UNIVERSITÉ DE LAUSANNE
FACULTÉ DES HAUTES ÉTUDES COMMERCIALES

# EMERGENCE OF COOPERATION ON STATIC AND DYNAMIC NETWORKS

THESE

Présentée à la Faculté des HEC
de l'Université de Lausanne

par

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# Abstract

> Knowledge rests not upon truth alone, but upon error also.
>
> Carl G. Jung

Game theory is a branch of applied mathematics used to analyze situation where two or more agents are interacting. Originally it was developed as a model for conflicts and collaborations between rational and intelligent individuals. Now it finds applications in social sciences, economics, biology (particularly evolutionary biology and ecology), engineering, political science, international relations, computer science, and philosophy.

Networks are an abstract representation of interactions, dependencies or relationships. Networks are extensively used in all the fields mentioned above and in many more. Many useful informations about a system can be discovered by analyzing the current state of a network representation of such system.

In this work we will apply some of the methods of game theory to populations of agents that are interconnected. A population is in fact represented by a network of players where one can only interact with another if there is a connection between them.

In the first part of this work we will show that the structure of the underlying network has a strong influence on the strategies that the players will decide to adopt to maximize their utility. We will then introduce a supplementary degree of freedom by allowing the structure of the population to be modified along the simulations. This modification allows the players to modify the structure of their environment to optimize the utility that they can obtain.

# Sommario

> La conoscenza poggia non solo sulla verità, bensí anche sull'errore.
>
> Carl G. Jung

La teoria dei giochi è una branca della matematica applicata utilizzata per analizzare la situazione in cui due o più agenti stanno interagendo. È stata originariamente sviluppata come un modello per i conflitti e le collaborazioni tra persone razionali e intelligenti; ora trova applicazioni in scienze sociali, economia, biologia (in particolare biologia evoluzionistica ed ecologia), ingegneria, scienze politiche, relazioni internazionali, informatica e filosofia.

Le reti sono una rappresentazione astratta di interazioni, dipendenze o relazioni e sono ampiamente utilizzate in tutti i settori di cui sopra e in molti altri ancora. Molte informazioni utili su un sistema possono essere carpite analizzando lo stato attuale della rappresentazione sotto forma di rete di tale sistema.

In questo lavoro applicheremo alcuni dei metodi della teoria dei giochi a una popolazione di agenti interconnessi, rappresentata da una rete di giocatori dove ognuno può interagire con un altro agente solo se esiste un collegamento tra di essi.

Nella prima parte di questo lavoro mostreremo che la struttura della rete ha una forte influenza sulle strategie che i giocatori adottano per massimizzare il loro guadagno.

In seguito introdurremo un ulteriore grado di libertà consentendo alla struttura della popolazione di essere modificata nel corso delle simulazioni; questo permetterà ai giocatori di agire sulla struttura del loro ambiente in modo da ottimizzare il guadagno che possono ottenere.

# Synopsis

> La connaissance repose non seulement sur la vérité mais aussi sur l'erreur.
>
> Carl G. Jung

La théorie des jeux est une branche des mathématiques appliquées utilisée pour analyser des situations où deux ou plusieurs agents interagissent. À l'origine elle a été développée comme un modèle pour les conflits et les collaborations entre des individus rationnels et intelligents. Maintenant elle trouve des applications dans les sciences sociales, l'économie, la biologie (en particulier la biologie évolutive et l'écologie), l'ingénierie, les sciences politiques, les relations internationales, l'informatique et la philosophie.

Les réseaux sont une représentation abstraite d'interactions, dépendances ou relations. Ils sont largement utilisés dans tous les domaines mentionnés ci-dessus et dans bien d'autres encore. Beaucoup d'informations utiles peuvent être découvertes par l'analyse de l'état actuel d'une représentation sous la forme d'un réseau d'un tel système.

Dans ce travail, nous allons appliquer certaines des méthodes de la théorie des jeux à des populations d'agents interconnectés. Une population est représentée par un réseau d'acteurs où un ne peut interagir avec un autre que s'ils sont connectés entre eux.

