show that Takamiya's result (Takamiya, 2001, Corollary 4.16) can indeed be extended to characterize cTTC for lexicographic preferences.

Theorem 3. *For multiple-type housing markets with lexicographic preferences, only cTTC is*

- *unanimous (or onto),*
- *individually rational,*
- *strategy-proof, and*
- *non-bossy.*

From Proposition 1 it follows that cTTC satisfies *unanimity* (or *ontoness*), *individual rationality*, *strategy-proofness*, and *non-bossiness*. Next, we explain the uniqueness part of the proof; the full proof that there is no other mechanism that satisfies the above properties is relegated to Appendices B and C.1.

First, we establish several auxiliary results for a mechanism f satisfying the properties of Theorem 3: invariance of f under (Maskin) monotonic transformations (Lemma 3) and *marginal individual rationality* (Lemma 4). Next, we assume that a mechanism f that is not equal to

[5]One agent multiple-type housing market problems are rather trivial since no trade occurs and for just one object type, we are back to the Shapley-Scarf housing market model.

cTTC, but has the same properties, exists. We then obtain a contradiction via a well-constructed sequence of preference profiles (by using the lexicographic nature of preferences).

Lemma 1 (Alva, 2017; Biró et al., 2022a) implies the following corollary.

Corollary 2. *For multiple-type housing markets with lexicographic preferences, only cTTC is*

- *unanimous (or onto),*
- *individually rational, and*
- *effectively group (or pairwise) strategy-proof.*

Note that even if one does not consider the domain of lexicographic preference profiles as an interesting or relevant preference profile domain for multiple-type housing markets, Theorem 3 serves as an important stepping stone to establish the corresponding characterization of cTTC for separable preferences, see Subsection 4.3.2. To the best of our knowledge, the technical tool of "lifting up" a result from lexicographic preferences to separable preferences is used here for the first time.

We establish the logical independence of the properties in Theorem 3 (Corollary 2) in Appendix C.3.

4.3.2. Characterizing cTTC for separable preferences

Note that for lexicographic preferences, under cTTC, the importance order of types plays no role because the allocation of each type only depends on the agents' marginal preferences of each type, i.e., for each market R and type t, $cTTC^t(R) = TTC(R^t_1, \dots, R^t_n)$. Thus, one could conjecture that Theorem 3 also holds for separable preferences. This conjecture is correct.

Theorem 4. *For multiple-type housing markets with separable preferences, only cTTC is*

- *unanimous (or onto),*
- *individually rational,*
- *strategy-proof, and*
- *non-bossy.*

From Proposition 1 it follows that cTTC on $\mathcal{R}^N_s$ satisfies *unanimity* (or *ontoness*), *individual rationality*, *strategy-proofness*, and *non-bossiness*. Next, we explain the uniqueness part of the proof; the full proof that there is no other mechanism that satisfies the above properties is relegated to Appendix C.2.

The uniqueness part of the proof works as follows. We assume that a mechanism is *unanimous* (or *onto*), *individually rational*, *strategy-proof*, and *non-bossy*. By Theorem 3, we know that if all agents happen to have lexicographic preferences, then the cTTC allocation is selected. Next, we consider a preference profile such that only one agent has separable and non-lexicographic preferences. We show that for this agent, if he (mis)reports lexicographic preferences without changing his marginal preferences, then he must receive the same allotment. According to Theorem 3, the allotment (in fact, the whole allocation) then equals the cTTC allotment (allocation). Hence, f selects the cTTC allocation if all but one agent have lexicographic preferences. By applying this preference replacement argument, one by one, for all other agents, we conclude that f equals cTTC on the domain of separable preference profiles.

Lemma 1 (Alva, 2017; Biró et al., 2022a) implies the following corollary.

Corollary 3. *For multiple-type housing markets with separable preferences, only cTTC is*

- *unanimous (or onto),*
- *individually rational, and*
- *effectively group (or pairwise) strategy-proof.*

The examples in Appendix C.3 are well-defined on the domain of separable preference profiles and establish the logical independence of the properties in Theorem 4 (Corollary 3).

4.3.3. Impossibility results for strict preferences

Note that cTTC is not well-defined for strict preferences since for non-separable preferences, marginal type preferences cannot be derived. Then, a natural question is if there exists an extension of cTTC to the domain of strict preference profiles that satisfies our properties. First, observe that the impossibility trinity result (Result 5) implies that for strict preferences, no mechanism satisfies *Pareto efficiency*, *individual rationality*, and *strategy-proofness*. Our next result shows that weakening *Pareto efficiency* to *unanimity* cannot resolve this impossibility.

Theorem 5. *For multiple-type housing markets with strict preferences, no mechanism is*

- *unanimous,*
- *individually rational, and*
- *strategy-proof.*

Proof. Without loss of generality, let $m = 2$. Suppose that there is a mechanism $f : \mathcal{R}^N \to X$ that is *unanimous*, *individually rational*, and *strategy-proof*. Let $x, y \in X \backslash \{e\}$ be such that at x agents 1 and 2 swap their endowments of type 2, i.e.,

$$\begin{aligned} & x_1 = (o_1^1, o_2^2, o_1^3, o_1^4, \ldots, o_1^m), \\ & x_2 = (o_2^1, o_1^2, o_2^3, o_2^4, \ldots, o_2^m), \\ \text{and for each } i = 3, \ldots, n, \quad & x_i = o_i \end{aligned}$$

and at y agents 1 and 2 swap their endowments of type 1, i.e.,

$$\begin{aligned} & y_1 = (o_2^1, o_1^2, o_1^3, o_1^4, \ldots, o_1^m), \\ & y_2 = (o_1^1, o_2^2, o_2^3, o_2^4, \ldots, o_2^m), \\ \text{and for each } i = 3, \ldots, n, \quad & y_i = o_i. \end{aligned}$$

Obviously, $x \neq y$.

Let $R \in \mathcal{R}^N$ be such that agents 1 and 2 prefer only their allotments at x and y to their endowments, they disagree on which allocation is the better one, and each other agent ranks his endowments highest, i.e.,

$$\begin{aligned} &R_1 : x_1, y_1, o_1, \ldots, \\ &R_2 : y_2, x_2, o_2, \ldots, \\ \text{and for each } i = 3, \ldots, n, \quad &R_i : o_i, \ldots. \end{aligned}$$

Note that $R \in \mathcal{R}^N \backslash \mathcal{R}_s^N$. There are only three *individually rational* allocations at R: x, y, and e.

Let

- $R'_1 : x_1, o_1, \ldots$,
- $R'_2 : y_2, o_2, \ldots$,
- $R''_1 : y_1, o_1, \ldots$, and
- $R''_2 : x_2, o_2, \ldots$.

Suppose that $f(R) = e$. Then, by *unanimity* of f, $f(R''_2, R_{-2}) = x$, which implies that agent 2 has an incentive to misreport R''_2 at R; contradicting *strategy-proofness* of f. Therefore, $f(R) \in \{x, y\}$.

Suppose that $f(R) = x$. Then, by *strategy-proofness* of f, $f_2(R'_2, R_{-2}) \neq y_2$ and hence by *individual rationality* of f, $f(R'_2, R_{-2}) = e$. However, by *unanimity* of f, $f(R''_1, R'_2, R_{-\{1,2\}}) = y$, which implies that agent 1 has an incentive to misreport R''_1 at (R'_2, R_{-2}); contradicting *strategy-proofness* of f.

Suppose that $f(R) = y$. Then, by *strategy-proofness* of f, $f_1(R'_1, R_{-1}) \neq x_1$ and hence, by *individual rationality* of f, $f(R'_1, R_{-1}) = e$. However, by *unanimity* of f, $f(R'_1, R''_2, R_{-\{1,2\}}) = x$, which implies that agent 2 has an incentive to misreport R''_2 at (R'_1, R_{-1}); contradicting *strategy-proofness* of f. □

Examples 7, 8, and 9 in Appendix C.3 are well-defined on the domain of strict preference profiles and establish the logical independence of the corresponding properties in Theorem 5.

Our next impossibility result is established by weakening *unanimity* to *ontoness* and by adding *non-bossiness*.

Corollary 4. *For multiple-type housing markets with strict preferences, no mechanism is*

- *onto,*
- *individually rational,*
- *strategy-proof, and*
- *non-bossy.*[6]

[6]Note that Lemma 1 (Alva, 2017; Biró et al., 2022a) implies that we can replace *strategy-proofness* and *non-bossiness* by *effective group* (or *pairwise*) *strategy-proofness*. In fact, on the domain of strict preference profiles, *strategy-proofness* and *non-bossiness* imply *group strategy-proofness*.

Proof. Lemma 2 (b) together with Theorem 5 implies Corollary 4. □

Examples 7, 8, and 9 in Appendix C.3 are well-defined on the domain of strict preference profiles and establish the logical independence of *ontoness*, *individual rationality*, and *strategy-proofness* in Corollary 4. The *non-bossiness* example, Example 10, in Appendix C.3 can be extended to strict preferences for Shapley-Scarf housing markets; for multiple-type housing markets and with separable preferences, the mechanism is extended by applying it coordinatewise to all object types. The latter extension method does not work for strict preferences and the independence of *non-bossiness* from the other properties in Corollary 4 is an **open problem** for multiple-type housing markets.

4.4. Discussion

Shapley-Scarf housing markets

Our results (Theorem 3 and Theorem 4) can be compared to Takamiya (2001, Corollary 4.16) for Shapley-Scarf housing markets. In the proof of Theorem 3 we make explicit use of the steps used by the TTC algorithm to compute the TTC allocation. In contrast, Takamiya's proof is not based on the TTC algorithm. Instead, his proof is based on *strict core-stability*, i.e., the absence of weak blocking coalitions and profitable coalitional deviations. His proof consists of two steps: (1) *strict core-stability* implies *group strategy-proofness* and (2) *group strategy-proofness* and *ontoness* together imply *Pareto efficiency*. Since cTTC neither satisfies *Pareto efficiency* nor *group strategy-proofness*, our results and proof strategy are logically independent. Moreover, Takamiya's proof strategy cannot be extended to multiple-type housing markets because weak blocking coalitions and profitable coalitional deviations need not coincide (see Theorems 1 and 2 for details).

Furthermore, comparing the classical TTC characterization (Result 3) with that of Takamiya (2001) yields the following result. For Shapley-Scarf housing markets, an *individually rational* and *strategy-proof* mechanism is *Pareto efficient* if and only if it is *unanimous* and *non-bossy*. However, this result does not extend to multiple-type housing markets, as illustrated in Example 6, which shows that *cTTC* is not Pareto efficient (recall that there, the no-trade allocation $cTTC(R) = ((H_1, C_1), (H_2, C_2))$ is Pareto dominated by the full-trade allocation $((H_2, C_2), (H_1, C_1))$).

Object allocation problems with multi-demand and without ownership

Our results can be compared to Monte and Tumennasan (2015) and Pápai (2001) for object allocation problems with multi-demand and without ownership, i.e., agents can consume more than one object, and the set of objects is a social endowment.

While Monte and Tumennasan (2015) still assume that objects are of different types and agents can only consume one object of each type, Pápai (2001) imposes no consumption restriction.[7] Although both models are slightly different, their characterization results are similar: the only mechanisms satisfying *Pareto efficiency*, *strategy-proofness*, and *non-bossiness* are sequential

[7] In Pápai (2001), agents can consume any set of objects, and their preferences are linear orders over all sets of objects.

dictatorships. Clearly, if agents, like in our model, have property rights, sequential dictatorship mechanisms will not satisfy *individual rationality*. Thus, their characterization results imply an impossibility result for our model, in line with our Theorem 5; however, note that our efficiency notion in Theorem 5, *unanimity*, is weaker than *Pareto efficiency*.

Object allocation problems with multi-demand and with ownership

Finally, we compare our results (Theorems 3 and 4) to Pápai (2003).

Similarly to Pápai (2001), Pápai (2003) considers a more general model of allocating objects to the set of agents who can consume any set of objects. In contrast to Pápai (2001), each object now is owned by an agent and each agent has strict preferences over all objects, and his preferences over sets of objects are monotonically responsive to these "objects-preferences."[8] In our model, we impose more structure by assuming that (i) the set of objects is partitioned into sets of exogenously given types and (ii) each agent owns and consumes one object of each type.

Pápai (2003) considers *strategy-proofness* and *non-bossiness* (as we do) and she introduces two additional (non-standard) properties: *trade sovereignty* and *strong individual rationality*. *Trade sovereignty* requires that every feasible allocation that consists of "admissible transactions" should be realized at some preference profile; it allows for trade restrictions and some objects never being traded and is hence weaker than *ontoness* (for details see Pápai, 2003). *Strong individual rationality* requires that for each agent and all preference relations with the same objects-preferences as the agent has, *individual rationality* holds (for details see Pápai, 2003). Note that *strong individual rationality* is stronger than *individual rationality*. For instance, if agent 1's endowment is (H_1, C_1), and his objects-preferences are $R_1 : H_2, H_1, C_1, C_2$, then allotment (H_2, C_2) is not *strongly individually rational*.[9]

Pápai (2003) shows that the set of mechanisms satisfying *trade sovereignty*, *strong individual rationality*, *strategy-proofness*, and *non-bossiness* coincides with the set of segmented trading cycle mechanisms. In this class of mechanisms, all objects are (endogenously) decomposed into different segments that can be expressed as the components of a trading possibility graph (which can express trading restrictions that can even mean that certain objects cannot be traded). Agents can own at most one object per segment and the TTC algorithm is then executed separately for each segment. The set of segmented trading cycle mechanisms is large and, for our model, would include cTTC, the no-trade mechanism, and many segmented trading cycles mechanisms with restricted trades.

cTTC is a specific segmented trading cycle mechanism in the sense that all segments are a priori determined by object types. Thus, our characterization result of cTTC can be seen as characterizing a specific segmented trading cycle mechanism while Pápai characterizes the whole

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[8]Formally, let O be a finite set of objects. A preference relation $\succeq$ over all non-empty sets of objects is *monotonically responsive* if (i) it is monotonic, i.e., for any two non-empty subsets of objects, $A, B \subseteq O$, $A \subseteq B$ implies that $B \succeq A$; and (ii) *responsive*, i.e., there exists a strict "objects-preference relation" over all objects, R, such that for any two distinct objects $o, o' \in O$, and a subset of objects $A \subseteq O \backslash \{o, o'\}$, $o \, P \, o'$ implies that $\{o\} \cup A \succ \{o'\} \cup A$. In our model, since agents' allotments have a fixed number of objects, monotonicity of preferences over sets of objects plays no role. Furthermore, given our constraint that each agent needs to receive an object of each type, responsiveness corresponds to separability.

[9]Let $\widetilde{\succ}_1 : (H_2, C_1), (H_1, C_1), (H_2, C_2), (H_1, C_2)$ and $\widehat{\succ}_1 : (H_2, C_1), (H_2, C_2), (H_1, C_1), (H_1, C_2)$. Note that both preferences are responsive to R_1. We see that $(H_2, C_2) \, \widehat{\succ}_1 \, (H_1, C_1)$ but $(H_1, C_1) \, \widetilde{\succ}_1 \, (H_2, C_2)$. Thus, (H_2, C_2) is *individually rational* at $\widehat{\succ}_1$ but *not individually rational* at $\widetilde{\succ}_1$.

class of segmented trading cycle mechanisms. On the one hand, we weaken *strong individual rationality* to *individual rationality* but strengthen *trade sovereignty* to *ontoness*. On the other hand, we consider two different preference domains that reflect some responsiveness through separability. Therefore, while there is a close connection between our models and results, there is no direct logical relation between Pápai (2003)'s result and ours (Theorems 3 and 4).

5. CHAPTER 3
Efficiency in Multiple-Type Housing Markets

5.1. Introduction

The assignment of indivisible scarce resources is a central problem in economics. This type of problem has been extensively studied under special, commonly used assumptions. For instance, many studies, such as those on auctions (Myerson, 1981), allow monetary transfers a priori. In cases where monetary transfers are not allowed, research tends to focus on unit-demand models, such as the Shapley-Scarf housing markets (Shapley and Scarf, 1974).

In many important situations, however, agents may desire more than one object (Huh et al., 2013; Manjunath and Westkamp, 2021). Thus, in this chapter, we relax the unit-demand assumption and focus on the multiple-type housing markets (Moulin, 1995), which is an extension of the Shapley-Scarf housing markets. In this model, objects are labeled by different types and assigned to a group of agents. Each agent owns one object of each type and consumes exactly one object of each type, with no monetary transfers during the reallocation process. As a result, agents have preferences over bundles, each consisting of one object of each type.

Similar to Shapley-Scarf housing markets, multiple-type housing markets apply to numerous real-world problems. To emphasize this, we provide two motivating examples below that can be modeled as multiple-type housing markets.[1]

Course reallocation. At many universities in Europe, Ph.D. candidates are hired as full-time employees and required to serve as teaching assistants for multiple courses. A common scenario is that they may be assigned to one undergraduate course and one graduate course per year. However, teaching assistants' interests may change over time, which means that the distribution of course assignments made this year may not be efficient in the next year. For example, a teaching assistant who previously taught an advanced game theory course may now wish to teach macro courses instead, and another teaching assistant who taught asset pricing this year may also want to switch his course. Therefore, reallocating courses among them may be Pareto improving.

Job rotation. Every doctor enrolled in a hospital residency program is obligated to take turns being on an emergency duty schedule every month. The schedule includes all doctors in the

[1] In the following two examples, although teaching assistants may not officially own certain courses, they often inherit the previously assigned curriculum, which can be considered as their ownership of the course. Similarly, doctors may also unofficially inherit their time slot from the previous year, which can be regarded as their ownership of the slot. Furthermore, some people may officially own certain duties in certain situations, see Klaus (2008) in detail.

hospital and covers an entire year. The schedule is automatically carried over to the next year, but can be modified at the beginning of each year. When making changes, the preferences of each doctor regarding their duty schedule in the new year can be taken into consideration.

When there is only one type, our model reduces to the Shapley-Scarf housing market model (Shapley and Scarf, 1974), and it is known that for such markets, *Pareto efficiency* is compatible with *individual rationality* and *strategy-proofness*. Furthermore, these three properties together uniquely identify the prominent top-trading-cycles mechanism (TTC) (Ma, 1994; Svensson, 1999). However, when there are multiple types, *Pareto efficiency* is incompatible with *individual rationality* and *strategy-proofness* (Konishi et al., 2001). Therefore, we aim to determine the level of efficiency that remains feasible by preserving *individual rationality* and *strategy-proofness*, i.e., we investigate which type of efficiency property is compatible with these two criteria. To address this question, we consider two weaker efficiency properties, *coordinatewise efficiency* and *pairwise efficiency*, on several domains of preference profiles: the strict preference domain, the (strictly) separable preference domain, and the lexicographic preference domain.

First, we show that for lexicographic and separable preferences, *coordinatewise efficiency* is compatible with *individual rationality* and *strategy-proofness*. We show that these three properties uniquely identify one mechanism, the coordinatewise top-trading-cycles mechanism (cTTC), which is an extension of TTC (Theorem 6). However, for strict preferences, *coordinatewise efficiency* is also incompatible with *individual rationality* and *strategy-proofness* (Theorem 7). We then turn to our second efficiency property, *pairwise efficiency*. We show that for lexicographic preferences, separable preferences, and strict preferences, only another TTC extension, the bundle top-trading-cycles mechanism (bTTC), satisfies *individual rationality*, *group strategy-proofness* (or the combination of *strategy-proofness* and *non-bossiness*), and *pairwise efficiency* (Theorems 8 and 9). Finally, we propose several variations of our efficiency properties. We find that each of them is either satisfied by cTTC or bTTC, or leads to an impossibility result (together with *individual rationality* and *strategy-proofness*). Therefore, our characterizations can be primarily interpreted as a compatibility test, where we determine whether certain efficiency properties are compatible with *individual rationality* and *strategy-proofness*. Loosely speaking, any "reasonable" efficiency property (defined by efficiency improvements) that is not satisfied by cTTC or bTTC is incompatible with *individual rationality* and *strategy-proofness*.[2]

5.1.1. Our contributions

Our results contribute to the following strands of literature.

Strategic robustness and efficiency. In economic design, the characterization of strategically robust and efficient mechanisms is an important issue (Holmström, 1979; Barberà et al., 1997; Moulin, 1980; Ma, 1994; Pycia and Ünver, 2017; Shinozaki, 2023). Standard properties of strategic robustness and efficiency are *strategy-proofness* and *Pareto efficiency*, respectively. However, when agents consume more than one object, the combination of *strategy-proofness* and *Pareto efficiency* essentially results in serially dictatorial mechanisms (Klaus and Miyagawa, 2002; Monte and Tumennasan, 2015). However, such mechanisms ignore property rights driven by the endowments, which violates *individual rationality*. In other words, in the presence

[2]It is worth noting that there might be a few restrictive efficiency properties compatible with these requirements that are not satisfied by cTTC or bTTC; however, such exceptions may not be particularly interesting to focus on. In Appendix D.2, we provide an example of one exception and demonstrate that it is uninteresting.

of *individual rationality*, *strategy-proofness* and *Pareto efficiency* are incompatible. This impossibility leads to a common trade-off between strategic robustness and efficiency in the literature (Arrow, 1950; Gibbard, 1973; Satterthwaite, 1975; Myerson and Satterthwaite, 1983; Alva and Manjunath, 2020).

We want to keep *individual rationality* since it induces voluntary participation. That is, agents may lose interest in participating in a *individually irrational* mechanism. To ensure strategic robustness, it is important for the social planner to have knowledge of agents' true preferences in order to efficiently allocate resources. For example, if agents report some fake preferences, then, some allocations that are efficient with respect to reported preferences, may not be efficient for their true preferences. To avoid such situations, a mechanism that is both *individually rational* and *strategy-proof* should be used. Therefore, we search for a plausible mechanism by relaxing *Pareto efficiency* while keeping *individual rationality* and *strategy-proofness*. There are various weakened efficiency properties that could be used, and some of them are satisfied by several mechanisms (Klaus, 2008; Anno and Kurino, 2016; Feng et al., 2022a). However, coordinatewise efficiency and *pairwise efficiency*, are unique in that (i) they are compatible with *individual rationality* and *strategy-proofness*, and (ii) they are only satisfied by two specific mechanisms, respectively. In this respect, our results are closely related to Pápai (2007), Klaus (2008), Anno and Kurino (2016), Nesterov (2017), Alva and Manjunath (2019), Shinozaki and Serizawa (2022), and Shinozaki (2022).

TTC based mechanisms. The top-trading-cycles (TTC) algorithm (due to David Gale) is commonly used for object allocation problems with unit demand. In particular, as we mentioned earlier, for Shapley-Scarf housing markets with strict preferences, only TTC satisfies *individual rationality*, *strategy-proofness*, and *Pareto efficiency*. Thus, one could conjecture that for multiple-type housing markets, some extensions of TTC would still satisfy some desirable, although perhaps not all of the three properties. We confirm this conjecture by proving several characterizations of two TTC extensions: cTTC and bTTC. Our characterizations successfully extend characterizations of TTC from one dimensional Shapley-Scarf housing markets to higher dimensional multiple-type housing markets. Moreover, such characterizations give strong support for the use of TTC extensions. Our characterizations are related to Feng et al. (2022a,b), Altuntaş et al. (2021), and Biró et al. (2022a). We provide a detailed discussion in Section 5.3.

Complementary preferences. As we mentioned earlier, our characterization for bTTC is also valid for strict preferences. The analysis on the domain of strict preference is demanding, because it allows agents' preferences to exhibit complementarity. Thus, our results also contribute to the literature on allocation problems with complements (Sun and Yang, 2006; Che et al., 2019; Rostek and Yoder, 2020; Jagadeesan and Teytelboym, 2021; Huang, 2023).

5.1.2. Organization

The chapter proceeds as follows. In the following section, we introduce two efficiency properties and state our results, and conclude with a discussion of our results and how they relate to the literature in Section 5.3. In Appendix C, we provide the proofs of our results that are not included in the main text. In Appendix D.3, we provide several examples to establish the logical independence of the properties in our characterizations.

5.2. Results

In this section, we consider two efficiency properties that (i) are weaker than *Pareto efficiency* and (ii) are compatible with *individual rationality* and *strategy-proofness*. Specifically, we will explore whether cTTC and bTTC satisfy these properties.

