

Safety Traps

By KENZA BENHIMA AND BAPTISTE MASSENOT*

Fear of risk provides a rationale for protracted economic downturns. We develop a real business cycle model where investors with decreasing relative risk aversion choose between a risky and a safe technology that exhibit decreasing returns. Because of a feedback effect from the interest rate to risk aversion, two equilibria can emerge: a standard equilibrium and a “safe” one in which investors invest in safer assets. We refer to the dynamics of this second equilibrium as a safety trap because it is self-reinforcing as investors accumulate more wealth and show it to be consistent with Japan’s lost decade.

Fear of risk is an oft-mentioned factor to explain protracted economic downturns such as Japan’s lost decade or the recent financial crisis because it makes investors seek safer and less profitable investments. The following evidence suggests that risk aversion may indeed have played an important role during the economic slowdown faced by Japan in the 1990s. First, anecdotal evidence suggests that the lost decade made the Japanese more anxious about the future. This fact is best illustrated by the following excerpt from an article of the New-York Times (“When consumers cut back: A lesson from Japan”, December 2, 2009, by Hiroko Tabuchi):

Today, years after the recovery, even well-off Japanese households use old bath water to do laundry, a popular way to save on utility bills. [...] Although the family has a comfortable nest egg, Hiroko Takigasaki carefully rations her vegetables. [...] Her husband has a well-paying job with the electronics giant Fujitsu, but “I don’t know when the ax will drop,” she says.

Second, Japanese households invested less and less in risky assets even though their total assets kept increasing (see Figures 1 and 2).

To account for this evidence, we introduce decreasing relative risk aversion in a business cycle model and let investors allocate their wealth between more or less risky assets. In addition to a standard equilibrium, our model can have a “safe” equilibrium with higher risk aversion and a lower interest rate. This

* Benhima: University of Lausanne, kenza.benhima@unil.ch. Massenot: University of Lausanne, baptiste.massenot@unil.ch. We are grateful to the editor and two anonymous referees for their helpful suggestions. We would also like to thank Philippe Bacchetta, Fabrice Collard, Lorenz Goette, Céline Poilly, Pascal Saint-Amour, Mathias Thoenig, and seminar participants at the University of Bern, University of Lausanne and at the SSES Annual Meeting 2012 for their comments and discussions. We gratefully acknowledge financial support from the National Centre of Competence in Research “Financial Valuation and Risk Management” (NCCR FINRISK).

equilibrium is characterized by investment in safer assets, which implies a greater misallocation of capital and thus a weaker production. It arises because of a feedback effect from risk aversion to the interest rate. On the one hand, with high risk aversion, investors invest more in safe assets, which decreases the interest rate. On the other, a low interest rate makes agents more risk averse because it decreases their safe revenues. To obtain these results, we assume that investors compare their consumption to a *reference* level and become more risk averse as their consumption gets closer to it.

The two equilibria have opposite predictions: the comovement between risky investment and wealth is positive in the standard equilibrium and negative in the safe equilibrium. When they become wealthier, investors want to keep a balanced portfolio by increasing their demand for both safe and risky assets. In the standard equilibrium, the stock of both assets increases. In the safe equilibrium, however, the return to safe assets decreases so much that it makes investors more risk-averse and they end up holding a *safer* portfolio even though their wealth has increased. As a consequence, when the economy switches from the standard to the safe equilibrium, its dynamics are characterized by a declining stock of risky assets and a rising wealth, a situation which we refer to as a safety trap. In the remainder of the paper, we show that the empirical evidence is consistent with interpreting Japan's lost decade as a safety trap. We also find that several alternative hypotheses fail to explain the same evidence.

Finally, we briefly discuss some policy implications of our model. We show that the government can crowd out the safe equilibrium by increasing the supply of bonds or by making transfers to investors. These two policies can reduce the adverse effect of the interest rate on risk aversion. A larger supply of bonds increases the interest rate and thus reduces risk aversion. Transfers to investors help them secure their level of reference consumption and thus make them less risk averse.

Although we illustrate our model with the lost decade, we believe it to have broader applications. First, regarding the methodology, we believe we are the first to show that decreasing relative risk aversion can be a source of multiple equilibria in a business cycle model. Second, whether the economy is in one equilibrium or the other is not linked to the fundamentals of the economy but is instead the result of the coordination between a large number of investors. The spell of this safer but less productive regime is potentially very long if agents do not coordinate out of it. Our mechanism could thus explain phenomena characterized by jumps and persistence that are difficult to reconcile with similar changes in the fundamentals and in which fear or anxiety are thought to play an important role. In particular, the prolonged recession in the US following the 2008 crisis is often compared to the lost decade in Japan.

According to Hayashi and Prescott (2002), the lost decade is mostly the result of a lower TFP. Our mechanism is consistent with this finding. The safety trap implies a lower TFP as investors invest increasingly in safer and less productive

assets. Another explanation for the lost decade which is also consistent with a lower TFP is the one proposed by Caballero, Hoshi and Kashyap (2008). They argue that banks kept on lending to unproductive firms – the zombies – during the lost decade. This prevented more productive firms to enter and decreased TFP.

Our framework can also be consistent with the liquidity trap view of Japan’s lost decade (Krugman, Dominguez and Rogoff, 1998). Although our model is a real model and hence has no room for monetary policy, it provides a possible explanation for low interest rate situations that lead to liquidity traps.

Preferences with decreasing relative risk aversion have received empirical support. For example, Carroll (2002) and Calvet, Campbell and Sodini (2009) find that wealthier investors hold riskier portfolios. Furthermore, our use of a reference level of consumption in the utility function relates our paper to the habit formation literature. Standard references in macroeconomics include Jermann (1998) and Boldrin, Christiano and Fisher (2001).¹ A first difference with the habit formation literature is that it uses a time-varying reference point, whereas it is constant in our framework. But this difference is only superficial, since similar results would obtain with sticky enough external habits.² A second difference is that we obtain a non-monotonic demand for capital and thus multiple equilibria. This additional result comes from our choice of using continuous-time finance methods to solve our model.³ This methodological choice allows us to derive closed-form solutions without resorting to local approximations.

The paper is organized as follows. Section 1 presents some salient facts of Japan’s lost decade. Section 2 introduces the model. Section 3 solves for the equilibrium. Section 4 analyzes the dynamics. Section 5 confronts the model’s predictions with Japan’s lost decade. Section 6 discusses some policy implications. Section 7 studies equilibrium stability. Section 8 concludes. All the proofs are in the Appendix.

I. Japan’s Lost Decade: Salient Facts

In this section, we describe some of the salient facts of Japan between 1981 and 2009. Figure 3 shows GDP. In the 1980s, the annual average growth rate of GDP is almost 4%. The period afterwards is famous for its much lower growth rate (an average of 1.5% from 1990 to 2007) and is referred to as the lost decade(s).

As suggested by Hayashi and Prescott (2002), this phenomenon can be consistent with a slowdown in TFP. Figure 4 indeed shows that TFP grows at a rate of about 2.4% per year before 1990. This rate falls to almost 0 afterwards.

During this slowdown, Figure 5 shows that the Japanese households reduced

¹See also Abel (1990) and Constantinides (1990) for early references in the asset pricing literature.

²In fact, extending our framework to habit formation by making the reference point more time-dependent would enable us to accommodate trend growth, which is absent in our model.

³See Merton (1998) for a presentation of these methods and Kraay et al. (2005), Angeletos and Panousi (2009), and Benhima (forthcoming) for examples of business cycle models using these methods.

their consumption growth from 3.4% per year before 1990 to 2.4% in the period 1990-95. Afterwards, this rate falls to almost 0.

Households not only consumed less, they also drastically reduced their holdings of risky assets. Figure 1 shows that prior to 1990 the Japanese households increased their stock of risky assets at a rate of 15% per year. In the 1990s, they reduced it at a rate of 1.5% per year. This is all the more striking that during the same period, Figure 2 shows that their stock of total assets kept increasing at a rate of 6.4% per year (against 13% before 1990). In the 2000s, they started to increase again their stock of risky capital until the crisis of 2007.

Finally, it is interesting to note that the declining share of risky assets in the 1990s cannot be explained by an increasing interest rate. Figure 6 shows indeed a declining interest rate during the 1990s. Nor can it be explained by a greater risk. Figure 7 shows that although the volatility of the Nikkei peaked in 1990 it quickly came back to its initial level and remained low afterwards.

To summarize, column 1 of Table 1 presents the correlations between these variables. We focus on comparing the periods 1981-1989 and 1990-1999 because they offer the starkest contrast. The salient features of the data are that the comovement between total and risky assets turns negative in 1990. The same happens to the comovement between total assets and TFP. By contrast, there does not appear to be such strong changes when looking at the interest rate and consumption.

The objective of the remainder of the paper is to build a model consistent with these comovements.

II. Model

This section builds a business cycle model with decreasing relative risk aversion and portfolio choice between a risky and a safe technology. The model predicts that the demand for risky capital can be a hump-shaped function of the interest rate, which can be a source of multiple equilibria, as we show in the next section.

A. Setup

Time is continuous, infinite, and indexed by t . The economy is populated by a continuum of investors of length one who allocate their wealth between safe and risky assets to maximize their utility.

The lifetime utility of investors is given by the expected sum of their consumption stream c discounted at the rate ρ :

$$(1) \quad W_t = E_t \int_t^\infty e^{-\rho s} u(c_s) ds,$$

with u being of the form:

$$(2) \quad u(c) = \ln(c - \tilde{c}),$$

where $\tilde{c} > 0$ is the level of reference consumption. This utility function has decreasing relative risk aversion. When consumption moves closer to its reference point, investors become more risk averse.

Investors allocate their wealth ω between risky capital k , safe capital z , and bonds b :

$$(3) \quad \omega_t = k_t + z_t + b_t,$$

Bonds deliver a return R_t^b and are in zero supply.⁴ The two types of capital k and z deliver a flow of capital income in units of final good of, respectively,

$$(4) \quad dk_t = (F(k_t, l_t^k) - w_t^k l_t^k)dt + \sigma k_t dv_t$$

and

$$(5) \quad dz_t = (G(z_t, l_t^z) - w_t^z l_t^z)dt,$$

where F and G are constant returns to scale functions, and are strictly increasing and concave in each of their arguments, l^k and l^z are capital-specific labor, w_k and w_z are the costs of labor, and dv_t is a macroeconomic shock to the return of the risky technology, where v_t is a standard Wiener process.

The budget constraint of investors is given by the following equation:

$$(6) \quad d\omega_t = dk_t + dz_t + (R_t^b b_t - c_t)dt.$$

Finally, the supply of each type of labor is inelastic and equal to one. We also assume that labor is not supplied by the main character of our story (the investor) but by a secondary protagonist (the workers) who plays no further role in our model. The main motivation for doing so is to generate external decreasing returns to capital in the simplest framework possible.⁵ The main advantage is that we do not have to carry around human wealth and this greatly simplifies both the presentation of the results and the resolution of the model. A possible interpretation is that workers are hand-to-mouth, that is, they consume all their labor income. This is a natural equilibrium outcome in a model where the

⁴Bonds thus play no role in the economy. However, they allow us to characterize the return to safe assets as the interest rate.

⁵Labor supply is inelastic in our model, but one might consider what happens with a more elastic supply. In that case, the external decreasing returns to capital, which are key to our results, will be less strong. However, one should expect that, as long as labor supply is sufficiently inelastic, our results will survive.

entrepreneurs' revenue is risky while the workers' revenue is not, and workers face a no-liability constraint. In such a setting, workers would end up hand-to-mouth as the result of a low interest rate.⁶

B. Profit Maximization

Investors choose the amounts of labor l^k and l^z that maximize Equations (4) and (5). Labor markets are competitive within each type of capital-specific labor and investors thus take the wages w_k and w_z as given. Optimal labor demands therefore equalize wages and the marginal productivities of labor: $w_t^k = F_2(\hat{k}_t, 1)$ and $w_t^z = G_2(\hat{z}_t, 1)$, where $\hat{k} = k/l^k$ and $\hat{z} = z/l^z$ are the capital-labor ratios. These equations define the optimal capital-labor ratios as functions of wages: $\hat{k}(w_t^k)$ and $\hat{z}(w_t^z)$.

Using the fact that the production functions exhibit constant returns to scale, we can show that the returns of the safe and risky technologies are linear functions of capital:

$$(7) \quad dk_t = r_t k_t dt + \sigma k_t dv_t$$

$$(8) \quad dz_t = R_t z_t dt$$

where $r_t = F_1(\hat{k}(w_t^k), 1)$ and $R_t = G_1(\hat{z}(w_t^z), 1)$. These private returns expressions are derived using the constant returns to scale assumption and the Euler theorem. They are taken as given by investors.

Arbitrage imposes that the return of bonds is equal to the return of the safe technology, that is, $R_t^b = R_t$. We can thus refer to R as the interest rate.

The budget constraint of investors therefore boils down to:

$$(9) \quad d\omega_t = (r_t k_t + R_t(z_t + b_t) - c_t)dt + \sigma k_t dv_t.$$

C. Utility Maximization

At each period t , investors consume c_t and allocate their wealth ω_t between bonds b_t , risky capital k_t , and safe capital z_t to maximize their lifetime utility (1), taking the returns r_t and R_t as given and under the constraints (3), (9) and

$$(10) \quad k_t \geq 0, \quad z_t \geq 0.$$

The solution to this problem is given by the following Proposition:

PROPOSITION 1: *For a given level of wealth ω_t and given returns r_t and R_t such that $R_t \in (\bar{c}/\omega_t, r_t]$, the solution $\{c_t, k_t, z_t, b_t\}$ to the problem of the investor*

⁶See for example Kiyotaki and Moore (2008).

is given by Equation (3) and:

$$(11) \quad c_t - \tilde{c} = \rho(\omega_t - \tilde{\omega}_t),$$

$$(12) \quad \frac{k_t}{\omega_t} = \frac{\Delta_t}{\sigma^2 \gamma_t},$$

where

$$(13) \quad \tilde{\omega}_t = \frac{\tilde{c}}{R_t},$$

$$(14) \quad \Delta_t = r_t - R_t.$$

$$(15) \quad \gamma_t = \frac{-\omega_t \partial^2 W_t / \partial \omega_t^2}{\partial W_t / \partial \omega_t} = \frac{\omega_t}{\omega_t - \tilde{\omega}_t},$$

The presence of reference consumption introduces the notion of reference wealth $\tilde{\omega}_t$, defined in Equation (13). $\tilde{\omega}_t$ is the quantity of safe assets necessary to secure the level of reference consumption \tilde{c} . It is decreasing in the interest rate. Indeed, when R_t is low, entrepreneurs need more safe assets to compensate for their lower return. We show in the following that this has important implications on savings and portfolio allocation.