Dans la première partie de ce travail nous allons montrer que la structure du réseau sous-jacent a une forte influence sur les stratégies que les joueurs décident d'adopter pour maximiser leur bénéfice.

Nous allons ensuite introduire un degré supplémentaire de liberté en permettant à la structure de la population d'être modifiée pendant les simulations. Cette modification permet aux joueurs d'agir sur la structure de leur environnement afin d'optimiser le bénéfice qu'ils peuvent obtenir.

# Acknowledgements

> Not all those who wander are lost.
>
> J.R.R. Tolkien

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And last but not least, a very special “Thank You” to my girlfriend Tamara, for supporting me along this whole work, for tolerating me when we shared a one room apartment, for being with me, despite my craziness, and for making my life so much more enjoyable.

Lausanne, November 2010

# Overview of the Thesis

> It may be that all games are silly. But then, so are humans.
>
> Robert Wilson Lynd

This Ph.D. thesis is organized in the form of a collection of articles. Parts II, III, and IV of this manuscript include seven peer-reviewed articles that have been published in international specialized journals. All these articles have not been edited, apart for the obvious formatting. Due to this there is some overlap between some articles and between articles and introductory parts. Also, for the same reasons, the style of figures and tables, and sometimes some notations are not coherent over the whole manuscript.

This work consists mainly in an empirical analysis of results obtained through computer simulations. These results are only presented and commented inside the articles. However, even if redundant with respect to the articles, an introduction to the methods used in the articles included in the three parts can be found in chapter 4 at the end of the first part of this manuscript. This has been done with the purpose of facilitating the reading of the thesis. The bibliography has been compiled in a unique section at the end of the manuscript, and contains all the publications cited in this work.

Now for a description of the structure of this work:

- Part I presents the motivations for this work as well as its background and theoretical foundations. Also, at the end of this part, there is an introduction to the methods used in the following parts of the dissertation.
- Parts II, III, and IV include the articles presenting the results obtained in these years.
- Finally, in part V general conclusion are drawn and some hints are given on possible directions for future works.

# Contents

# Part I

# Background & Definitions

# Chapter 1

# Motivation

> I was gratified to be able to answer promptly,
> and I did. I said I didn't know.
>
> Mark Twain

Game theory aims at resolving conflicting situations that appear commonly in socio-economic settings. However, standard game theory, being based on perfectly rational, payoff-maximizing agents, has to face agent's real-world limitations such as bounded rationality and limited computational capabilities.

A first step toward a more realistic model for social interaction using game theory has been the introduction of Evolutionary Game Theory (EGT) that uses populations of players that individually have limited capabilities. This assumption and an evolution model inspired by biological evolution has allowed a great improvement.

More recently, in 1992, Martin Nowak and Robert May [84], made another fundamental improvement when they introduced an underling structure, a network, to represent the relationships between the players.

In this work we try to understand the role played by characteristics of several kind of networks in sustaining (or not) the cooperation of agents in a population.

Other than exploring the benefits of the population structure we also take a closer look at some mechanics that can modify the behavior of the agents and thus change the stable state of the system and the solutions to the problem.

Here we enumerate the aspects that are covered in this manuscript and our hypothesis about their effect on the dynamical system we use in this work.

I. We analyzed the effects of different updates rules, the changes introduced by using different methods to calculate the utility of an agent, and the effects of the timing of the operations.

   In our opinion all of these mechanics have a great influence on the amount of cooperation that can be established and maintained in a population.

II. When deciding on their next strategy, agents often use their wellness and the apparent wellness of other individuals and try to adapt their strategy to that of the fittest. However that is not always the case. In some situations other techniques are used by the agents to improve their place in the society and one of these techniques is conformity, the tendency of humans to imitate locally common behaviors.

In this work we looked at the effects that the introduction of small amounts of conformism has on the evolution of a networked society.

