

THE VALUATION OF ASIAN OPTIONS FOR MARKET MODELS OF EXPONENTIAL LÉVY TYPE

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ABSTRACT. We give a brief survey on some recent developments in pricing and hedging of European-style arithmetic average options given that the underlying asset price process is of exponential Lévy type.

1. INTRODUCTION

During the last decade it has been realized that the strong assumptions of the classical Black-Scholes model for the stochastic behavior over time of stock prices and indices are usually not fulfilled in practical applications. Among the major deficiencies of the Black-Scholes model are the normality assumption of log returns across all time scales and the assumption of a non-stochastic volatility (see e.g. [36]). In this survey, we will consider finance market models of exponential Lévy type which are able to capture the empirically observed distributional behavior of log returns and thus overcome some imperfections of the Black-Scholes model. In particular, we will discuss the pricing and hedging of Asian options under these market models. It turns out that by exploiting the independent and stationary increments property of Lévy processes, one can derive quick and rather accurate approximations of Asian option prices for arbitrary risk-neutral pricing measures (Section 4). In Section 5 a simple static super-hedging strategy for the payoff of Asian options in terms of a portfolio of European options is discussed. Its performance can be optimized by utilizing comonotonicity theory. This hedging strategy can even be applied to market models including stochastic volatility.

2. THE EMPIRICAL BEHAVIOR OF LOG RETURNS

One of the crucial assumptions of the Black-Scholes model is that log returns across all time scales follow a normal distribution. However, this is clearly unrealistic in practice, in particular for short time horizons. As an illustrative example, Figure 1 shows a kernel density estimator of daily log-returns of the Austrian stock index (ATX) based on data over a time span of more than three years in comparison to a maximum-likelihood fit of a normal distribution as proposed by the Black-Scholes model. Clearly the normal distribution does not reflect the empirical distribution, lacking mass in the center and in the tails. Various alternatives have been proposed for fitting log returns, among them the generalized hyperbolic (GH) distribution (cf. [21]), the Meixner distribution (cf. [35]) and the CGMY distribution [13, 14].

Figure 2 shows a maximum-likelihood fit of the GH distribution to the same data set and it can be observed that the GH distribution is able to capture both the

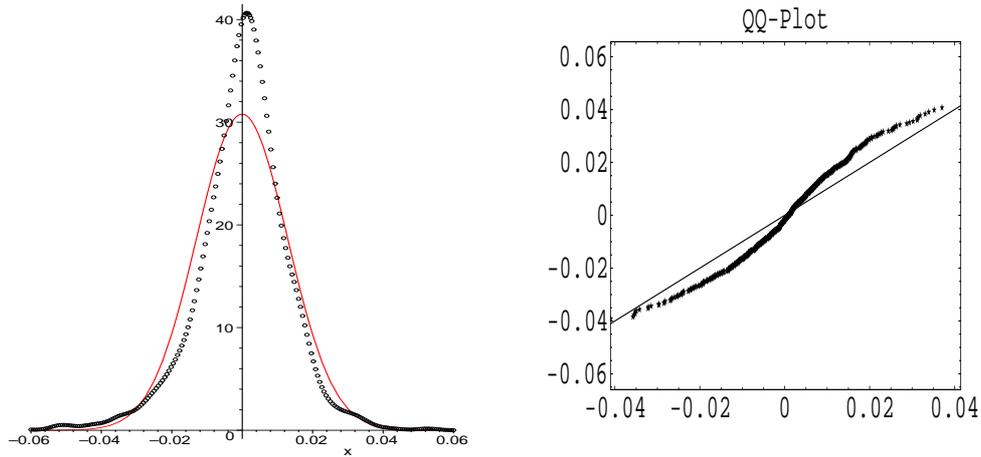


FIGURE 1. kernel density estimator (dotted line) and maximum-likelihood fit of a normal distribution (solid line) for daily ATX log returns (Jan 97 - June 00) and corresponding QQ-plot.