Equation (11) describes the saving behavior. Consumption decreases with reference wealth and thus increases with the interest rate. In other words, investors cut on consumption when the interest rate is low in order to compensate for the lower returns on savings and maintain their reference level of consumption. Therefore, reference consumption introduces a strong income effect of the interest rate on savings. This is consistent with Muellbauer and Murata (2011) who document a positive correlation between the interest rate and consumption in Japan after controlling for wealth.

Equation (12) is a standard portfolio allocation rule. It states that the share of wealth invested in risky capital is increasing with the excess return Δ_t , defined in Equation (14), decreasing with risk σ and with risk aversion γ_t . Equation (15) shows that γ_t decreases with wealth ω_t and with the interest rate R_t . Both a higher wealth and a higher interest rate indeed enable households to better cope with adverse shocks and secure reference consumption.

The main insight from Equation (12) is that the interest rate R_t has two contradictory effects on the demand for risky capital. The first effect goes through the excess return Δ_t and is negative. This is a standard arbitrage effect in asset pricing theory. The second effect is positive and goes through risk aversion γ_t .

As already noted, a higher interest rate makes investors less risk averse and this increases their demand for risky capital. We show below that the risk aversion effect dominates for a low interest rate while the excess return effect dominates for a high interest rate. The demand for risky capital is thus a hump-shaped function of the interest rate.

Finally, the policy functions expressed in Proposition 1 are well-defined only if the solution is an interior one, that is if the constraints described in Equation (10) are satisfied, and if the value function is well-defined, that is, if $c > \tilde{c}$. These policy functions are therefore valid only if:

$$(16) \quad R_t \in (\tilde{c}/\omega_t, r_t].$$

Equation (11) indeed shows that a lower R_t would violate the constraint $c > \tilde{c}$, in which case lifetime utility W is not well-defined. Equation (12) further shows that a higher R_t would violate the constraint $k \geq 0$.

D. Equilibrium

Note that entrepreneurs are heterogeneous but their policy functions are linear in wealth, which implies that the properties of the aggregate variables are the same as the individual ones described in the previous section. We therefore keep the same notations, but we refer from now on to the aggregate variables instead of the individual ones.

The equilibrium is defined as follows:

DEFINITION 1 (Equilibrium): *For a given initial aggregate wealth ω_0 , an equilibrium is a sequence of wealth $\{\omega_t\}_{t \geq 0}$ of risk aversion $\{\gamma_t\}_{t \geq 0}$, of excess return $\{\Delta_t\}_{t \geq 0}$, of reference wealth $\{\tilde{\omega}_t\}_{t \geq 0}$, of vector of aggregate quantities $\{c_t, k_t, z_t, b_t, l_t^k, l_t^z\}_{t \geq 0}$ and of vector of prices $\{r_t, R_t, R_t^b, w_t^k, w_t^z\}_{t \geq 0}$ such that, for each period $t \geq 0$:*

- (i) *The excess supply of bonds is equal to zero: $b_t = 0$;*
- (ii) *Workers supply $l_t^k = 1$ and $l_t^z = 1$;*
- (iii) *For given returns r_t and R_t , investors' aggregate consumption c_t , risky capital k_t , safe capital z_t , reference wealth $\tilde{\omega}_t$, excess return Δ_t and risk aversion γ_t are such that Equations (3) and (11)-(13) are satisfied and that inequalities (10) and $c_t > \tilde{c}$ hold.*
- (iv) *$w_t^k = F_2(k_t, 1)$, $w_t^z = G_2(z_t, 1)$, $R_t = R_t^b$, $r_t = F_1(k_t, l_t^k)$ and $R_t = G_1(z_t, l_t^z)$.*

In what follows, the analysis is split in two steps. We first analyze the period- t equilibrium: for a given inherited aggregate wealth ω_t , we characterize the possible equilibrium allocations and prices. Second, we solve for the dynamics of ω_t , which will determine the evolution of the economy.

III. Period-t Equilibrium

This section defines and solves for the period-t equilibrium. We drop the t index from now on unless confusing.

A. Equilibrium Returns

Labor supply is equal to one for each type of labor. Then, solving for the equilibrium wages, the equilibrium returns on capital become:

$$(17) \quad r = r(k)$$

$$(18) \quad R = R(z),$$

where $r(k) = F_1(k, 1)$ and $R(z) = G_1(z, 1)$. Given the properties of F and G , these marginal returns r and R are strictly positive and decreasing with, respectively, k and z . Intuitively, a higher demand for capital increases the demand for labor and, thus, the equilibrium wage. This decreases the equilibrium return to capital. We further restrict F and G so that r and R satisfy the following assumption:

ASSUMPTION 1: *The functions r and R satisfy the following conditions:*

- (i) $r(k)$ and $R(z)$ are strictly decreasing, with $\lim_{k \rightarrow +\infty} r(k) = \lim_{z \rightarrow +\infty} R(z) = 0$, $\rho < \lim_{k \rightarrow 0} r(k) < +\infty$ and $\rho < \lim_{z \rightarrow 0} R(z) < +\infty$;
- (ii) $r(k) \geq R(k)$ for any given k ;
- (iii) There exists $\omega_{min} > 0$ such that $r(k) = R(\omega_{min} - k)$ and $R(\omega_{min} - k)\omega_{min} = \tilde{c}$ are both satisfied for some $k \geq 0$;
- (iv) $r(\omega_{min}) < R(0)$;
- (v) $r(k)k$ and $R(z)z$ are strictly increasing, with $\lim_{k \rightarrow +\infty} r(k)k = \lim_{z \rightarrow +\infty} R(z)z = +\infty$ and $\lim_{k \rightarrow 0} r(k)k = \lim_{z \rightarrow 0} R(z)z = 0$.

This assumption will guarantee that the problem is well-defined. Condition (i) is close to the implication of Inada conditions on F and G . The only deviation from the Inada conditions is that the limit of marginal returns goes to a finite number when the capital stock goes to zero in (i). This will ensure that the excess return to risky capital remains finite. Conditions (ii) to (iv) ensure that r and R are well-behaved. These conditions imply that for $\omega \geq \omega_{min}$, there is a unique interior solution to $r(k) = R(\omega - k)$, where ω_{min} defines the minimum level of wealth that can sustain a level of consumption at least equal to \tilde{c} . Finally, condition (v) imposes some of the Inada conditions on $r(k)k$ and $R(z)z$, the equilibrium revenues for entrepreneurs.

B. *The Aggregate Demand for Risky Capital*

The next step we take to solve the equilibrium is to replace the equilibrium returns derived below into the demand for capital given by Equation (12). This section analyzes the properties of the resulting demand for risky capital.

Bonds b are in zero supply and thus the whole demand for safe investment is absorbed by safe assets $z = w - k$. Using Equation (3), we can also rewrite $R(z)$ as $R(\omega - k)$. We obtain the demand for risky capital D , which only depends on k , ω , and σ :

$$(19) \quad D(k, \omega, \sigma) = \frac{\Delta(k, \omega)}{\sigma^2 \gamma(k, \omega)} \omega,$$

with $\Delta(k, \omega) = r(k) - R(\omega - k)$ and $\gamma(k, \omega) = \frac{\omega}{\omega - \tilde{c}/R(\omega - k)}$. An equilibrium stock of risky capital k then satisfies:

$$(20) \quad D(k, \omega, \sigma) = k$$

In other words, the actual stock of risky capital k should be consistent in equilibrium with the level of desired risky capital $D(k, \omega, \sigma)$.

To understand formally the behavior of D , we take the derivative of D with respect to k :

$$(21) \quad D_1(k, \omega) = \frac{r'(k) + R'(\omega - k)}{\sigma^2 \gamma(k, \omega)} \omega - \frac{\tilde{c} R'(\omega - k) \Delta(k, \omega)}{\sigma^2 (R(\omega - k))^2}$$

The two terms summarize the two contradictory effects of k on D . First, a negative *excess return effect* through a lower Δ (first term). The function D is indeed increasing in the excess return, which is itself reduced by a higher k . Second, a positive *risk aversion effect* through a lower γ (second term). The risk aversion effect is strictly positive if $\tilde{c} > 0$ and if returns are decreasing ($R' < 0$), which is implied by Assumption 1. A higher stock of risky capital means that fewer resources are invested in safe assets, which increases the interest rate, makes investors less risk-averse and thus more willing to hold risky assets. For a high enough value of k (R close to r), Δ is close to zero, which implies that the excess return effect dominates. For a low value of k (R close to \tilde{c}/ω), γ goes to infinity, so the risk aversion effect dominates. The function D is therefore hump-shaped in k , as represented in Figure 8. Finally, note that for low enough values of \tilde{c} , the risk aversion effect vanishes and the demand for risky capital becomes monotonically decreasing.

As illustrated in Figure 8, D is equal to zero for two values, $\underline{k}(\omega)$ and $\bar{k}(\omega)$, where $\bar{k}(\omega)$ is the solution to $\Delta = 0$, that is:

$$(22) \quad r(k) = R(\omega - k)$$

and $\underline{k}(\omega)$ is the solution to $\gamma \rightarrow +\infty$, that is:

$$(23) \quad R(\omega - k)\omega = \tilde{c}$$

so D is positive only for $\underline{k}(\omega) \leq k \leq \bar{k}(\omega)$. If ω decreases, then the range over which D is positive becomes smaller, as represented by the curve with $\omega = \omega_1$. We show that $k \in (\underline{k}(\omega), \bar{k}(\omega)]$ is actually a necessary condition for the demand for risky capital to be well-defined. The conditions on k that ensure a well-behaved D are summarized in the following Lemma:

LEMMA 1: *Define ω_{max} as the unique $\omega > \omega_{min}$ such that $\underline{k}(\omega) = 0$. Under Assumption 1:*

- (i) *if $\omega > \omega_{min}$, then $\bar{k}(\omega)$ and $\underline{k}(\omega)$ are uniquely defined, with $\underline{k}(\omega) < \bar{k}(\omega)$;*
- (ii) *if $\omega_{min} < \omega \leq \omega_{max}$, then $\underline{k}(\omega) \geq 0$ and D is well-defined for $k \in (\underline{k}(\omega), \bar{k}(\omega)]$;*
- (iii) *if $\omega > \omega_{max}$, then $\underline{k}(\omega) < 0$ and D is well-defined for $k \in [0, \bar{k}(\omega)]$;*

This result is derived as follows. For a given ω , the restriction (16) can be rewritten in equilibrium as $R(\omega - k) \in (\tilde{c}/\omega, r(k)]$. This new restriction can be rewritten as a range of $k \in (\underline{k}(\omega), \bar{k}(\omega)]$. The demand for risky capital D is only well defined in this range, which requires that $k > 0$ and $c > \tilde{c}$. Indeed, these are the conditions under which the policy functions of Proposition 1 are valid. Lemma 1 tells us that this range is non-trivial for $\omega > \omega_{min}$, that is, if the economy is not too poor. Besides, if ω is large enough, then this range is wide enough to include 0. This case is illustrated in Figure 9 by the case with $\omega = \omega_2$. The intuition is that when the economy is rich, any equilibrium with $k \geq 0$ can sustain $c > \tilde{c}$.

C. Multiple Equilibria

An equilibrium stock of risky capital k is a solution to Equation (20). The following proposition gives the solution:

PROPOSITION 2: *Under Assumption 1, for a small σ and $\omega_{min} < \omega \leq \omega_{max}$:*

- (i) *there are two well-defined solutions to Equation (20), denoted $k^L(\omega, \sigma)$ and $k^H(\omega, \sigma)$, with $k^L(\omega, \sigma) < k^H(\omega, \sigma)$;*
- (ii) *these solutions converge respectively to $\underline{k}(\omega)$ and $\bar{k}(\omega)$ as σ goes to zero.*

Let us first describe part (i). If a well-defined equilibrium exists, it must satisfy (20). This relationship defines implicitly the equilibrium stock of capital as a function of ω and σ . Because the function D can be hump-shaped, there can exist up to two non-degenerate equilibria: A *standard* and a *safe* one. These

equilibria are given by the intersection of the 45 degree line and of the demand curve in Figure 8.⁷

The standard equilibrium corresponds to the part of the demand curve that is downward-sloping and where the standard excess return effect dominates. It is characterized by a relatively large stock of risky capital that we denote $k^H(\omega, \sigma)$, where the superscript H refers to the equilibrium variables associated with the standard regime.

The safe equilibrium corresponds to the part of the demand curve that is upward-sloping and where the unconventional risk aversion effect dominates. It is characterized by a smaller stock of risky capital than the standard equilibrium that we denote k^L , where we use superscript L to denote the equilibrium variables associated with the safe regime. This implies a lower interest rate, a higher risk aversion, and a lower consumption than in the standard equilibrium.