5.2.1. Coordinatewise efficiency

Here, we consider a natural modification of *Pareto efficiency* for multiple-type housing markets, *coordinatewise efficiency*, which rules out Pareto improvements within a single type. Let $R \in \mathcal{R}^N$. An allocation $y \in X$ is a *coordinatewise improvement* of allocation $x \in X$ at R if (i) y is a *Pareto improvement* of x, and (ii) y and x only differ in one type $t \in T$, i.e., $y^t \neq x^t$ and for each $\tau \in T \setminus \{t\}$, $y^\tau = x^\tau$. An allocation is *coordinatewise efficient* at R if there is no *coordinatewise improvement*.

Definition (Coordinatewise efficiency).
A mechanism on $\mathcal{R}^N$ is *coordinatewise efficient* if it only selects *coordinatewise efficient* allocations.

One easily verifies that *Pareto efficiency* implies *coordinatewise efficiency*, and *coordinatewise efficiency* implies *unanimity*.

Remark 5. Coordinatewise efficiency for subdomain
For a multiple-type housing market with separable preferences, *coordinatewise efficiency* simply means that the selected allocation of each type is *Pareto-efficient* for agents' marginal preferences for the type. Formally, $f : \mathcal{R}_s^N \to X$ is *coordinatewise efficient* if for each $R \in \mathcal{R}_s^N$ and each $t \in T$, $f^t(R)$ is *Pareto efficient* at R^t.

Moreover, since TTC is *Pareto efficient* for Shapley-Scarf housing markets, it is easy to see that for each $R \in \mathcal{R}_s^N$ and each $t \in T$, $cTTC^t(R)$ is *Pareto efficient* at R^t, and hence cTTC is *coordinatewise efficient*. ◇

We first characterize cTTC using *coordinatewise efficiency*.

Theorem 6. *For multiple-type housing markets with (i) lexicographic preferences and (ii) separable preferences, only cTTC satisfies*

- *individual rationality,*
- *strategy-proofness, and*
- *coordinatewise efficiency.*

We prove Theorem 6 in Appendix D.1. It is known that cTTC satisfies (i) *individual rationality* and *strategy-proofness* (Proposition 1), and (ii) *coordinatewise efficiency* (see Remark 5). For uniqueness, the proof consists of two steps. Given a mechanism f on the domain of separable preferences satisfying all our three properties. We first consider a restricted domain of preference profiles. We show that f always selects the cTTC allocation on this restricted domain (Proposition 1). Then by replacing agents' preferences, one by one, from our restricted domain to the lexicographic preference domain and separable preference domain, respectively, we extend this result to the domain of (i) lexicographic preference profiles and (ii) separable preference profiles.

We would like to make two additional remarks to emphasize the significance of our result.

Remark 6. Second-best incentive compatibility
For multiple-type housing markets with separable preferences, Klaus (2008) weakens *Pareto efficiency* to another efficiency property, *second-best incentive compatibility*. She shows that cTTC is *second-best incentive compatible*, i.e., there exists no other strategy-proof mechanism that Pareto dominates cTTC. However, she also shows that there exists another mechanism that is *individually rational*, *strategy-proofness*, and *second-best incentive compatible*. In a follow-up work, Anno and Kurino (2016) investigate *second-best incentive compatibility* for *independent* mechanisms, i.e., the mechanisms that treat each submarket independently and separately. In other words, under an *independent* mechanism, the selected allocation of each type only depends on agents' marginal preferences for each type. They also show that cTTC is not the unique *independent* mechanism that satisfies these properties. Thus, Theorem 6 complement to Klaus (2008) and Anno and Kurino (2016): by strengthening *second-best incentive compatibility* to *coordinatewise efficiency*, we find that cTTC is the only *independent* mechanism that satisfies *individual rationality*, *strategy-proofness*, and *coordinatewise efficiency*. $\diamond$

Remark 7. Individual rationality
Although one might view Theorem 6 as a trivial extension of Result 3, we want to stress that our finding actually adds novelty to the field. In particular, a major challenge with multiple-type housing markets is that *individual rationality* is weakened considerably. For instance, when agents lexicographically prefer one type over others, if an agent receives a better object than his endowment for his most important object type, then *individual rationality* of the allotment is respected even if we ignore the endowments of the other object types. Let us consider a small domain where there are two types of objects, houses and cars, and all agents lexicographically prefer houses over cars. In this domain, an alternative mechanism exists that differs from cTTC and still satisfies *individual rationality*, *coordinatewise efficiency*, and *strategy-proofness*:

Step 1: First apply TTC to houses. If agent 1 received a new house at Step 1 (and hence improves upon his own house), then move his endowed car to the bottom of his marginal preferences for cars (in terms of cTTC, agent 1 is not allowed to point at his own car until the end of the algorithm).

Step 2: Apply TTC to cars with the adjusted preferences. $\diamond$

Since cTTC is not well-defined for strict preferences, a natural question is whether there exists an extension of cTTC for strict preferences that satisfies our desired properties. Our answer is no.

Theorem 7. *For multiple-type housing markets with strict preferences, no mechanism satisfies*

- *individual rationality,*
- *strategy-proofness, and*
- *coordinatewise efficiency.*

Theorem 7 is implied by Theorem 5 as *coordinatewise efficiency* is stronger than *unanimity*, and hence we omit the proof.

Theorem 7 also leads to a new question: whether there is an efficiency property that is compatible with *individual rationality* and *strategy-proofness* for strict preferences. We will address this question in the next subsection.

5.2.2. Pairwise efficiency

Unanimity is a weak efficiency property. However, for strict preferences, it is still incompatible with *individual rationality* and *strategy-proofness* (Theorem 5). Therefore, for strict preferences, it seems difficult to find an efficiency property that is compatible with *individual rationality* and *strategy-proofness*. To establish a suitable efficiency property, we consider efficiency improvements that only involve a small number of agents (Goldman and Starr, 1982). To be precise, here we consider *pairwise efficiency* that rules out efficiency improvements by pairwise reallocation (Ekici, 2022). Let $R \in \mathcal{R}^N$. An allocation $x \in X$ is *pairwise efficient* at R if there is no pair of agents $\{i, j\} \subseteq N$ such that $x_j \, P_i \, x_i$ and $x_i \, P_j \, x_j$.

Definition (Pairwise efficiency).
A mechanism on $\mathcal{R}^N$ is *pairwise efficient* if it only selects *pairwise efficient* allocations.

For Shapley-Scarf housing markets, the result related to *pairwise efficiency*, that is pertinent for our analysis of multiple-type housing markets is the following.

Result 6 (Ekici, 2022).
For Shapley-Scarf housing markets, only TTC is *individually rational*, *strategy-proof*, and *pairwise efficient*.

By using arguments similar to arguments in Result 6, we also obtain that bTTC inherits *pairwise efficiency* from the underlying top trading cycles algorithm for the restricted market $R|^e$. Also, it is known that bTTC is *individually rational* and *strategy-proof* (Feng et al., 2022b). Hence, based on Result 6, one could now conjecture that for multiple-type housing markets, bTTC is identified by these three properties. That conjecture is nearly correct, but to fully support it, we need to strengthen *strategy-proofness* to *group strategy-proofness* (or the combination of *strategy-proofness* and *non-bossiness*: recall that for strict preferences, *group strategy-proofness* coincides with the combination of *strategy-proofness* and *non-bossiness*).

Theorem 8. *For multiple-type housing markets with strict preferences, only bTTC satisfies*

- *individual rationality,*
- *group strategy-proofness (or the combination of strategy-proofness and non-bossiness), and*
- *pairwise efficiency.*

Quite interestingly, this characterization is also valid for (i) lexicographic preferences and (ii) separable preferences, even if we weaken *group strategy-proofness* to the combination of *strategy-proofness* and *non-bossiness* (recall that for separable preferences, *group strategy-proofness* is stronger than the combination of *strategy-proofness* and *non-bossiness*).

Theorem 9. *For multiple-type housing markets with (i) lexicographic preferences and (ii) separable preferences, only bTTC satisfies*

- *individual rationality,*
- *strategy-proofness,*
- *non-bossiness, and*
- *pairwise efficiency.*

Corollary 5. *For multiple-type housing markets with (i) lexicographic preferences and (ii) separable preferences, only bTTC satisfies*

- *individual rationality,*
- *group strategy-proofness, and*
- *pairwise efficiency.*

We prove Theorems 8 and 9 in Appendix D.2. Here we only explain the intuition of the uniqueness part of the proof. We first consider lexicographic preferences. On this domain, consider a top trading cycle that forms at the first step of bTTC. We first show that, by *individual rationality*, *strategy-proofness*, *non-bossiness*, and *pairwise efficiency*, agents in this top trading cycle receive their bTTC allotments. We can then consider agents who trade at the second step of bTTC by following the same arguments as for first step trading cycles, and so on. Thus, we find that on this domain, only bTTC satisfies *individual rationality*, *strategy-proofness*, *non-bossiness*, and *pairwise efficiency*. Then, following a similar approach as in the proof of Theorem 6, we extend this result to the domain of separable preference profiles and strict preference profiles.

We provide four additional remarks to facilitate the reader's understanding.

Remark 8. Interpretation of Theorems 8 and 9

How to interpret our characterization of bTTC? There are three ways to explain it. First, as a positive result, Theorems 8 and 9 demonstrate that bTTC is identified by a list of properties. Consequently, the social planner should select bTTC if he cares about these properties. Second, Theorems 8 and 9 reveal the trade-off between efficiency and strategic robustness (*group strategy-proofness*) in the presence of *individual rationality*: if the social planner wishes to achieve stronger efficiency, he needs to weaken *group strategy-proofness*. Third, Theorems 8 and 9 suggest that bTTC can be used as a benchmark for the reallocation of multiple-type housing markets, in the sense that no mechanism should perform worse than bTTC. ◇

Remark 9. Independence of Theorems 8 and 9

Theorem 8 is not a more general result or a trivial extension of Theorem 9 (Corollary 5), and Theorem 9 (Corollary 5) is not a direct implication of Theorem 8. There is a logical independence between proving a characterization on different domains.[3] On the one hand, we may have a characterization on some domain but not on a subdomain. For instance, for Shapley-Scarf housing markets, the characterization of TTC for strict preferences (Result 3) is not necessarily

[3]For a detailed discussion of the role of domains in characterizations, see Thomson (2022, Section 11.3).

valid on some subdomain (Bade, 2019). Conversely, a mechanism satisfying a given list of properties may not exist on a larger domain, even if it does exist on a smaller domain. For instance, while our characterization of cTTC (Theorem 6) is valid on the domain of separable preference profiles, it may not hold true in certain superdomains (Theorem 7). ⋄

Remark 10. Constraints and efficiency
Trading constraints frequently occur in reality (Shinozaki and Serizawa, 2022). On the one hand, constraints may exclude some desirable outcomes, on the other hand, they may help us to guarantee positive results (Raghavan, 2015). For instance, to ensure the existence of the core, Kalai et al. (1978) impose restrictions on trades among certain agents and Pápai (2007) restricts the set of feasible trades. However, this also raises a new question for the mechanism designer: which constraint should be enforced? In other words, if any constraint is admissible, which constraint is the most suitable? Theorems 8 and 9 partially answer this question: if we still want to achieve some efficiency, then, without loss of other properties, allowing agents to trade their endowments completely is sufficient and necessary to achieve *pairwise efficiency*. ⋄

Remark 11. Non-bossiness
It is surprising that in our characterization of bTTC, *non-bossiness* is also involved, in comparison with Result 6. Why is this the case? For Shapley-Scarf housing markets, each agent only demands one object. Thus, each agent will trade within only one coalition. Therefore, *non-bossiness* is redundant since each agent can only influence the coalition in which he is involved. However, the same is not true for multiple-type housing markets because each agent may trade different objects with different coalitions. To see this point, refer to Example 18 in Appendix D.3. ⋄

Moreover, based on our results (Theorems 6, 8, and 9), we also have the following observation, which essentially shows a trade-off between our two efficiency properties in the presence of *individual rationality* and *strategy-proofness*.

Observation 1. For multiple-type housing markets, even with separable preferences, an *individually rational* and *strategy-proof* mechanism cannot satisfy both *coordinatewise efficiency* and *pairwise efficiency*.

5.2.3. Other efficiency properties

We will now discuss other efficiency properties that are derived from *coordinatewise efficiency* and *pairwise efficiency*.

First, we consider a weaker version of *coordinatewise efficiency* that only involves two agents.

Let $R \in \mathcal{R}^N$. An allocation $y \in X$ is a *pairwise coordinatewise improvement* of allocation $x \in X$ at R if (i) y is a *coordinatewise improvement* of x, and (ii) y and x only differ in two agents $i, j \in N$ at one type $t \in T$, i.e., for some $t \in T$ and for some distinct $i, j \in N$, $y^t_i = x^t_j$, $y^t_j = x^t_i$, for each $k \in N \setminus \{i, j\}$, $y^t_k = x^t_k$, and for each $\tau \in T \setminus \{t\}$, $y^\tau = x^\tau$. An allocation is *pairwise coordinatewise efficient* at R if there is no *pairwise coordinatewise improvement*.

Definition (Pairwise coordinatewise efficiency).
A mechanism on $\mathcal{R}^N$ is *pairwise coordinatewise efficient* if it only selects *pairwise coordinatewise efficient* allocations.

Our results in Theorems 6 and 7 are still true if we replace *coordinatewise efficiency* with *pairwise coordinatewise efficiency*.

Theorem 10. *For multiple-type housing markets*

- *with (i) lexicographic preferences and (ii) separable preferences, only cTTC satisfies individual rationality, strategy-proofness, and pairwise coordinatewise efficiency.*
- *with strict preferences, no mechanism satisfies individual rationality, strategy-proofness, and pairwise coordinatewise efficiency.*

The proofs are the same as in Theorems 6 and 7 and hence we omit them. Note that Theorem 10 implies that *pairwise coordinatewise efficiency* and *pairwise efficiency* are logically independent: cTTC satisfies the former but violates the latter, and bTTC satisfies the latter but violates the former.

Second, we consider a stronger version of *pairwise efficiency* that involves larger coalitions: *coalitional efficiency* (Tierney, 2022).[4] This property says that the selected allocation cannot be improved by the reallocation of allotments, keeping bundled allotments intact.

Let $R \in \mathcal{R}^N$. An allocation $x \in X$ is *coalitionally efficient* at R if there is no coalition $S \equiv \{i_1, i_2, \ldots, i_K\} \subseteq N$ such that for each $i_\ell \in S$, $x_{i_\ell} \, P_{i_\ell} \, x_{i_{\ell+1}}$ (mod K).[5]

Definition (**Coalitional efficiency**).
A mechanism on $\mathcal{R}^N$ is *coalitionally efficient* if it only selects *coalitionally efficient* allocations.

Our characterization of bTTC is still valid if we replace *pairwise efficiency* with *coalitional efficiency*, because bTTC satisfies *coalitional efficiency* from the underlying TTC algorithm for the restricted market $R|^e$.

Theorem 11. *For multiple-type housing markets (i) with lexicographic preferences, (ii) with separable preferences, and (iii) with strict preferences, only bTTC satisfies individual rationality, group strategy-proofness (or the combination of strategy-proofness and non-bossiness), and coalitional efficiency.*

All the efficiency properties discussed above have certain constraints on efficiency improvements. For example, *coordinatewise efficiency* and *pairwise coordinatewise efficiency* only consider efficiency improvements within one type ($\{o_1^t, \ldots, o_n^t\}$), while *pairwise efficiency* and *coalitional efficiency* only consider efficiency improvements within the full endowments of a coalition ($S \subseteq N$ with $\{e_i\}_{i \in S}$). A natural question is whether it is possible to consider something in between, such as efficiency improvements for more than one type but less than all types.

Let $R \in \mathcal{R}^N$. An allocation $x \in X$ is *T'-types pairwise efficient* at R if there is no pair of agents $\{i, j\} \subseteq N$ and a strict subset of types $T' \subsetneq T$ such that $y_i \, P_i \, x_i$ and $y_j \, P_j \, x_j$, where $y_i = ((x_j^t)_{t \in T'}, (x_i^t)_{t \in T \setminus T'})$ and $y_j = ((x_i^t)_{t \in T'}, (x_j^t)_{t \in T \setminus T'})$.

Definition (**T'-types pairwise efficiency**).
A mechanism on $\mathcal{R}^N$ is *T'-types pairwise efficient* if it only selects *T'-types pairwise efficient* allocations.

[4] Tierney (2022) originally refers to it as *conditional optimality*.
[5] For Shapley-Scarf housing markets, *coalitional efficiency* is equivalent to *Pareto efficiency*.

Remark 12. Restriction on $|T'|$
By the definition of *T'-types pairwise efficiency*, it is easy to see that *T'-types pairwise efficiency* is stronger than *pairwise coordinatewise efficiency*. If we do not assume that $T' \subsetneq T$, i.e., $|T'| = m$ is also possible, then this new property is also stronger than *pairwise efficiency*. By Observation 1, we know that no *individually rational* and *strategy-proof* mechanism satisfies it.

Given $T' \subsetneq T$, if there are only two types, i.e., $|T| = m = 2$, then *T'-types pairwise efficiency* coincides with *pairwise coordinatewise efficiency*. ◇

The following result reveals the strength of *T'-types pairwise efficiency*.

Theorem 12. *If $|T| = m > 2$, then even for multiple-type housing markets with lexicographic preferences, no mechanism satisfies* individual rationality, strategy-proofness, *and* T'-types pairwise efficiency.

Proof. Let $N = \{1, 2\}$ and $T = \{1, 2, 3\}$. Let $R \in \mathcal{R}_l^N$ be such that

$$R_1 : o_2^1, \boldsymbol{o_1^1}, \boldsymbol{o_1^3}, o_2^3, o_2^2, \boldsymbol{o_1^2}.$$
$$R_2 : o_1^1, \boldsymbol{o_2^1}, o_1^3, \boldsymbol{o_2^3}, o_1^2, \boldsymbol{o_2^2}.$$

So, agent 1 would like to trade type-1 and type-2 but not type-3, and agent 2 would like to trade all types.

Let f be *individually rational* and *T'-types pairwise efficient*. Thus, by *T'-types pairwise efficiency* of f, two agents trade in type-1 and type-2, i.e., $f_1^1(R) = o_2^1, f_2^1(R) = o_1^1$ and $f_1^2(R) = o_2^2, f_2^2(R) = o_1^2$. We only need to consider the allocation in type-3. There are two cases.

Case 1. Agents trade in type-3, i.e., $f_1^3(R) = o_2^3$ and $f_2^3(R) = o_1^3$. Then $f_1(R) = (o_2^1, o_2^2, o_2^3)$.

Let $R_1' : \boldsymbol{o_1^3}, o_2^3, o_2^1, \boldsymbol{o_1^1}, o_2^2, \boldsymbol{o_1^2}$. By *individual rationality* of f, agent 1 receives his type-3 endowment, i.e., $f_1^3(R_1', R_2) = o_1^3$. By *T'-types pairwise efficiency* of f, agents 1 and 2 still trade in type-1 and type-2 at (R_1', R_2), i.e., $f_1^1(R_1', R_2) = o_2^1, f_2^1(R_1', R_2) = o_1^1$ and $f_1^2(R_1', R_2) = o_2^2, f_2^2(R_1', R_2) = o_1^2$. Thus, $f_1(R_1', R_2) = (o_2^1, o_2^2, o_1^3)\ P_1\ (o_2^1, o_2^2, o_2^3) = f_1(R)$, which implies that f is not *strategy-proof*.

Case 2. Agents do not trade in type-3, i.e., $f_1^2(R) = o_1^2$ and $f_2^2(R) = o_2^2$. Then $f_2(R) = (o_1^1, o_1^2, o_2^3)$.

Let $y = (y_1, y_2) = ((o_2^1, o_1^2, o_2^3), (o_1^1, o_2^2, o_1^3))$, i.e., agents only trade in type-1 and type-3.

Let $R_2' : o_1^3, \boldsymbol{o_2^3}, \boldsymbol{o_2^1}, o_1^1, \boldsymbol{o_2^2}, o_1^2$. If agents do not trade in type-3 at (R_1, R_2'), then by *individual rationality* of f, $f_2(R_1, R_2') = o_2$ and hence $f(R_1, R_2') = e$. This contradicts *T'-types pairwise efficiency* of f, since $y_1\ P_1\ o_1$ and $y_2\ P_2\ o_2$. Thus, type-3 is traded at (R_1, R_2'). Therefore, by *individual rationality* of f, type-1 is also traded, otherwise $o_1 P_1 f_1(R_1, R_2')$. Thus, $f_2^1(R_1, R_2') = o_1^1$ and $f_2^3(R_1, R_2') = o_1^3$. So, $f_2(R_1, R_2')\ P_2\ (o_1^1, o_1^2, o_2^3) = f_2(R)$, which implies that f is not *strategy-proof*. □

We conclude this section with an important remark.

Remark 13. Constrained efficiency improvements
Based on Result 5, we know that for an efficiency property based on efficiency improvements, constraints on efficiency improvements must be made in order to ensure compatibility with *individual rationality* and *strategy-proofness*.[6] In this section, we examine two categories of

[6] Recall that *Pareto efficiency* is an efficiency property based on efficiency improvements without any constraints over improvements.

Table 5.1.: efficiency properties and their implicit restriction conditions

	CE	**pE**	**pCE**	**cE**	T'**-pE**
constrained improvements for one type	+	−	+	−	−
constrained improvements for all types	−	+	−	+	−
constrained improvements for two agents	−	+	+	−	+
compatibility with IR and SP	+ (T6)	+ (T8)	+ (T10)	+ (T11)	− (T12)

We present a summary of our results in the table above. The first row describes five efficiency properties that we consider in this section. In the first column, the first three items represent three restriction conditions that we discussed in Remark 13, and the last item means the compatibility with *individual rationality* and *strategy-proofness*.
The notation "+" ("−") in a cell means that the property satisfies (violates) the condition. The notation for the first three rows is determined by the definition of our efficiency properties. The compatibility with *individual rationality* and *strategy-proofness* for the last row is obtained from Theorems 6, 8, 10, 11, and 12, respectively.
Abbreviations in the first row respectively refer to:
CE stands for *coordinatewise efficiency*,
pE stands for *pairwise efficiency*,
pCE stands for *pairwise coordinatewise efficiency*,
cE stands for *coalitional efficiency*, and
T'**-pE** stands for T'-*types pairwise efficiency*.

constraints, those regarding the constrained improvements with a certain number of object types, and those pertaining to the constrained improvements with a certain number of agents, to guarantee compatibility with *individual rationality* and *strategy-proofness*. On the one hand, when it comes to restrictions over object types, only two constraints prove to be useful: (i) improvements for one type only and (ii) improvements for all types together. On the other hand, restrictions over agents may be unnecessary. To be more precise, even if we only consider improvements for two agents, we can only achieve impossibility as long as there are no restrictions on object types. ◇

5.3. Discussion

We finish the chapter with a discussion of other characterizations of TTC extensions.

Another characterization of cTTC

Here, we discuss the relation between our characterization of cTTC in this chapter (Theorem 6) and the characterizations presented in the previous chapter on the basis of *individual rationality*, *strategy-proofness*, *non-bossiness*, and *unanimity* (Theorems 3 and 4). To distinguish between them, note that Theorems 3 and 4 are established by weakening *Pareto efficiency* to *unanimity* and strengthening *strategy-proofness* to the combination of *strategy-proofness* and *non-bossiness*, whereas in Theorem 6 we do not require *non-bossiness*. On the other hand, in

Theorem 6 we use *coordinatewise efficiency*, which is stronger than *unanimity*. Hence, the incentive property in Theorem 6 is weaker while the efficiency property is stronger than in Theorems 3 and 4, and these two characterizations are logically independent.

Another characterization of bTTC

Here, we discuss the relation between our characterization of bTTC (Theorems 8 and 9) and Feng et al. (2022b)'s characterization of bTTC by means of *individual rationality*, *group strategy-proofness* and *anonymity*.[7] Since (i) there is no logical relation between *anonymity* and *pairwise efficiency*, and (ii) the incentive property in our characterization for separable preferences is weaker, our results are logically independent.

Object allocation problems with multi-demands and with ownership

Next, we compare our results to Altuntaş et al. (2021) and Biró et al. (2022a). Each considers an extension of Shapley-Scarf housing markets.