III. The structure of the network also plays an important role in the emergence of social norms. By using several kind of networks, from simple networks to complex social networks, we hope to discover the role played by the characteristic of the structure on the strategies of the agents. We believe that the structure of social networks, and in particular the existence of social groups and communities have a great impact on the quality of the social interactions in the population.

IV. Finally the network of a social group is, in reality, an evolving complex system. We hypothesize that the behavior of players can push the structure of dynamical networked population to evolve from a regular or random topology to a more functional structure to optimize the wellness of the whole population.

# Chapter 2

# Game theory

> God does not play dice with the universe; He plays an ineffable game of His own devising, which might be compared, from the perspective of any of the other players, to being involved in an obscure and complex version of poker in a pitch-dark room, with blank cards, for infinite stakes, with a Dealer who won't tell you the rules, and who smiles all the time.
>
> Terry Pratchett

Game theory is a mathematical framework for the analysis of models of conflict and cooperation between rational and intelligent individuals. It provides tools for the analysis of situations involving two or more agents, where each agent's actions will influence the welfare of the others. This field was founded in 1944 with the publication of the book "Theory of Games and Economic Behavior" by von Neumann and Morgenstern [124]. It was initially developed to analyze situations where the success of an agent implied the failure of another, also known as *zero sum* games. However now "game theory is a sort of umbrella or 'unified field' theory for the rational side of social science, where 'social' is interpreted broadly, to include human as well as non-human players (computers, animals, plants)" [5]. The development of game theory was extensive during the 1950s thanks to the contributions of several scholars. In the 1970s the theory was explicitly applied to biology and its now a recognized and important tool in many fields to the point that eight games theorists have won the Nobel Prize in Economics.

Many situations where decisions have to be taken in interaction with other parties or in conflicting situation can be found in everyday life and can be interpreted as "games" using the tools provided by game theory. Some examples of such situations can be: animals competing for some resources, driving, paying taxes, playing chess, etc. Clearly the representation used in game theory does not include all the details and the complexity that many situations involve. Sometimes the solution proposed by the theory does not correspond to the one suggested by common-sense. For example when taking a train, one have to chose whether to buy a ticket or not. If the control is absent (or rare enough) the best solution is to take a free ride. Rationality is the key to the interpretation of this solution. Rationality that has to be interpreted in the *homo economicus* sense, i.e. an individual with very specific goals trying to maximize his utility at the least possible cost. The english term *economic man* goes back to John Stuart Mill, in 1836 he defined the economic man as someone who "is concerned with him solely as a being who desires to possess wealth, and who is capable of judging the comparative efficacy of means for obtaining that end." [68]. The latin name is probably due to Pareto [87]. This concept has been used by Savage, and von Neumann and Morgenstern [105, 124]. According to game theory all the agents are *rationals* and *intelligent* which means that:

- they know the possible choices in a given situation
- they can associate a *utility* to each choice and to its consequences
- they take decisions aimed to *maximise* the utility
- they know that every agent is also rational and intelligent (common knowledge of rationality).

The utility is a numerical value that represents the value of the outcome of every possible decision, this utility is also often called *payoff*. As said, a rational agent tries to maximize this utility, every time an individual is confronted with a decision he choses the strategy that will lead to the highest possible outcome. If there is uncertainty or incomplete knowledge, the agent chooses the maximum *expected payoff*. It's important to point out that the actual numerical values are not important, only the ordering of the payoff of the different strategies is taken into account. Many criticisms have been directed to this approach and to the assumption that an agent has unbounded rationality: this is obviously not possible in the real world and others models where players have limited rationality have been proposed, however the standard models allows the use of exact mathematical formalizations.

All the works included in this manuscript make use of *non-cooperative game theory*, i.e. the agents aim is totally selfish and they try to maximize their own utility no matter what. In the *cooperative game theory* players are allowed to form coalitions, this has not been taken into account in these works.
Also the games used here are all *one-shot* games. Which means that even when two player have to play together for a second time, they will have no memory of their previous encounters. When considering *iterated* or *repeated* games, players may develop more complex strategies. Both situations are very common in society; however, we focused our attention on the former.