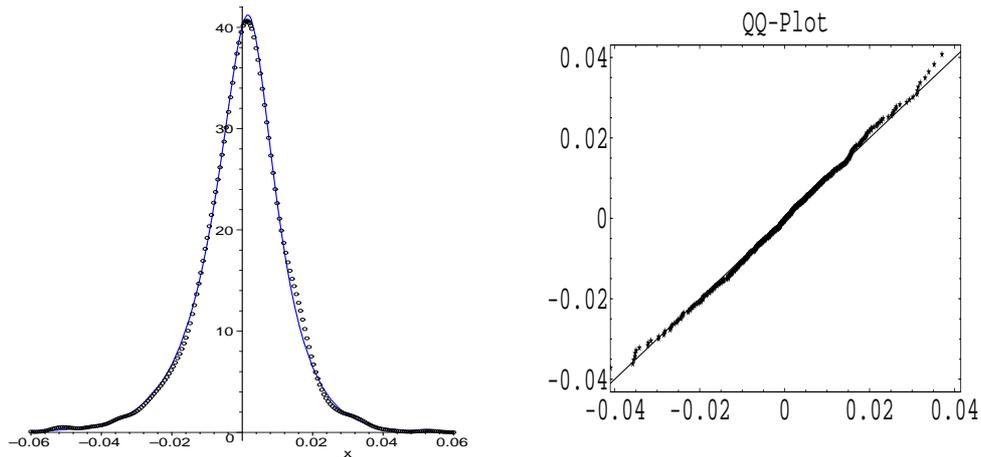


FIGURE 2. kernel density estimator (dotted line) and maximum-likelihood fit of a GH distribution (solid line) for daily ATX log returns (Jan 97 - June 00) and corresponding QQ-plot.

behavior at the center and in the (semi-heavy) tails quite well. The GH distribution (originally introduced by Barndorff-Nielsen [5]) has five parameters and contains the normal, the normal inverse Gaussian and the variance gamma distribution as a special case (for a detailed discussion see [8]). For further investigations on the suitability of these distributions for fitting financial data we refer to [6, 24, 29, 31, 34]. GH, Meixner and CGMY distributions all are infinitely divisible and thus generate a Lévy process as described in the next section.

3. THE EXPONENTIAL LÉVY MODEL

Any infinitely divisible distribution X generates a Lévy process $(Z_t)_{t \geq 0}$, i.e. a stochastic process with stationary and independent increments, $Z_0 = 0$ a.s. and $Z_1 \sim^d X$. It is always possible to choose a càdlàg version of the Lévy process. According to the construction, increments of length 1 have distribution X , but in general none of the increments of length different from 1 has a distribution of the same class. Exceptions are the cases where X is a normal inverse Gaussian, a variance gamma, a Meixner or a CGMY distribution; due to their respective convolution properties, each increment $Z_t - Z_s$ ($t > s \geq 0$) is then of the same class with new parameters depending on $t - s$, which makes these distributions natural and particularly attractive candidates for fitting the marginal distributions.

An exponential Lévy model for the price process $(S_t)_{t \geq 0}$ of an asset (a stock or an index) is now defined by

$$(1) \quad S_t = S_0 \exp(Z_t).$$

Thus the Brownian motion with drift of the Black-Scholes model is replaced by a Lévy process. Indeed, the distribution of log returns over time t is now given by the distribution of Z_t . Implications of the model choice (1) on the dynamics of the asset price process are e.g. discussed in [20].

There is some empirical evidence that the Lévy triplet of realistic Lévy models does not contain a Brownian "diffusion" component, so that the price process $(S_t)_{t \geq 0}$ is purely discontinuous (with finitely or infinitely many jumps in every finite interval, see e.g. [13]). The model (1) assumes a constant volatility, but the volatility smile effect of the Black-Scholes model is considerably reduced (cf. [24]). Time-consistency of Lévy models was investigated in [22]. For an up-to-date survey on exponential Lévy models we refer to [20, 36].