Altuntaş et al. (2021) consider a general model for allocating objects to agents who can consume any set of objects. Each object is owned by an agent, but now each agent has strict preferences over all objects and his preferences over sets of objects are monotonically responsive to these "objects-preferences". In our model, we impose more structure by assuming that (i) the set of objects is partitioned into exogenously given types and (ii) each agent owns and wishes to consume one object of each type. For this more general model, Altuntaş et al. (2021) consider another TTC extension: the "generalized top trading cycles mechanism (gTTC)," which satisfies *individual rationality* and *Pareto efficiency* but violates *strategy-proofness*. By strengthening *individual rationality* and weakening *strategy-proofness*, they provide a characterization of gTTC for lexicographic preferences. Thus, their results complement ours: if we exclude *strategy-proofness*, then there exists another TTC extension, which performs better than our mechanisms in terms of efficiency.

Biró et al. (2022a) consider another extension where each agent owns a set of homogeneous and agent-specific objects, and they consider a modification of bTTC to their model with the "endowments quota constraint". This constraint means that for each agent, the number of objects he can consume is the same as the number of objects he is endowed with. They show that this modification is neither *Pareto efficient* nor *strategy-proof*. Thus, their results show the limitation of bTTC in their model, while in our model, bTTC is *group strategy-proof*.

[7] *Anonymity* says that the mechanism is defined independently of the names of the agents. And they show that for separable preferences and for strict preferences, only the class of hybrid mechanisms between the no-trade mechanism and bTTC, satisfies all of their properties.

6. Appendices

A. The generalized indivisible goods allocation model

A *generalized indivisible goods allocation problem* (as first introduced by Sönmez, 1999) is a list $(N, \omega, \mathcal{A}^f, R)$ where $N = \{1, \ldots, n\}$ is a finite *set of agents* and for each $i \in N$, $\omega(i)$ denotes the *endowment of agent* i; we will interpret $\omega(i)$ as a set of objects. An *allocation* is a multi-valued function $x : N \rightrightarrows \bigcup_{i \in N} \omega(i)$ such that (i) for any $i, j \in N$, $i \neq j$, $x(i) \cap x(j) = \emptyset$ (no two agents can receive the same object) and (ii) $\bigcup_{i \in N} x(i) = \bigcup_{i \in N} \omega(i)$ (there is no free disposal); $\mathcal{A}$ denotes the set of all allocations. Next, a subset $\mathcal{A}^f \subseteq \mathcal{A}$ is exogenously fixed as the set of all *feasible allocations*. For each agent $i \in N$, agent i's preference relation R_i is a *complete* and *transitive* binary relation on $\mathcal{A}^f$. The set $\mathcal{R}$ of preferences over allocations in $\mathcal{A}^f$ is assumed to satisfy the following properties.

Assumption A (Sönmez, 1999): An agent is indifferent between an allocation and the endowment allocation if and only if he keeps his endowment, i.e., for each $i \in N$, each $R_i \in \mathcal{R}$, and each $x \in \mathcal{A}^f$,

$$x \, I_i \, \omega \text{ if and only if } x(i) = \omega(i).$$

Assumption B (Sönmez, 1999): For any preference relation R_i ($i \in N$), and any allocation x that is at least as good as the endowment allocation, there exists a preference relation R'_i such that (i) all allocations that are better than x at R_i are better than x at R'_i, (ii) all allocations that are worse than x at R_i are worse than x at R'_i, and (iii) the endowment allocation ranks right after (or indifferent to) x. Formally, for each $i \in N$, each $R_i \in \mathcal{R}$, and each $x \in \mathcal{A}^f$ with $x \, R_i \, \omega$, there is R'_i such that

1. for all $y \in \mathcal{A}^f \setminus \{x\}$, $y \, R_i \, x$ if and only if $y \, R'_i \, x$,
2. for all $y \in \mathcal{A}^f \setminus \{x\}$, $x \, R_i \, y$ if and only if $x \, R'_i \, y$,
3. for all $y \in \mathcal{A}^f \setminus \{x\}$, $x \, P_i \, y$ if and only if $x \, P'_i \, y$ and $x \, R'_i \, \omega \, R'_i \, y$.

Assumption B is a *preference domain richness condition.*

The above general model can be specified to a range of well-known models such as Shapley-Scarf housing markets (Shapley and Scarf, 1974), marriage and roommate markets (Gale and Shapley, 1962), hedonic coalition formation problems (Banerjee et al., 2001), and network formation problems (Jackson and Wolinsky, 1996).

Apart from domain richness Assumption B, our multiple-type housing market problems also fits the generalized indivisible goods allocation model as introduced by Sönmez (1999): for each $i \in N$, $\omega(i) = \{o_i^1, \ldots, o_i^m\}$, $\mathcal{A}^f = X$, and agents' preferences over allotments are extended straightforwardly to preferences over feasible allocations by assuming that there are no consumption externalities, i.e., any agent is indifferent between two allocations at which he receives the same allotment. Furthermore, by our assumption that agents' preferences over allotments are strict, Assumption A is satisfied. However, the preference domain richness Assumption B is violated for our marginal type-preference based domains, even for the larger preference domain of separable preferences:

Consider $R \in \mathcal{R}_s^N$ with $N = \{1, 2\}$, $T = \{H(ouse), C(ar)\}$, $O = \{H_1, H_2, C_1, C_2\}$, each agent i's endowment is (H_i, C_i), and agent 1's marginal preferences are

$$R_1^H : H_2, H_1$$

and

$$R_1^C : C_2, C_1.$$

Thus, separability implies that either

$$\boldsymbol{R_1} : (H_2, C_2),\ (H_1, C_2),\ (H_2, C_1),\ \boldsymbol{(H_1, C_1)},\ \cdots$$

or

$$\boldsymbol{R_1} : (H_2, C_2),\ (H_2, C_1),\ (H_1, C_2),\ \boldsymbol{(H_1, C_1)},\ \cdots .$$

It is not possible to derive separable preferences R_1' over agent 1's allotments such that (H_2, C_2) is the best allotment (this is only the case when both objects are acceptable and ranked first in the marginal preferences) and the endowment is ranked right behind, i.e.,

$$\boldsymbol{R_1'} : (H_2, C_2),\ \boldsymbol{(H_1, C_1)},\ \cdots$$

is not possible. This implies that no separable preference relation R_1' over allocations satisfying Assumption B can be derived.

Takamiya (2009) also considered generalized indivisible goods allocation problems but he imposed the following conditions on preferences, which are slightly different from Assumptions A and B in Sönmez (1999).

Condition A (Takamiya, 2009):
(i) There are no consumption externalities, i.e., for each $i \in N$, each $R_i \in \mathcal{R}$, and each $x, y \in \mathcal{A}^f$, if $x(i) = y(i)$ then $x\ I_i\ y$;
(ii) For any allocation x, each agent i can rank allocations that contain $x(i)$ as his most preferred allocations, and rank allocations that contain his endowment right behind, i.e., for each $i \in N$ and each $x \in \mathcal{A}^f$, there exists a preference relation $R_i \in \mathcal{R}$ such that
(a) for all $y \in \mathcal{A}^f$ such that $y(i) \neq x(i)$, $x\ P_i\ y$; and
(b) for all $y \in \mathcal{A}^f$ such that $y(i) \notin \{x(i), \omega(i)\}$, $\omega\ P_i\ y$.[1]

Condition A(ii) is a *preference domain richness condition.*

Condition B (Takamiya, 2009): If a coalition S is autarkic at two feasible allocations x and y, which means that in both allocations they reallocated their endowments among themselves, then a new feasible allocation is obtained by allocating allotments according to x to all agents in S and by allocating allotments according to y to all agents in $N \setminus S$, i.e., if there are two feasible allocations $x, y \in \mathcal{A}^f$ and a coalition $S \subseteq N$ such that $\cup_{i \in S} x(i) = \cup_{i \in S} y(i) = \cup_{i \in S} \omega(i)$, then $((x(i))_{i \in S}, (y(i))_{i \in N \setminus S}) \in \mathcal{A}^f$.

[1] In our model, where Takamiya's Condition A(i) is satisfied, we can restate Condition A in terms of strict preferences over allotments as follows: for each $i \in N$ and each allotment x_i, there exists a preference relation $R_i \in \mathcal{R}$ such that for all allotments $x_i' \notin \{o_i, x_i\}$, $x_i\ R_i\ o_i\ R_i\ x_i'$.

Condition B is a richness condition on the set of feasible allocations.

Apart from domain richness Condition A(ii), our multiple-type housing market problems also fits the generalized indivisible goods allocation model considered by Takamiya (2009): we extend preferences over allotments to feasible allocations by assuming Condition A(i) and since coalitions can freely reallocate endowments among themselves, Condition B is satisfied as well. The same example that shows that for separable preferences Sönmez's preference domain richness Assumption B is not satisfied, shows that Takamiya's preference domain richness Condition A(ii) is not satisfied either.

To summarize, in our model with separable preferences, both of the domain richness conditions that were used before give the flexibility to position an agent's endowment right behind an allotment. The intuitive use of this condition in proofs is to be able to truncate an agent's preferences right behind a specific allotment and guarantee that by individual rationality, the agent receives his endowment, his previously received allotment, or a better allotment. In our model, neither preference domain richness condition is satisfied; hence, the corresponding results of Sönmez (1999) and Takamiya (2009) need not hold anymore. We show how the lack of domain richness in our model changes the results compared to Takamiya's results.

B. Auxiliary properties and results

In Appendix B, we introduce auxiliary properties and prove results that help us to prove Theorem 3 in Appendix C.1. While some of the results in Appendix B can also be proven for separable preferences, we focus on lexicographic preferences because Theorem 3 deals with such preferences.

We introduce the well-known property of *(Maskin) monotonicity*, i.e., the invariance under monotonic transformations of preferences at a selected allocation.

Let $i \in N$. Given preferences $R_i \in \mathcal{R}_l$ and an allotment x_i, let $L(x_i, R_i) = \{y_i \in \Pi_{t\in T} O^t \mid x_i \; R_i \; y_i\}$ be the *lower contour set of R_i at x_i*. Preference relation $R'_i \in \mathcal{R}_l$ is a *monotonic transformation of R_i at x_i* if $L(x_i, R_i) \subseteq L(x_i, R'_i)$. Similarly, given a preference profile $R \in \mathcal{R}_l^N$ and an allocation x, a preference profile $R' \in \mathcal{R}_l^N$ is a *monotonic transformation of R at x* if for each $i \in N$, R'_i is a monotonic transformation of R_i at x_i.

Monotonicity: For each $R \in \mathcal{R}_l^N$ and each monotonic transformation $R' \in \mathcal{R}_l^N$ of R at $f(R)$, $f(R') = f(R)$.

We show that *strategy-proofness* and *non-bossiness* imply *monotonicity*.

Lemma 3. *If a mechanism on $\mathcal{R}_l^N$ is strategy-proof and non-bossy, then it is monotonic.*

Proof. The proof is a straightforward extension of Takamiya (2001, Theorem 4.12) and Pápai (2001, Lemma 1). Suppose mechanism f on $\mathcal{R}_l^N$ is *strategy-proof* and *non-bossy*. Let $R \in \mathcal{R}_l^N$ and let $x = f(R)$. Let $R' \in \mathcal{R}_l^N$ be a monotonic transformation of R at x. Let $i \in N$ and $y = f(R'_i, R_{-i})$. By *strategy-proofness* of f, we have $x_i \, R_i \, y_i$, which implies that $y_i \in L(x_i, R_i) \subseteq L(x_i, R'_i)$. However, by *strategy-proofness* of f, we also have $y_i \, R'_i \, x_i$. Thus, since $y_i \in L(x_i, R'_i)$, $x_i = y_i$. Then, by *non-bossiness* of f, we have $x = y$. By applying this argument sequentially for all agents in $N \backslash \{i\}$, we find that $f(R) = x = f(R')$. □

The converse of Lemma 3 is not true: the domain of lexicographic preferences is not rich enough to satisfy Alva's (2017) preference domain richness condition *two-point connectedness*.

Next, we introduce a "marginal version" of a monotonic preference transformation. Let $i \in N$. Given preferences $R_i \in \mathcal{R}_l$ and an allotment x_i, for each type t, consider the associated marginal preferences R_i^t and marginal allotment x_i^t. Let $L(x_i^t, R_i^t) = \{y_i^t \in O^t \mid x_i^t \; R_i^t \; y_i^t\}$ be the lower contour set of R_i^t at x_i^t. Marginal preference relation $\hat{R}_i^t$ is a monotonic transformation of R_i^t at x_i^t if $L(x_i^t, R_i^t) \subseteq L(x_i^t, \hat{R}_i^t)$.

Fact 1. Let x_i be an allotment. Let $R_i, \hat{R}_i$ be lexicographic preferences such that (1) $\pi_i = \hat{\pi}_i$ and (2) for each $t \in T$, $\hat{R}_i^t$ is a monotonic transformation of R_i^t at x_i^t. Then, $\hat{R}_i$ is a monotonic transformation of R_i at x_i.

Proof. We show that $L(x_i, R_i) \subseteq L(x_i, \hat{R}_i)$. Let $y_i \in L(x_i, R_i)$ with $y_i \neq x_i$. Then, $x_i \; P_i \; y_i$. Restate y_i and x_i as $y_i^{\pi_i} = (y_i^{\pi_i(1)}, \dots, y_i^{\pi_i(m)})$ and $x_i^{\pi_i} = (x_i^{\pi_i(1)}, \dots, x_i^{\pi_i(m)})$, respectively. Let k be the first type for which x_i and y_i assign different objects, i.e., for all $l < k$, $y_i^{\pi_i(l)} = x_i^{\pi_i(l)}$ and $y_i^{\pi_i(k)} \neq x_i^{\pi_i(k)}$. Since $x_i \; P_i \; y_i$ and preferences are lexicographic, we have $x_i^{\pi_i(k)} \; P_i^{\pi_i(k)} \; y_i^{\pi_i(k)}$. Thus, $y_i^{\pi_i(k)} \in L(x_i^{\pi_i(k)}, R_i^{\pi_i(k)}) \subseteq L(x_i^{\pi_i(k)}, \hat{R}_i^{\pi_i(k)})$, which implies that $x_i^{\pi_i(k)} \; \hat{P}_i^{\pi_i(k)} \; y_i^{\pi_i(k)}$. Then, since $\pi_i = \hat{\pi}_i$, $x_i \; \hat{P}_i \; y_i$, i.e., $y_i \in L(x_i, \hat{R}_i)$. □

Therefore, by *monotonicity*, if an agent always receives an allotment and shifts each of its objects up in the marginal preferences (without changing his importance order), he still receives that allotment and the allotments of the other agents do not change either.

C. Proofs in Chapter 2

Now, for lexicographic preferences, we introduce a new property, *marginal individual rationality*, which is a stronger property than *individual rationality*.

Definition (Marginal individual rationality).
A mechanism f on $\mathcal{R}_l^N$ is *marginally individually rational* if for each $R \in \mathcal{R}_l^N$, each $i \in N$, and each $t \in T$, $f_i^t(R) \; R_i^t \; o_i^t$.

Lemma 4. *A mechanism on $\mathcal{R}_l^N$ is unanimous, individually rational, strategy-proof, and non-bossy, then it is marginally individually rational.*

Proof. Suppose mechanism f on $\mathcal{R}_l^N$ is *unanimous*, *individually rational*, *strategy-proof*, *non-bossy*, and not *marginally individually rational*, i.e., there exist a preference profile $R \in \mathcal{R}_l^N$, an agent $i \in N$, and a type $t \in T$ such that $o_i^t \; P_i^t \; f_i^t(R)$. Then, by *individual rationality* of f, we know that $t \neq \pi_i(1)$.

Let $x \equiv f(R)$. Consider a preference profile $\hat{R} \in \mathcal{R}_l^N$ such that

for agent i,

- $\hat{R}_i^t : o_i^t, x_i^t, \dots,$
- for each $\tau \in T \backslash \{t\}$, $\hat{R}_i^\tau : x_i^\tau, \dots,$ and
- $\hat{\pi}_i = \pi$;

and for each agent $j \in N \backslash \{i\}$,

- for each $\tau \in T$, $\hat{R}_j^\tau : x_j^\tau, \ldots$, and
- $\hat{\pi}_j = \pi_j$.

Note that, by Fact 1, $\hat{R}$ is a monotonic transformation of R at x. By Lemma 3, f is *monotonic*. Thus, $f(\hat{R}) = x$.

Next, consider a preference profile $(\bar{R}_i, \hat{R}_{-i}) \in \mathcal{R}_l^N$, where $\bar{R}_i$ is such that

- for each $\tau \in T$, $\bar{R}_i^\tau = \hat{R}_i^\tau$, and
- $\bar{\pi}_i(1) = t$.

Note that $\bar{R}_i$ can be interpreted as a linear order over all objects such that $\bar{R}_i : o_i^t, \ldots$, i.e., object o_i^t is the most preferred object.

Let $y \equiv f(\bar{R}_i, \hat{R}_{-i})$. By *individual rationality* of f, $y_i^t = o_i^t$. Thus, $y_i \neq x_i$. By *strategy-proofness* of f, $x_i = f(\hat{R}_i, \hat{R}_{-i}) \, \hat{P}_i \, f(\bar{R}_i, \hat{R}_{-i}) = y_i$. Since agent i gains in type t by misreporting at $\hat{R}$ (i.e., $y_i^t = o_i^t \, \hat{P}_i^t \, f_i^t(\hat{R}) = x_i^t$), he must lose in some other more important type according to $\hat{\pi}_i$. That is, there is a type $t' \neq t$ such that (1) $\hat{\pi}_i^{-1}(t') < \hat{\pi}_i^{-1}(t)$ and (2) $x_i^{t'} \, \hat{P}_i^{t'} \, y_i^{t'}$. In particular, $x_i^{t'} \neq y_i^{t'}$.

Next, consider a preference profile $\bar{R} \equiv (\bar{R}_i, \bar{R}_{-i})$ such that

for each agent $j \in N \backslash \{i\}$,

- $\bar{R}_j^t : y_j^t, \ldots$,
- for each $\tau \in T \backslash \{t\}$, $\bar{R}_j^\tau = \hat{R}_i^\tau$, and
- $\bar{\pi}_j = \hat{\pi}_j$.

Note that the only relevant difference between $\bar{R}$ and $(\bar{R}_i, \hat{R}_{-i})$ is that under $\bar{R}$, each agent $j \neq i$ positions y_j^t as his most preferred type-t object. Thus, $\bar{R}$ is a monotonic transformation of $(\bar{R}_i, \hat{R}_{-i})$ at y. Therefore, by *monotonicity* of f, $f(\bar{R}) = y$.

However, under $\bar{R}$, for each agent $k \in N$, his most preferred allotment is $z_k = (x_k^1, \ldots, x_k^{t-1}, y_k^t, x_k^{t+1}, \ldots, x_k^m)$. Note that $z = (z_k)_{k \in N} \in X$ is an allocation because z is a mixture of y (for type t) and x (for other types). Thus, by *unanimity* of f, $f(\bar{R}) = z$. So, $y = z$. However, for type t', $z_i^{t'} = x_i^{t'} \neq y_i^{t'}$, a contradiction. □

C.1. Proof of Theorem 3: uniqueness

Proof of Theorem 3: uniqueness. Suppose that there is a mechanism $f : \mathcal{R}_l^N \to X$, different from the *cTTC* mechanism, that satisfies the properties listed in Theorem 3 (by Lemma 2, *ontoness* and *unanimity* can be used interchangeably). Then, there is a market R such that $y \equiv f(R) \neq cTTC(R) \equiv x$. In particular, there is a type t such that $(y_1^t, \ldots, y_n^t) \neq (x_1^t, \ldots, x_n^t)$.

By Lemma 3, both mechanisms, f and *cTTC*, are *monotonic*. By Lemma 4, both mechanisms, f and *cTTC*, are *marginally individually rational*. Since both mechanisms are *marginally individually rational*, for each $i \in N$ and each $\tau \in T$, $y_i^\tau R_i^\tau o_i^\tau$ and $x_i^\tau R_i^\tau o_i^\tau$. So, we can define a preference profile $\hat{R} \in \mathcal{R}_l^N$ such that for each agent $i \in N$,

- $\hat{R}^t_i : \begin{cases} x^t_i, y^t_i, o^t_i, \ldots \text{ if } & x^t_i \, R^t_i \, y^t_i \\ y^t_i, x^t_i, o^t_i, \ldots \text{ if } & y^t_i \, R^t_i \, x^t_i \end{cases}$

- for each $\tau \in T \backslash \{t\}$, $\hat{R}^\tau_i : y^\tau_i, o^\tau_i, \ldots$, and

- $\hat{\pi}_i = \pi_i$.

Note that, by Fact 1, $\hat{R}$ is a monotonic transformation of R at y. Since f is *monotonic*, $f(\hat{R}) = y$. Furthermore, since $\hat{R}^t$ is a monotonic transformation of R^t at x^t, *monotonicity* of the *TTC* mechanism implies $cTTC^t(\hat{R}) = TTC(\hat{R}^t) = x^t$.

Next, consider a preference profile $\bar{R} \in \mathcal{R}^N_l$ such that

for each agent $i \in N$,

- $\bar{R}^t_i : x^t_i, o^t_i, \ldots$,

- for each $\tau \in T \backslash \{t\}$, $\bar{R}^\tau_i = \hat{R}^\tau_i$, and

- $\bar{\pi}_i = \pi_i$.

Note that the only relevant difference between $\bar{R}$ and $\hat{R}$ is that under $\bar{R}$, each agent $i \in N$ positions x^t_i as his most preferred type-t object and his endowment o^t_i as his second preferred.

Under $\bar{R}$, each agent i's most preferred allotment is $z_i \equiv (y^1_i, \ldots, y^{t-1}_i, x^t_i, y^{t+1}_i, \ldots, y^m_i)$. Note that $z = (z_i)_{i \in N} \in X$ is an allocation because z is a mixture of x (for type t) and y (for other types). Thus, by *unanimity* of f, $f(\bar{R}) = z$.

Recall that since $(x^t_1, \ldots, x^t_n) = cTTC^t(\hat{R}) = TTC(\hat{R}^t)$, $(x^t_1, \ldots, x^t_n)$ is obtained by applying the TTC algorithm to preference profile $\hat{R}^t$. For each $i \in N$, let s_i be the step of the TTC algorithm at which agent i receives object x^t_i. Without loss of generality, assume that if $i < i'$ then $s_i \leq s_{i'}$.

Next, we will show that $f(\hat{R}) = z$ by using that $f(\bar{R}) = z$ and replacing, step-by-step, each $\bar{R}_i$ with $\hat{R}_i$. More specifically, we will replace the individual preferences in the order $n, n-1, \ldots, 1$.

We first show that $f(\bar{R}_{-n}, \hat{R}_n) = z$. Suppose $x^t_n \, \hat{R}^t_n \, y^t_n$. Then, $(\bar{R}_{-n}, \hat{R}_n)$ is a monotonic transformation of $\bar{R}$ at z. By *monotonicity* of f, $f(\bar{R}_{-n}, \hat{R}_n) = f(\bar{R}) = z$.

Now suppose $y^t_n \, \hat{P}^t_n \, x^t_n$. Let $\tau \in T$ such that $\pi_n(\tau) = 1 < \pi_n(t)$ (if $\pi_n(t) = 1$, then skip this step). Since f is *strategy-proof*, preferences are lexicographic, and τ is the most important type for agent n, we have $f^\tau_n(\bar{R}_{-n}, \hat{R}_n) \, \hat{R}^\tau_n \, f^\tau_n(\bar{R})$. Since $\tau \neq t$, $f^\tau_n(\bar{R}) = z^\tau_n = y^\tau_n$ and $f^\tau_n(\bar{R}_{-n}, \hat{R}_n) \, \hat{R}^\tau_n \, y^\tau_n$. Since $\tau \neq t$, it follows from the definition of $\hat{R}^\tau_n$ that y^τ_n is the best type-τ object. So, $f^\tau_n(\bar{R}_{-n}, \hat{R}_n) = y^\tau_n$. Now one can, sequentially, from more to less important types, apply similar arguments to show that

$$\text{for each type } t' \in T \text{ with } \pi_n(t') < \pi_n(t), \; f^{t'}_n(\bar{R}_{-n}, \hat{R}_n) = y^{t'}_n = f^{t'}_n(\bar{R}). \tag{6.1}$$

Since f is *marginally individually rational*, $f^t_n(\bar{R}_{-n}, \hat{R}_n) \in \{x^t_n, y^t_n, o^t_n\}$. Suppose $f^t_n(\bar{R}_{-n}, \hat{R}_n) = o^t_n$ and $o^t_n \neq x^t_n$. Then, $f^t_n(\bar{R}) = z^t_n = x^t_n \, \hat{P}^t_n \, o^t_n = f^t_n(\bar{R}_{-n}, \hat{R}_n)$, which together with (6.1) would contradict the *strategy-proofness* of f. Hence, $f^t_n(\bar{R}_{-n}, \hat{R}_n) \in \{x^t_n, y^t_n\}$.