The multiplicity of equilibria comes from decreasing risk aversion. Indeed, with a low \tilde{c} , only the standard equilibrium exists. In the limit, when \tilde{c} goes to zero relative to ω , the risk aversion effect disappears in Equation (21) and the function D is downward sloping, so the equilibrium is unique. This case is represented by the grey line in Figure 9 ($\omega = \omega_3$). By contrast, with a higher \tilde{c} relative to ω , the demand for risky capital becomes upward sloping for a low k . This case is represented by the solid and dashed lines in Figure 9 ($\omega = \omega_1$ or ω_2). A second equilibrium with few risky assets then results from a feedback from risk aversion to the interest rate. If the interest rate is low, then risk aversion is high and so is the demand for safe assets, which in turn decreases the interest rate. Finally, there exists no equilibrium with a positive k if the demand for capital is too low, for example, because there is too much risk or because wealth is too small. This case is represented by the dotted line in Figure 9 ($\omega = \omega_0$).

Part (ii) of Proposition 2 describes the limit case where σ goes to zero, for which we can obtain closed-form solutions. The standard equilibrium corresponds to the solution of a standard Ramsey problem. Absent risk, the risky technology has no excess return. In the safe equilibrium, investors are infinitely risk-averse (which is implied by $\omega = \tilde{\omega}$) and only consume their level of reference consumption. They invest a larger share of their wealth in safe capital, which implies a positive excess return. Note that this equilibrium is not well-defined, since it implies $\omega = \tilde{\omega}$ and $c = \tilde{c}$. However, it is useful to consider it because it can be interpreted as a limit equilibrium to which the economy converges when σ moves closer to 0. Upper-bar letters denote variables in the limit standard equilibrium while lower-bar letters denote variables in the limit safe equilibrium.

⁷Actually, $k = 0$ defines a third equilibrium, but this equilibrium is not well-defined since in that case we would have $c \leq \tilde{c}$. Indeed, if we abstract from the concern that c must be higher than \tilde{c} , an equilibrium must satisfy $k = \max\{0, D(k, \omega, \sigma)\}$. The function $k \rightarrow \max\{0, D(k, \omega, \sigma)\}$ defines a continuous mapping from the compact $[0, +\infty]$ to itself. As expected for this type of fixed point problem, the number of equilibria is odd: there are three solutions 0, $k^L(\omega, \sigma)$ and $k^H(\omega, \sigma)$. However, because of the value function definition problem for $c \leq \tilde{c}$, our problem only admits two well-defined solutions.

D. Comparative statics

Before turning to the dynamics, we show analytically in the limit case $\sigma \rightarrow 0$ that our two equilibria have opposite implications regarding the effect of wealth on the stock of risky capital: it is positive in the standard equilibrium and negative in the safe equilibrium. This is a key implication of our model because it helps us discriminate between the two equilibria. We state it formally in the following proposition:

PROPOSITION 3: *Under Assumption 1, for $\omega_{min} < \omega \leq \omega_{max}$:*

- (i) \bar{k} is increasing in ω ;
- (ii) \underline{k}/ω is decreasing in ω ; \underline{k} is decreasing in ω in the neighborhood of ω_{max} .

The intuition of this result is as follows. A lower wealth switches the economy from the curve $D(k, \omega_2, \sigma)$ to the curve $D(k, \omega_1, \sigma)$ in Figure 9. A higher wealth increases the demand for risky capital, everything else equal. In the standard regime, this increases the equilibrium level of risky capital. In the safe equilibrium, paradoxically, the equilibrium level of risky assets *decreases* when agents are wealthier. This is because a lower interest rate makes agents *even more risk averse*.

We also examine the implied comovements of wealth with consumption, the interest rate and the misallocation of resources. We define a measure of misallocations as $y(\omega) - y^*(\omega)$, where $y^*(\omega)$ is the maximum level of production that can be technically achieved when aggregate wealth is equal to ω . The following Proposition shows how these variables comove with ω in the limit standard and safe equilibria along the transitional dynamics:

PROPOSITION 4: *Under Assumption 1, for $\omega_{min} < \omega \leq \omega_{max}$:*

- (i) \bar{R} is decreasing in ω ; $\bar{y} = y^*$ so $\bar{y} - y^*$ is invariant in ω ; \bar{c} is increasing in ω if $\bar{c} < \frac{R^2|r'+R'|}{|r'R'|}$;
- (ii) \underline{R} is decreasing in ω , $\underline{y} < y^*$ and $\underline{y} - y^*$ is decreasing in ω ; \underline{c} is invariant in ω in the neighborhood of ω_{max} .

The interest rate is decreasing in wealth in both regimes. This is not surprising since investment in the safe technology increases with wealth in both regimes. In the standard regime, it is optimal to dispatch investment in the two technologies. In the safe regime, agents invest even more in the safe technology, as they invest less in the risky technology.

When σ goes to zero, resources are always allocated optimally in the standard regime, in the sense that, given available resources, the maximal level of production is reached. By contrast, in the safe regime, the risky technology has a positive excess return, which means that a higher production level could be reached by reallocating capital from the safe to the risky technology. For a given level of

aggregate wealth, production is therefore lower in the safe regime. In this regime, resources are more and more misallocated as investors become richer, because they are more and more unwilling to invest in the risky technology.

In the standard equilibrium, consumption is increasing in wealth, as long as \tilde{c} is not too large. Indeed, if \tilde{c} was too large, then the income effect of the interest rate would be large and would depress consumption as the interest rate falls with wealth. In the calibrated version of the model, \tilde{c} is not so large that it would generate this pathological behavior for consumption in the standard regime. In the safe regime, consumption is not affected by wealth since investors are stuck at their reference level in the limit case where σ goes to zero.

IV. Dynamics

The dynamics of the economy is obtained by replacing Equation (11) into Equation (9). This gives the following expression for the expected growth rate of wealth:

$$(24) \quad g_\omega = \frac{\dot{\omega}}{\omega} = \frac{R - \rho}{\gamma} + \Delta \frac{k}{\omega}.$$

Wealth grows as the result of two components: an aggregate saving component (first term) and a saving composition component (second term). The first term gives the dynamics of wealth in the absence of excess return to capital. It is positive if the return of savings R is higher than the propensity to consume out of wealth ρ , and it is lower when wealth is close to its reference level (γ is high), as agents cut on savings to maintain consumption at its reference level. The second term represents the additional increase in wealth that comes from the excess return on risky capital.

Notice that, for a given regime $i \in \{L, H\}$, g_ω is entirely determined by the state variable ω , so we can denote $g_\omega^i(\omega)$. It is useful to define a steady state and a stable steady state, towards which the economy naturally converges:

DEFINITION 2 (Steady state): *For a given regime $i \in \{L, H\}$, a steady state is defined by a level of wealth ω such that $g_\omega^i(\omega) = 0$.*

DEFINITION 3 (Stable steady state): *For a given regime $i \in \{L, H\}$, a stable steady state is defined by a steady-state level of wealth ω such that $\partial g_\omega^i(\omega) / \partial \omega < 0$.*

We first describe the dynamics of the standard equilibrium and then the dynamics of the safe equilibrium, which we refer to as the safety trap.

A. The Standard Dynamics

The dynamics of the standard equilibrium is given by the following proposition:

PROPOSITION 5: *Under Assumption 1 and starting from an initial level of wealth $\omega_0 > \omega_{min}$:*

- (i) *If σ goes to zero, there are two possible steady-state levels of wealth ω^* and ω_{min} . They are the solutions to $\bar{R}(\omega) = \rho$ and $\bar{R}(\omega)\omega = \tilde{c}$, respectively. If $\omega_{min} \geq \omega^*$, the economy converges to ω_{min} , which is the only stable steady state. If $\omega^* \geq \omega_{min}$, the economy converges to ω^* , which is the only stable steady state.*
- (ii) *If σ is small and $\omega^* \geq \omega_{min}$, then ω^* is increasing in σ .*

First consider the limit case with no risk. In the standard regime, we have $\bar{\Delta} = 0$, so the second term in Equation (24) disappears and wealth is thus exclusively driven by the saving behavior of agents:

$$(25) \quad \bar{g}_\omega = \frac{\bar{R} - \rho}{\bar{\gamma}}.$$

The steady-state ω^* is the solution to $\bar{R} = \rho$. It corresponds to the long-run level of wealth to which an economy with no reference consumption would converge. The steady state ω_{min} is the solution to $1/\bar{\gamma} = 0$. It corresponds to the case where agents do not save because they are stuck at their reference consumption.

If $\omega_{min} < \omega^*$, the problem is well defined around $\omega = \omega^*$, and the economy converges to that level. The dynamics is represented in Figure 10 by the solid lines. The top-left panel represents \bar{g}_ω as a function of ω . It shows that starting from $\omega < \omega^*$, the growth rate of wealth is positive until the economy reaches ω^* and the economy can thus be thought of as moving towards the right. By contrast, the growth rate of wealth is negative if $\omega > \omega^*$. The economy thus moves towards the left in this region.

We can now infer the dynamics of consumption, the stock of risky assets, and the interest rate from the three remaining panels, which show how wealth affects these variables. Starting from a low level of wealth, consumption and the stock of risky assets increase as wealth increases, while the interest rate decreases at the same time. This is consistent with the standard view that when investors become wealthier, they further diversify their portfolio by increasing their holdings of both safe and risky assets. As the stock of safe assets increases, the interest rate also decreases because of decreasing returns.

If $\omega^* \leq \omega_{min}$, an economy starting from a level of wealth $\omega \geq \omega_{min}$ converges to ω_{min} . This happens if \tilde{c} is large or if the two technologies are not very productive. In that case, the economy is not productive enough to sustain a level of consumption greater than reference consumption in the long run. In the remainder of the paper, we will discard this case and only consider the situation where $\omega_{min} < \omega^*$.

In the case $\sigma > 0$ (not represented), the dynamics is qualitatively very similar to the case without risk. The only difference is that entrepreneurs accumulate more wealth in the long-run, which is a standard effect of risk.

B. The Safety Trap

The dynamics of the safe equilibrium which we refer to as the safety trap is given by the following proposition:

PROPOSITION 6: *Under Assumption 1 and starting from an initial level of wealth $\omega_{min} < \omega_0 < \omega_{max}$:*

- (i) *If σ goes to zero, ω_{min} and ω_{max} are the two unique steady-state levels of wealth in the safe regime. The economy converges to ω_{max} , which is the only stable steady state.*
- (ii) *If σ is small, then the economy converges to ω_{max} , but never reaches ω_{max} .*

Consider the limit case with no risk. In the safe regime, γ is infinite, so the first term in Equation (24) disappears and the growth rate of wealth is driven exclusively by the composition of savings:

$$(26) \quad \underline{g}_\omega = \underline{\Delta} \frac{k}{\omega}.$$

Agents are stuck at their reference consumption and do not save, so wealth increases over time only because of the excess return on capital. The possible steady states are the solutions to $\underline{g}_\omega = 0$, that is, either $\underline{\Delta} = 0$, which corresponds to ω_{min} , or $\underline{k} = 0$, which corresponds to ω_{max} .

The economy converges to a steady state ω_{max} if it starts at $\omega < \omega_{max}$. The dynamics is represented in Figure 10 by the dashed lines. The top-left panel shows that the growth rate of wealth is always positive. Thus, the economy moves towards the right along the convergence path.

The three other panels show how wealth affects consumption, the stock of risky capital, and the interest rate. As wealth increases towards ω_{max} , consumption moves closer to \tilde{c} , the stock of risky assets falls to 0, and the interest rate decreases.⁸ Investors hold a positive level of risky capital that yields a positive excess return, and their wealth keeps increasing as long as that level is strictly positive.

However, the structure of investment changes as wealth increases. The wealthier the economy, the lower their investment in risky capital, as already predicted by Proposition 3. Here, the entrepreneurs are in a *safety trap*: they accumulate wealth, but this accumulation does not make them less risk-averse. Indeed, because of decreasing returns in the safe technology, the interest rate decreases along the convergence path, which compensates for the effect of a higher wealth on risk aversion. As the level of risky capital decreases, the growth rate of wealth falls until ω reaches ω_{max} where $k = 0$ and $\dot{\omega} = 0$.

⁸Here, we represent the limit case where σ goes to 0, which makes the level of consumption look constant. A case with a higher level of risk would make this curve look more decreasing.

V. Japan's Lost Decade: A Safety Trap?

In this section, we argue that Japan's lost decade resembles a safety trap. We also show that several alternative shocks on fundamentals are difficult to reconcile with this episode.

A. Our Story

We confront the stylized facts of Japan's lost decade with a calibrated version of our model. We first show that the correlation between aggregate wealth and investment in risky capital in the safety trap contrasts sharply with the standard dynamics, as predicted by Proposition 3. This provides us with a simple test that helps us characterize the lost decade as a safety trap. Further checks are implemented by considering other variables. Finally, we simulate the transitory dynamics following a switch from the standard to the safe equilibrium, which replicates the dynamics observed during the lost decade.

CALIBRATION. — We first describe the calibration that we use for our simulation exercise. Our objective here is not to obtain realistic quantitative predictions as our model is far too simple but rather to verify that our story can make sense for reasonable parameter values. We use as production functions $F(k, l^k) = (k + al^k)^\alpha (l^k)^{1-\alpha}$ and $G(z, l^z) = (z + al^z)^\alpha (l^z)^{1-\alpha}$ with $\alpha = .3$ and $a = 0.1$. When $a = 0$, these production functions are Cobb-Douglas, but do not satisfy Assumption 1, as the implied $r(k)$ and $R(z)$ go to infinity when k and z go to zero. We therefore choose a small, but positive a in order to mimic the Cobb-Douglas technology without violating Assumption 1. We set $\rho = .05$, which implies that the interest rate in the standard equilibrium with no risk is 5%. We also set $\sigma = .02$, implying a volatility of 2% which is consistent with the volatility of output in Japan in the 1980s. The remaining parameter \tilde{c} is matched such that in the standard equilibrium the ratio of reference consumption to steady-state consumption is equal to .73. This corresponds to the calibration chosen by Boldrin, Christiano and Fisher (2001) and implies that investors' reference consumption is equal to 73% of their steady-state consumption. This gives us $\tilde{c} = 5.7$.