The market model (1) is in general incomplete (cf. Cherny [19]), and there exist infinitely many equivalent martingale measures \mathbb{Q} so that in order to price derivative securities one has to choose one particular candidate. One mathematically tractable choice is the so-called Esscher equivalent measure, essentially obtained by exponential tilting of the original measure. It was first introduced to mathematical finance by Madan and Milne [29]; see also Gerber and Shiu [25]. Let f_t denote the density of the marginal distribution Z_t , then the Esscher transform of f_t is defined by

$$(2) \quad f_t(x; \theta) = \frac{e^{\theta x} f_t(x)}{\int_{-\infty}^{\infty} e^{\theta y} f_t(y) dy}$$

with $\theta \in \mathbb{R}$. One can now define another Lévy process $(Z_t^\theta)_{t \geq 0}$ such that its one-dimensional marginal distributions are the Esscher transforms of the corresponding marginals of $(Z_t)_{t \geq 0}$ (for details see Raible [32]) and the parameter θ is chosen in such a way, that the discounted stock price process $(e^{-rt} S_t^\theta)_{t \geq 0}$ is a \mathbb{Q} -martingale (where r denotes the risk-free interest rate). It turns out that for normal inverse Gaussian and variance gamma Lévy processes the switch to the Esscher measure

just amounts to a shift in the parameters (cf. [2, 3]), which makes the analysis particularly tractable. There have been attempts to justify this particular choice for \mathbb{Q} both within utility and equilibrium theory; however, the topic is still controversial (cf. [11, 18, 26]).

Another natural approach is to shift the drift of the Lévy process in such a way that a risk-neutral framework is obtained. However, in this case the resulting risk-neutral measure is in general not equivalent to the physical measure. One way to circumvent this problem is to start out immediately with a risk-neutral model for S_t defined by

$$S_t = S_0 \frac{\exp(rt)}{E[\exp(Z_t)]} \exp(Z_t)$$

and then calibrating the parameters from current option prices observed in the market rather than from historical log-returns. This approach is quite common in practice, see e.g. [36].

Note that the techniques presented in Sections 4 and 5 are applicable for any risk-neutral pricing measure \mathbb{Q} and thus the choice of \mathbb{Q} is not discussed any further.

4. PRICING OF ASIAN OPTIONS

Let us now consider the price of a European-style arithmetic average call option at time t under exponential Lévy models given by

$$(3) \quad \text{AA}_t = \frac{e^{-r(T-t)}}{n} \mathbb{E}^{\mathbb{Q}} \left[\left(\sum_{k=0}^{n-1} S_{T-k} - nK \right)^+ \middle| \mathcal{F}_t \right],$$

where n is the number of averaging days, K the strike price, T the time to expiration, \mathcal{F}_t the information available at time t and \mathbb{Q} any risk-neutral pricing measure. For convenience we will restrict ourselves to the case $t = 0$ and $n = T$, so that the averaging starts at time 1 (the other cases can be handled in a completely analogous way).

The main difficulty in evaluating (3) is to determine the distribution of the dependent sum $\sum S_k$, for which in general no explicit analytical expression is available. There are several approaches to the problem: one can use Monte Carlo simulation techniques to obtain numerical estimates of the price, which can be achieved by adapting procedures developed for the Black-Scholes case (see e.g. [9, 10, 33, 37]). Recently Večer and Xu [40] developed a partial integro-differential equation approach that is applicable for exponential Lévy models, which transforms the problem into finding numerical solution of these equations. Both approaches are rather time consuming. For an approach based on Fast Fourier Transforms, see [7, 17]. Another alternative is to use approximations of the distribution of the average, which sometimes leads to closed-form expressions for the price approximation. In the sequel we will discuss an adaption of such approximation techniques developed for the Black-Scholes case ([28, 39, 41]) to our exponential Lévy setting.