Suppose that $f_n^t(\bar{R}_{-n}, \hat{R}_n) = y_n^t$. By the definition of the TTC algorithm, x_n^t is agent n's most preferred type-t object among the remaining objects at Step s_n of the TTC algorithm at preference profile $\hat{R}^t$. Therefore, object y_n^t is removed (i.e., assigned to some agent) at some Step $s^* < s_n$ of the TTC algorithm at preference profile $\hat{R}^t$.

Let C be the trading cycle of the TTC algorithm at preference profile $\hat{R}^t$ that contains y_n^t. Suppose C only contains one agent, say $j \neq n$. Then, among all objects present at Step s^*, agent j most prefers his own endowment, i.e., $o_j^t = y_n^t$. Hence, $x_j^t = cTTC_j^t(\hat{R}) = TTC_j(\hat{R}^t) = y_n^t = o_j^t$. So, by definition of $\bar{R}$, we have that at $(\bar{R}_{-n}, \hat{R}_n)$ agent j's marginal preferences of type t are given by $\bar{R}_j^t : o_j^t, \ldots$. By *marginal individual rationality* of f, $f_j^t(\bar{R}_{-n}, \hat{R}_n) = y_n^t$, which contradicts $f_n^t(\bar{R}_{-n}, \hat{R}_n) = y_n^t$.

Hence, C consists of agents $i_1, i_2, \ldots, i_K$ (with $K \geq 2$) and type-t objects $o_{i_1}^t, \ldots, o_{i_K}^t$ such that $n \notin \{i_1, \ldots, i_K\}$ and $y_n^t \in \{o_{i_1}^t, \ldots, o_{i_K}^t\}$. Without loss of generality, the cycle C is ordered $(i_1, i_2, \ldots, i_K)$. Note that at $(\bar{R}_{-n}, \hat{R}_n)$, for each $i_k \in \{i_1, \ldots, i_K\}$, agent i_k's marginal preferences of type t are $\bar{R}_{i_k}^t : o_{i_{k+1}}^t (= x_{i_k}^t), o_{i_k}^t, \ldots$ (modulo K). Without loss of generality, assume that $y_n^t = o_{i_1}^t$. It follows from $f_n^t(\bar{R}_{-n}, \hat{R}_n) = y_n^t$ and *marginal individual rationality* of f that $f_{i_K}^t(\bar{R}_{-n}, \hat{R}_n) = o_{i_K}^t$. Subsequently, for each agent $i_k \in \{i_2, \ldots, i_K\}$, $f_{i_k}^t(\bar{R}_{-n}, \hat{R}_n) = o_{i_k}^t$. Therefore, $f_{i_1}^t(\bar{R}_{-n}, \hat{R}_n) \neq o_{i_2}^t$. Moreover, $f_{i_1}^t(\bar{R}_{-n}, \hat{R}_n) \neq o_{i_1}^t$ because $f_n^t(\bar{R}_{-n}, \hat{R}_n) = y_n^t = o_{i_1}^t$. Thus, $o_{i_1}^t \; \bar{P}_{i_1} \; f_{i_1}^t(\bar{R}_{-n}, \hat{R}_n)$, which violates *marginal individual rationality* of f. Therefore, $f_n^t(\bar{R}_{-n}, \hat{R}_n) \neq y_n^t$. Hence,

$$f_n^t(\bar{R}_{-n}, \hat{R}_n) = x_n^t = f_n^t(\bar{R}). \tag{6.2}$$

Having established (6.1) and (6.2), one can use arguments similar to those for (6.1) to show that

$$\text{for each type } t' \in T \text{ with } \pi_n(t') > \pi_n(t),\; f_n^{t'}(\bar{R}_{-n}, \hat{R}_n) = y_n^{t'} = f_n^{t'}(\bar{R}). \tag{6.3}$$

From (6.1), (6.2), and (6.3) it follows that for each type $\tau \in T$, $f_n^\tau(\bar{R}_{-n}, \hat{R}_n) = f_n^\tau(\bar{R})$. Hence, $f_n(\bar{R}_{-n}, \hat{R}_n) = f_n(\bar{R})$. By *non-bossiness* of f, $f(\bar{R}_{-n}, \hat{R}_n) = f(\bar{R}) = z$.

By applying repeatedly the same arguments for agents $i = n-1, \ldots, 1$, we can sequentially replace each $\bar{R}_i$ with $\hat{R}_i$, and conclude that $f(\hat{R}) = f(\bar{R}) = z$. However, since $(y_1^t, \ldots, y_n^t) \neq (x_1^t, \ldots, x_n^t)$, there exists an agent j such that $y_j^t \neq x_j^t$. Hence, $f_j^t(\hat{R}) = y_j^t \neq x_j^t = z_j^t$, a contradiction. □

C.2. Proof of Theorem 4: uniqueness

Proof of Theorem 4: uniqueness. Suppose that mechanism $f : \mathcal{R}_s^N \to X$ satisfies the properties listed in Theorem 4 (by Lemma 2, *ontoness* and *unanimity* can be used interchangeably). We will show that for each $R \in \mathcal{R}_s^N$, $f(R) = cTTC(R)$. We introduce the following notation. For any agent $i \in N$ and any two separable preferences $R_i, \bar{R}_i \in \mathcal{R}_s$, we write $R_i \sim \bar{R}_i$ if they induce the same marginal preferences, i.e., for each $t \in T$, $R_i^t = \bar{R}_i^t$.

Let $R \in \mathcal{R}_s^N$ such that each agent has lexicographic preferences, i.e., $R \in \mathcal{R}_l^N$. Since the restriction of f to $\mathcal{R}_l^N$ satisfies the properties listed in Theorem 3, it immediately follows from Theorem 3 that $f(R) = cTTC(R)$.

Let $R \in \mathcal{R}_s^N$ such that only one agent does not have lexicographic preferences. We can assume, without loss of generality, that $R_1 \in \mathcal{R}_s \backslash \mathcal{R}_l$ and for each agent $j \neq 1$, $R_j \in \mathcal{R}_l$. Let $y \equiv f(R)$.

For each $t \in T$, define $R'_1(t) \in \mathcal{R}_l$ such that $R'_1(t) \sim R_1$ and the most important type of $R'_1(t)$ is type t. Since $R_1 \sim R'_1(1) \sim R'_1(2) \sim \cdots \sim R'_1(m)$, it follows from the definition of $cTTC$ that $x \equiv cTTC(R) = cTTC(R'_1(1), R_{-1}) = cTTC(R'_1(2), R_{-1}) = \cdots = cTTC(R'_1(m), R_{-1})$. We will show that $y = x$.

Let $t \in T$. From the case where each agent has lexicographic preferences, it follows that $f(R'_1(t), R_{-1}) = cTTC(R'_1(t), R_{-1}) = x$. By *strategy-proofness* of f when moving from $(R'_1(t), R_{-1})$ to (R_1, R_{-1}), $x_1 = f_1(R'_1(t), R_{-1})\; R'_1(t)\; f_1(R_1, R_{-1}) = y_1$. Then, since $R'_1(t) \sim R_1$ and $R'_1(t)$ is a lexicographic preference relation where t is the most important type, $x^t_1\; R^t_1\; y^t_1$.

Since for each $t \in T$, $x^t_1\; R^t_1\; y^t_1$ and since $R_1 \in \mathcal{R}_s$, we have $x_1\; R_1\; y_1$. By *strategy-proofness* of f when moving from (R_1, R_{-1}) to $(R'_1(t), R_{-1})$, we have that $y_1 = f_1(R_1, R_{-1})\; R_1\; f_1(R'_1(t), R_{-1}) = x_1$. Hence, $x_1 = y_1$. By *non-bossiness* of f, we have that $y = f(R_1, R_{-1}) = f(R'_1(t), R_{-1}) = x$.

Let $R \in \mathcal{R}^N_s$ such that exactly two agents do not have lexicographic preferences. We can assume, without loss of generality, that $R_1, R_2 \in \mathcal{R}_s \backslash \mathcal{R}_l$ and for each agent $j \neq 1, 2$, $R_j \in \mathcal{R}_l$. Let $y \equiv f(R)$.

For each $t \in T$, define $R'_2(t) \in \mathcal{R}_l$ such that $R'_2(t) \sim R_2$ and the most important type of $R'_2(t)$ is type t. Since $R_2 \sim R'_2(1) \sim R'_2(2) \sim \cdots \sim R'_2(m)$, it follows from the definition of $cTTC$ that $x \equiv cTTC(R) = cTTC(R'_2(1), R_{-2}) = cTTC(R'_2(2), R_{-2}) = \cdots = cTTC(R'_2(m), R_{-2})$. We will show that $y = x$.

Let $t \in T$. At preference profile $(R'_2(t), R_{-2})$, only agent 1 has non-lexicographic preferences. Thus, from the previous case, $f(R'_2(t), R_{-2}) = cTTC(R'_2(t), R_{-2}) = cTTC(R) = x$. By *strategy-proofness* of f when moving from $(R'_2(t), R_{-2})$ to (R_2, R_{-2}), we have that $x_2 = f_2(R'_2(t), R_{-2})\; R'_2(t)\; f_2(R_2, R_{-2}) = y_2$. Then, since $R'_2(t) \sim R_2$ and $R'_2(t)$ is a lexicographic preference relation where t is the most important type, $x^t_2\; R^t_2\; y^t_2$.

Since for each $t \in T$, $x^t_2\; R^t_2\; y^t_2$ and since $R_2 \in \mathcal{R}_s$, we have $x_2\; R_2\; y_2$. By *strategy-proofness* of f when moving from (R_2, R_{-2}) to $(R'_2(t), R_{-2})$, $y_2 = f_2(R_2, R_{-2})\; R_2\; f_2(R'_2(t), R_{-2}) = x_2$. Hence, $x_2 = y_2$. By *non-bossiness* of f, we have that $y = f(R_2, R_{-2}) = f(R'_2(t), R_{-2}) = x$.

We can apply repeatedly the same arguments to obtain that for each $k = 0, 1, \ldots, n$ and each preference profile $R \in \mathcal{R}^N_s$ where exactly k agents have non-lexicographic preferences, $f(R) = cTTC(R)$. Thus, for each $R \in \mathcal{R}^N_s$, $f(R) = cTTC(R)$. □

C.3. Independence of properties in Chapter 2

The following examples establish the logical independence of the properties in Theorem 3 (Corollary 2) on $\mathcal{R}^N_l$. We label the examples by the property/properties that is/are not satisfied.

Example 7 (***Ontoness* and *unanimity***).
The no-trade mechanism that always assigns the endowment allocation to each market is *individually rational*, *(group) strategy-proof*, and *non-bossy*, but neither *onto* nor *unanimous*. ◇

The no-trade mechanism in Example 7 is well-defined on $\mathcal{R}^N_l$, $\mathcal{R}^N_s$, and $\mathcal{R}^N$.

Example 8 (***Individual rationality***).
By ignoring property rights that are established via the endowments, we can easily adjust the well-known mechanism of serial dictatorship to our setting: based on an ordering of agents, we let agents sequentially choose their allotments. Serial dictatorship mechanisms have been shown in various resource allocation models to satisfy *Pareto efficiency* (and hence *ontoness*

and *unanimity*), *strategy-proofness*, and *non-bossiness*; since property rights are ignored, they violate *individual rationality* (e.g., see Monte and Tumennasan, 2015, Theorem 1). ◇

The serial dictatorship mechanism in Example 8 is well-defined on $\mathcal{R}_l^N$, $\mathcal{R}_s^N$, and $\mathcal{R}^N$.

Example 9 (***Strategy-proofness***).
We adapt so-called Multiple-Serial-IR mechanisms introduced by Biró et al. (2022b) for their circulation model to our multiple-type housing markets model. A Multiple-Serial-IR mechanism is determined by a fixed order of the agents. At any preference profile and following the order, the mechanism lets each agent pick his most preferred allotment from the available objects such that this choice together with previous agents' choices is compatible with an *individually rational* allocation. Formally,

Input. An order $\delta = (i_1, \ldots, i_n)$ of the agents and a multiple-type housing market $R \in \mathcal{R}_l^N$.

Step 0. Let $Y(0)$ be the set of individually rational allocations in X.

Step 1. Let Y_1 be the set of agent i_1's allotments that are compatible with some allocation in $Y(0)$, i.e., Y_1 consists of all $y_{i_1} \in \Pi_{t\in T} O^t$ for which there exists an allocation $x \in Y(0)$ such that $x_{i_1} = y_{i_1}$.
Let $y^*_{i_1}$ be agent i_1's most preferred allotment in Y_1, i.e., for each $y_{i_1} \in Y_1$, $y^*_{i_1}\ R_i\ y_{i_1}$.
Let $Y(1) \subseteq Y(0)$ be the set of allocations in $Y(0)$ that are compatible with $y^*_{i_1}$, i.e., $Y(1)$ consists of all $x \in Y(0)$ with $x_{i_1} = y^*_{i_1}$.

Step $\boldsymbol{k = 2, \ldots, n}$. Let Y_k be the set of agent i_k's allotments that are compatible with some allocation in $Y(k-1)$.
Let $y^*_{i_k}$ be agent i_k's most preferred allotment in Y_k.
Let $Y(k) \subseteq Y(k-1)$ be the set of allocations in $Y(k-1)$ that are compatible with $y^*_{i_k}$.

Output. The allocation of the Multiple-Serial-IR mechanism associated with δ at R is $MSIR(\delta, R) \equiv (y^*_1, y^*_2, \ldots, y^*_n)$.
Given an order δ, the associated Multiple-Serial-IR mechanism Δ assigns to each market R the allocation $\Delta(R) \equiv MSIR(\delta, R)$.

Biró et al. (2022b) show that Multiple-Serial-IR mechanisms are *individually rational* and *Pareto efficient*.

Next, we show that Multiple-Serial-IR mechanisms are *non-bossy*. Let $\delta = (i_1, \ldots, i_n)$ be an order of the agents and let Δ denote the associated Multiple-Serial-IR mechanism.

Let $R \in \mathcal{R}_l^N$, $i \in N$, and $R'_i \in \mathcal{R}_l$. Let $R' \equiv (R'_i, R_{-i})$, $x \equiv \Delta(R)$, and $y \equiv \Delta(R')$. Assume $y_i = x_i$. We show that $y = x$.

Let $i_k \equiv i$. Since $y_i = x_i$ and for each $\ell = 2, \ldots, k-1, k+1, \ldots, n$, $R'_{i_\ell} = R_{i_\ell}$, agent i_1's choice at Step 1 under R' is restricted in the same way as agent i_1's choice at Step 1 under R. Thus, since $R'_{i_1} = R_{i_1}$, we have $y_{i_1} = x_{i_1}$. Similar arguments show that for each $\ell = 2, \ldots, k-1, k+1, \ldots, n$, $y_{i_\ell} = x_{i_\ell}$. Hence, Δ is non-bossy.

In the context of multiple-type housing markets, Konishi et al. (2001) show that there is no mechanism that is *Pareto efficient*, *individually rational*, and *strategy-proof*. Since Multiple-Serial-IR mechanisms are *Pareto efficient* and *individually rational*, they are not *strategy-proof*. We include a simple illustrative example for $n = 2$ agents and $m = 2$ types for completeness.

Let $N = \{1, 2\}$ and $T = \{H(ouse), C(ar)\}$. For each $i \in N$, let (H_i, C_i) be agent i's endowment. Let $R \in \mathcal{R}_l^N$ be given by

$$\boldsymbol{R_1} : H_2, \boldsymbol{H_1}, C_2, \boldsymbol{C_1},$$

$$\boldsymbol{R_2} : H_1, \boldsymbol{H_2}, \boldsymbol{C_2}, C_1.$$

Consider the Multiple-Serial-IR mechanism Δ induced by $\delta = (1,2)$, i.e., agent 1 moves first (note that since there are only two agents, when agent 1 picks his allotment, the final allocation is completely determined). Since allocation $x \equiv ((H_2, C_2), (H_1, C_1))$ is *individually rational* at R and $x_1 = (H_2, C_2)$ is agent 1's most preferred allotment, $\Delta(R) = x$.

Next, consider $\boldsymbol{R'_2} : \boldsymbol{C_2}, C_1, H_1, \boldsymbol{H_2}$. Note that at (R_1, R'_2), only $y \equiv ((H_2, C_1), (H_1, C_2))$ and e are *individually rational*. Thus, agent 1 can only pick y_1 or o_1. Since $y_1 \, R_1 \, o_1$, agent 1 picks y_1 and hence $\Delta(R_1, R'_2) = y$. Finally, we see that $y_2 \, R_2 \, x_2$, which implies that agent 2 has an incentive to misreport R'_2 at R. Hence, the Multiple-Serial-IR mechanism induced by $\delta = (1,2)$ is not *strategy-proof*. ◇

The mechanism in Example 9 is well-defined on $\mathcal{R}_l^N$, $\mathcal{R}_s^N$, and $\mathcal{R}^N$.

Note that if $n = 2$, then any mechanism is *non-bossy*. Thus, for our last independence example, we assume $n > 2$.

Example 10 (***Non-bossiness***)**.**
We first provide an example of a mechanism for $n = 3$ and $m = 1$. Let $N = \{1,2,3\}$ and $T = \{H(ouse)\}$. Let $R \in \mathcal{R}^N$. We say that agents 1 and 3 are *in conflict* if H_2 is the most preferred object for both R_1 and R_3. Similarly, we say that agents 1 and 2 are *in conflict* if H_3 is the most preferred object for both R_1 and R_2. Let mechanism f be defined as follows: for each $R \in \mathcal{R}^N$,

- **(a)** if agents 1 and 2 are in conflict, then (i) transform R_2 to $\bar{R}_2$ by dropping H_3 to the bottom, i.e., $\bar{R}_2 : \ldots, H_3$, while keeping the relative order of H_1 and H_2, and (ii) set $f(R) \equiv TTC(R_1, \bar{R}_2, R_3)$;
- **(b)** if agents 1 and 3 are in conflict, then (i) transform R_3 to $\bar{R}_3$ by dropping H_2 to the bottom, i.e., $\bar{R}_3 : \ldots, H_2$, while keeping the relative order of H_1 and H_3, and (ii) set $f(R) \equiv TTC(R_1, R_2, \bar{R}_3)$;
- **(c)** if agent 1 is not in conflict with either agent 2 or agent 3, then $f(R) \equiv TTC(R)$.

It is easy to verify that f is *individually rational* and *unanimous*. We prove that f is *strategy-proof* on the next page. To see that f is *bossy*, let R be such that

$$\boldsymbol{R_1} : H_3, \boldsymbol{H_1}, H_2,$$
$$\boldsymbol{R_2} : H_3, \boldsymbol{H_2}, H_1,$$
$$\boldsymbol{R_3} : H_2, \boldsymbol{H_3}, H_1.$$

Then, since agents 1 and 2 are in conflict, we have $\boldsymbol{\bar{R}_2} : \boldsymbol{H_2}, H_1, H_3$ and $f(R) = TTC(\bar{R}_2, R_{-2})$. In particular, for each $i = 1,2,3$, $f_i(R) = H_i$. Next consider $\boldsymbol{R'_1} : \boldsymbol{H_1}, \ldots$. Then, $f(R'_1, R_{-1}) = TTC(R'_1, R_{-1})$. In particular, $f_1(R'_1, R_{-1}) = H_1$, $f_2(R'_1, R_{-1}) = H_3$, and $f_3(R'_1, R_{-1}) = H_2$. Therefore, $f_1(R'_1, R_{-1}) = H_1 = f_1(R)$, $f_2(R'_1, R_{-1}) = H_3 \neq H_2 = f_2(R)$, and $f_3(R'_1, R_{-1}) = H_2 \neq H_3 = f_3(R)$. Hence, f is *bossy* (and not *Pareto efficient*).

Next, we extend mechanism f from $n = 3$ to any $n > 3$. Let $n > 3$ and recall that $m = 1$. An object $o \in O$ is *acceptable* for agent $i \in N$ if $o \, R_i \, H_i$. Let mechanism g be defined as follows: for each $R \in \mathcal{R}^N$,

Case (A) if some agent $i \in \{4, \dots, n\}$ finds some object different from his endowment acceptable, then set $g(R) \equiv TTC(R)$;
Case (B) if each agent $i \in \{4, \dots, n\}$ only finds his own endowment acceptable, then

- let $N' \equiv \{1, 2, 3\}$ and for each $i \in N'$, let $g_i(R) \equiv f_i(R_{|N'})$ where $R_{|N'}$ denotes the preferences of agents in N' restricted to $\{H_1, H_2, H_3\}$;
- for each agent $i \in \{4, \dots, n\}$, $g_i(R) \equiv H_i$.

Since f and TTC are *individually rational* and *unanimous*, g is *individually rational* and *unanimous*. Since f is *bossy*, g is *bossy* as well.

Next, we show that g is *strategy-proof*. First, we verify that no agent $i \in \{4, \dots, n\}$ can profitably misreport his preferences. If R is in case (A), then a misreport by agent i that creates another profile in case (A) does not lead to a more preferred allotment because TTC is *strategy-proof*; a misreport that creates a profile in case (B) assigns endowment H_i to agent i. In either case, the misreport does not yield a more preferred allotment for agent i. If R is in case (B), then each agent $i \in \{4, \dots, n\}$ obtains his most preferred object (his own endowment) and hence cannot gain by misreporting his preferences.

Second, no agent in $\{1, 2, 3\}$ can "move" R from case (A) to case (B) nor from case (B) to case (A). If R is in case (A), no agent in $\{1, 2, 3\}$ can profitably misreport his preferences because TTC is *strategy-proof*. If R is in case (B), no agent in $\{1, 2, 3\}$ can profitably misreport his preferences because f is *strategy-proof*. Hence, g is *strategy-proof*.

Finally, we extend mechanism g from Shapley-Scarf housing markets to multiple-type housing markets with lexicographic (or separable) preferences by applying it coordinatewise to all object types. Let h be the mechanism that assigns the objects of each type t according to g. Then, h is *unanimous* (and hence *onto*), *individually rational*, and *strategy-proof*, but *bossy*. ◇

The mechanism in Example 10 is well-defined on $\mathcal{R}_l^N$ and $\mathcal{R}_s^N$ (but not on $\mathcal{R}^N$).

Proof of strategy-proofness in Example 10

We show that mechanism f on $\mathcal{R}^N$ defined in Example 10 for $n = 3$ and $m = 1$ is *strategy-proof*.

Proof. Let $R \in \mathcal{R}^N$. We consider three cases.
Case 1. Preferences of agent 1 are $R_1 : H_1, \dots$.
By *individual rationality* of f, $f_1(R) = H_1$ and since this is his most preferred object, agent 1 cannot gain by misreporting his preferences.

Let R_2' be some misreport of agent 2. Since agents 1 and 2 (nor 1 and 3) are not in conflict at R nor at (R_1, R_2', R_3), mechanism f yields the corresponding TTC allocations at R and (R_1, R_2', R_3). Hence, by *strategy-proofness* of TTC, agent 2 does not have a profitable deviation at R. Similarly, agent 3 does not have a profitable deviation at R.

Case 2. Preferences of agent 1 are $R_1 : H_2, H_1, H_3$. (Since agents 2 and 3 play a symmetric role in the definition of f, similar symmetric arguments work for H_3, H_1, H_2.)
Agents 1 and 2 are not in conflict. Hence, by *strategy-proofness* of TTC, agent 2 does not have a profitable deviation at R.

Next, we verify that agent 1 does not have a profitable deviation at R.