## 2.1 Representation of games

Game theory is a mathematical theory, and the games studied are also well-defined mathematical objects. A finite game $\Gamma$ has a finite number of players $N$, a finite set of strategies $S_i$ for each player $i \in N$ and utility functions $u_i : \bigtimes_{j \in N} S_j \to \mathbb{R}$ for all the players $j \in N$.

Two forms are used to represent non-cooperative games: the *Extensive form* and the *Strategic form.*

### 2.1.1 Extensive form

The extensive form is particularly useful when the ordering of the action of the players is important. The representation of a game in its extensive form requires:

- The set of players $N = \{0, 1, ..., n\}$ where 0 usually represent *Nature.*
- For every player, the set of moments in the game where this player has to take an action.
- The possible actions for each player at each choice point she has.
- The information set for each player at each choice point.
- The payoff for each possible outcome, i.e. for each possible combination of moves of the players.

In this form games can be presented as trees where each vertex is a choice to one of the players and each branch corresponds to the consequences of that choice. At each leaf of the tree, where the game ends, there are the corresponding payoff values for each player. Let take an easy game for an example. In this two-players game, player one will choose first between two strategies *top* (T) and *bottom* (B). After that, player 2 will choose between *right* (R) and $left$ (L) without knowing the action of player 1. The extensive form of this game is shown in figure 2.1.

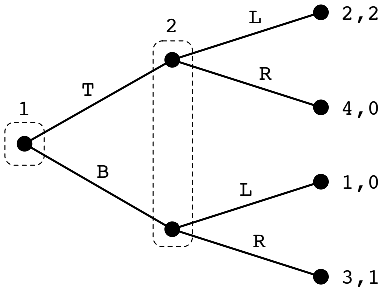


Figure 2.1: Extensive form for the first version of the game. The dashed line around the two choice points for player 2 represent his unawareness of the situation when is called to chose his strategy.

The dashed lines represent the information a player has when it is her turn to decide, this is called information set. Player 1 information set contains only one state because the game is just beginning and there is only one possible state. On the other hand, as player 2 does not know the strategy of player 1, she cannot distinguish between the two possible state of the game, therefor these states are in the same information set. The fact that player 2 cannot distinguish in which state the game is can be interpreted as a missing information, in this case we will be describing a game with *imperfect information* [71], or also the situation where player's 2 move is simultaneous to player's 1. In this situation, player 1 is better off choosing $T$ as she will be better off whatever player 2 decide. Player 2, knowing the options for player one, has to choose $L$ in order to obtain a better payoff. When we modify the game by introducing *perfect information* the situation changes, as shown in figure 2.2.

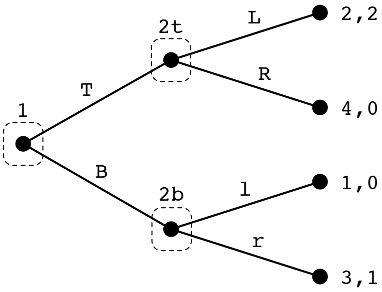


Figure 2.2: Extensive form for the second version of the game. In this case player two knows the action of player 1, so she can distinguish between the two choice points.

In this case player 2 knows what player 1 has done. This is represented by the two different information sets (2t) and (2b) near the choices for player 2. This development will also change the strategy for player 1, as she now knows that if she plays $B$, player 2 will play $r$ to obtain a better payoff.

This kind of representation can also be used for complex games like chess. Clearly in that case each player will have a plethora of choices and the tree will practically not be drawable.

### 2.1.2 Strategic form

The strategic form, also called *Normal form*, represents a game using fewer elements compared to the previous form.