The basic idea is to determine moments of the dependent sum in (3) and then replace it by a more tractable distribution with identical first moments. Due to

the independence and stationarity of increments of Lévy processes, one can derive a simple algorithm to derive the m th moment of the dependent sum $A_n := \sum_{k=1}^n S_k$:

Let us define

$$R_i = \frac{S_i}{S_{i-1}}, \quad i = 1, \dots, n$$

and

$$\begin{aligned} L_n &= 1 \\ L_{i-1} &= 1 + R_i L_i, \quad i = 2, \dots, n. \end{aligned}$$

Then we have

$$\sum_{k=1}^n S_k = S_0(R_1 + R_1 R_2 + \dots + R_1 R_2 \dots R_n) = S_0 R_1 L_1.$$

Thus it remains to determine $\mathbb{E}^{\mathbb{Q}}[(R_1 L_1)^m] = \mathbb{E}^{\mathbb{Q}}[R_1^m] \mathbb{E}^{\mathbb{Q}}[L_1^m]$ (the last equality follows from the independence of the increments). But

$$(4) \quad \mathbb{E}^{\mathbb{Q}}[R_i^k] = \mathbb{E}^{\mathbb{Q}}[\exp(k Z_1)] = \mathbb{E}^{\mathbb{Q}}[\exp(k X)],$$

so that one just has to evaluate the risk-neutral moment generating function of X at k , given it exists (recall that X is the generating distribution of the Lévy process). Furthermore we have

$$(5) \quad \mathbb{E}^{\mathbb{Q}}[L_{i-1}^m] = \mathbb{E}^{\mathbb{Q}}[(1 + L_i R_i)^m] = \sum_{k=0}^m \binom{m}{k} \mathbb{E}^{\mathbb{Q}}[L_i^k] \mathbb{E}^{\mathbb{Q}}[R_i^k].$$

Starting with $\mathbb{E}^{\mathbb{Q}}[L_n^k] = 1 \forall k \in \{0, \dots, m\}$, one can then apply recursion (5) together with (4) to obtain $\mathbb{E}^{\mathbb{Q}}[L_1^m]$ and subsequently $\mathbb{E}^{\mathbb{Q}}[(A_n)^m] = S_0^m \mathbb{E}^{\mathbb{Q}}[R_1^m] \mathbb{E}^{\mathbb{Q}}[L_1^m]$.

These moments can now be used to approximate $A_n = \sum_{k=1}^n S_k$ by another more tractable distribution with identical first moments. If A_n is approximated by a log-normal distribution, then one obtains an explicit formula for the approximated price resembling the Black-Scholes price of a European option. Higher moments of A_n can then be used to improve the approximation in terms of an Edgeworth series expansion (this approach is known as the Turnbull-Wakeman approximation). Another natural and usual effective choice is to approximate A_n by a distribution of the same class as X . All these approximations have been worked out in detail for the normal inverse Gaussian Lévy model in [2] and for the variance gamma Lévy model in [3]. They turn out to be a quick and accurate alternative to other numerical pricing techniques, the approximation error typically being less than 0.5% (for an extensive numerical study we refer to Albrecher and Predota [2, 3]).

Note that whereas the effectiveness of most of the other numerical techniques depends quite strongly on the structure of the marginal distributions of the Lévy process, the above approach is applicable for arbitrary risk-neutral measures and arbitrary exponential Lévy models as long as the risk-neutral moment-generating