Case 2.a. Preferences of agent 2 are $R_2 : H_2, \ldots$.
Note that by *individual rationality* of f we have $f_2(R) = H_2$. So, $f_1(R) = H_1$. Reporting any other preferences will not give him H_2 either. So, agent 1 does not have a profitable deviation at R.

Case 2.b. Preferences of agent 2 are $R_2 : H_3, H_2, H_1$ and preferences of agent 3 are $R_3 : H_2, H_3, H_1$.
Agents 1 and 3 are in conflict and one easily verifies that $f(R)$ is the no-trade allocation. In particular, agent 1 receives his endowment H_1 at R. Obviously, misreporting $R_1' : H_1, \ldots$ gives him H_1. Any other misreport of agent 1's preferences yields the no-trade allocation. So, agent 1 does not have a profitable deviation at R.

Case 2.c. Preferences of agent 2 are $R_2 : H_1, \ldots$ or preferences of agent 2 are $R_2 : H_3, H_1, H_2$ or [preferences of agent 2 are $R_2 : H_3, H_2, H_1$ and preferences of agent 3 are *not* $R_3 : H_2, H_3, H_1$]. It is easy but cumbersome to verify that $f_1(R) = H_2$, i.e., agent 1 receives his most preferred object H_2. So, agent 1 does not have a profitable deviation at R.

Finally, we verify that agent 3 does not have a profitable deviation at R.

Case 2.I. Preferences of agent 3 are $R_3 : H_3, \ldots$.
By *individual rationality* of f, $f_3(R) = H_3$ and since this is his most preferred object, agent 3 cannot gain by misreporting his preferences.

Case 2.II. Preferences of agent 3 are $R_3 : H_1, \ldots$.
Agents 1 and 3 are not in conflict and by *strategy-proofness* of TTC, agent 3 does not have a profitable deviation at R.

Case 2.III. Preferences of agent 3 are $R_3 : H_2, H_3, H_1$.
Agents 1 and 3 are in conflict and one easily verifies that $f_3(R) = H_3$. Any possible profitable misreport of preferences by agent 3 requires that H_2 is acceptable and appears in second position. Hence, the only possible candidate for a profitable deviation is $R_3' : H_1, H_2, H_3$. However, if $R_2 : H_1, \ldots$ or $R_2 : H_2, \ldots$, then $f_3(R_1, R_2, R_3') = H_3$; and if $R_2 : H_3, \ldots$, then $f_3(R_1, R_2, R_3') = H_1$. So, agent 3 does not have a profitable deviation at R.

Case 2.IV.i. Preferences of agent 3 are $R_3 : H_2, H_1, H_3$ and preferences of agent 2 are $R_2 : H_1, \ldots$ or $R_2 : H_2, \ldots$.
Agents 1 and 3 are in conflict and for any possible deviation R_3', $f_3(R_1, R_2, R_3') = H_3 = f_3(R)$. Hence, agent 3 does not have a profitable deviation at R.

Case 2.IV.ii. Preferences of agent 3 are $R_3 : H_2, H_1, H_3$ and preferences of agent 2 are $R_2 : H_3, \ldots$.
Agents 1 and 3 are in conflict and one easily verifies that $f_3(R) = H_1$. Any possible profitable misreport of preferences by agent 3 requires that H_2 is acceptable and appears in second position. Hence, the only possible candidate for a profitable deviation is $R_3' : H_1, H_2, H_3$. However, $f_3(R_1, R_2, R_3') = H_1$. So, agent 3 does not have a profitable deviation at R.

Case 3. Preferences of agent 1 are $R_1 : H_2, H_3, H_1$. (Since agents 2 and 3 play a symmetric role in the definition of f, similar symmetric arguments work for H_3, H_2, H_1.)
Agents 1 and 2 are not in conflict. Hence, by *strategy-proofness* of TTC, agent 2 does not have a profitable deviation at R.

Next, we verify that agent 1 does not have a profitable deviation at R.

Case 3.a. Preferences of agent 2 are $R_2 : H_1, \ldots$.

One immediately verifies that $f_1(R) = H_2$, which is his most preferred object. So, agent 1 does not have a profitable deviation at R.

Case 3.b. Preferences of agent 2 are $R_2 : H_2, \ldots$ and preferences of agent 3 are $R_3 : H_1, \ldots$ or $R_3 : H_2, H_1, H_3$.
Then, for any possible deviation R'_1, $f_1(R'_1, R_2, R_3) = H_3 = f_1(R)$. Hence, agent 1 does not have a profitable deviation at R.

Case 3.c. Preferences of agent 2 are $R_2 : H_2, \ldots$ and preferences of agent 3 are $R_3 : H_3, \ldots$ or $R_3 : H_2, H_3, H_1$;
or
Case 3.d. Preferences of agent 2 are $R_2 : H_3, H_2, H_1$ and preferences of agent 3 are $R_3 : H_3, \ldots$ or $R_3 : H_2, H_3, H_1$.
In cases 3.c and 3.d, we have that for any possible deviation R'_1, $f_1(R'_1, R_2, R_3) = H_1 = f_1(R)$. Hence, agent 1 does not have a profitable deviation at R.

Case 3.e. Preferences of agent 2 are $R_2 : H_3, \ldots$ and preferences of agent 3 are $R_3 : H_1, \ldots$;
or
Case 3.f. Preferences of agent 2 are $R_2 : H_3, H_2, H_1$ and preferences of agent 3 are $R_3 : H_2, H_1, H_3$;
or
Case 3.g. Preferences of agent 2 are $R_2 : H_3, H_1, H_2$ and preferences of agent 3 are $R_3 : H_2, \ldots$;
or
Case 3.h. Preferences of agent 2 are $R_2 : H_3, H_1, H_2$ and preferences of agent 3 are $R_3 : H_3, \ldots$;
In cases 3.e, 3.f, 3.g, and 3.h, $f_1(R) = H_2$, i.e., agent 1 receives his most preferred object H_2. So, agent 1 does not have a profitable deviation at R.

Finally, we verify that agent 3 does not have a profitable deviation at R. Cases 3.I, 3.II, and 3.III below are as 2.I, 2.II, and 2.III. There is a small difference between cases 2.IV and 3.IV.

Case 3.I. Preferences of agent 3 are $R_3 : H_3, \ldots$.
By *individual rationality* of f, $f_3(R) = H_3$ and since this is his most preferred object, agent 3 cannot gain by misreporting his preferences.

Case 3.II. Preferences of agent 3 are $R_3 : H_1, \ldots$.
Agents 1 and 3 are not in conflict and by *strategy-proofness* of TTC, agent 3 does not have a profitable deviation at R.

Case 3.III. Preferences of agent 3 are $R_3 : H_2, H_3, H_1$.
Agents 1 and 3 are in conflict and one easily verifies that $f_3(R) = H_3$. Any possible profitable misreport of preferences by agent 3 requires that H_2 is acceptable and appears in the second position. Hence, the only possible candidate for a profitable deviation is $R'_3 : H_1, H_2, H_3$. However, if $R_2 : H_1, \ldots$, then $f_3(R_1, R_2, R'_3) = H_3$; and if $R_2 : H_3, \ldots$ or $R_2 : H_2, \ldots$, then $f_3(R_1, R_2, R'_3) = H_1$. So, agent 3 does not have a profitable deviation at R.

Case 3.IV.i. Preferences of agent 3 are $R_3 : H_2, H_1, H_3$ and preferences of agent 2 are $R_2 : H_1, \ldots$.
Agents 1 and 3 are in conflict and for any possible deviation R'_3, $f_3(R_1, R_2, R'_3) = H_3 = f_3(R)$. Hence, agent 3 does not have a profitable deviation at R.

Case 3.IV.ii. Preferences of agent 3 are $R_3 : H_2, H_1, H_3$ and preferences of agent 2 are $R_2 : H_2, \ldots$ or $R_2 : H_3, \ldots$.
Agents 1 and 3 are in conflict and one easily verifies that $f_3(R) = H_1$. Any possible profitable misreport of preferences by agent 3 requires that H_2 is acceptable and appears in second position.

Hence, the only possible candidate for a profitable deviation is $R'_3 : H_1, H_2, H_3$. However, $f_3(R_1, R_2, R'_3) = H_1$. So, agent 3 does not have a profitable deviation at R. $\square$

D. Proofs in Chapter 3

We list one useful result based on *strategy-proofness*, *non-bossiness*, and *monotonicity*.

Fact 2. Let f be a *strategy-proof* and *non-bossy* mechanism. Let $R \in \mathcal{R}_l^N$, $x \equiv f(R)$, $i \in N$, and $R_i^* \in \mathcal{R}_l$ be preferences that only differ from R_i in the marginal preference of the most important type (type-t), i.e., (1) $\pi_i = \pi_i^*$ where $\pi_i^*(t) = 1$, and (2) for each $\tau \neq t$, $R_i^\tau = R_i^{*\tau}$.
If $f_i^t(R_i^*, R_{-i}) = x_i^t$, then $f(R_i^*, R_{-i}) = x$.

Proof. It is without loss of generality to assume that $t = 1$ and $\pi_i : 1, \ldots, m$. Let $y \equiv f(R_i^*, R_{-i})$ and assume $y_i^1 = x_i^1$. By *strategy-proofness* of f, $x_i \, R_i \, y_i$ and $y_i \, R_i^* \, x_i$. Since R_i are lexicographic preferences, $x_i \, R_i \, y_i$ implies $x_i^2 \, R_i^2 \, y_i^2$. Similarly, since R_i^* are lexicographic preferences, $y_i \, R_i^* \, x_i$ implies $y_i^2 R_i^{*2} x_i^2$. Since $R_i^2 = R_i^{*2}$, we find that $x_i^2 = y_i^2$. Applying the same argument sequentially for type-τ marginal preferences with $\tau = 3, \ldots, m$ yields $x_i = y_i$. By *non-bossiness* of f, $x = y$. $\square$

D.1. Proof of Theorem 6

Here we only show the uniqueness.

Let $f : \mathcal{R}_l^N \to X$ be a mechanism satisfying *individual rationality*, *strategy-proofness*, and *coordinatewise efficiency*.

A result for restricted preferences

We first consider a restricted domain $\mathcal{R}_\pi^N \subsetneq \mathcal{R}_l^N$ such that all agents share the same importance order π. It is without loss of generality to assume that $\pi : 1, \ldots, m$.

Proposition 2. *For each* $R \in \mathcal{R}_\pi^N$, $f(R) = cTTC(R)$.

The proof of Proposition 2 consists of three claims.

First, we show that for each market with restricted preferences, f assigns the cTTC allocation of type-1.

Claim 1. For each $R \in \mathcal{R}_\pi^N$, $f^1(R) = cTTC^1(R)$.

Proof. Let C be a first step top trading cycle under TTC^1 at R involving a set of agents $S_C \subseteq N$. We first show that C is executed at $f(R)$, i.e., for each $i \in S_C$, $f_i^1(R) = cTTC_i^1(R)$, by induction on $|S_C|$.
Induction basis. $|S_C| = 1$. In this case, agent $i \in S_C$ points to his type-1 endowed object, i.e., $C = (i \to o_i^1 \to i)$. Since preferences are lexicographic, agent i would be strictly worse off if he received any other type-1 object. Thus, by *individual rationality* of f, C is executed.
Induction hypothesis. Let $K \in \{2, \ldots, n\}$. Suppose that C is executed when $|S_C| < K$.

Induction step. Let $|S_C| = K$. Without loss of generality, assume that $S_C = \{1, \ldots, K\}$ and $C = (1 \to o_2^1 \to 2 \to o_3^1 \to \ldots \to K \to o_1^1 \to 1)$.

By contradiction, assume that C is not executed. Thus, there is an agent $i \in S_C$ who does not receive o_{i+1}^1, i.e., $f_i^1(R) \neq o_{i+1}^1$. It is without loss of generality to assume that $i = 2$. We proceed by contradiction in two steps.

Step 1. Let $\hat{R}_2 \in \mathcal{R}_\pi$ be such that for agent 2 and type-1 objects, only $o_3^1 (= cTTC_2^1(R))$ is acceptable (apart from his type-1 endowment), i.e.,

$$\hat{R}_2^1 : o_3^1, o_2^1, \ldots,$$
$$\text{for each } t \in T \setminus \{1\} : \hat{R}_2^t = R_2^t, \text{ and}$$
$$\hat{\pi}_2 = \pi : 1, \ldots, m.$$

Let

$$\hat{R} \equiv (\hat{R}_2, R_{-2}).$$

Since $\hat{R}_2 \in \mathcal{R}_l$ and f is *individually rational*, $f_2^1(\hat{R}) \in \{o_3^1, o_2^1\}$. By *strategy-proofness* of f, $f_2^1(R) \neq o_3^1$ implies that $f_2^1(\hat{R}) \neq o_3^1$, otherwise instead of R_2, agent 2 has an incentive to misreport $\hat{R}_2$ at R. Thus, $f_2^1(\hat{R}) = o_2^1$. Thus, agent 1 cannot receive $o_2^1 (= cTTC_1^1(R))$ from agent 2 because it is assigned to agent 2. Overall, we find that

$$f_2^1(\hat{R}) = o_2^1 \neq o_3^1 \text{ and } f_1^1(\hat{R}) \neq o_2^1. \tag{6.4}$$

Step 2. Let $\hat{\hat{R}}_1 \in \mathcal{R}_\pi$ be such that for agent 1 and type-1 objects, only o_2^1 and o_3^1 are acceptable (apart from his type-1 endowment), i.e.,

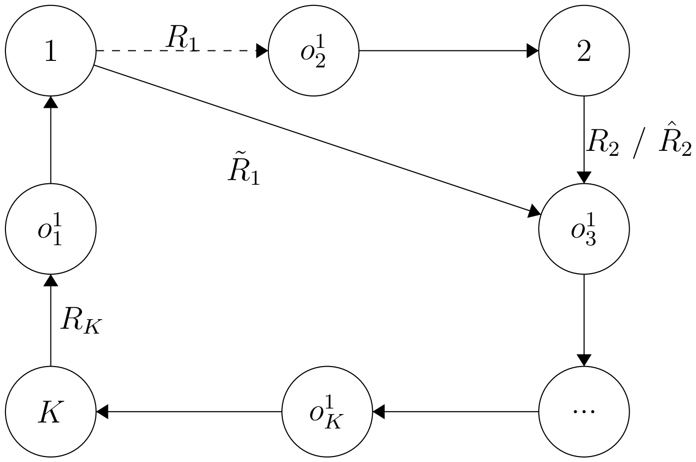
$$\hat{\hat{R}}_1^1 : o_2^1, o_3^1, o_1^1, \ldots, \text{ (if } K = 2 \text{ then here we have } \hat{\hat{R}}_1^1 : o_2^1, o_1^1, \ldots)$$
$$\text{for each } t \in T \setminus \{1\} : \hat{\hat{R}}_1^t = R_1^t, \text{ and}$$
$$\hat{\hat{\pi}}_1 = \pi : 1, \ldots, m.$$

Let

$$\hat{\hat{R}} \equiv (\hat{\hat{R}}_1, \hat{R}_{-1}) = (\hat{\hat{R}}_1, \hat{R}_2, R_3, \ldots, R_n).$$

By *individual rationality* of f, $f_1^1(\hat{\hat{R}}) \in \{o_3^1, o_2^1, o_1^1\}$. By *strategy-proofness* of f, $f_1^1(\hat{R}) \neq o_2^1$ (see (6.4)) implies that $f_1^1(\hat{\hat{R}}) \neq o_2^1$, otherwise instead of $\hat{R}_1 (= R_1)$, agent 1 has an incentive to misreport $\hat{\hat{R}}_1$ at $\hat{R}$.

We then show that $f_1^1(\hat{\hat{R}}) = o_3^1$. Let $\tilde{R}_1^1 : o_3^1, o_1^1, \ldots$ and let $\tilde{R}_1$ be obtained from R_1 by replacing type-1 marginal preferences R_1^1 with type-1 marginal preferences $\tilde{R}_1^1$. That is, $\tilde{R}_1 = (\tilde{R}_1^1, R_1^2, \ldots, R_1^m, \pi)$. At $(\tilde{R}_1, \hat{\hat{R}}_{-1})$, there is a top trading cycle $C' = (1 \to o_3^1 \to 3 \to \ldots \to K \to o_1^1 \to 1)$ that only involves $K - 1$ agents. Thus, by the induction hypothesis, C' is executed and $f_1^1(\tilde{R}_1, \hat{\hat{R}}_{-1}) = o_3^1$. See the figure below.



Therefore, by *strategy-proofness* of f, $f_1^1(\hat{\hat{R}}) = o_3^1$, otherwise instead of $\hat{\hat{R}}_1$, agent 1 has an incentive to misreport $\tilde{R}_1$ at $\hat{\hat{R}}$.

Moreover, $f_1^1(\hat{\hat{R}}) = o_3^1$ implies that $f_2^1(\hat{\hat{R}}) \neq o_3^1$. By *individual rationality* of f, $f_2^1(\hat{\hat{R}}) = o_2^1$. Overall, we find that

$$f_1^1(\hat{\hat{R}}) = o_3^1 \text{ and } f_2^1(\hat{\hat{R}}) = o_2^1. \tag{6.5}$$

However, this equation implies that f is not *coordinatewisely efficient* since agents 1 and 2 can be better off by swapping their type-1 allotments. That is, for $y \equiv f(\hat{\hat{R}})$, there is a coordinatewise improvement z such that (i) $z_1^1 = y_2^1 (= o_2^1), z_2^1 = y_1^1 (= o_3^1)$, and (ii) all others are the same as y. Thus, we conclude that C is executed when $|S_C| = K$.

It suffices to show that C is executed at $f(R)$ because once we have shown that agents who trade at the first step of TTC (of type-1) always receive their TTC allotments of type-1 under f, we can consider agents who trade at the second step of TTC (for type-1) by following the same proof arguments as in the first step trading cycles, and so on. Thus, the proof of Claim 1 is completed. □

Note that at step 1 and step 2 of the proof of Claim 1, we only require that agents in S_C have restricted preferences in $\mathcal{R}_\pi$, i.e., if $R_{S_C} \in \mathcal{R}_\pi^{S_C}$ then for any $R_{-S_C} \in \mathcal{R}_l^{-S_C}$, $f_{S_C}^1(R_{S_C}, R_{-S_C}) = cTTC_{S_C}^1(R_{S_C}, R_{-S_C})$. Therefore, Claim 1 implies the following fact.

Fact 3 (Restricted preferences)**.**
For each $R \in \mathcal{R}_\pi^N$, let $\mathbb{C} \equiv \{C_1, C_2, \ldots, C_I\}$ be the set of top trading cycles that are obtained via the TTC algorithm of type-1 at R^1. Moreover, for each top trading cycle $C_i \in \mathbb{C}$, assume that C_i is executed at step s_i, and without loss of generality, assume that if $i < i'$ then $s_i \leq s_{i'}$.

For each $C_i \in \mathbb{C}$, if all agents in $S_{C_1}, S_{C_2}, \ldots, S_{C_{i-1}}, S_{C_i}$ have restricted preferences, then $C_1, \ldots,$
C_{i-1}, C_i are executed, regardless of the preferences of other agents in $C_{i+1}, \ldots, C_I$. That is, for each $C_i \in \mathbb{C}$, let $S' \equiv \cup_{k=1}^{i} S_{C_k}$. If R is such that for each $j \in S'$, $R_j \in \mathcal{R}_\pi$, then $f_{S'}^1(R) = cTTC_{S'}^1(R)$.

Next, we show that f is "coordinatewisely individually rational" at type-2.

Claim 2. For each $R \in \mathcal{R}_\pi^N$ and each $i \in N$, $f_i^2(R)\, R_i^2\, o_i^2$.

Proof. By contradiction, assume that there exist $R \in \mathcal{R}_\pi^N$ and $i \in N$ such that $o_i^2 \, P_i^2 \, f_i^2(R)$.

Let $y \equiv f(R)$. Recall that by Claim 1, $y^1 = cTTC^1(R) = TTC^1(R^1)$. It is without loss of generality to assume that $i = 1$. Since $\hat{R}_1 \in \mathcal{R}_l$ and f is *individually rational*,

$$y_1^1 \neq o_1^1. \tag{6.6}$$

Let $\hat{R}_1 \in \mathcal{R}_\pi$ be such that

$$\hat{R}_1^2 : o_1^2, y_1^2, \ldots,$$
$$\text{for each } t \in T \setminus \{2\}, \hat{R}_1^t : y_1^t, o_1^t, \ldots, \text{ and}$$
$$\hat{\pi}_1 = \pi : 1, \ldots, m.$$

By *strategy-proofness* of f, $f_1(\hat{R}_1, R_{-1}) = y_1$. Note that $\hat{\pi}_1 = \pi : 1, \ldots, m$ and $(\hat{R}_1, R_{-1}) \in \mathcal{R}_\pi^N$. By Claim 1, $f^1(\hat{R}_1, R_{-1}) = cTTC^1(\hat{R}_1, R_{-1}) = y^1$.

Let $\bar{R}_1 \in \mathcal{R}_l$ be such that $\bar{R}_1$ and $\hat{R}_1$ only differ in the importance order, where the orders of type-1 and type-2 are switched, i.e.,

$$\text{for each } t \in T, \bar{R}_1^t = \hat{R}_1^t, \text{ and}$$

$$\bar{\pi}_1 : 2, 1, 3, \ldots, m.$$

By *individual rationality* of f, $f_1^2(\bar{R}_1, R_{-1}) = o_1^2$ and hence $f_1^1(\bar{R}_1, R_{-1}) \in \{y_1^1, o_1^1\}$. Since R_1 is lexicographic, any allotment z_1 with $z_1^1 = y_1^1$ and $z_1^2 = o_1^2$ is strictly better than y_1 at R_1.

By *strategy-proofness* of f, $f_1^1(\bar{R}_1, R_{-1}) \neq f_1^1(\hat{R}_1, R_{-1}) = y_1^1$; otherwise agent 1 has an incentive to misreport $\bar{R}_1$ at $(\hat{R}_1, R_{-1})$. Thus,

$$f_1^1(\bar{R}_1, R_{-1}) = o_1^1. \tag{6.7}$$

Next, we show that (6.7) contradicts *coordinatewise efficiency* of f.

let ℓ be the step of the TTC algorithm at which agent 1 receives his type-1 object y_1^1. Let C be the corresponding top trading cycle that involves agent 1, i.e., $1 \in S_C$.

Note that by Claim 1 and Fact 3, all top trading cycles that are obtained before step ℓ are executed at $f(\bar{R}_1, R_{-1})$. Thus, by the definition of TTC, for each agent in S_C, the object that he pointed at in C is his most preferred type-1 object among the unassigned type-1 objects, i.e., for each $i \in S_C$, all better type-1 objects for him, are assigned to someone else via some top trading cycles that are obtained before step ℓ.

Since $y_1^1 \neq o_1^1$ (see (6.6)), $|S_C| > 1$. We show a contradiction by induction on $|S_C|$.

Induction basis. $|S_C| = 2$. Without loss of generality, let $C = (1 \to o_2^1 \to 2 \to o_1^1 \to 1)$. Since $f_1^1(\bar{R}_1, R_{-1}) = o_1^1$ (see (6.7)), agent 2 does not receive his most (feasible) preferred object o_1^1. Thus, $f(\bar{R}_1, R_{-1})$ is not *coordinatewisely efficient* since there is a *coordinatewise improvement* such that agent 1 receives o_2^1, agent 2 receives o_1^1, and all others are the same as $f(\bar{R}_1, R_{-1})$.

The following induction arguments for $K > 2$ are similar to the proof of Claim 1.

Induction hypothesis. Let $K \in \{2, \ldots, n\}$. Suppose that C is executed when $|S_C| < K$.

Induction step. Let $|S_C| = K$. Without loss of generality, assume that $S_C = \{1, \ldots, K\}$ and hence $C = (1 \to o_2^1 \to 2 \to \ldots \to K-1 \to o_K^1 \to K \to o_1^1 \to 1)$.