Importantly, we find that $\omega_{min} < \omega^* < \omega_{max}$. This means that, when the economy is at the standard steady state, there is the possibility that the investors coordinate on the safe equilibrium (see Proposition 2).

CORRELATIONS. — Table 1 gives the correlations between total households' wealth and risky assets, TFP, consumption c , and the interest rate R , both in the data and in our model. Recall that Column (1) gives the empirical correlations in 1981-1989, which corresponds to the period before the lost decade, and in 1990-1999, which corresponds to the lost decade period. Column (2) gives the correlations

obtained for the calibrated model. These correlations correspond to the comovement of variables along the convergence to the steady state. More precisely, the correlations are computed by simulating a 30-year convergence to the steady state starting from a level of wealth ω_0 equal to 90% of the steady-state level, without aggregate shocks dv_t . In that sense, they correspond to the comovements described in Proposition 3, but for the calibrated version of the model. This column only presents the signs of the correlations as our model is not rich enough to produce reliable quantitative results.⁹

In the pre-lost decade period, risky assets are positively correlated with wealth, while they are negatively correlated during the lost decade. According to Column (2), the predictions of Proposition 3 survive in the calibrated model: this correlation is positive in the standard dynamics, while it is negative in the safety trap dynamics. The dynamics of risky assets therefore help us identify the lost decade as a safety trap.

Table 1 also helps us to check whether our model is consistent with the dynamics of other key variables. First, we consider the comovement of TFP and wealth. TFP is measured as $y/(k+z)^\alpha$, which is how it is measured in practice, that is under the hypothesis that the aggregate production function is Cobb-Douglas. Interestingly, the correlation between TFP and wealth changes during the lost decade: it switches from positive to negative. This is consistent with our calibrated model, which predicts that the correlation in the standard equilibrium is positive and negative in the safe equilibrium. This is because, in the standard equilibrium, investors become less risk-averse when they become richer, which makes them invest more in the risky technology. On the opposite, in the safe equilibrium, they become more risk-averse and less willing to take risks, which increases the misallocation of resources.

Second, the behavior of the interest rate is also consistent with our hypothesis since its correlation with wealth is negative in both episodes and in both regimes.

Finally, consider the correlation between consumption and wealth. This time, the predicted correlation during a safety trap, which is negative, is at odds with the empirical correlation during the lost decade, which is positive. One possible reason is that our model is too simple and for example does not distinguish between the intertemporal elasticity of substitution and the coefficient of risk aversion as in Epstein and Zin (1989). In the model, as investors become more risk averse, their intertemporal elasticity of substitution decreases. This makes the income effect dominant: since the interest rate decreases as wealth increases, investors cut heavily on consumption. We show in the following that a simpler way to correct for this is to introduce explicitly the probability to switch from one regime to the other in the model. In that case, the dynamics of consumption in each regime is “contaminated” by the other.

We extend our model to allow for the possibility to switch from one regime to

⁹The signs were robust to alternative values of initial wealth (95% and 80% of the steady-state level) and to the inclusion of aggregate shocks dv_t .

the other.¹⁰ These switches represent changes in the expectations of investors and can be interpreted as animal spirits or sunspot shocks. Let p^L be the probability to switch from the standard to the safe regime and p^H the probability to switch from the safe to the standard regime. We calibrate p^L and p^H so that a safety trap occurs every 50 years and lasts 5 years on average. The resulting correlations are represented in Column (3). Now, the two equilibria contaminate each other and the correlation between consumption and wealth is positive in the safe regime as well, while the signs of the other correlations remain unchanged. The safety trap now matches all the empirical correlations of Japan's lost decade presented in Table 1.

This result is not too sensitive to the calibration of the switching probabilities. If we increase the average duration of the safety trap by decreasing p^H , consumption becomes less sensitive to wealth because investors expect to stay longer in the safe equilibrium. We have to make safety traps last more than 20 years on average to make the comovement between consumption and wealth turn negative. Although this may look like a reasonable value for Japan's lost decade, what matters is not the objective probability but rather the subjective probability used by investors when making their economic decisions. In the 1990s, it is doubtful that the Japanese expected this recession to last more than a decade and this is why we believe our initial calibration makes more sense.

EXPERIMENT. — So far, we have examined the correlations of some key variables with wealth around the safe and standard steady states. Here, we go further and examine the following hypothesis: the dynamics of the lost decade is the result of a switch from the standard to the safe equilibrium.

Figure 11 shows the simulations corresponding to this experiment within our calibrated model with positive switching probabilities. Starting from the standard steady state, agents coordinate on the safe equilibrium, entering the safety trap. The simulations show that at this point wealth starts increasing while the stock of risky capital decreases at the same time. This implies that the demand for safe assets increases and thus that the interest rate decreases. This reallocation from risky to safe capital also implies a lower TFP. Finally, we find that consumption increases with wealth on the convergence path, but drops on impact. Note that these dynamics imply that output falls on impact as TFP drops. However, as wealth subsequently increases while TFP decreases, production can either increase or decrease, depending on the parameter values. In our parametrization, it increases along the transition path.¹¹

These simulations are overall consistent with the Figures 1 to 7 described in Section I. Again, we fail to completely match the behavior of consumption. We

¹⁰This extension is presented in more details in a supplementary material, available on request.

¹¹This is due to our conservative parametrization of the relative productivity of the safe and risky technologies. Namely, the safe technology is as productive as the risky one ($F = G$). For a sufficiently higher productivity of the risky technology, output decreases on the transition path, for example when we set $F = 2G$.

get it right along the convergence path but wrong on impact. The initial drop in consumption occurs because the elasticity of intertemporal substitution falls on impact, due to the increase in the coefficient of relative risk aversion. In a more general model where these two coefficients would be distinct, this effect could be avoided.

B. Alternative Stories

We have shown that Japan's lost decade can be consistent with a switch from the standard to the safe equilibrium. In this section, we investigate whether it could alternatively be consistent with shocks on fundamentals in the standard equilibrium. We simulate shocks on wealth, the productivity of safe assets, the productivity of risky assets, and on risk. Although some of these shocks can generate a negative correlation between wealth and risky assets on impact, this correlation always becomes positive on the transition path to the new steady-state, as predicted by Proposition 3. Figure 12 reports the results of these simulations using the same calibration as the previous section.

The first shock we consider is a negative shock on wealth, which corresponds to a negative dv in our model. This shock moves the economy away from its steady state, and induces both risky and safe capital to fall on impact, and the interest rate to increase. After impact, the variables of the model go back to the initial steady state, which implies that both total wealth and risky capital increase while the interest rate decreases. Here, the increase in risky capital on the convergence path is at odds with the empirical evidence.

We then consider an unexpected permanent increase in the productivity of safe assets. This makes investors increase their holdings of safe capital and decrease their investment in risky capital on impact. As a result, the interest rate increases on impact. In the long run, wealth increases and investors hold more and more both risky and safe assets. As a consequence, the interest rate decreases. Here, the increase in risky capital on the convergence path contradicts the actual Japanese experience during the lost decade.

We next examine a permanent lower productivity of risky assets. Our model predicts a drop in risky capital and a lower interest rate on impact. In the long run, investors become poorer which further decreases the stock of risky assets and increases the interest rate. The decreasing wealth is at odds with the data.

The dynamics generated by a permanent positive shock on risk decreases the stock of risky capital on impact. As risk increases, risky capital becomes less attractive and investors allocate more of their wealth to safe assets, which decreases the interest rate. They thus start accumulating more wealth because of the income effect. Thus, the stock of risky capital increases in the long run. To be reconciled with the data, we would need a steady rise in volatility. This explanation is, however, difficult to reconcile with Figure 7, which shows that although the volatility of the Nikkei initially increased in 1990, it then quickly faded away.

VI. Policy Implications

The safety trap mechanism relies on a market failure between agents who coordinate on the safe equilibrium. This section discusses two policies that can potentially crowd out the safe equilibrium: the supply of bonds and transfers.

The existence of the safe equilibrium relies on the possibility of a low interest rate since it increases risk aversion. By supplying a sufficient amount of public bonds $b^G > 0$, the government can sustain a minimum interest rate and hence crowd out the safe equilibrium, as represented in Figure 13. Indeed, in that context, the equilibrium investment in the safe technology z is equal to $\omega - k - b^G$, which means that the equilibrium interest rate is at least $R(\omega - b^G)$. By maintaining a lower bound on the interest rate, the government prevents risk aversion from being too high, and therefore crowds out the safe equilibrium. This policy implication is reminiscent of Woodford (1990) and Holmström and Tirole (1997)'s conclusion that public liquidity can crowd in investment.

However, this prediction has to be taken with a grain of salt. Our model points to the need to guarantee minimal returns to investors but does not incorporate the issue that firms or borrowers would benefit from a lower interest rate. This trade-off is absent in our framework and would require a more complete model to be addressed. Besides, this is only a partial equilibrium analysis. In general equilibrium, the source of financing for government debt would be crucial. If government debt was financed exclusively through taxes on investors, then the net effect of government debt would be neutral, because the Ricardian equivalence would hold in such a model. If it was financed through taxes on both investors and workers, then government debt, which is held by investors only, could have real effects.

Another possible policy that would help crowd out the safe equilibrium is to make transfers to investors. To see that, consider the budget constraint (9), where we include constant transfers τ from the government:

$$d\omega_t = (r_t k_t + R_t(z_t + b_t) - \hat{c}_t)dt + \sigma k_t dv_t$$

where $\hat{c}_t = c_t - \tau$. Instantaneous utility can be rewritten as $\ln(\hat{c}_t - \hat{c})$, with $\hat{c} = \tilde{c} - \tau < \tilde{c}$. The program of investors is the same as before, but with a lower apparent reference consumption. This is because part of the actual reference consumption is guaranteed to households by government transfers. As illustrated in Figure 13, the government can crowd out the safe equilibrium with sufficient transfers.

Again, one has to be cautious about this prediction, because it is a partial equilibrium analysis. For transfers to investors to have a real effect, it would be important that they are financed at least partially by other sources than investors themselves. An alternative could be a redistributive tax from rich to poor investors that would guarantee that their revenues do not go below a certain level. Such an optimal taxation problem is, however, beyond the scope of this paper.

VII. Equilibrium Stability

One possible limitation of our model is that only the standard equilibrium is locally stable under a market process based on Walrasian tâtonnement. We show in this section, however, that local stability can be obtained with a more general market process in which the convergence to equilibrium is costly.¹² Although we are able to show that our safe equilibrium can be locally stable under some conditions, we acknowledge that this may not always be true.

For a given period t , aggregate wealth ω_t is fixed. Aggregate investment is determined through an intra-period market process, starting from an arbitrary initial value. The stability or instability of a given equilibrium depends on the specific market process. We first show that in a world with Walrasian tâtonnement, only the equilibrium $k^H(\omega, \sigma)$ is stable. The given initial aggregate level of risky capital k_{init} defines the aggregate returns $r(k_{init})$ and $R(\omega - k_{init})$. Agents then adjust their demand for these given prices until an equilibrium is reached. This demand then evolves according to:

$$\dot{k} + k = D(k)$$

where $D(k)$ refers to $D(k, \omega, \sigma)$.

When we linearize this dynamic equation around $k^i(\omega, \sigma)$, $i \in \{L, H\}$, we obtain:

$$\frac{(k - k^i(\omega, \sigma))}{k - k^i(\omega, \sigma)} = D'(k^i(\omega, \sigma)) - 1$$

with $D'(k^H(\omega, \sigma)) - 1 < 0$ and $D'(k^L(\omega, \sigma)) - 1 > 0$, which means that only $k^H(\omega, \sigma)$ is a stable equilibrium. But this dynamics hinges on the hypothesis that agents incur no cost when they change their plans.

We now assume that agents have a convex disutility both when they do not satisfy their notional demand and when they adjust it and show that $k^L(\omega, \sigma)$ becomes locally stable under certain conditions. We assume that agent j has the following intra-period objective:

$$(27) \quad \min_{k_s^j, s > 0} \int_0^\infty \left[\frac{1}{2} (D(k_s) - k_s^j - \dot{k}_s^j)^2 + C(\dot{k}_s^j, k_s) \right] ds,$$

where $C(\dot{k}^j, k)$ is strictly convex in \dot{k}^j and k , with $C_{12} > 0$ and $C(0, k) = 0$ for all k , where k refers to the aggregate stock of capital and k^j to the individual stock of capital. The first term corresponds to the disutility agents get from not satisfying their notional demand $D(k)$, which depends on the aggregate level of risky capital through prices. The second term C is an external diseconomy of scale. It can be seen as the result of a matching friction in the market for risky

¹²See also Graham and Temple (2006).

capital: when the aggregate demand for capital is large, it becomes more and more difficult for investors to adjust their capital stock. We discard the discount rate because the market process takes place within one period. A stationary solution to this problem must satisfy $D(k) = k$, which is the case for $k = k^H(\omega, \sigma)$ and $k = k^L(\omega, \sigma)$.

We establish the following proposition:

PROPOSITION 7 (Equilibrium stability): *Consider the market process that solves Equation (27):*

- (i) *If $C = 0$, then $\dot{k} = D(k) - k$;*
- (ii) *if $C_{12}(0, k^L) < D'(k^L)$, then only k^H is locally stable;*
- (iii) *if $C_{12}(0, k^L) > D'(k^L)$, then both k^H and k^L are locally stable.*

Part (i) establishes that the Walrasian tâtonnement is the solution to the households' intra-period objective when there is no matching friction. Parts (ii) and (iii) state that k^H is a locally stable equilibrium, independently of the matching friction, but k^L becomes stable if diseconomies of scale are strong enough. However, k^L never exhibits saddle-path stability, and the market dynamics around k^L can be quite complicated, as illustrated in Figure 14. We assume in the paper that the condition that ensures the stability of k^L , that is $C_{12}(0, k^L) > D'(k^L)$, is satisfied for all the relevant values of ω .