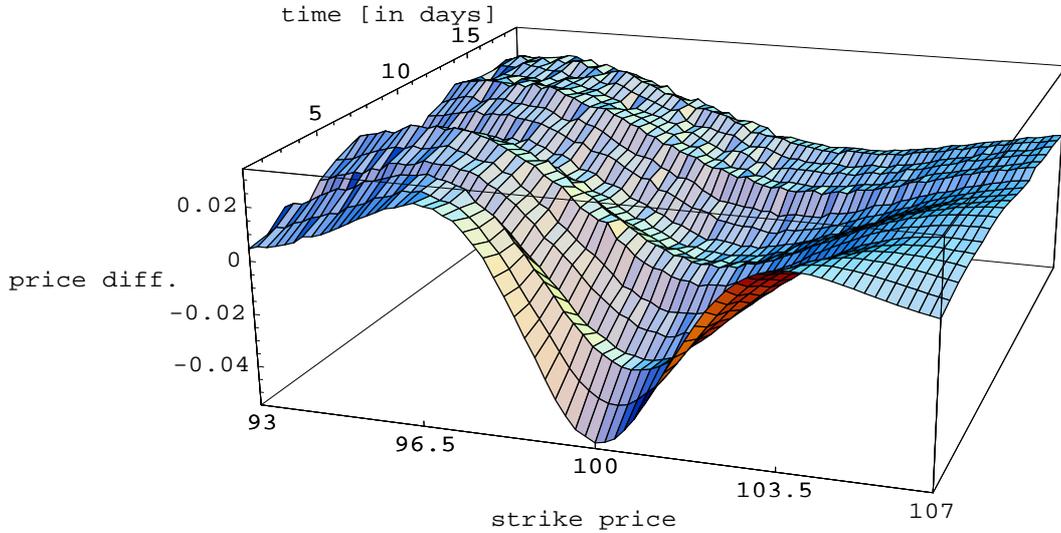


FIGURE 3. ${}^{(\text{NIG})}\text{AA}_0 - {}^{(\text{BS})}\text{AA}_0$ (Asian arithmetic option)

function of the log returns exists in the interval $[0, k]$.

The sensitivity of the price of an Asian option on the underlying market model has been investigated in [2, 3]). As an illustrative example, Figure 3 (taken from [2]) depicts the difference of Asian call option prices for a Black-Scholes model (in which \mathbb{Q} is unique) and the Esscher price in a normal inverse Gaussian Lévy model across different strikes and maturities, where the two models were fitted to historical data of OMV daily log returns ($S_0 = 100$, daily averaging and the prices were determined by Monte Carlo simulation).

The behavior of the price difference in Figure 3 is quite typical. At the money, where most of the volume is traded, the Black-Scholes price is too high. In and out of the money, it is too low. This is intuitively clear since the Black-Scholes model underestimates the risk of larger asset price moves. If the option is very deep in or out of the money, the option price is more or less model independent. The difference in option prices becomes less pronounced for increasing maturity. A comparison with the corresponding sensitivity of European option prices on the underlying model shows that the effects are quite similar, see e.g. [3, 21].

5. HEDGING OF ASIAN OPTIONS

In many circumstances the availability of a hedging strategy for a financial product is far more important than the determination of its price (note that in view of the incompleteness of the market, there exists a whole interval of no-arbitrage prices for the product depending on the particular choice of the risk-neutral measure \mathbb{Q} , which limits the explanatory power of a "price"). Moreover, hedging strategies are utilized as devices for representing risk in standard reports. Even in the Black-Scholes world,

hedging an Asian option is far from trivial. One approach is to derive upper and lower analytic bounds for the option price based on conditioning of random variables (for instance conditioning on the geometric average) and then to apply delta-hedging in terms of these bounds (see e.g. [30, 38]). Since these conditioning techniques are based on the simple structure of the log-normal distribution of the Black-Scholes model, it does not seem feasible to extend this approach to arbitrary exponential Lévy models. Another possibility is to apply a log-normal approximation to the dependent sum in (3) using the moment-matching technique discussed in Section 4 and then use the resulting closed-form expression of the price for delta-hedging. However, it is difficult to keep track of the implied hedging error in this case and the latter can be quite substantial since the log-normal fit of A_n may be quite poor. Moreover, delta-hedging itself is to be considered with care, since, while producing stable payoffs under idealized conditions (no limit on frequency of rehedging, no transaction costs), it produces highly variable payoffs under realistic conditions (limitations on the hedging liquidity, transaction costs). Therefore it is desirable to develop static hedging strategies where the initial hedge is kept in place for the whole lifetime of the product (or quasi-static strategies with only a small number of hedge adjustments).

In the sequel we will discuss a simple static superhedging strategy for fixed-strike Asian call options which was developed in Albrecher et al. [1]. It is based on a buy-and-hold strategy consisting of European call options maturing with and before the Asian option. For that purpose let us consider the following upper bound for the price given in (3): $\forall K_1, \dots, K_n \geq 0$ with $K = \sum_{k=1}^n K_k$, we have

$$\left(\sum_{k=1}^n S_{t_k} - nK \right)^+ = \left((S_{t_1} - nK_1) + \dots + (S_{t_n} - nK_n) \right)^+ \leq \sum_{k=1}^n (S_{t_k} - nK_k)^+$$