Recall that by (6.7), agent 1 receives his type-1 endowment and hence agent K does not receive his most (feasible) preferred object o_1^1. Let $\hat{\hat{R}}_K \in \mathcal{R}_\pi$ be such that for agent K and type-1 objects, only o_1^1 and o_2^1 are acceptable (apart from his type-1 endowment), i.e.,

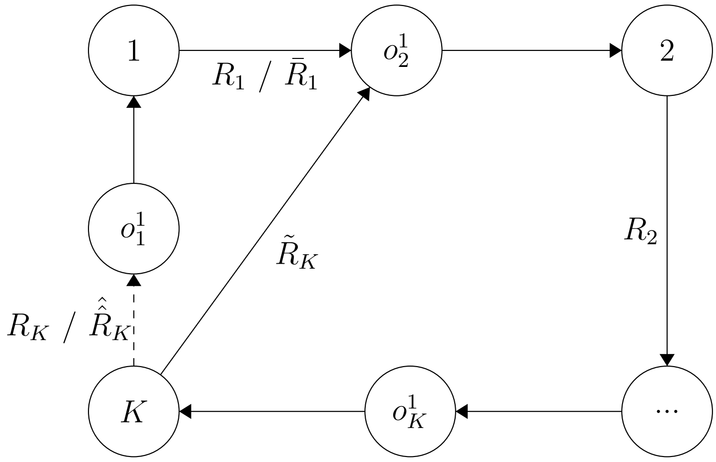
$$\hat{\hat{R}}^K_1 : o^1_1, o^1_2, o^1_K, \ldots,$$

$$\text{for each } t \in T \setminus \{1\} : \hat{\hat{R}}^t_K = R^t_K, \text{ and}$$

$$\hat{\hat{\pi}}_K = \pi : 1, \ldots, m.$$

Let $\hat{\hat{R}} \equiv (\hat{\hat{R}}_K, \bar{R}_1, R_{N\setminus\{1,K\}})$. By *individual rationality* of f, $f^1_K(\hat{\hat{R}}) \in \{o^1_1, o^1_2, o^1_K\}$. By *strategy-proofness* of f, $f^1_K(\hat{R}) \neq o^1_1$ (see (6.7)) implies that $f^1_K(\hat{\hat{R}}) \neq o^1_1$, otherwise instead of R_K, agent K has an incentive to misreport $\hat{\hat{R}}_K$ at $(\bar{R}_1, R_{-1})$.

We then show that $f^1_K(\hat{\hat{R}}) = o^1_2$. Let $\tilde{R}^1_K : o^1_2, o^1_K, \ldots$ and let $\tilde{R}_K$ be obtained from R_K by replacing type-1 marginal preferences R^1_K with type-1 marginal preferences $\tilde{R}^1_K$. That is, $\tilde{R}_K = (\tilde{R}^1_K, R^2_K, \ldots, R^m_K, \pi)$. At $(\tilde{R}_K, \hat{\hat{R}}_{-K})$, there is a top trading cycle $C' = (2 \to \ldots \to K \to o^1_2 \to 2)$ that only involves $K-1$ agents. Thus, by Fact 3 and the induction hypothesis, C' is executed and $f^1_K(\tilde{R}_K, \hat{\hat{R}}_{-1}) = o^1_2$. See the figure below.



Therefore, by *strategy-proofness* of f, $f^1_K(\hat{\hat{R}}) = o^1_2$, otherwise instead of $\hat{\hat{R}}_K$, agent K has an incentive to misreport $\tilde{R}_K$ at $\hat{\hat{R}}$. Moreover, $f^1_K(\hat{\hat{R}}) = o^1_2$ implies that $f^1_1(\hat{\hat{R}}) \neq o^1_2$. By *individual rationality* of f, $f^1_1(\hat{\hat{R}}) = o^1_1$. Overall, we find that

$$f^1_K(\hat{\hat{R}}) = o^1_2 \text{ and } f^1_1(\hat{\hat{R}}) = o^1_1. \tag{6.8}$$

However, this equation above implies that f is not *coordinatewisely efficient* since agents 1 and K can be better off by swapping their type-1 allotments. □

Next, we show that f also selects the cTTC allocation of type-2.

Claim 3. For each $R \in \mathcal{R}^N_\pi$, $f^2(R) = cTTC^2(R)$.

***Proof*.** The proof is similar to Claim 1, the main difference is that instead of *individual rationality*, we use Claim 2.

Let C be a first step top trading cycle at $cTTC^2(R)$ involving a set of agents $S_C \subseteq N$.

Similar to Claim 1, we only show that C is executed at $f(R)$ by induction on $|S_C|$. It suffices to show that C is executed at $f(R)$ because once we have shown that agents who trade at the first step of the TTC algorithm (of type-2) always receive their TTC allotments of type-2 under f, we can consider agents who trade at the second step of the TTC (of type-2) by following the same proof arguments as for first step trading cycles, and so on.

Induction basis. $|S_C| = 1$. In this case, agent $i \in S_C$ points to his type-2 endowed object, i.e., $C = (i \to o_i^2 \to i)$. Since preferences are lexicographic, agent i would be strictly worse off if he received any other type-2 object. Thus, by Claim 2, C must be executed.

Induction hypothesis. Let $K \in \{2, \dots, n\}$. Suppose that C is executed when $|S_C| < K$.

Induction step. Let $|S_C| = K$. Without loss of generality, assume that $S_C = \{1, \dots, K\}$ and $C = (1 \to o_2^2 \to 2 \to o_3^2 \to \dots \to K \to o_1^2 \to 1)$.

By contradiction, assume that C is not executed. Thus, there is an agent $i \in S_C$ who does not receive o_{i+1}^2, i.e., $f_i^2(R) \neq o_{i+1}^2$. It is without loss of generality to assume that $i = 2$. We proceed by contradiction in two steps.

Step 1. Let $\hat{R}_2 \in \mathcal{R}_\pi$ be such that for agent 2 and type-2 objects, only o_3^2 is acceptable (apart from his type-2 endowment), i.e.,

$$\hat{R}_2^2 : o_3^2, o_2^2, \dots,$$

$$\text{for each } t \in T \setminus \{2\} : \hat{R}_2^t = R_2^t, \text{ and}$$

$$\hat{\pi}_2 = \pi : 1, \dots, m.$$

By Claim 2, $f_2^2(\hat{R}) \in \{o_3^2, o_2^2\}$. By *strategy-proofness* of f, $f_2^2(R) \neq o_3^1$ implies that $f_2^2(\hat{R}) \neq o_3^2$, otherwise instead of R_2, agent 2 has an incentive to misreport $\hat{R}_2$ at R. Thus, $f_2^2(\hat{R}) = o_2^2$. Thus, agent 1 cannot receive o_2^2 from agent 2 because it is assigned to agent 2. Overall, we find that

$$f_2^2(\hat{R}) = o_2^2 \neq o_3^2 \text{ and } f_1^2(\hat{R}) \neq o_2^2. \tag{6.9}$$

Step 2. Let $\hat{\hat{R}}_1 \in \mathcal{R}_\pi$ be such that for agent 1 and type-2 objects, only o_2^2 and o_3^2 are acceptable (apart from his type-2 endowment), i.e.,

$$\hat{\hat{R}}_1^2 : o_2^2, o_3^2, o_1^2, \dots, \text{ (if } K = 2 \text{ then here we have } \hat{\hat{R}}_1^2 : o_2^2, o_1^2, \dots)$$

$$\text{for each } t \in T \setminus \{1\} : \hat{\hat{R}}_1^t = R_1^t, \text{ and}$$

$$\hat{\hat{\pi}}_1 = \pi : 1, \dots, m.$$

Let $\hat{\hat{R}} \equiv (\hat{\hat{R}}_1, \hat{R}_{-1}) = (\hat{\hat{R}}_1, \hat{R}_2, R_3, \dots, R_n)$. By Claim 2, $f_1^2(\hat{\hat{R}}) \in \{o_3^2, o_2^2, o_1^2\}$. By *strategy-proofness* of f, $f_1^2(\hat{R}) \neq o_2^2$ (see (6.9)) implies that $f_1^2(\hat{\hat{R}}) \neq o_2^2$, otherwise instead of $\hat{R}_1 (= R_1)$, agent 1 has an incentive to misreport $\hat{\hat{R}}_1$ at $\hat{R}$.

We then show that $f_1^2(\hat{\hat{R}}) = o_3^2$. To see it, consider $\tilde{R}_1^2 : o_3^2, o_1^2, \dots$ and $\tilde{R}_1 = (R_1^1, \tilde{R}_1^2, R_1^3, \dots, R_1^m, \pi)$. At $(\tilde{R}_1, \hat{\hat{R}}_{-1})$, there is a top trading cycle $C' = (1 \to o_3^2 \to 3 \to \dots \to K \to o_1^2 \to 1)$ that only involves $K-1$ agents. Thus, by the induction hypothesis, C' is executed and $f_1^2(\tilde{R}_1, \hat{\hat{R}}_{-1}) = o_3^2$. Therefore, by *strategy-proofness* of f, $f_1^2(\hat{\hat{R}}) = o_3^2$, otherwise instead of $\hat{\hat{R}}_1$, agent 1 has an incentive to misreport $\tilde{R}_1$ at $\hat{\hat{R}}$. Moreover, $f_1^2(\hat{\hat{R}}) = o_3^2$ implies that $f_2^2(\hat{\hat{R}}) \neq o_3^2$. By Claim 2, $f_2^2(\hat{\hat{R}}) = o_2^2$. Overall, we find that

$$f_1^2(\hat{\hat{R}}) = o_3^2 \text{ and } f_2^2(\hat{\hat{R}}) = o_2^2. \tag{6.10}$$

However, this equation above implies that f is not *coordinatewisely efficient* since agents 1 and 2 can be better off by swapping their type-2 allotments.

Thus, the proof of Claim 3 is completed. $\square$

By Claim 1 and Claim 3, we know that for each $R \in \mathcal{R}_\pi^N$, the allocations of type-1 and type-2 under f are the same as $cTTC$ allocation, i.e., $f^1(R) = cTTC^1(R)$ and $f^2(R) = cTTC^2(R)$. By applying similar arguments, we can also show that $f^3(R) = cTTC^3(R)$ and so on. Thus, we conclude that for each $R \in \mathcal{R}_\pi^N$, and each $t \in T$, $f^t(R) = cTTC^t(R)$, which completes the proof of Proposition 2.

Proof of Theorem 6

The proof of Theorem 6 will be shown by extending Proposition 2 to the domain of separable preference profiles.

Let $\bar{S} \subseteq N$ and R be such that only agents in $\bar{S}$ do no have restricted (but separable) preferences, i.e., $R_{-\bar{S}} \in \mathcal{R}_\pi^{-\bar{S}}$ and for each $i \in \bar{S}$, $R_i \in \mathcal{R}_s \setminus \mathcal{R}_\pi$. We show that $f(R) = cTTC(R)$ by induction on $|\bar{S}|$.

We first consider the case that only one agent i does no have restricted (but separable) preferences, i.e., $\bar{S} = \{i\}$. We will show that f still selects the cTTC allocation.

Lemma 5. *For each $R \in \mathcal{R}_s^N$, each $i \in N$ with $R_i \in \mathcal{R}_s \setminus \mathcal{R}_\pi$ and $R_{-i} \in \mathcal{R}_\pi^{-i}$, $f(R) = cTTC(R)$.*

The proof of Lemma 5 consists of four claims.

It is without loss of generality to assume that $i = 1$. Thus, $R_1 \in \mathcal{R}_s \setminus \mathcal{R}_\pi$. Let $y \equiv f(R)$ and $x \equiv cTTC(R)$.

We first show that agent 1 still receives his cTTC allocation at R, i.e., $y_1 = x_1$.

Claim 4. $y_1 = x_1$.

Proof. By contradiction, suppose that $y_1 \neq x_1$.

Let $\bar{R}_1 \in \mathcal{R}_\pi$ be such that $\bar{R}_1$ and R_1 share the same marginal preferences, i.e., for each $t \in T$, $\bar{R}_1^t = R_1^t$. Note that $(\bar{R}_1, R_{-1}) \in \mathcal{R}_\pi^N$ and hence by Proposition 2, $f(\bar{R}_1, R_{-1}) = cTTC(R) = x$.

Note that if for each $t \in T$, $x_1^t \, R_1^t \, y_1^t$, then $x_1 \, P_1 \, y_1$ as $x_1 \neq y_1$. However, this implies that agent 1 has an incentive to misreport $\bar{R}_1$ at R. Thus, by *strategy-proofness* of f, there exists one type $\tau \in T$ such that $y_1^\tau \, P_1^\tau \, x_1^\tau$.

By the definition of $cTTC$, $x_1^\tau \, R_1^\tau \, o_1^\tau$ and hence $y_1^\tau \neq o_1^t$. Overall, we have

$$y_1^\tau \, P_1^\tau \, x_1^\tau \, R_1^\tau \, o_1^\tau. \tag{6.11}$$

Let $\hat{R}_1 \in \mathcal{R}_\pi$ be such that for each type $t \in T$, agent 1 positions y_1^t first and o_1^t second, i.e.,

$$\text{for each } t \in T : \hat{R}_1^t : y_1^t, o_1^t, \ldots, \text{ and}$$

$$\hat{\pi}_1 = \pi : 1, \ldots, m.$$

Let

$$\hat{R} \equiv (\hat{R}_1, R_{-1}).$$

By *strategy-proofness* of f, $f_1(\hat{R}) = f_1(R) = y_1$; otherwise agent 1 has an incentive to misreport R_1 at $\hat{R}$. Since $\hat{R} \in \mathcal{R}_\pi^N$, by Proposition 2, $f(\hat{R}) = cTTC(\hat{R})$. In particular, $y_1^\tau = cTTC_1^\tau(\hat{R})$.

Next, to obtain the contradiction, we show that $cTTC_1^\tau(\hat{R}) = o_1^\tau$. By the definition of $cTTC$,

$$cTTC_1^\tau(\hat{R}) = TTC_1^\tau(\hat{R}^\tau) \in \{y_1^\tau, o_1^\tau\}. \tag{6.12}$$

Recall that $cTTC^\tau(R) = TTC^\tau(R^\tau) = x^\tau$ and $y_1^\tau \; P_1^\tau \; x_1^\tau \; R_1^\tau \; o_1^\tau$ (see (6.11)). Thus, by *strategy-proofness* of TTC, $x_1^\tau = TTC_1^\tau(R^\tau) \; R_1^\tau \; TTC_1^\tau(\hat{R}^\tau)$. Together with (6.12), we conclude that $TTC_1^\tau(\hat{R}^\tau) = o_1^\tau$. It implies that $cTTC_1^\tau(\hat{R}_1, R_{-1}) = o_1^\tau \neq y_1^\tau$. □

Note that Claim 4 implies that for each $R = (R_1, R_{-1}) \in (\mathcal{R}_s \setminus \mathcal{R}_\pi) \times \mathcal{R}_\pi^{N\setminus\{1\}}$, $f_1(R) = cTTC_1(R)$.

Next, we show that $y = x$ by applying similar arguments in Claims 1, 2, and 3.

Claim 5. For each $R = (R_1, R_{-1}) \in (\mathcal{R}_s \setminus \mathcal{R}_\pi) \times \mathcal{R}_\pi^{N\setminus\{1\}}$, $f^1(R) = cTTC^1(R)$.

Proof. Let ℓ be the step of the TTC algorithm at which agent 1 receives type-1 object $y_1^1 (= x_1^1 = cTTC_1^1(R))$. Let C be the corresponding top trading cycle that involves agent 1, i.e., $1 \in S_C$.

Note that by Claim 1 and Fact 3, all top trading cycles that are obtained before step ℓ are executed at $f(R)$. Moreover, if C is executed, then again by Claim 1 and Fact 3, all remaining top trading cycles are also executed. Thus, it suffices to show that C is executed, i.e., for each $i \in S_C$, $f_i^1(R) = cTTC_i^1(R)$.

Since all top trading cycles that are obtained before step ℓ are executed, by the definition of TTC, we know that for each agent in S_C, the object that he pointed at in C is his most preferred type-1 object among the unassigned type-1 objects, i.e., for each $i \in S_C$, all better type-1 objects for him, are assigned to someone else via top trading cycles that are obtained before step ℓ.

Similar to Claim 1, we show that C is executed at $f(R)$ by induction on $|S_C|$.

Induction basis. $|S_C| = 1$. In this case, $S_C = \{1\}$. By Claim 4, $f_1^1(R) = cTTC_1^1(R)$.

Induction hypothesis. Let $K \in \{2, \ldots, n\}$. Suppose that C is executed when $|S_C| < K$.

Induction step. Let $|S_C| = K$. Without loss of generality, assume that $S_C = \{1, \ldots, K\}$ and $C = (1 \to o_2^1 \to 2 \to \ldots \to K \to o_1^1 \to 1)$.

By contradiction, assume that C is not executed. Thus, there is an agent $i \in S_C \setminus \{1\}$ who does not receive o_{i+1}^1, i.e., $f_i^1(R) \neq o_{i+1}^1$. We proceed by contradiction in two steps.

Step 1. Let $\hat{R}_i \in \mathcal{R}_\pi$ be such that for agent i and type-1 objects, only o_{i+1}^1 is acceptable (apart from his type-1 endowment), i.e.,

$$\hat{R}_i^1 : o_{i+1}^1, o_i^1, \ldots,$$

$$\text{for each } t \in T \setminus \{1\} : \hat{R}_i^t = R_i^t, \text{ and}$$

$$\hat{\pi}_1 = \pi : 1, \ldots, m.$$

Note that at $\hat{R}_i$, if i does not receive o_{i+1}^1, then by *individual rationality* of f, he must receive his type-1 endowment o_i^1.

Let $\hat{R} \equiv (\hat{R}_i, R_{-i})$. Since $\hat{R}_i \in \mathcal{R}_l$ and f is *individually rational*, $f_i^1(\hat{R}) \in \{o_{i+1}^1, o_i^1\}$. By *strategy-proofness* of f, $f_i^1(R) \neq o_{i+1}^1$ implies that $f_2^i(\hat{R}) \neq o_{i+1}^1$, otherwise instead of R_i, agent

i has an incentive to misreport $\hat{R}_i$ at R. Thus, $f_i^1(\hat{R}) = o_i^1$. Thus, agent $i-1$ cannot receive o_i^1 from agent i because it is assigned to agent i. Overall, we find that

$$f_i^1(\hat{R}) = o_i^1 \neq o_{i+1}^1 \text{ and } f_{i-1}^1(\hat{R}) \neq o_i^1. \tag{6.13}$$

Step 2. Let $\hat{\hat{R}}_{i-1} \in \mathcal{R}_\pi$ be such that for agent $i-1$ and type-1 objects, only o_i^1 and o_{i+1}^1 are acceptable (apart from his type-1 endowment), i.e.,

$$\hat{\hat{R}}_{i-1}^1 : o_i^1, o_{i+1}^1, o_{i-1}^1, \ldots, \text{ (if } K = 2 \text{ then here we have } \hat{\hat{R}}_{i-1}^1 : o_i^1, o_{i-1}^1, \ldots)$$

$$\text{for each } t \in T \setminus \{1\} : \hat{\hat{R}}_{i-1}^t = R_{i-1}^t, \text{ and}$$

$$\hat{\hat{\pi}}_{i-1} = \pi : 1, \ldots, m.$$

Let $\hat{\hat{R}} \equiv (\hat{\hat{R}}_{i-1}, \hat{R}_{-1})$. By *individual rationality* of f, $f_{i-1}^1(\hat{\hat{R}}) \in \{o_{i+1}^1, o_i^1, o_{i-1}^1\}$. By *strategy-proofness* of f, $f_{i-1}^1(\hat{R}) \neq o_i^1$ (see (6.13)) implies that $f_{i-1}^1(\hat{\hat{R}}) \neq o_i^1$, otherwise instead of $\hat{R}_{i-1}$, agent $i-1$ has an incentive to misreport $\hat{\hat{R}}_{i-1}$ at $\hat{R}$.

We then show that $f_{i-1}^1(\hat{\hat{R}}) = o_{i+1}^1$. To see it, consider $\tilde{R}_{i-1}^1 : o_{i+1}^1, o_{i-1}^1, \ldots$ and $\tilde{R}_{i-1} = (\tilde{R}_{i-1}^1, R_{i-1}^2, \ldots, R_{i-1}^m, \pi)$. At $(\tilde{R}_{i-1}, \hat{\hat{R}}_{-(i-1)})$, there is a top trading cycle $C' = (i-1 \to o_{i+1}^1 \to i+1 \to o_{i+2}^1 \to \ldots \to i-2 \to o_{i-1}^1 \to i-1)$ that only involves $K-1$ agents. Thus, by the induction hypothesis, C' is executed and $f_{i-1}^1(\tilde{R}_1, \hat{\hat{R}}_{-1}) = o_{i+1}^1$. Therefore, by *strategy-proofness* of f, $f_{i-1}^1(\hat{\hat{R}}) = o_{i+1}^1$, otherwise instead of $\hat{\hat{R}}_{i-1}$, agent $i-1$ has an incentive to misreport $\tilde{R}_{i-1}$ at $\hat{\hat{R}}$. Moreover, $f_{i-1}^1(\hat{\hat{R}}) = o_{i+1}^1$ implies that $f_i^1(\hat{\hat{R}}) \neq o_{i+1}^1$. By *individual rationality* of f, $f_i^1(\hat{\hat{R}}) = o_i^1$. Overall, we find that

$$f_{i-1}^1(\hat{\hat{R}}) = o_{i+1}^1 \text{ and } f_i^1(\hat{\hat{R}}) = o_i^1. \tag{6.14}$$

However, this equation implies that f is not *coordinatewisely efficient* since agents $i-1$ and i can be better off by swapping their type-1 allotments. □

The next two claims, Claims 6 and 7, can be proven by a similar way to Claim 2 and 3, respectively. Thus, we omit the proofs. Note that the key point is that since agent 1 still receives his cTTC allocation, we only need to show that agents who still have restricted preferences, will also receive their cTTC allocation. Thus the proofs of Claim 2 and 3 are still valid for the case where only agent 1 does not have restricted preferences.

Claim 6. For each $R = (R_1, R_{-1}) \in (\mathcal{R}_s \setminus \mathcal{R}_\pi) \times \mathcal{R}_\pi^{N\setminus\{1\}}$, and each $i \in N$, $f_i^2(R)\ R_i^2\ o_i^2$.

Claim 7. For each $R = (R_1, R_{-1}) \in (\mathcal{R}_s \setminus \mathcal{R}_\pi) \times \mathcal{R}_\pi^{N\setminus\{1\}}$, $f^2(R) = cTTC^2(R)$.

Thus, similar to the proof of Proposition 2, by Claims 4, 5, 6, and 7, we conclude that for each $R \in \mathcal{R}_s^N$, and each $\bar{S} \subseteq N$, such that $|\bar{S}| = 1$ and $R_{-\bar{S}} \in \mathcal{R}_\pi^{-\bar{S}}$, $f(R) = cTTC(R)$. Thus, the proof of Lemma 5 is completed.

Now, we are ready to prove Theorem 6. Let $R \in \mathcal{R}_l^N$ and $\bar{S} \subseteq N$ be such that exactly only agents in $\bar{S}$ have non restricted (but separable) preferences, we show that $f(R) = cTTC(R)$.

Lemma 6. *For each $R \in \mathcal{R}_s^N$ and each $\bar{S} \subseteq N$ such that $R_{-\bar{S}} \in \mathcal{R}_\pi^{-\bar{S}}$, $f(R) = cTTC(R)$.*

The proof of Lemma 6 is showing by induction on $|\bar{S}|$.

Induction basis. $|\bar{S}| = 1$. This is done by Lemma 5.

Induction hypothesis. Let $K \in \{2, \ldots, n\}$. Suppose that $f(R) = cTTC(R)$ when $|\bar{S}| < K$.

Induction step. Let $|\bar{S}| = K$. Similar to Lemma 5, the proof of this part consists of four claims.

We first show that agents in $\bar{S}$ still receive their cTTC allotments.

Claim 8. For each $R = (R_{\bar{S}}, R_{-\bar{S}}) \in (\mathcal{R}_s \setminus \mathcal{R}_\pi)^{\bar{S}} \times \mathcal{R}_\pi^{-\bar{S}}$, and each $i \in \bar{S}$, $f_i(R) = cTTC_i(R)$.

Proof. Let $y \equiv f(R)$ and $x = cTTC(R)$. By contradiction, assume that there is an agent $i \in \bar{S}$ who does not receive his cTTC allotment x_i. Without loss of generality, assume that $i = 1$.