VIII. Conclusion

In this paper, we study the role of risk aversion as a determinant of poor economic performance. Our general mechanism is that risk averse investors invest in safer and less profitable projects, which generates a lower TFP. We build a business cycle model with endogenous risk aversion and portfolio choice and illustrate it with Japan's lost decade.

Our model predicts a hump-shaped demand for risky capital. For low interest rates, investors become very risk averse and thus demand safer assets while for high interest rates investing in risky capital is not profitable anymore. This implies that our model can have two equilibria. The first one is standard and corresponds to the Ramsey solution. In the second equilibrium, the safe one, investors are more risk-averse and as a consequence consume less and invest in safer assets. This second equilibrium generates a safety trap because investors increasingly invest in safe assets even though they become wealthier.

The predictions of the model when the economy is in a safety trap are consistent with the empirical evidence on Japan's lost decade. In a switch from the standard to the safe equilibrium, consumption, TFP, and the interest rate decrease, and investors invest a larger share of their wealth in safe assets, the more so, the higher their wealth.

More generally, our model provides a framework to think about episodes of poor economic performance that are difficult to reconcile with parallel changes in fundamentals and that are thought to be related to fear of risk.

Finally, our mechanism relies on exogenous switches between the two equilibria, that reflect changes in anticipations and which we could interpret as animal spirits. However, we do not provide a more fundamental explanation for why these switches happen and we believe this could be an interesting avenue for future research.

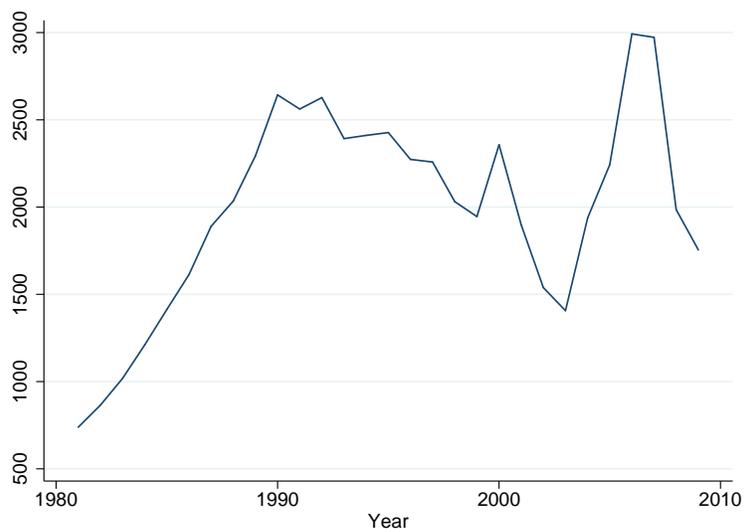
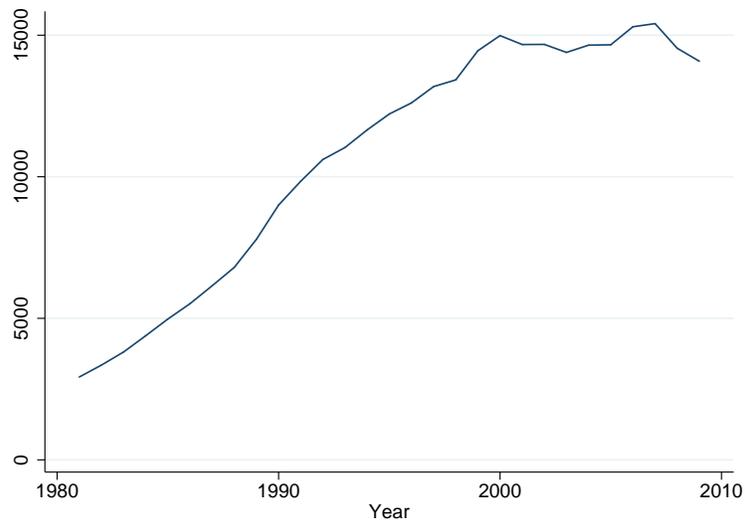


FIGURE 1. STOCK OF RISKY ASSETS HELD BY JAPANESE HOUSEHOLDS

Note: Risky assets include the categories “Shares and other equities” and “Securities other than shares”. Unit: trillion real yen. Source: Bank of Japan. Real yen are yen adjusted for inflation as measured by the GDP deflator.
Source: Bank of Japan and WDI.

FIGURE 2. TOTAL ASSETS HELD BY JAPANESE HOUSEHOLDS



Note: Unit: trillion real yen. Source: Bank of Japan. Real yen are yen adjusted for inflation as measured by the GDP deflator.
Source: Bank of Japan and WDI.

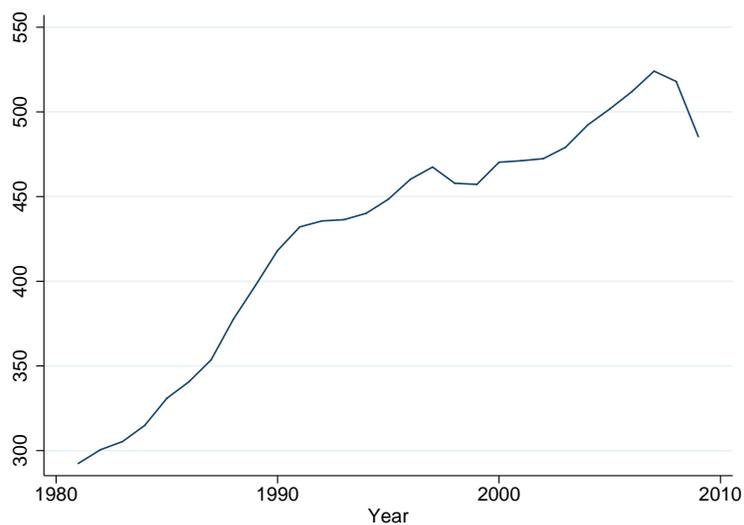


FIGURE 3. GDP OF JAPAN

Note: Unit: trillion real yen. Real yen are yen adjusted for inflation as measured by the GDP deflator.

Source: WDI.

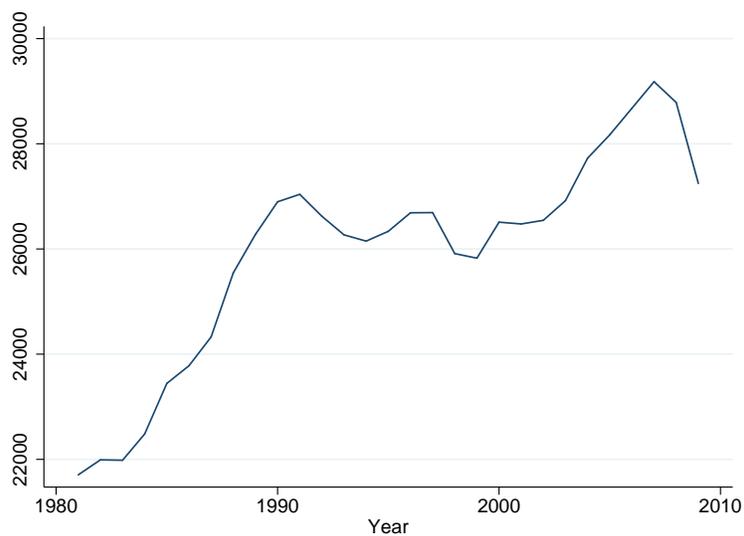


FIGURE 4. TFP OF JAPAN

Note: TFP is computed using the formula: $\ln(\text{TFP}) = \ln(\text{GDP}) - .33\ln(K) - .67\ln(L)$, where GDP is in constant yen; K is derived from the formula $K_{t+1} = .93K_t + I_t$, where I is the gross capital formation in constant yen, $K_{1970} = I_{1970}/.07$, and L is the total labor force.

Source: WDI.

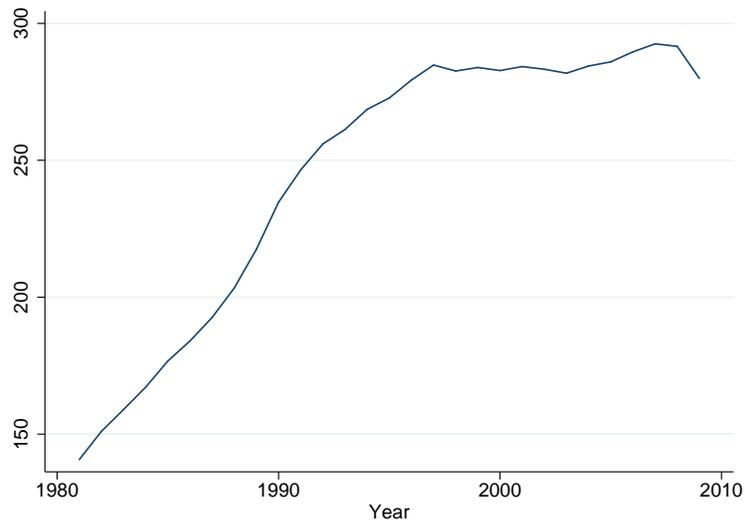


FIGURE 5. HOUSEHOLD FINAL CONSUMPTION EXPENDITURE IN JAPAN

Note: Unit: trillion real yen. Real yen are yen adjusted for inflation as measured by the GDP deflator.

Source: WDI.

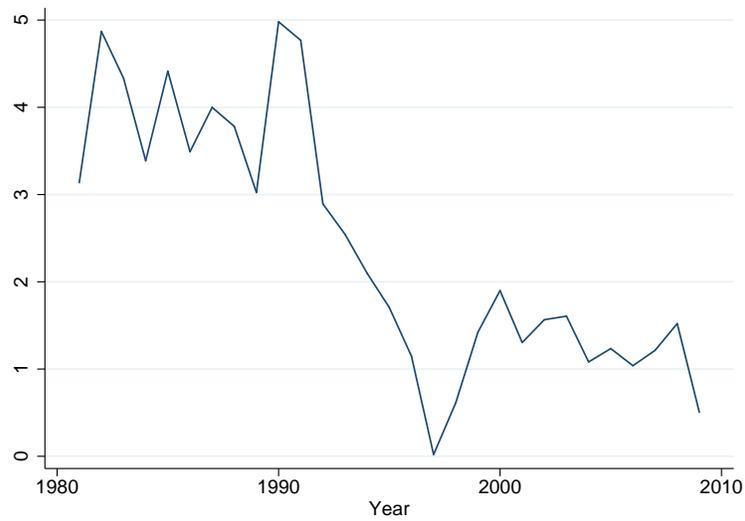


FIGURE 6. REAL INTEREST RATE IN JAPAN

Note: Unit: %. This is the annual average real interest rate over all the certificates of deposits reported by banks to the Bank of Japan. The real value is obtained using the GDP deflator.

Source: WDI and Bank of Japan.

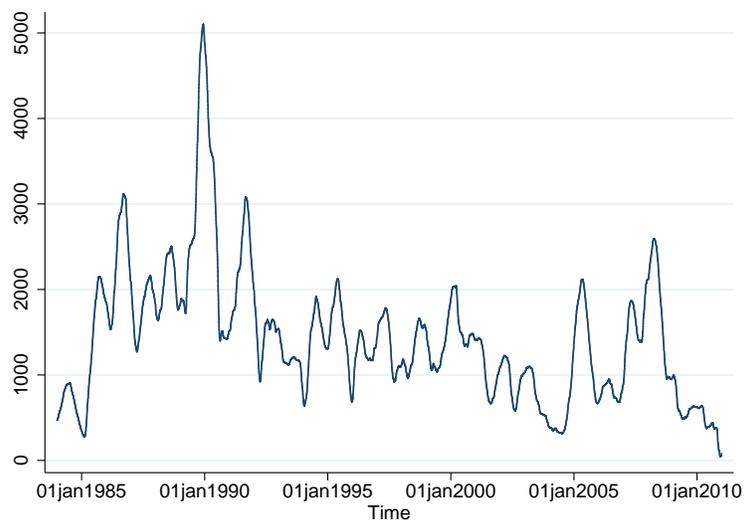


FIGURE 7. VOLATILITY OF THE NIKKEI

Note: Standard deviation of the Nikkei over the past year.

Source: Yahoo finance.

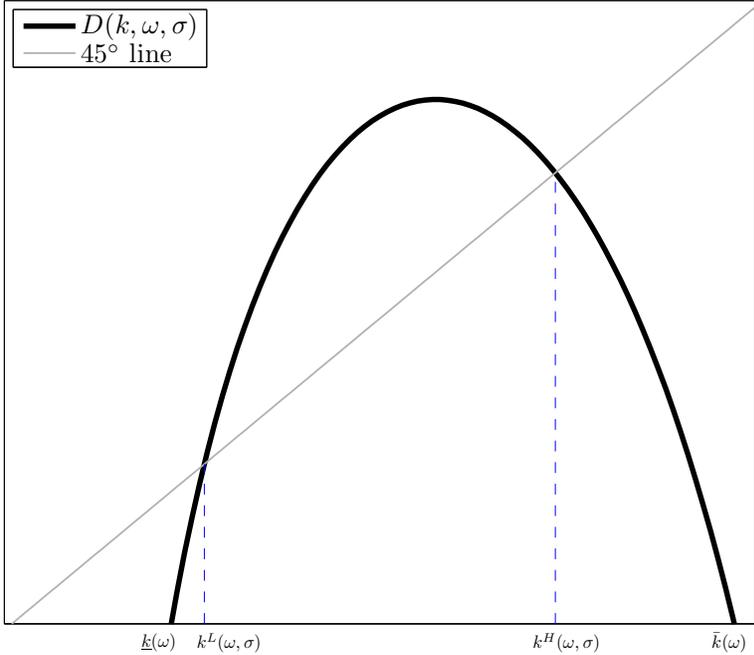


FIGURE 8. THE EQUILIBRIUM STOCK OF RISKY CAPITAL

Note: $D(k, \omega, \sigma)$ is the demand for risky capital k as a function of actual aggregate risky capital k , wealth ω and risk σ . The equilibrium stock of risky capital are the solutions to $k = D(k, \omega, \sigma)$.