implying

$$(6) \quad AA_0 \leq \frac{\exp(-rT)}{n} \sum_{k=1}^n E^{\mathbb{Q}} \left[(S_{t_k} - nK_k)^+ \middle| \mathcal{F}_0 \right] = \frac{1}{n} \sum_{k=1}^n \exp(r(T - t_k)) EC_0(\kappa_k, t_k),$$

where $EC_0(nK_k, t_k)$ is the price of a European call option at time 0 with strike nK_k and maturity t_k . One observes that buying $\exp(-r(T - t_k))/n$ European call options at time $t = 0$ (with strike κ_k , maturity t_k) ($k = 1, \dots, n$), holding them until their expiry and putting their payoff on the bank account represents a static superhedging strategy for this Asian option.

One still has the freedom to choose values K_k such that $\sum_{k=1}^n K_k = K$. A trivial choice is $K_k = K/n$ ($k = 1, \dots, n$). Since $\forall K \geq 0$ one has $EC_0(K, t) \leq EC_0(K, T)$ ($0 \leq t \leq T$) (note that this inequality even holds if we allow for a dividend rate q as long as $q \leq r$), leading to $AA_0 \leq EC_0$, so that an Asian call option with strike K and maturity T is always dominated by a European call option

with same strike and maturity. This result holds for arbitrary arbitrage-free market models; for the Black-Scholes setting it was already derived by Kemna and Vorst [27], see also [30].

Since the aim is to optimize the performance of the superhedge, one needs to determine the combination of K_k that minimizes (6). In the Black-Scholes model, this has been achieved by Nielsen and Sandmann [30] using Lagrange functions. In the general case, it turns out that comonotonicity theory leads to the optimal choice of the strike prices. Let $F(x_k; t_k) = P^{\mathbb{Q}}(S_{t_k} \leq x_k | \mathcal{F}_0)$ ($x_k, t_k > 0$) denote the marginal distribution function of S_{t_k} . Then the optimal choice of strike prices is given by

$$n K_k = F^{-1}(F_{S^c}(n K); t_k), \quad k = 1, \dots, n,$$

where F_{S^c} is the distribution function of the comonotone sum of S_{t_1}, \dots, S_{t_n} determined by $F_{S^c}^{-1}(x) = \sum_{k=1}^n F^{-1}(x; t_k)$. These values can be determined within less than a minute on a normal PC for the entire hedge portfolio. Note that in (3) we have $t_k = k$ ($k = 1, \dots, n$). Whereas the upper bound $AA_0 \leq EC_0$ (leading itself to a trivial super-hedge) is model-independent, the performance of the superhedge (6) can thus be optimized by specifying a market model and a risk-neutral measure \mathbb{Q} . For a proof of the optimality we refer to [1], where one can also find a numerical study of the performance of this superhedging strategy for normal inverse Gaussian, variance gamma and Meixner Lévy models (with the mean-correcting measure used for \mathbb{Q}). The numerical results indicate that this strategy is quite effective, in particular for low values of the strike price K . For an option with moneyness of 80%, the difference between the hedging cost and the estimated option price is typically around 1.5%, whereas the classical hedge with the European call leads to a difference of almost 10%. For options out of the money, the difference increases, but in view of the easy and cheap way in which this hedge can be implemented in practice, the comonotonic approach seems to be competitive also in these cases. Furthermore, the European call options needed for the hedge are typically available on the market and quite liquidly traded. In addition, static hedging is not exposed to the risk inherent in dynamic hedging, namely that at times of large market moves liquidity may dry up making rebalancing impossible. But especially in these situations effective hedging is needed (for further discussions on the topic, we refer to [4, 12, 15, 16]). Finally, the proposed hedging strategy works whenever an approximation of the risk-neutral density is available and can thus also be applied to stochastic volatility models using Fast Fourier transforms.

Remark: The results presented in this survey were formulated for fixed-strike arithmetic average call options. However, many of them translate immediately to put options and floating-strike options (using put-call parity and symmetries of floating and fixed strike Asian options recently established for exponential Lévy models in [23]). The inclusion of dividend payments in the model is also merely a matter of notation. Furthermore, the approximation technique of Section 4 can be adapted to geometric average rate options (cf. [2, 3]).

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