Let $\bar{R}_1 \in \mathcal{R}_\pi$ be such that $\bar{R}_1$ and R_1 share the same marginal preferences, i.e., for each $t \in T$, $\bar{R}_1^t = R_1^t$. Let $\bar{R} \equiv (\bar{R}_1, R_{-1})$.

Note that at $\bar{R}$, there are only $K-1$ agents (in $\bar{S}\setminus\{1\}$) who have non restricted (but separable) preferences. Thus, by the induction hypothesis and the definition of $cTTC$, $f(\bar{R}) = cTTC(\bar{R}) = cTTC(R) = x$. Then, the remaining proof is exactly the same as the proof of Claim 4 and hence we omit it. □

The following three claims can be proven by a similar way to Claims 5, 6, and 7, respectively. Thus, we omit the proofs. Note that the key point is that since agents in $\bar{S}$ still receive their cTTC allocation, we only need to show that agents who still have restricted preferences, will also receive their cTTC allocation. Thus the proofs are still valid for the case where only agents in $\bar{S}$ do not have restricted preferences.

Claim 9. For each $R = (R_{\bar{S}}, R_{-\bar{S}}) \in (\mathcal{R}_s \setminus \mathcal{R}_\pi)^{\bar{S}} \times \mathcal{R}_\pi^{-\bar{S}}$, $f^1(R) = cTTC^1(R)$.

Claim 10. For each $R = (R_{\bar{S}}, R_{-\bar{S}}) \in (\mathcal{R}_s \setminus \mathcal{R}_\pi)^{\bar{S}} \times \mathcal{R}_\pi^{-\bar{S}}$, and each $i \in N$, $f_i^2(R)\ R_i^2\ o_i^2$.

Claim 11. For each $R = (R_{\bar{S}}, R_{-\bar{S}}) \in (\mathcal{R}_s \setminus \mathcal{R}_\pi)^{\bar{S}} \times \mathcal{R}_\pi^{-\bar{S}}$, $f^2(R) = cTTC^2(R)$.

Hence, we conclude that for each $R \in \mathcal{R}_s^N$, and each $\bar{S} \subseteq N$, such that $|\bar{S}| = K$ and $R_{-\bar{S}} \in \mathcal{R}_\pi^{-\bar{S}}$, $f(R) = cTTC(R)$. Thus, the proof of Lemma 6 is completed. Therefore, Theorem 6 is proven by applying Lemma 6 with $\bar{S} = N$. Note that the proof for separable preferences and the proof for lexicographic preferences are exactly same, thus we only prove one of them.

D.2. Proofs of Theorems 8 and 9

We only show the uniqueness here. Before doing it, we redefine bTTC for lexicographic preferences as follows.

Alternative definition of bTTC

We restate bTTC for lexicographic preferences by adjusting the multiple-type top trading cycles (mTTC) algorithm from Feng and Klaus (2022).

The bundle top trading cycles (bTTC) algorithm / mechanism.

Input. A multiple-type housing market (N, e, R) with $R \in \mathcal{R}_l^N$.

Step 1. ***Building step.*** Let $N(1) = N$ and $U(1) = O$. We construct a directed graph $G(1)$ with the set of nodes $N(1) \cup U(1)$. For each $o \in U(1)$, we add an edge from the object to its owner and for each $i \in N(1)$, we add an edge from the agent to his most preferred object in O (according to the linear representation of R_i). For each edge $(i, o) \in N \times O$ we say that agent i points to object o.

Implementation step. A *trading cycle* is a directed cycle in graph $G(1)$. Given the finite number of nodes, at least one trading cycle exists. We assign to each agent i in a trading cycle the object that he pointed to, and denote the object assigned to him in this step by $a_i(1)$. Moreover, let $e_i(1)$ be the whole endowment of object $a_i(1)$'s owner, and assign the allotment $x_i(1) = \{e_i(1)\}$ to agent i. If agent $i \in N$ was not part of a trading cycle, then $x_i(1) = \emptyset$.

Removal step. We remove all agents and objects that were assigned in the implementation step, let $N(2)$ and $U(2)$ be the remaining agents and objects, respectively. Go to Step 2.

In general, at Step q (≥ 2) we have the following:

Step *q*. If $U(q)$ (or equivalently $N(q)$) is empty, then stop; otherwise do the following.

Building step. We construct a directed graph $G(q)$ with the set of nodes $N(q) \cup U(q)$. For each $o \in U(q)$, we add an edge from the object to its owner and for each $i \in N$, we add an edge from the agent to his most preferred feasible continuation object in $U_i(q)$ (according to the linear representation of R_i).

Implementation step. A *trading cycle* is a directed cycle in graph $G(q)$. Given the finite number of nodes, at least one trading cycle exists. We assign to each agent i in a trading cycle the object that he pointed to, and denote the object assigned to him in this step by $a_i(q)$. Moreover, let $e_i(q)$ be the whole endowment of object $a_i(q)$'s owner, and assign the allotment $x_i(q) = \{e_i(q)\}$ to agent i. If agent $i \in N$ was not part of a trading cycle, then $x_i(q) = \emptyset$.

Removal step. We remove all agents and objects that were assigned in the implementation step, let $N(q+1)$ and $U(q+1)$ be the remaining agents and objects, respectively. Go to Step $q+1$.

Output. The bTTC algorithm terminates when all objects in O are assigned (it takes at most n steps). Assume that the final step is Step q^*. Then, the final allocation is $x(q^*) = \{x_1(q^*), \ldots, x_n(q^*)\}$.

The *bundle top trading cycles mechanism (bTTC)*, $bTTC$, assigns to each market $R \in \mathcal{R}_l^N$ the allocation $x(q^*)$ obtained by the bTTC algorithm.

Example 11 (bTTC).
Consider $R \in \mathcal{R}_l^N$ with $N = \{1, 2, 3\}$, $T = \{H(ouse), C(ar)\}$, $O = \{H_1, H_2, H_3, C_1, C_2, C_3\}$, and

$$\boldsymbol{R_1} : H_2, H_3, \boldsymbol{H_1}, C_3, C_2, \boldsymbol{C_1},$$

$$\boldsymbol{R_2} : C_1, \boldsymbol{C_2}, C_3, H_3, \boldsymbol{H_2}, H_1,$$

$$\boldsymbol{R_3} : H_2, H_1, \boldsymbol{H_3}, C_1, \boldsymbol{C_3}, C_2.$$

The bTTC allocation at R is obtained as follows.

Step 1. ***Building step.*** $G(1) = (N \cup O, E(1))$ with set of directed edges
$E(1) = \{(H_1, 1), (H_2, 2), (H_3, 3), (C_1, 1), (C_2, 2), (C_3, 3), (1, H_2), (2, C_1), (3, H_2)\}$.

Implementation step. The trading cycle $1 \to H_2 \to 2 \to C_1 \to 1$ forms. Then, $a_1(1) = H_2$, $a_2(1) = C_1$, and $e_1(1) = \{H_2, C_2\}$, $e_2(1) = \{H_1, C_1\}$; thus, $x_1(1) = \{H_2, C_2\}$, $x_2(1) = \{H_1, C_1\}$, and $x_3(1) = \emptyset$.

Removal step. $N(2) = 3$, $U(2) = \{H_3, C_3\}$.

Step 2. ***Building step.*** $G(2) = (N(2) \cup U(2), E(2))$ with set of directed edges $E(2) = \{(H_3, 3), (C_3, 3), (3, H_3)\}$.

Implementation step. The trading cycle $3 \to H_3 \to 3$ forms. Then, $a_3(2) = H_3$ and $e_3(2) = \{H_3, C_3\}$; $x_1(2) = \{H_2, C_2\}$, $x_2(2) = \{H_1, C_1\}$, and $x_3(2) = \{H_3, C_3\}$.

Removal step. $N(3) = \emptyset$ and $U(3) = \emptyset$.

Thus, the bTTC algorithm computes the allocation $x = ((H_2, C_2), (H_1, C_1), (H_3, C_3))$. $\diamond$

Let $R \in \mathcal{R}_l^N$, let $\mathcal{C}(R)$ be a set of top trading cycles that are obtained at step 1 of the re-defined bTTC above at (R). We say that a trading cycle C is a *first step top trading cycle* if $C \in \mathcal{C}(R)$. For each first step top trading cycle C, let $S_C \subseteq N$ be the set of agents who are involved in C, and for each $i \in S_C$, let c_i be the object that agent i points at in C, and t_i be the type of object c_i, i.e., $c_i \in O^{t_i}$. We say that a trading cycle C is *executed* at $f(R)$ if for each $i \in S_C$, agent i receives c_i at $f(R)$. Moreover, for each $i \in S_C$ and $c_i \in O$, let i' be the owner of c_i. Since i and i' are involved in C, $i' \in SC$. We say that a trading cycle C is *fully executed* at $f(R)$ if for each $i \in S_C$, agent i receives $e_{i'}$ at $f(R)$, i.e., $f_i(R) = e_{i'}$.

Next, we show the first part of Theorem 9: the characterization of bTTC for lexicographic preferences.

Theorem 13. *A mechanism $f : \mathcal{R}_l^N \to X$ is individually rational, strategy-proof, non-bossy, and pairwise efficient if and only if it is bTTC.*

Proof of Theorem 13

Let $f : \mathcal{R}_l^N \to X$ be *individually rational*, *strategy-proof*, *non-bossy*, and *pair-efficient*. Note that by Lemma 3, f is *monotonic*.

We first explain the intuition of the proof. Consider a first step top trading cycle that forms at the first step of bTTC. First, we show that if this first-step top trading cycle is formed by only one or two agents, it is fully executed under f (Lemma 7) . Then, we extend this result to any number of agents under f (Lemma 8). Once we have shown that agents who trade at the first step of bTTC always receive their bTTC allotments under f, we can consider agents who trade at later steps of bTTC. The full execution of second step top trading cycles can be shown by following the same proof arguments as for first step top trading cycles; etc. The formal proof for first step top trading cycles now follows.

Lemma 7. *If a mechanism $f : \mathcal{R}_l^N \to X$ is individually rational, strategy-proof, non-bossy, and pair-efficient, then for each $R \in \mathcal{R}_l^N$, each first step top trading cycle $C(\in \mathcal{C}(R)$ with $|S_C| \leq 2$, C is fully executed at $f(R)$.*

Proof. Let $C \in \mathcal{C}(R)$ be a first step top trading cycle that consists of agents S_C with $|S_C| \leq 2$.

We show it by two steps. First, we show that C is executed.

Claim 12. C is executed.

Proof. When $|S_C| = 1$. In this case, agent $i \in S_C$ points to one of his endowed object, i.e., $c_i = o_i^{t_i}$ and hence $C = (i \to c_i \to i)$. Since preferences are lexicographic, i.e., $R_i \in \mathcal{R}_l$, agent i will be strictly worse off if he receives any other type-t_i objects. Thus, C must be executed by *individual rationality* of f.

When $|S_C| = 2$. Without loss of generality, assume that $S_C = \{1, 2\}$. By contradiction, assume that C is not executed. Thus, there is an agent $i \in S_C$ does not receive his most preferred object c_i. Without loss of generality, let $i = 2$.

Let $\hat{R}_2$ be such that agent 2 only wants to receive type-t_2 object c_2 and no other objects, i.e.,

$$\hat{R}_2^{t_2} : c_2(= o_1^{t_2}), o_2^{t_2}, \ldots,$$

$$\text{for each } t \in T \setminus \{t_2\} : \hat{R}_2^t : o_2^t, \ldots, \text{ and}$$

$$\hat{\pi}_2 = \pi_2 : t_2, \ldots.$$

Note that at $\hat{R}_2$, if 2 does not receive c_2, then from individual rationality of f, he must receive his full endowment $e_2 = (o_2^1, \ldots, o_2^m)$. Let

$$\hat{R} \equiv (\hat{R}_2, R_{-2}).$$

By *individual rationality* of f, $f_2^{t_2}(\hat{R}) \in \{c_2, o_2^{t_2}\}$. By *strategy-proofness* of f, $f_2^{t_2}(R) \neq c_2$ implies that $f_2^{t_2}(\hat{R}) \neq c_2$, otherwise instead of R_2, agent 2 has an incentive to misreport $\hat{R}_2$ at R. Thus, $f_2^{t_2}(\hat{R}) = o_2^{t_2}$. Then, by *individual rationality* of f, $f_2(\hat{R}) = e_2$. Thus, agent 1 cannot receive $c_1 (\in e_2)$ from agent 2 because it is assigned to agent 2.

Let $\bar{R}_1$ be such that be such that agent 1 only wants to receive type-t_1 object c_1 and no other objects, i.e.,

$$\bar{R}_1^{t_1} : c_1(= o_2^{t_1}), o_1^{t_1}, \ldots,$$

$$\text{for each } t \in T \setminus \{t_1\} : \bar{R}_1^t : o_1^t, \ldots \text{ and}$$

$$\bar{\pi}_1 = \hat{\pi}_1 = \pi_1 : t_1, \ldots.$$

Note that at $\bar{R}_1$, if agent 1 does not receive c_1, then from individual rationality of f, he must receive his full endowment e_1. Let

$$\bar{R} \equiv (\bar{R}_1, \hat{R}_2, \hat{R}_3, \ldots, \hat{R}_n) = (\bar{R}_1, \hat{R}_2, R_3, \ldots, R_n).$$

By *individual rationality* of f, $f_1^{t_1}(\bar{R}) \in \{c_1, o_1^{t_1}\}$. By *strategy-proofness* of f, $f_1^{t_1}(\hat{R}) \neq c_1$ implies that $f_1^{t_1}(\bar{R}) \neq c_1$, otherwise instead of $\hat{R}_1$, agent 1 has an incentive to misreport $\bar{R}_1$ at $\hat{R}$. Thus, $f_1^{t_1}(\bar{R}) = o_1^{t_1}$. Then, by *individual rationality* of f, $f_1(\bar{R}) = e_1$, and in particular, $f_1^{t_2}(\bar{R}) = o_1^{t_2} = c_2$. Moreover, by *individual rationality* of f, $f_2(\bar{R}) = e_2$, and in particular, $f_2^{t_1}(\bar{R}) = o_2^{t_1} = c_1$. This implies that $f_2^{t_1}(\bar{R}) \, P_1^{t_1} \, f_1^{t_1}(\bar{R})$ and $f_1^{t_2}(\bar{R}) \, P_2^{t_2} \, f_2^{t_2}(\bar{R})$ and hence $f_2(\bar{R}) \, P_1 \, f_1(\bar{R})$ and $f_1(\bar{R}) \, P_2 \, f_2(\bar{R})$, in which contradicts with *pair-efficiency* of f.

Overall, by the contradiction above we show that at $\bar{R}$, agent 1 receives c_1. Thus, by *strategy-proofness* of f, he also receives c_1 at $\hat{R}$; otherwise he has an incentive to misreport $\bar{R}_1$ at $\hat{R}$. Together with *individual rationality* of f, it implies that agent 2 receives c_2 at $\hat{R}$. Therefore, by *strategy-proofness* of f, agent 2 also receives c_2 at R; otherwise he has an incentive to misreport $\hat{R}_1$ at R. □

Next, we show that C is fully executed. There are two cases.

Case 1. $|S_C| = 1$. In this case, agent $i \in S_C$ points to one of his endowed object, i.e., $c_i \in e_i$. Without loss of generality, assume that $S_C = \{1\}$ and $\pi_1 : t_1, \ldots$. Thus, $C = (1 \to o_1^{t_1} \to 1)$.

Let $y \equiv f(R)$. By contradiction, suppose that $y_1 \neq e_1$. Note that by Claim 12, $y_1^{t_1} = o_1^{t_1}$. Let $t \in T \setminus \{t_i\}$ be such that $y_1^t \neq o_1^t$. Without loss of generality, assume that agent 1 receives agent 2's endowment of type-t at y, i.e., $y_1^t = o_2^t$.

Let $\hat{R} \in \mathcal{R}_l^N$ be such that each agent j positions y_j at the top and changes his importance order as π_1, i.e., for each agent $j \in N$, (i) $\hat{\pi}_j = \pi_1 : t_1, \ldots$, and (ii) for each $\tau \in T$, $\hat{R}_j^\tau : y_j^\tau, \ldots$

It is easy to see that $\hat{R}$ is a monotonic transformation of R at y. Thus, by monotonicity of f, $f(\hat{R}) = y$.

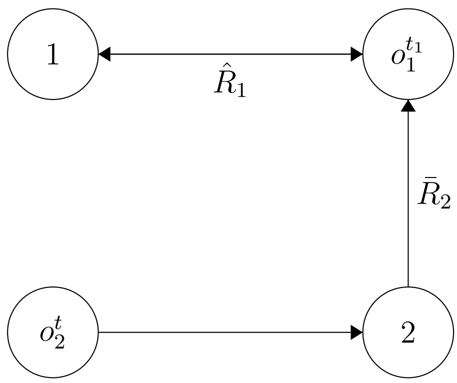
Let $\bar{R}_2$ be such that

$$\bar{\pi}_2 = \hat{\pi}_2 (= \pi_1),$$
$$\bar{R}_2^{t_1} : o_1^{t_1}, y_2^{t_1}, \ldots, \text{ and}$$
$$\text{For each } \tau \in T \setminus \{t_1\}, \bar{R}_2^\tau = \hat{R}_2^\tau.$$

Let

$$\bar{R} \equiv (\bar{R}_2, \hat{R}_{-2}).$$

Note that by strategy-proofness of f, for type-t_1, agent 2 either receives $o_1^{t_1}$ or $y_2^{t_1}$; otherwise he has an incentive to misreport $\hat{R}_2$ at $\bar{R}$. Moreover, C is still a first top trading cycle at $\bar{R}$, i.e., $C \in \mathcal{C}(\bar{R})$. Thus, by Claim 12, C is executed and hence agent 1 receives $o_1^{t_1}$ at $f(\bar{R})$. See the figure below.



Thus, agent 2 still receives $y_2^{t_1}$, and hence by Fact 2,

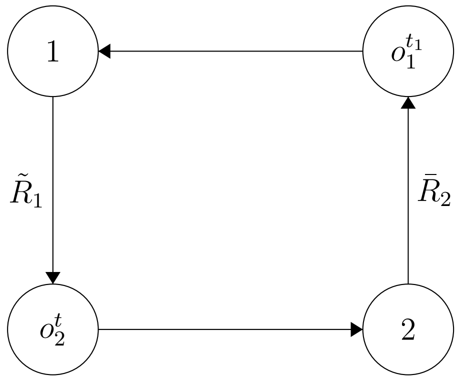
$$f(\bar{R}) = f(\hat{R}) = y \text{ and particularly, } f_1^{t_1}(\tilde{R}) = y_1^{t_1} = o_1^{t_1}. \tag{6.15}$$

Let $\tilde{R}_1$ be such that agent 1 only changes his importance order as t is the most important, i.e., $\tilde{\pi}_1 : t, \ldots$ and $\tilde{R}_1 = (\hat{R}_1^1, \ldots, \hat{R}_1^m, \tilde{\pi}_1)$.

Let

$$\tilde{R} \equiv (\tilde{R}_1, \bar{R}_{-1}).$$

By monotonicity of f, $f(\tilde{R}) = f(\bar{R}) = y$. However, at $\tilde{R}$, there is a first step top trading cycle $C' \in \mathcal{C}(\tilde{R})$ consisting of agents 1 and 2, i.e., $C' = (1 \to o_2^t (= y_1^t) \to 2 \to o_1^{t_1} \to 1)$. See the figure below.



By Claim 12, C' is executed at $f(\tilde{R})$. Thus, $f_2^{t_1}(\tilde{R}) = o_1^{t_1}$, which contradicts with the fact that $f_1^{t_1}(\tilde{R}) = y_1^{t_1} = o_1^{t_1}$ (see (6.15)).

Case 2. $|S_C| = 2$. Without loss of generality, assume that $S_C = \{1, 2\}$. Thus, $C = (1 \to c_1(= o_2^{t_1}) \to 2 \to c_2(= o_1^{t_2}) \to 1)$. By contradiction, assume that C is not executed. Without loss of generality, assume that agent 1 does not receive agent 2's full endowments, i.e., $f_1(R) \neq e_2$. Note that by Claim 12, $f_1^{t_1}(R) = c_1 = o_2^{t_1}$. Thus, there is a type $t \in T \setminus \{t_1\}$ such that $f_1^t(R) \neq o_2^t$. Without loss of generality, assume that agent 1 receives agent i's endowment of type-t, i.e., $f_1^t(R) = o_i^t$. Let $y \equiv f(R)$. There are two sub-cases.

Sub-case 1. $i = 1$. Let $\hat{R}_1$ be such that

$$\text{for each } t \in T, \hat{R}_1^t : y_1^t, \ldots, \text{ and}$$

$$\hat{\pi}_1 : t, \ldots$$

By *monotonicity* of f, $f(\hat{R}_1, R_{-1}) = f(R) = y$. Then, we are back to Case 1.

Sub-case 2. $i \neq 1$. Without loss of generality, assume that $i = 3$. Thus, $y_2^{t_2} = o_1^{t_2}$, $y_1^{t_1} = o_2^{t_1}$, and $y_1^t = o_3^t$. We will obtain a contradiction to complete the proof of this sub-case. Let $\hat{R}_3$ be such that

$$R_3^{t_2} : o_1^{t_2}, y_3^{t_2}, \ldots$$

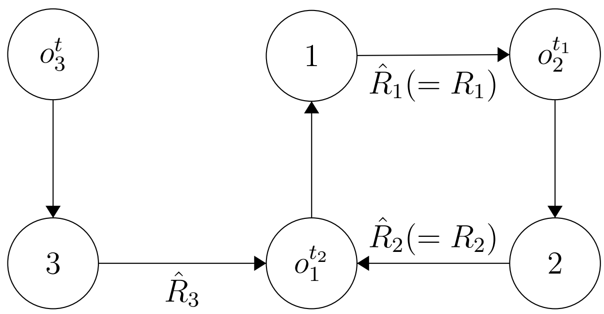
$$\text{for each } t \in T \setminus \{t_2\}, \hat{R}_3^t : y_3^t, \ldots, \text{ and}$$

$$\hat{\pi}_3 : t_2, \ldots$$

Let

$$\hat{R} \equiv (\hat{R}_3, R_{-3}).$$

Note that C is still a first step top trading cycle at $\hat{R}$, i.e., $C \in \mathcal{C}(\hat{R})$. Thus, by Claim 12, C is executed. See the figure below.



Hence, agent 3 cannot receive $o_1^{t_2}(= c_2)$. Thus, by *strategy-proofness* of f, he still receives y_3, i.e., $f_3(\hat{R}) = y_3$. Therefore, by *non-bossiness* of f, $f(\hat{R}) = y$.

Let $\bar{R}_1$ be such that

$$\text{for each } t \in T, \bar{R}_1^t : y_1^t, \ldots, \text{ and}$$

$$\bar{\pi}_1 : t, \ldots$$

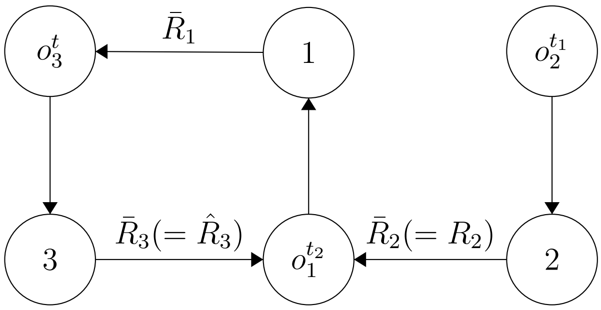
Let

$$\bar{R} \equiv (\bar{R}_1, \hat{R}_{-1}).$$

Then, since $\bar{R}_1$ is a monotonic transformation of $\hat{R}_1$ at y, $f(\bar{R}) = y$. In particular,

$$f_2^{t_1}(\bar{R}) = o_1^{t_2}(= c_2). \tag{6.16}$$

Note that at $\bar{R}$, there is a first step top trading cycle $C' = (1 \to y_1^t(= o_3^t) \to 3 \to o_1^{t_2} \to 1)$ that involves two agents. Thus, by Claim 12, C' is executed. See the figure below.