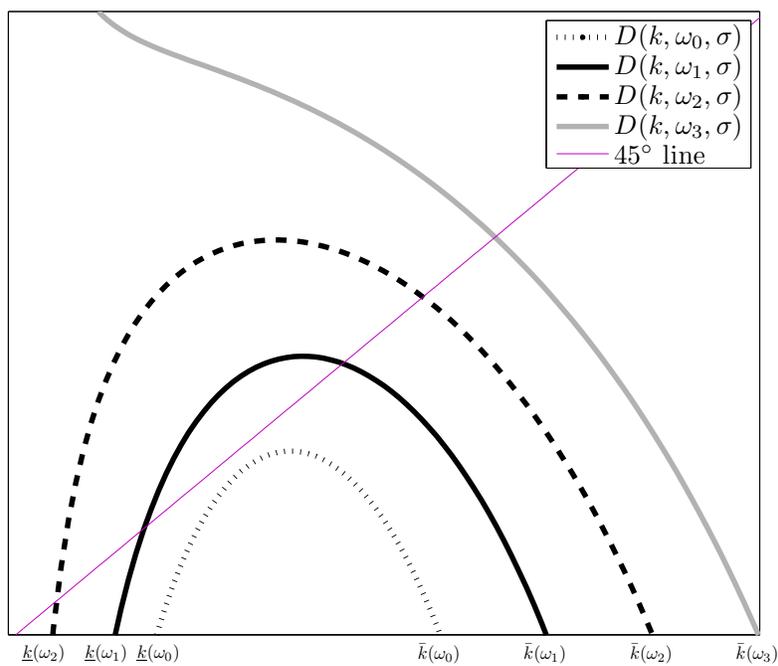


FIGURE 9. THE DEMAND FOR RISKY CAPITAL

Note: $D(k, \omega, \sigma)$ is the demand for risky capital k as a function of risk σ and for different values of wealth ω , with $\omega_0 < \omega_1 < \omega_2 < \omega_3$. The equilibrium stock of risky capital are the solutions to $k = D(k, \omega, \sigma)$.

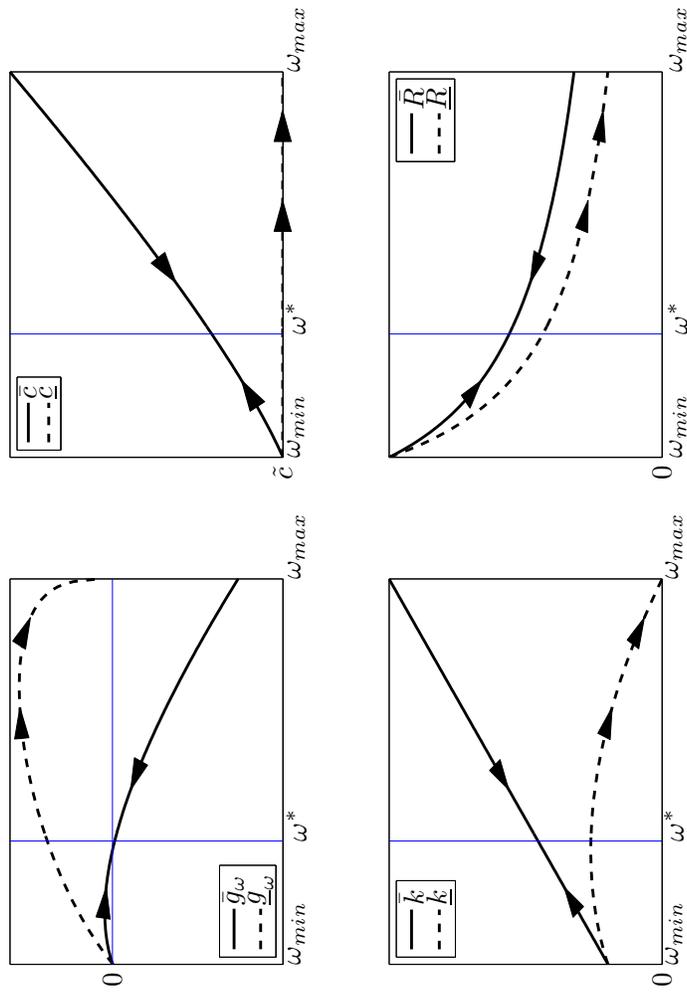


FIGURE 10. DYNAMICS

Note: We represent the growth rate of wealth g_ω , consumption c , and the interest rate R as a function of wealth ω . The superscripts H and L refer respectively to the standard and safe equilibria.

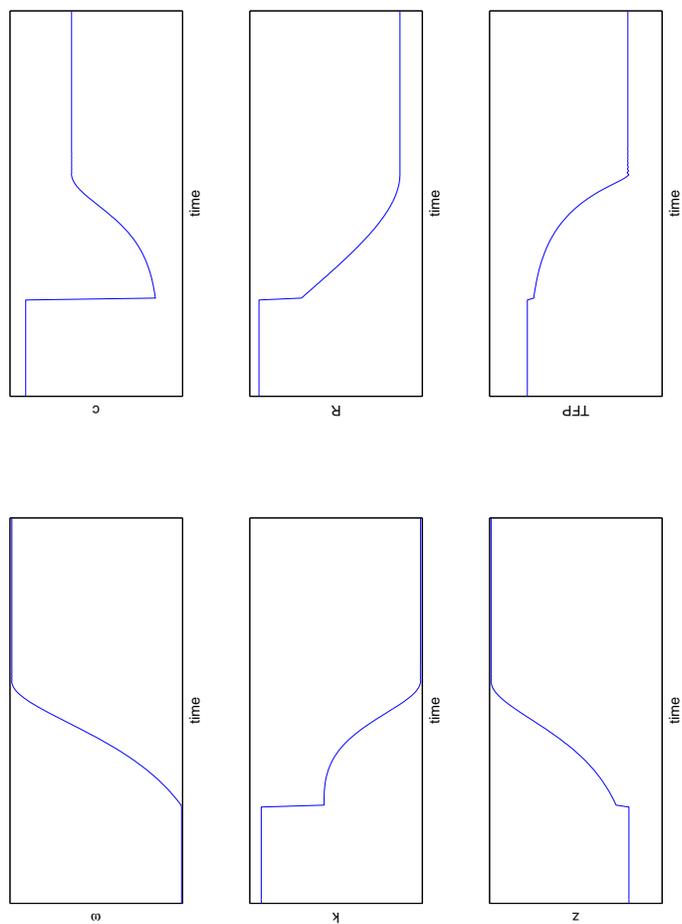


FIGURE 11. SIMULATION OF A SWITCH FROM THE STANDARD TO THE SAFE EQUILIBRIUM

Note: The simulations represent the evolution of the wealth ω , the consumption c , the stock of risky capital k , the interest rate R , the stock of safe capital z , and the TFP .

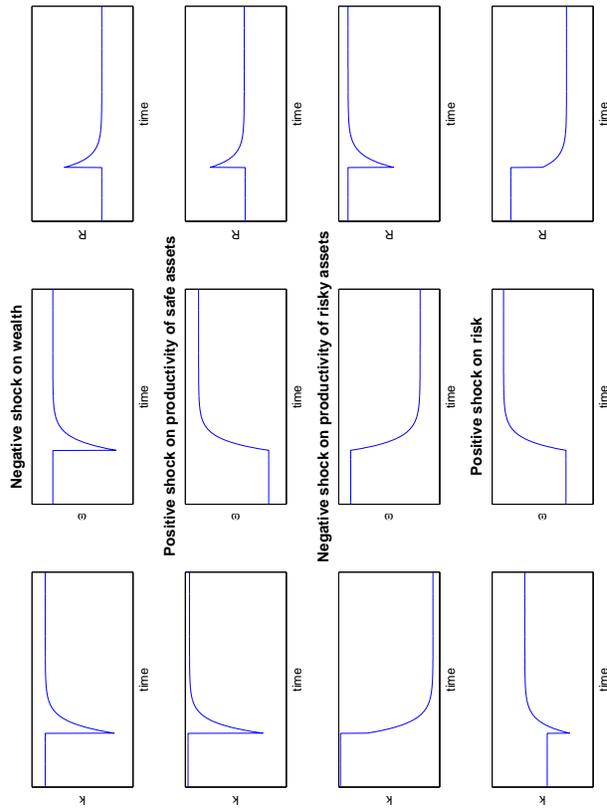


FIGURE 12. SIMULATION OF DIFFERENT SHOCKS IN THE STANDARD EQUILIBRIUM

Note: The simulations represent the evolution the stock of risky capital k , the wealth ω , and the interest rate R .

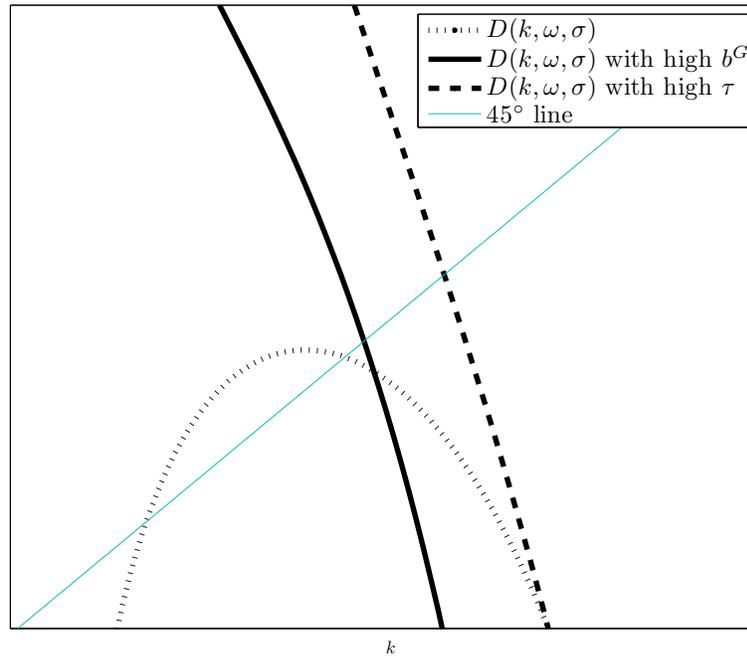
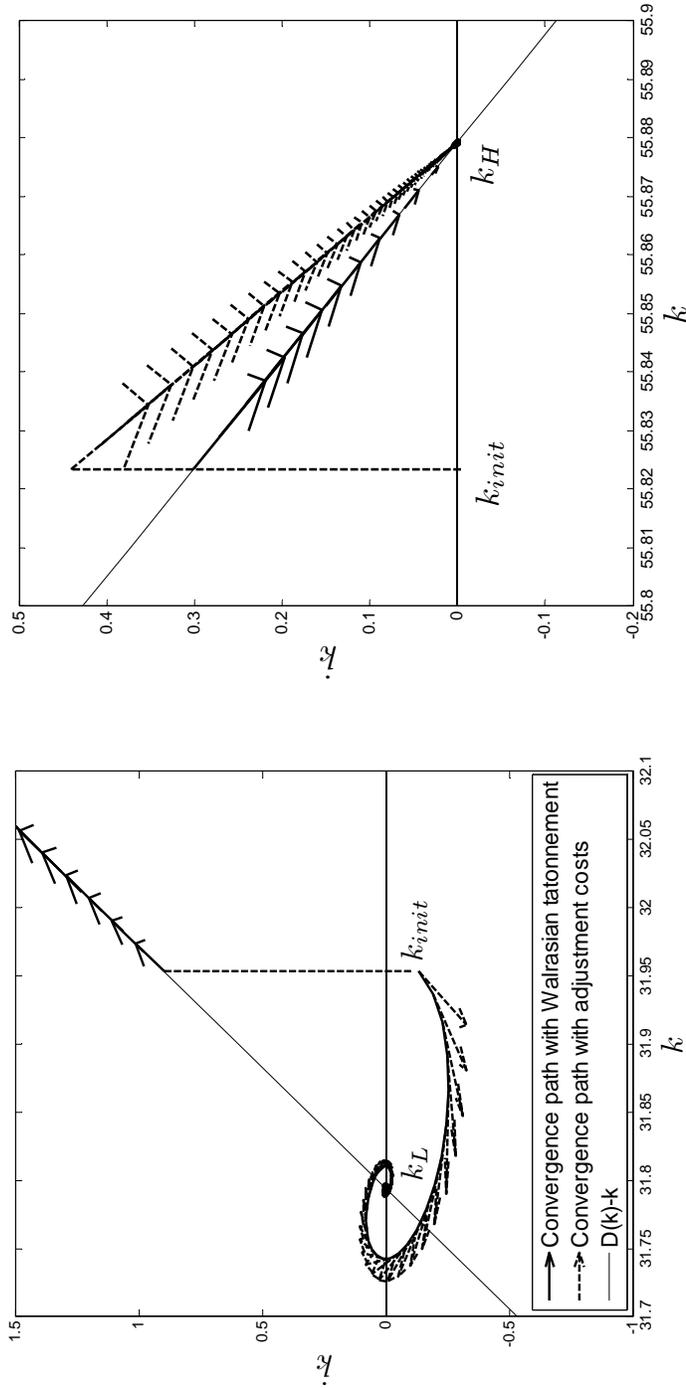


FIGURE 13. POLICY ANALYSIS

Note: In the “High b ” case, $D(k, \omega, \sigma)$ is the demand for risky capital k as a function of wealth ω and risk σ , when government supplies a positive amount of government bonds b^G . In the “High τ ” case, $D(k, \omega, \sigma)$ is the demand for risky capital k as a function of wealth ω and risk σ , with $\bar{c} - \tau$ the reference level of consumption.



(b) Local stability of k^H

(a) Local stability of k^L

Notes: Convergence to (or divergence from) k^L (or k^H) when starting from k^{init} close to k^L (or k^H). Arrows denote velocity vectors.

FIGURE 14. EQUILIBRIUM STABILITY

TABLE 1—COMPARING THE DATA AND THE IMPLICATIONS OF THE MODEL.

	(1)		(2)		(3)	
	Data		Model		Model w/ switching	
	1981-89	1990-99	Standard	Safe	Standard	Safe
$Corr(\omega, k)$.99*	-.90*	+	-	+	-
$Corr(\omega, TFP)$.97*	-.71*	+	-	+	-
$Corr(\omega, R)$	-.30	-.92*	-	-	-	-
$Corr(\omega, c)$.97*	.97*	+	-	+	+

Note: * $p < 0.05$

The model correlations are the correlations over the convergence path, starting from a level of initial wealth ω_0 equal to 90% of the steady-state level. The convergence is simulated over 30 years. ω refers to the log of the total assets held by Japanese households in real yen. k refers to the logarithm of the stock of risky assets held by Japanese households in real yen. Risky assets include the categories “Shares and other equities” and “Securities other than shares”. R refers to the annual real interest rate of Japan in %. It is the annual real average interest rate over all the certificates of deposits reported by banks to the Bank of Japan. c refers to the the log of the household final consumption expenditure in Japan in real yen. Real values are adjusted for inflation as measured by the GDP deflator. TFP is computed using the formula: $TFP = \ln(GDP) - .33\ln(K) - .67\ln(L)$, where GDP is in real yen; K is derived from the formula $K_{t+1} = .93K_t + I_t$, where I is the gross capital formation in real yen, $K_{1970} = I_{1970}/.07$, and L is the total labor force.

Source: Bank of Japan and WDI.

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Appendix: Proofs

Proof of Proposition 1

Let $W(\omega, R, r)$ be the value function of the agent. R and r evolve according to $\partial R = \mu_R(t)\partial t + \sigma_R(t)\partial dv$ and $\partial r = \mu_r(t)\partial t + \sigma_r(t)\partial dv$. The parameters $\mu_R(t)$, $\mu_r(t)$, $\sigma_R(t)$ and $\sigma_r(t)$ might depend on the equilibrium values for k and z but the agents are infinitesimal so they do not take this into account for their maximization problem.

We first assume that $c > \tilde{c}$ so the value function is well-defined and that the non-negativity constraint on k is satisfied. Using Ito's Lemma, we can derive the following Bellman equation:

$$\rho W(\omega, R, r) = \max_{c, k} \left\{ \ln(c - \tilde{c}) + \frac{\partial W}{\partial \omega}(rk + R(\omega - k) - c) + \frac{\partial W}{\partial R}\mu_g(t) + \frac{\partial W}{\partial r}\mu_f(t) \right. \\ \left. + \frac{\partial^2 W}{\partial \omega \partial R}\sigma\sigma_R k + \frac{\partial^2 W}{\partial \omega \partial r}\sigma\sigma_r k + \frac{\partial^2 W}{\partial \omega^2}\frac{\sigma^2 k^2}{2} + \frac{\partial^2 W}{\partial R^2}\frac{\sigma_R^2}{2} + \frac{\partial^2 W}{\partial r^2}\frac{\sigma_r^2}{2} \right\}$$

The first-order conditions with respect to c and k associated with this Bellman equation are:

$$\frac{1}{c - \tilde{c}} = \frac{\partial W}{\partial \omega}$$

$$\frac{\partial W}{\partial \omega}(r - R) + \frac{\partial^2 W}{\partial \omega \partial R}\sigma\sigma_R + \frac{\partial^2 W}{\partial \omega^2}\sigma^2 k = 0$$

An educated guess of the general form of the value function is:

$$W = \alpha \ln(\omega - \tilde{\omega}) + \beta(R, r),$$

The parameters α and $\tilde{\omega}$, along with Equations (11) and (12), are found by substituting the value function into the first-order conditions:

$$\alpha = 1/\rho,$$

$$\tilde{\omega} = \tilde{c}/R.$$

Second, we examine whether the value function is well-defined over a relevant set of prices. If $R > \tilde{c}/\omega$, then $\omega > \tilde{\omega}$, which implies that the value-function is well-defined and that $c > \tilde{c}$. Besides, $\omega > \tilde{\omega}$ implies that the demand for capital is of the same sign as $r - R$. Therefore, if $r \geq R$, then $k \geq 0$, which means that the non-negativity constraint on k is satisfied. As a result, $R \in (\tilde{c}/\omega, r]$ ensures that the solution is interior and therefore that the policy functions are well-defined.

Proof of Lemma 1

PROOF OF (I) Consider $\omega > \omega_{min}$.

First, according to Assumption 1, $r(k) - R(\omega - k)$ is strictly decreasing on $[0, \omega]$, with $r(k) - R(\omega - k)$ strictly negative for $k = 0$ and strictly positive for $k = \omega/2$. There is therefore a unique $\bar{k}(\omega) > 0$ that satisfies $r(\bar{k}(\omega)) - R(\omega - \bar{k}(\omega)) = 0$.

Second, since, according to Assumption 1, R is strictly decreasing, then $R(\omega - \underline{k}(\omega))\omega - \tilde{c} = 0$ defines $\underline{k}(\omega)$ uniquely as $\omega - R^{-1'}(\tilde{c}/\omega)$.

We now prove $\bar{k}(\omega) > \underline{k}(\omega)$. Since $r(\bar{k}(\omega)) = R(\omega - \bar{k}(\omega))$, then $R(\omega - \bar{k}(\omega))\omega = r(\bar{k}(\omega))\bar{k}(\omega) + R(\omega - \bar{k}(\omega))(\omega - \bar{k}(\omega))$. Given that $r(k)k$ and $R(z)z$ are increasing functions, and that $\bar{k}(\omega)$ and $\omega - \bar{k}(\omega)$ are both increasing in ω , this implies that $R(\omega - \bar{k}(\omega))\omega$ is increasing in ω . Therefore, $R(\omega - \bar{k}(\omega))\omega > R(\omega_{min} - \bar{k}(\omega_{min}))\omega_{min}$. Using the definition of ω_{min} given in Assumption 1, we derive: $R(\omega_{min} - \bar{k}(\omega_{min}))\omega_{min} = \tilde{c}$. As a result, we have $R(\omega - \bar{k}(\omega))\omega > \tilde{c}$. Besides, by definition of $\underline{k}(\omega)$, $\tilde{c} = R(\omega - \underline{k}(\omega))\omega$. Consequently, we have $R(\omega - \bar{k}(\omega))\omega > R(\omega - \underline{k}(\omega))\omega$, which is equivalent to $R(\omega - \bar{k}(\omega)) > R(\omega - \underline{k}(\omega))$. Since, according to Assumption 1, R is a strictly decreasing function, this implies that $\bar{k}(\omega) > \underline{k}(\omega)$.

PROOF OF (II) AND (III) We first have to prove that there exists a unique ω_{max} such that $\underline{k}(\omega_{max}) = 0$ and that $\omega_{max} > \omega_{min}$. $\underline{k}(\omega_{max}) = 0$ is equivalent to $R(\omega_{max})\omega_{max} = \tilde{c}$. According to Assumption 1, $R(\omega)\omega$ is a strictly increasing function with values between 0 and $+\infty$, so there exists a unique ω_{max} satisfying $\underline{k}(\omega_{max}) = 0$.

We now prove that $\omega_{max} > \omega_{min}$. ω_{min} must satisfy $R(\omega_{min} - \bar{k}(\omega_{min}))\omega_{min} = \tilde{c}$. Besides, according to Assumption 1, and given that $\bar{k}(\omega_{min})$ is uniquely defined, we have $\bar{k}(\omega_{min}) \geq 0$. Since R is a decreasing function, this implies $R(\omega_{min})\omega_{min} < R(\omega_{min} - \bar{k}(\omega_{min}))\omega_{min}$. Finally, since $R(\omega_{min} - \bar{k}(\omega_{min}))\omega_{min} = \tilde{c} = R(\omega_{max})\omega_{max}$, then we have $R(\omega_{min})\omega_{min} < R(\omega_{max})\omega_{max}$, which imposes $\omega_{max} > \omega_{min}$.

Now, we show that $k \in [0, \bar{k}(\omega)] \cap (\underline{k}(\omega), \bar{k}(\omega)]$ is a necessary condition for D to be well-defined, that is, for the condition $R(\omega - k) \in (\tilde{c}/\omega, r(k)]$ to be satisfied. $\bar{k}(\omega)$ is the solution to $r(k) - R(\omega - k) = 0$, with $r(k) - R(\omega - k)$ strictly decreasing in k . Therefore, $r(k) \geq R(\omega - k)$ imposes $k \leq \bar{k}(\omega)$. $\underline{k}(\omega)$ is the solution to $R(\omega - k)\omega - \tilde{c} = 0$, with $R(\omega - k)\omega - \tilde{c}$ strictly increasing in k . Therefore, $R(\omega - k)\omega > \tilde{c}$ imposes $k > \underline{k}(\omega)$. The condition $k \geq 0$ that D is therefore well-defined for $k \in [0, \bar{k}(\omega)] \cap (\underline{k}(\omega), \bar{k}(\omega)]$.

Finally, we have $\partial \underline{k} / \partial \omega(\omega_{max}) = 1 - \epsilon_z(\omega_{max})$, where $\epsilon_z(z) = R(z) / |R'(z)z|$. Since $R(z)z$ is increasing in z according to Assumption 1, we should have $R(z) + R'(z)z > 0$, which implies that $\epsilon_z > 1$, so $\partial \underline{k} / \partial \omega(\omega_{max}) < 0$. By continuity, since ω_{max} is the only solution to $\underline{k}(\omega) = 0$, a level of $\omega > \omega_{max}$ would then imply $\underline{k}(\omega) < 0$. Conversely, $\omega < \omega_{max}$ would imply $\underline{k}(\omega) > 0$. Therefore, $k \geq 0$ that D is well-defined for $k \in [0, \bar{k}(\omega)] \cap (\underline{k}(\omega), \bar{k}(\omega)] = (\underline{k}(\omega), \bar{k}(\omega)]$ when $\omega < \omega_{max}$ and for $k \in [0, \bar{k}(\omega)] \cap (\underline{k}(\omega), \bar{k}(\omega)] = [0, \bar{k}(\omega)]$ when $\omega \geq \omega_{max}$.

Proof of Proposition 2

PROOF OF (II) The limit solutions when σ goes to zero must be the solutions to $D(k, \omega, 0) = k$, which, according to Equation (19), is equivalent to $\Delta(k, \omega)/\gamma(k, \omega)\omega = \sigma^2 k = 0$. Since $\omega > \omega_{min} \geq 0$, they should be the solutions to $\Delta(k, \omega)/\gamma(k, \omega) = 0$. $\bar{k}(\omega)$ and $\underline{k}(\omega)$ correspond respectively to $\Delta(k) = 0$ and $1/\gamma(k) = 0$. Besides, according to Lemma 1, $\bar{k}(\omega)$ and $\underline{k}(\omega)$ are uniquely defined for $\omega > \omega_{min}$. If, additionally, $\omega \leq \omega_{max}$, then $\underline{k}(\omega)$ satisfies the non-negativity constraint (see Lemma 1).

PROOF OF (I) Equation (20) is equivalent to:

$$(A1) \quad \sigma^2 k = \left(r(k) - R(\omega - k) \right) \left(\omega - \frac{\tilde{c}}{R(\omega - k)} \right).$$

According to (i), for a given $\omega \in (\omega_{min}, \omega_{max}]$, the two limit solutions when σ goes to zero are $\bar{k}(\omega)$ and $\underline{k}(\omega)$. Given that $r(k) - R(\omega - k)$ is decreasing in k and that $\omega - \frac{\tilde{c}}{R(\omega - k)}$ is increasing in k , the right-hand side of Equation (A1) is strictly positive if and only if $\underline{k}(\omega) < k < \bar{k}(\omega)$, where, according to Lemma 1, the demand for k is well-defined. By continuity, for a small enough σ , there exist two interior solutions within $(\underline{k}(\omega), \bar{k}(\omega))$.

Proof of Proposition 3

PROOF OF (I) Differentiating Equation (22) with respect to ω yields $\partial \bar{k} / \partial \omega = R' / (r' + R') > 0$. This proves (i).

PROOF OF (II) Differentiating (23), we obtain $\partial \underline{k} / \partial \omega = 1 + R(\underline{z}) / \omega R'(\underline{z}) = 1 - \epsilon_z(\underline{z}) \underline{z} / \omega$, with $\epsilon_z(z) = R(z) / |R'(z)z| > 1$. First, it implies that:

$$\frac{\partial(\underline{k}/\omega)}{\partial \omega} = \frac{\partial \underline{k} / \partial \omega}{\omega} - \frac{\underline{k}}{\omega^2} = -\frac{1}{\omega} (\epsilon_z(\underline{z}) - 1) \left(\frac{\underline{z}}{\omega} \right) < 0$$

Second, since, $\underline{k}(\omega_{max}) = 0$, it implies that $\partial \underline{k}(\omega_{max}) / \partial \omega = 1 - \epsilon_z(\omega_{max}) < 0$. This proves (ii).

Proof of Proposition 4

PROOF OF (I) First, we have shown in the proof of Proposition 3 that $\partial \bar{k} / \partial \omega = R' / (r' + R') \geq 0$, which implies that $\partial \bar{z} / \partial \omega = r' / (r' + R') \geq 0$. \bar{z} is then increasing in ω . Since $\bar{R} = R(\bar{z})$ is a decreasing function of \bar{z} , then \bar{R} is increasing in ω .

Second, using $\partial\bar{z}/\partial\omega = r'/(r' + R')$, we get $\partial\bar{c}/\partial\omega = \rho \left(1 + \frac{\bar{c}}{\bar{z}} \frac{r'R'}{(r'+R')R^2}\right) = \rho \left(1 - \frac{\bar{c}}{\bar{z}} \frac{|r'R'|}{|r'+R'|R^2}\right)$. Therefore, \bar{c} is increasing in ω if $\bar{c} < \frac{R^2|r'+R'|}{|r'R'|}$.