It implies that $f_3^{t_2}(\bar{R}) = o_1^{t_2}$, which contradicts with $f(\bar{R}) = y = f(\hat{R})$ and (6.16). □

Lemma 8. *If a mechanism $f : \mathcal{R}_l^N \to X$ is individually rational, strategy-proof, non-bossy, and pair-efficient, then for each $R \in \mathcal{R}_l^N$, each first step top trading cycle $C(\in \mathcal{C}(R))$ is fully executed under f at R.*

Proof. Let $C \in \mathcal{C}(R)$ be a first step top trading cycle that consists of agents $S_C \subseteq N$. We prove this lemma by induction on $|S_C|$.

Induction Basis. $|S_C| \leq 2$. This is done by Lemma 7.

Induction hypothesis. Let $K \in \{3, \ldots, n\}$. Suppose that C is fully executed when $|S_C| < K$.

Induction step. Let $|S_C| = K$. Without loss of generality, assume that $S_C = \{1, \ldots, K\}$ and $C = (1 \to c_1 \to 2 \to c_2 \to \ldots \to K \to c_K \to 1)$.

Similar to Lemma 7, we first show that C is executed.

Claim 13. C is executed.

Proof. By contradiction, assume that C is not executed. Thus, there is an agent $i \in S_C$ who does not receive c_i, i.e., $f_i^{t_i}(R) \neq c_i$. Without loss of generality, let $i = 2$.

Let $\hat{R}_2$ be such that agent 2 only wants to receive type-t_2 object c_2 and no other objects, i.e.,

$$\hat{R}_2^{t_2} : c_2(= o_3^{t_2}), o_2^{t_2}, \ldots,$$

$$\text{for each } t \in T \setminus \{t_2\} : \hat{R}_2^t : o_2^t, \ldots, \text{ and}$$

$$\hat{\pi}_2 = \pi_2 : t_2, \ldots.$$

Note that at $\hat{R}_2$, if agent 2 does not receive c_2, then from individual rationality of f, he must receive his full endowment e_2.

Let $\hat{R} \equiv (\hat{R}_2, R_{-2})$. We proceed in two steps.

Step 1. We show that agent 2 receives c_2 under f at $\hat{R}$, i.e., $f_2^{t_2}(\hat{R}) = c_2$.

By *individual rationality* of f, $f_2^{t_2}(\hat{R}) \in \{c_2, o_2^{t_2}\}$. By *strategy-proofness* of f, $f_2^{t_2}(R) \neq c_2$ implies that $f_2^{t_2}(\hat{R}) \neq c_2$, otherwise instead of R_2, agent 2 has an incentive to misreport $\hat{R}_2$ at R. Thus, $f_2^{t_2}(\hat{R}) = o_2^{t_2}$. Then, by *individual rationality* of f, $f_2(\hat{R}) = e_2$. Thus, agent 1 cannot receive $c_1 (\in e_2)$ from agent 2 because it is assigned to agent 2.

Let $y \equiv f(\hat{R})$. Overall, we find that

$$y_2 = e_2 \text{ and } y_1^{t_1} \neq c_1 (= o_2^{t_1}). \tag{6.17}$$

Let $\bar{R}_1$ be such that

$$\bar{R}_1^{t_1} : c_1(= o_2^{t_1}), o_3^{t_1}, o_1^{t_1}, \ldots,$$
$$\text{for each } t \in T \setminus \{t_1\} : \bar{R}_1^t := \hat{R}_1^t (= R_1^t) \text{ and}$$
$$\bar{\pi}_1 = \hat{\pi}_1 = \pi_1 : t_1, \ldots$$

Note that $\bar{R}_1$ and $\hat{R}_1$ only differ in type-t_1 marginal preferences. Let

$$\bar{R} \equiv (\bar{R}_1, \hat{R}_2, \hat{R}_3, \ldots, \hat{R}_n).$$

To obtain the contradiction, we want to show that at $\bar{R}$, agent 1 receives c_1 and agent 2 receives c_2, i.e., $f_1^{t_1}(\bar{R}) = c_1 = o_2^{t_1}$ and $f_2^{t_2}(\bar{R}) = c_2 = o_3^{t_2}$.

By *individual rationality* of f, $f_1^{t_1}(\bar{R}) \in \{o_2^{t_1}, o_3^{t_1}, o_1^{t_1}\}$. By *strategy-proofness* of f, $f_1^{t_1}(\hat{R}) \neq o_2^{t_1}$ implies that $f_1^{t_1}(\bar{R}) \neq o_2^{t_1}$, otherwise instead of $\hat{R}_1$, agent 1 has an incentive to misreport $\bar{R}_1$ at $\hat{R}$.

Thus, $f_1^{t_1}(\bar{R}) \in \{o_3^{t_1}, o_1^{t_1}\}$. Next, we show that $f_1(\bar{R}) = e_3$ and hence $f_1^{t_2}(\bar{R}) = o_3^{t_2} = c_2$.

Let $\tilde{R}_1$ be such that

$$\tilde{R}_1^{t_1} : o_3^{t_1}, o_1^{t_1}, \ldots,$$
$$\text{for each } t \in T \setminus \{t_1\} : \tilde{R}_1^t := \bar{R}_1^t (= R_1^t) \text{ and}$$
$$\tilde{\pi}_1 = \bar{\pi}_1 = \hat{\pi}_1 = \pi_1 : t_1, \ldots$$

Since $f_1^{t_1}(\bar{R}) \neq c_1$, $\tilde{R}_1$ is a monotonic transformation of $\bar{R}_1$ at $f(\bar{R})$. Thus,

$$f(\bar{R}) = f(\tilde{R}_1, \bar{R}_{-1}). \tag{6.18}$$

Note that at $(\tilde{R}_1, \bar{R}_{-1})$, there is a first step top trading cycle $C' = (1 \to o_3^{t_1} \to 3 \to c_3 \to \ldots \to K \to c_K \to 1)$. Since $C' \in \mathcal{C}(\tilde{R}_1, \bar{R}_{-1})$ and $|S_{C'}| = K - 1$, by the Induction hypothesis, C' is fully executed. Thus, $f_1(\tilde{R}_1, \bar{R}_{-1}) = e_3$, and in particular, $f_1^{t_2}(\tilde{R}_1, \bar{R}_{-1}) = o_3^{t_2} = c_2$. Together with (6.18), we conclude that $f_1^{t_2}(\bar{R}) = o_3^{t_2} = c_2$ and $f_1(\bar{R}) = e_3$. Therefore, $f_2^{t_2}(\bar{R}) \neq o_3^{t_1} = c_2$. Hence, by *individual rationality* of f, $f_2(\bar{R}) = e_2$, and in particular, $f_2^{t_1}(\bar{R}) = o_2^{t_1} = c_1$.

This implies that $c_1 = f_2^{t_1}(\bar{R}) \; \bar{P}_1^{t_1} \; f_1^{t_1}(\bar{R})$ and $c_2 = f_1^{t_2}(\bar{R}) \; \bar{P}_2^{t_2} \; f_2^{t_2}(\bar{R})$. Hence, $f_2(\bar{R}) \; \bar{P}_1 \; f_1(\bar{R})$ and $f_1(\bar{R}) \; \bar{P}_2 \; f_2(\bar{R})$, in which contradicts with *pair-efficiency* of f.

Overall, by contradiction we show that at $\bar{R}$, agent 1 receives c_1. Together with *individual rationality* of f, it implies that agent 2 receives c_2 at $\bar{R}$. Subsequently, by *strategy-proofness* of f, agent 1 also receives c_1 at $\hat{R}$; otherwise he has an incentive to misreport $\bar{R}_1$ at $\hat{R}$. Again, together with *individual rationality* of f, it implies that agent 2 receives c_2 at $\hat{R}$.

Step 2. We show that agent 2 receives c_2 under f at R, i.e., $f_2^{t_2}(R) = c_2$.

Note that c_2 is agent 2's most preferred type-t_2 object at R_2. By *strategy-proofness* of f, $f_2(R)\, R_2\, f_2(\hat{R})$. Hence, $f_2^{t_2}(R)\, R_2^{t_2}\, f_2^{t_2}(\hat{R})$, which implies that $f_2^{t_2}(R) = c_2$. □

Next, we show that C is fully executed at $f(R)$. Let $x \equiv bTTC(R)$, $y \equiv f(R)$. Note that if C is fully executed, then for each $i \in S_C$, $y_i = f_i(R) = x_i$.

By contradiction, suppose that there is an agent $i \in S_C$ such that $y_i \neq x_i$. Without loss of generality, let $i = 1$. By Claim 13, C is executed under f at R. In particular,

$$y_1^{t_1} = o_2^{t_1} = x_1^{t_1} \text{ and } y_K^{t_K} = o_1^{t_K} = x_K^{t_K}. \tag{6.19}$$

Since $y_1 \neq x_1 (= e_2)$, there is a type $t \in T \setminus \{t_1\}$ and an agent $j \neq 2$ such that $y_1^t = o_j^t$. There are two cases.

Case 1: $j \in S_C$. Let $\hat{R}_1$ such that agent 1 positions y_1 at the top and moves t to the most important, i.e., (i) $\hat{\pi}_1 : t, \ldots$; and (ii) for each $\tau \in T$, $\hat{R}_1^\tau : y_1^\tau, \ldots$. Since $\hat{R}_1$ is a monotonic transformation of R_1 at y, we have

$$f(\hat{R}_1, R_{-1}) = f(R) = y \text{ and particularly, } f_1^{t_1}(\hat{R}_1, R_{-1}) = o_2^{t_1}. \tag{6.20}$$

Note that there is a first step top trading cycle $C' \equiv (1 \to o_j^t \to j \to o_{j+1}^{t_j} \to j+1 \to \cdots \to K \to o_1^{t_K} \to 1)$ at $(\hat{R}_1, R_{-1})$. i.e., $C' \in \mathcal{C}(\hat{R}_1, R_{-1})$. Since $j \neq 2$, C' contains less than K agents. Thus, by the induction hypothesis, C' is fully executed at $f(\hat{R}_1, R_{-1})$. Therefore, $f_1(\hat{R}_1, R_{-1}) = e_j$ and hence $f_1^{t_1}(\hat{R}_1, R_{-1}) = o_j^{t_1}$, which contradicts with the fact that $f_1^{t_1}(\hat{R}_1, R_{-1}) = o_2^{t_1}$ (see (6.20)).

Case 2: $j \notin S_C$. Let $\hat{R}_j$ be such that

$$\hat{R}_j^{t_K} : o_1^{t_K}, y_j^{t_K}, \ldots$$

$$\text{for each } \tau \in T \setminus \{t_K\}, \hat{R}_j^\tau : y_j^\tau, \ldots, \text{ and}$$

$$\hat{\pi}_j : t_K, \ldots$$

Let

$$\hat{R} \equiv (\hat{R}_j, R_{-j}).$$

Note that C is still a first step top trading cycle at $\hat{R}$, and hence, by Claim 13, C is executed. In particular, with (6.19), we have

$$f_K^{t_K}(\hat{R}) = y_K^{t_K} = o_1^{t_K}. \tag{6.21}$$

So, agent j does not receive $o_1^{t_K}$ at $f(\hat{R})$. So, by strategy-proofness of f, $f_j(\hat{R}) = y_j$; otherwise he has an incentive to misreport R_j at $\hat{R}$. So, by non-bossiness of f, $f(\hat{R}) = y$.

Let $\bar{R}_1$ be such that agent 1 positions y_1 at the top and moves t to the most important, i.e.,

$$\text{for each } \tau \in T, \bar{R}_1^\tau : y_1^\tau, \ldots \text{ and}$$

$$\bar{\pi}_1 : t, \ldots$$

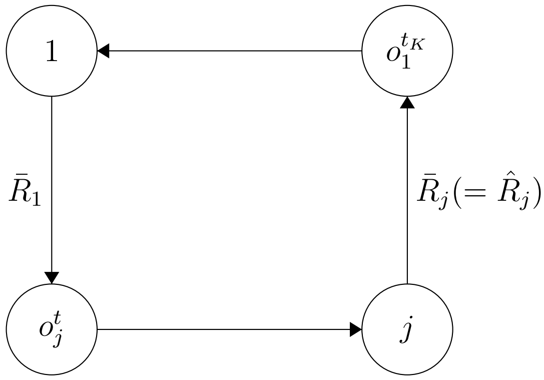
Let

$$\bar{R} \equiv (\bar{R}_1, \hat{R}_{-1}).$$

Since $\bar{R}_1$ is a monotonic transformation of $\hat{R}_1(= R_1)$ at y, we have

$$f(\bar{R}) = y \text{ and particularly, } f_K^{t_K}(\bar{R}) = o_1^{t_K}. \tag{6.22}$$

Note that $C' \equiv (1 \to y_1^t(= o_j^t) \to j \to o_1^{t_K} \to 1)$ is a first step top trading cycle at $\bar{R}$, i.e., $C' \in \mathcal{C}(\bar{R})$. See the figure below.



Thus, by Claim 13, cycle C' is executed at $f(\bar{R})$. Therefore, $f_j^{t_K}(\bar{R}) = o_1^{t_K}$. Since $j \notin S_C$, this contradicts with the fact that $f_K^{t_K}(\hat{R}) = o_1^{t_K}$ (see (6.22)). □

By Lemma 8, we have shown that agents who trade at step 1 of the bTTC algorithm always receive their bTTC allotments under f. Next, we can consider agents who trade at step 2 of the bTTC algorithm by following the same proof arguments as for first step trading cycles, and so on. Thus, the proof of Theorem 13 is completed.

Proof of Theorems 8 and 9

To complete the proof, next, we extend Theorem 13 from lexicographic preferences to separable preferences.

Let $f : \mathcal{R}_s^N \to X$ be *individually rational*, *strategy-proof*, *non-bossy*, and *bundle endowments-swapping proof*. Note that by Lemma 3, f is *monotonic*.

Let $S \subseteq N$ and $R \in \mathcal{R}_s^N$ be such that only agents in S do no have lexicographic preferences, i.e., $R_S \notin \mathcal{R}_l^S$ and $R_{-S} \in \mathcal{R}_l^{-S}$. We show that $f(R) = bTTC(R)$ by induction on $|S|$.
We first consider $S = \{i\}$, i.e., $|S| = 1$ as the induction basis.

Let $x \equiv f(R)$ and $y \equiv bTTC(R)$.

Let $\hat{R}_i \in \mathcal{R}_l$ be such that for each $t \in T$, $\hat{R}_i^t : x_i^t, \ldots$

By *monotonicity* of f, $f(\hat{R}_i, R_{-i}) = x$. Note that $(\hat{R}_i, R_{-i}) \in \mathcal{R}_l^N$. Thus, by Theorem 13, f coincides with bTTC, i.e., $bTTC(\hat{R}_i, R_{-i}) = f(\hat{R}_i, R_{-i}) = x$.

Let $\bar{R}_i \in \mathcal{R}_l$ be such that (a) $\bar{\pi}_i = \hat{\pi}_i$; and (b) for each $t \in T$, $\hat{R}_i^t : y_i^t, \ldots$

By *monotonicity* of $bTTC$, $bTTC(\bar{R}_i, R_{-i}) = y$. Note that $(\bar{R}_i, R_{-i}) \in \mathcal{R}_l^N$. Thus, again by Theorem 13, $f(\bar{R}_i, R_{-i}) = bTTC(\bar{R}_i, R_{-i}) = y$.

By *strategy-proofness* of $bTTC$, $y_i = bTTC_i(R)\, R_i\, bTTC_i(\hat{R}_i, R_{-i}) = x_i$; by *strategy-proofness* of f, $x_i = f_i(R)\; R_i\; f_i(\bar{R}_i, R_{-i}) = y_i$. Thus, $x_i = y_i$. Subsequently, by *non-bossiness* of $bTTC$, $x = bTTC(\hat{R}_i, R_{-i}) = bTTC(\bar{R}_i, R_{-i}) = y$.

We can apply repeatedly the same argument to obtain that for $|S| \in \{2, \ldots, n\}$, and for each profile $R \in \mathcal{R}_s^N$ where exactly $|S|$ agents have non-lexicographic preferences, $f(R) = bTTC(R)$. Thus, for each $R \in \mathcal{R}_s^N$, $f(R) = bTTC(R)$. Thus, the proof of Theorem 9 is completed.

Theorem 8 can be proven in exactly the same way above and hence we omit it.

Example for footnote 2 in Chapter 3

The following efficiency property is an adaptation of Pápai (2007)'s *restricted efficiency*.[2]

Let $y \in X$ and $Y \equiv \{y, e\}$. $f : \mathcal{R}^N \to Y$ is *Y-restricted efficient* if for each $R \in \mathcal{R}^N$, there does not exist $x \in Y$ such that x Pareto dominates $f(R)$ at R.

This is a weak efficiency property since it rules out the extremely inefficient no-trade mechanism. Also, it is easy to see that cTTC and bTTC do not satisfy this efficiency property. However, due to its restriction (feasibility of only two allocations), this property is uninteresting. Next, we show that this property is compatible with *individual rationality* and *group strategy-proofness*.

Let $f : \mathcal{R}^N \to Y$ be such that for each $R \in \mathcal{R}^N$, if for each $i \in N$, $y\ R_i\ e$, then $f(R) = y$, otherwise $f(R) = e$. This mechanism resembles a form of unanimous voting, i.e., it always selects the status quo allocation (e) unless all agents unanimously prefer y to e. By the definition of f, it is easy to see it is *individually rational*, *group strategy-proof*, and *Y-restricted efficient*.

D.3. Independence of the properties in Chapter 3

We provide several examples to establish the logical independence of the properties in our characterizations.

Theorems 6 and 7

The following examples establish the logical independence of the properties in Theorem 6. We label examples by the property that is not satisfied.

Example 12 (***Coordinatewise efficiency***)**.**
As in Example 7, the no-trade mechanism that always assigns the endowment allocation to each market is *individually rational*, *group strategy-proof* (and hence *strategy-proof* and *non-bossy*), but not *coordinatewisely efficient*. ◇

Example 13 (***Individual rationality***)**.**
As in Example 8, serial dictatorship mechanisms satisfy *Pareto efficiency* (and hence *coordinatewise efficiency* and *pairwise efficiency*), *group strategy-proofness* (and hence *strategy-proofness* and *non-bossiness*); but violate *individual rationality*. ◇

Example 14 (***Strategy-proofness***)**.**
As in Example 9, Multiple-Serial-IR mechanisms are *individually rational* and *Pareto efficient* (and hence *coordinatewisely efficient*), but not *strategy-proof*. ◇

The three examples above are well-defined for strict preferences and establish the logical independence of the properties in Theorems 7.

[2] In Pápai (2007), *restricted efficiency* is defined originally as follows. Let $X' \subsetneq X$ and $f : \mathcal{R}^N \to X'$. f is *restricted efficient* if for each $R \in \mathcal{R}^N$, there does not exist $x \in X'$ such that x Pareto dominates $f(R)$ at R.

Theorems 8, 9, and 13

The following examples establish the logical independence of the properties in Theorem 13.

Example 15 (***Pairwise efficiency***).
Same as Example 7, the no-trade mechanism is *individually rational*, *strategy-proof*, and *non-bossy*, but not *pairwise efficient*. ◇

Example 16 (***Individual rationality***).
Same as Example 8, serial dictatorship mechanisms are *group strategy-proof* (hence *strategy-proof* and *non-bossy*), and *Pareto efficient* (hence *pairwise efficient*), but not *individually rational*. ◇

Example 17 (***Strategy-proofness***).
Same as Example 9, multiple-Serial-IR mechanisms satisfy *individual rationality*, *Pareto efficiency* (and hence *pairwise efficiency*), and *non-bossiness*, but violate *strategy-proofness*. ◇

Example 18 (***Non-bossiness***).
Note that when there are only two agents, *non-bossiness* is trivially satisfied. Thus, we need at least three agents. So, consider markets with three agents and two types, i.e., $N = \{1, 2, 3\}$ and $T = \{1, 2\}$.

Let $\hat{\mathcal{R}} \subsetneq \mathcal{R}_l^N$ be a set of markets such that for each $R \in \hat{\mathcal{R}}$, $R_1|^e$ is such that for some agent $i \in \{2, 3\}$, agent 1 positions agent i's full endowment at the top, i.e., for each $t \in T$, $R_1^t : o_i^t, \ldots$.

Let $y \in X$ be such that (i) $y_1 = e_i$, (ii) $y_i = (e_1^1, e_j^2)$ and $y_j = (e_j^1, e_1^2)$, where $\{i, j\} = \{2, 3\}$.

Let f be such that

$$f(R) = \begin{cases} y, & (R) \in \hat{\mathcal{R}} \text{ and } y \text{ Pareto dominates } bTTC(R), \\ bTTC(R), & \text{otherwise.} \end{cases}$$

Note that for $R \in \hat{\mathcal{R}}$, if $f_1(R) \neq e_i$, then there is an agent $k \in \{2, 3\}$ (possibly $k = i$) who receives e_i and prefers e_i to y_k, i.e., $f_k(R) = bTTC_k(R) = e_i \; P_k \; y_k$.

It is easy to see that f inherits *individually rational* and *pairwise efficiency* from bTTC. By the definition of f, one can verify that f is *bossy*. We show that f is *strategy-proof*.

We first show that agent 1 has no incentive to misreport. For $R \notin \hat{\mathcal{R}}$, agent 1 positions his full endowment at the top. Thus, $f_1(R) = bTTC_1(R) = o_1$. Clearly, for any misreport $R_1' \neq R_1$, $e_1 \; R_1 \; f_1(R_1', R_2, R_3)$.

For $R \in \hat{\mathcal{R}}$, by the definition of f, o_i is agent 1's most preferred allotment among $\{e_1, e_2, e_3\}$, and $f_1(R) \in \{e_1, e_i\}$. If $f_1(R) = e_i$ then clearly $f_1(R) = e_i \; R_1 \; bTTC_1(R)$.

If $f_1(R) = e_1 (= bTTC_1(R))$ then there is an agent $k \in \{2, 3\}$ such that $bTTC_k(R) \; P_k \; y_k$ and $f_k(R) = bTTC_k(R) \in \{e_2, e_3\}$. Let $R_1' \neq R_1$ be a misreporting. By the definition of $bTTC$, $bTTC(R_1', R_{-1}) = bTTC(R)$. Thus, $bTTC_k(R_1', R_{-1}) \; P_k \; y_k$ and hence, $f(R_1', R_{-1}) = bTTC(R_1', R_{-1})$, which implies that $f_1(R_1', R_{-1}) = e_1$. Therefore, if $(R) \in \hat{\mathcal{R}}$, then $f_1(R) \; R_1 \; bTTC_1(R)$.

Next, we show that agents 2 and 3 have no incentive to misreport.

For $R \notin \hat{\mathcal{R}}$, $f_2(R) = bTTC_2(R)$ and $f_3(R) = bTTC_3(R)$. Since $bTTC$ is *strategy-proof*, agents 2 and 3 have no incentive to misreport.

For $R \in \hat{\mathcal{R}}$, there are two cases.

Case 1. $f(R) = bTTC(R)$. By the definition of f, there is an agent $k \in \{2, 3\}$ such that $bTTC_k(R)\, P_k\, y_k$. Let $R'_k \neq R_k$ be a misreporting. Then, $f_k(R'_k, R_{-k}) \in \{bTTC_k(R'_k, R_{-k}), y_k\}$. Since $bTTC$ is *strategy-proof*, $bTTC_k(R)\, R_k\, bTTC_k(R'_k, R_{-k})$, and hence $bTTC_k(R) = f_k(R)\, R_k$ $f_k(R'_k, R_{-k})$.
Case 2. $f(R) = y$. By the definition of f, $y_2\, R_2\, bTTC_2(R)$ and $y_3\, R_3\, bTTC_3(R)$. Let $k \in \{2, 3\}$ and $R'_k \neq R_k$ be a misreporting. Since $bTTC$ is *strategy-proof*, $y_k\, R_k\, bTTC_k(R)\, R_k$ $bTTC_k(R'_k, R_{-k})$. By the definition of f, $f_k(R'_k, R_{-k}) \in \{bTTC_k(R'_k, R_{-k}), y_k\}$. Thus, $f_k(R) = y_k\, R_k\, f_k(R'_k, R_{-k})$. $\diamond$

The examples above are well-defined on the domain of separable preference (strict preference) profiles and establish the logical independence of the properties in Theorem 9 (Theorem 8).

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