Third, by definition, we have $\bar{r} = \bar{R}$, which means that $f'(\bar{k}) = g'(\omega - \bar{k})$. This means that $\bar{k}(\omega)$ and $\bar{z}(\omega) = \omega - \bar{z}(\omega)$ maximize aggregate production for ω . We therefore have $y^*(\omega) = f(\bar{k}(\omega)) + g(\bar{z}(\omega)) = \bar{y}(\omega)$, and $\bar{y} - y^* = 0$, which is independent of ω .

PROOF OF (II) First, $\partial\bar{z}/\partial\omega = 1 - \partial\bar{k}/\partial\omega$. According to Proposition 3, we have $\partial\bar{k}/\partial\omega < 0$ in the neighborhood of ω_{max} , so $\partial\bar{z}/\partial\omega > 0$.

Second, when $k = \bar{k}$, we have $c = \bar{c}$, which is independent of ω .

Third, when $k = \bar{k}$, we have $\partial(y - y^*)/\partial\omega = (\underline{r} - \underline{R})\partial\bar{k}/\partial\omega + (\underline{R} - \bar{R})$, where we have used the fact that $y^* = \bar{y}$ and $\bar{r} = \bar{R}$. We know that $\underline{r} - \underline{R} > 0$ and we have shown in Proposition 3 that $\partial\bar{k}/\partial\omega < 0$ in the neighborhood of ω_{max} , so the first term is strictly negative. Similarly, $\bar{k} > \underline{k}$, so $\bar{z} < \underline{z}$ and $\bar{R} > \underline{R}$. As a result, the second term is negative as well. Therefore, $\partial(y - y^*)/\partial\omega < 0$ in the neighborhood of ω_{max} .

Proof of Proposition 5

PROOF OF (I) We first show that ω_{min} is the unique steady state such that $\bar{R}(\omega)\omega = \bar{c}$. Since $\bar{r}(\omega) = \bar{R}(\omega)$, we have $\bar{R}(\omega)\omega = \bar{r}(\omega)\bar{k}(\omega) + \bar{R}(\omega)\bar{z}(\omega)$. Given that $r(k)k$ and $R(z)z$ are increasing functions, this implies that $\bar{R}(\omega)\omega = r(\bar{k}(\omega))\bar{k}(\omega) + R(\bar{z}(\omega))\bar{z}(\omega)$ is increasing in ω . Besides, for $\omega = 0$, the non-negativity constraint on k and z implies that $k = z = 0$, so $\bar{R}(0)0 = f(\bar{k}(0))\bar{k}(0) + g(\bar{z}(0))\bar{z}(0) = 0$. Since $\bar{k} \geq 0$ and R is decreasing, we can write $\bar{R}(\omega)\omega = R(\omega - \bar{k}(\omega))\omega \geq R(\omega)\omega$, which goes to $+\infty$ when ω goes to $+\infty$. Therefore, $\bar{R}(\omega)\omega$ goes to infinity as well. To summarize, $\bar{R}(\omega)\omega$ is increasing with values between 0 and $+\infty$ for ω between 0 and $+\infty$, so ω_{min} is the unique steady state such that $\bar{R}(\omega)\omega = \bar{c}$.

Second, $\bar{R}(\omega) = R(\bar{z}(\omega))$ is decreasing with value between 0 and ∞ . Therefore, there exists a unique steady state ω^* such that $\bar{R}(\omega^*) = \rho$.

If $\omega^* \geq \omega_{min}$, ω_{min} and ω^* are the two steady states and only ω^* is stable. Indeed, for $\omega_{min} < \omega < \omega^*$, $\bar{R}(\omega)\omega > \bar{c}$ and $\bar{R}(\omega) > \rho$, so $\dot{\omega} > 0$. For $\omega > \omega^*$, we still have $\bar{R}(\omega)\omega > \bar{c}$ but $\bar{R}(\omega) < \rho$, so $\dot{\omega} < 0$. If $\omega^* < \omega_{min}$, ω_{min} is the unique well-defined steady state. This steady state is stable since $\bar{R}(\omega)\omega > \bar{c}$ and $\bar{R}(\omega) < \rho$ for $\omega > \omega_{min}$.

Finally, we check whether the following transversality condition is satisfied:

$$(A2) \quad \lim_{t \rightarrow \infty} E_t \omega_t u'(c_t) e^{-\rho t} = 0$$

In the standard regime, when the steady state is w^* , c_t converges to a level that is strictly higher than \bar{c} , so the marginal utility converges to a finite level,

which ensures that the transversality condition is satisfied. When the steady state is ω_{min} , c converges to \tilde{c} . For the transversality condition to be satisfied, the marginal utility must diverge to infinity at a lower rate than ρ . In order to infer the growth rate of marginal utility, consider the Euler equation with respect to safe capital:

$$(A3) \quad \frac{E_t \partial u'(c_t) / \partial t}{u'(c_t)} = \rho - R_t$$

Since z is finite and strictly positive in steady state, R is strictly positive. Therefore, the convergence rate of marginal utility is strictly lower than ρ . This ensures that the transversality condition holds in the long-run.

PROOF OF (II) Differentiating Equation (24) with respect to σ^2 , we get the following:

$$(A4) \quad \frac{\partial \dot{\omega}}{\partial \sigma^2} = \frac{\partial k}{\partial \sigma^2} \left(r(k) - R(\omega - k) + r'(k)k + R'(\omega - k)(\omega - k) + \frac{\rho \tilde{c} R'(\omega - k)}{R(\omega - k)^2} \right)$$

In the standard regime, for σ close to 0, we have $r(k) = R(\omega - k)$, so this simplifies to:

$$\frac{\partial \dot{\omega}}{\partial \sigma^2} = \frac{\partial k}{\partial \sigma^2} \left(r'(k)k + R'(\omega - k)(\omega - k) + \frac{\rho \tilde{c} R'(\omega - k)}{R(\omega - k)^2} \right) > 0$$

The fact that the steady-state wealth ω^* is increasing in σ when $\omega^* > \omega_{min}$ then derives from the fact that $\dot{\omega}$ is increasing in σ and that ω^* is a stable steady state according to (i). This proves (ii).

A similar argument as in (i) proves that the transversality condition is satisfied.

Proof of Proposition 6

PROOF OF (I) A first possible steady-state is the solution to $\underline{\Delta}(\omega) = 0$. This implies $\underline{R}(\omega)\omega = \tilde{c}$, the solution of which is ω_{min} . The second possible steady-state is the solution to $\underline{k}(\omega) = 0$, which is ω_{max} . These are the two unique steady states.

We have $\underline{k}(\omega) > 0$ and $\underline{\Delta}(\omega) \geq 0$ for all $\omega \in [\omega_{min}, \omega_{max}]$ (see Lemma 1). Therefore, ω_{max} is stable while ω_{min} is not.

PROOF OF (II) We first prove that ω_{max} is still a steady state for $\sigma > 0$. In the safe regime, we cannot derive the sign of the derivative in Equation (A4) for all $\omega > \omega_{min}$. However, for $\omega = \omega_{max}$, $\partial k / \partial \sigma^2 = 0$, so $\partial \dot{\omega} / \partial \sigma^2 = 0$. This proves the first part of (ii). Next, we prove that ω_{max} is an inaccessible boundary in the safe regime. In the neighborhood of $\omega = \omega_{max}$, using $\omega_{max} - \tilde{\omega}(\omega_{max}) = \underline{k}(\omega_{max}) = 0$,

the following approximation holds:

$$\begin{aligned} \partial\omega = & \left[(R(\omega_{max}) - \rho) \left(1 - \frac{\tilde{c}R'(\omega_{max})}{(R(\omega_{max}))^2} \right) \right. \\ & \left. + (r(0) - R(\omega_{max})) \left(1 + \frac{\tilde{c}}{\omega_{max}^2 R'(\omega_{max})} \right) \right] (\omega - \omega_{max}) \partial t \\ & + \sigma \left(1 + \frac{\tilde{c}}{\omega_{max}^2 R'(\omega_{max})} \right) (\omega - \omega_{max}) \partial v \end{aligned}$$

Using:

$$\epsilon_z(\omega_{max}) = -R(\omega_{max})/\omega_{max}R'(\omega_{max}) = -R(\omega_{max})^2/\tilde{c}R'(\omega_{max}) = -\tilde{c}/\omega_{max}^2 R'(\omega_{max}),$$

we derive:

$$\begin{aligned} \partial\omega = & \left[(R(\omega_{max}) - \rho) \left(1 + \frac{1}{\epsilon_z(\omega_{max})} \right) + (r(0) - R(\omega_{max})) (1 - \epsilon_z(\omega_{max})) \right] (\omega - \omega_{max}) \partial t \\ & + \sigma (1 - \epsilon_z(\omega_{max})) (\omega - \omega_{max}) \partial v. \end{aligned}$$

Since $\epsilon_z(\omega_{max}) > 1$, we can write:

$$\partial\omega = \gamma(\omega - \omega_{max})\partial t + \sigma\kappa(\omega - \omega_{max})\partial v$$

where γ and κ are strictly positive constants. This implies:

$$\partial x = \gamma\partial t + \sigma\kappa\partial v$$

with $x = \log(\omega_{max} - \omega)$ defined on $(-\infty, \log(\omega_{min}))$. Cox and Miller (2001) have shown that such a process with $\gamma > 0$ has a non-degenerate steady state and that $-\infty$ is an inaccessible boundary. Since $x = \log(\omega_{max} - \omega)$, this implies that ω_{max} is an inaccessible boundary for ω . This proves the second part of (ii).

In the safe regime, consumption converges to \tilde{c} , which implies that the marginal utility goes to infinity. For the transversality condition to be satisfied, the marginal utility must diverge to infinity at a lower rate than ρ . The dynamics of ω_t is of the form:

$$\partial\omega = R(\omega_t)\omega_t\partial t + [r(\omega_t) - R(\omega_t)]k(\omega_t) + \sigma k(\omega_t)\partial v$$

Using the Euler equation and Ito's Lemma, we can show that the dynamics of $u'(c_t)$ is the following:

$$\partial u'(c_t) = (\rho - R_t)u'(c_t)\partial t + \sigma k(\omega_t)\rho \left(1 + \frac{\tilde{c}R'(z(\omega_t))}{R(z(\omega_t))^2} \right) \partial v$$

Using Ito's Lemma, we can show that:

$$\frac{\partial \omega_t u'(c_t)}{\omega_t u'(c_t)} = \frac{R(\omega_t)}{\omega_t} + \rho - R_t + \sigma^2 k_t^2 \rho \left(1 + \frac{\tilde{c} R'(z(\omega_t))}{R(z(\omega_t))^2} \right)$$

When ω_t goes to ω_{max} , since $k(\omega_{max}) = 0$ and $R(\omega_{max}) = 0$, the growth rate of $\omega_t u'(c_t)$ converges to $\rho - R_t$, which is strictly lower than ρ . The transversality condition is therefore satisfied.

Proof of Proposition 7

Consider first the case where there are no adjustment costs, that is $C = 0$. The optimality condition for households' intra-period objective is the following:¹³

$$D(k) - k = \dot{k} D'(k) - \ddot{k}$$

The initial demand k_{init} is given and the following "transversality" condition must be satisfied:

$$\lim_{s \rightarrow \infty} [D(k_s) - k_s - \dot{k}_s] = 0$$

The market process of k follows a second-order differential equation with an initial and a terminal condition. It has therefore a unique solution. We can check that the Walrasian tâtonnement, which is characterized by $\dot{k} = D(k) - k$, satisfies both the optimality and the transversality conditions. This process is locally stable only for k^H . This proves (i).

Consider now the case where there are adjustment costs. $C(\dot{k}^j, k)$ is positive and strictly convex in \dot{k}^j and k , with $C_{12} > 0$ and $C(0, k) = 0$ for all k . The optimality condition for households is now:

$$D(k) - k = \dot{k} [D'(k) - C_{12}(\dot{k}, k)] - \ddot{k} [1 + C_{11}(\dot{k}, k)]$$

The initial demand k_{init} is given and the following "transversality" condition must be satisfied:

$$\lim_{s \rightarrow \infty} [D(k_s) - k_s - \dot{k}_s - C_1(\dot{k}_s, k_s)] = 0$$

This second-order differential equation in k can be rewritten as system of two first-order differential equations in k and x , where $x = \dot{k}$:

$$(A5) \quad \dot{x} = \frac{D'(k) - C_{12}(\dot{k}, k)}{1 + C_{11}(\dot{k}, k)} x - [D(k) - k]$$

$$(A6) \quad \dot{k} = x$$

¹³See Howitt and McAfee (1988).

A linear approximation of this system around $(x, k) = (0, k^i)$, $i \in \{L, H\}$ yields the following linear dynamic system:

$$\begin{pmatrix} \dot{x} \\ \dot{k} \end{pmatrix} = \begin{pmatrix} \frac{D'(k^i) - C_{12}(0, k^i)}{1 + C_{11}(0, k^i)} & -[D'(k^i) - 1] \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ k \end{pmatrix}$$

The roots (λ_1, λ_2) of this dynamic system must satisfy:

$$\lambda_1 + \lambda_2 = \frac{D'(k^i) - C_{12}(0, k^i)}{1 + C_{11}(0, k^i)}$$

$$\lambda_1 \lambda_2 = D'(k^i) - 1$$

Saddle-stability occurs if and only if there are two real roots with opposite signs, that is if $\lambda_1 \lambda_2 < 0$. This is the case only if $i = H$. This means that for $i = L$, the equilibrium is not a saddle point. However, it will be locally stable if $\lambda_1 + \lambda_2 < 0$, which is equivalent to $D'(k^i) < C_{12}(0, k^i)$.