Formally a finite game Γ with complete information in the normal form can be specified as follows:

$$\Gamma = (N, S_i, u_i) \; \forall_i \in N.$$

In this case each strategy includes all the actions a player will take at every choice point she has. The normal form can be represented as the matrix of the payoffs associated with every possible combination of strategies. Clearly if more than two players are involved in the game the matrix representation is not adapted. The normal form is used when the actions are simultaneous or when players decide their strategy without knowing the action of the others. Otherwise the extensive form is generally used. The representation of the first of the two games presented in section 2.1.1, is shown in table 2.1.

Table 2.1: The normal form for the first version of the game game presented in section 2.1.1. The two players decide their strategy without knowing what the other will do.

| | | Player 2 | |
|---|---|---|---|
| | | *L* | *R* |
| Player 1 | T | $(2,2)$ | $(4,0)$ |
| | B | $(1,0)$ | $(3,1)$ |

Also the second version of this game can be represented in this form. However in this case the possible strategies for player 2 are not only $\{L, R\}$. The strategy set is modified to take into account the two possible information sets, in this way player 2 will be able to decide his strategy before knowing the choice of player 1. The new strategy set will be $\{Ll, Lr, Rl, Rr\}$, where for each pair, the first action correspond to the reply for player 1 playing T and the second for her playing B. The corresponding strategic form representation is in table 2.2.

Table 2.2: The normal form for the second version of the game game presented in section 2.1.1. Here too the two players decide their strategy without knowing what the other will do, but the strategy for player 2 includes the reply for each one of the moves for player 1.

| | | Player 2 | | | |
|---|---|---|---|---|---|
| | | *Ll* | *Lr* | *Rl* | *Rr* |
| Player 1 | T | $(2,2)$ | $(2,2)$ | $(4,0)$ | $(4,0)$ |
| | B | $(1,0)$ | $(3,1)$ | $(1,0)$ | $(3,1)$ |

## 2.2 Equilibrium

The equilibria represent the most common *solution concept* used in game theory. A solution concept is a formal rule for predicting how the game will be played. The predictions are called solutions and describe what strategy will be selected by each player that takes part in the game. A *pure strategy* $s_i$ for player $i$ is a complete plan of action that describes what a player will do every time he can chose between multiple options. The strategy set $S_i$ of a player $i$ is the set of all possible *pure strategies* that this player can use.

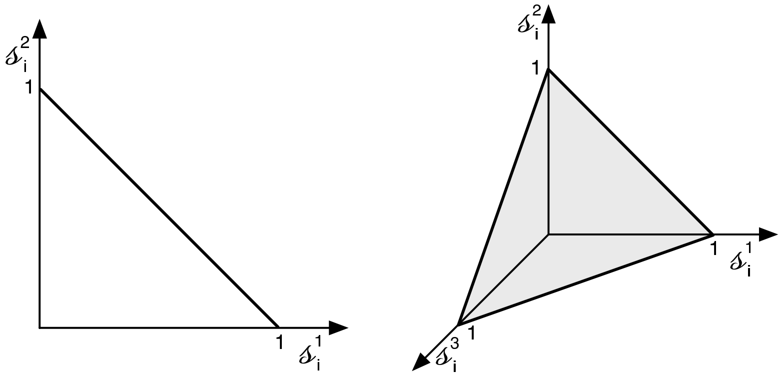
Solution concepts, in most games, will find more than one possible solution. Some of these solutions use *pure strategies*, some of them use *mixed strategies*.

### 2.2.1 Mixed strategy

A *mixed strategy* $\sigma_i$ for a player $i$ is a set of probabilities $\Delta(S_i)$ assigned to each pure strategy $s_i \in S_i$ available to that player. Any strategy $s_i$ will be selected with a nonnegative probability $\sigma(s_i)$. This allows a player to randomly select one pure strategy. The set of mixed strategies available to a player is infinite, assuming that that player has at least two possible pure strategies. The sum of the probabilities in a mixed strategy must be exactly 1:

$$\sum_{s_i \in S_i} \sigma(s_i) = 1.$$

The probability set of a mixed strategy can be represented as a vector in $\mathbb{R}^{m_i}$ where $m_i$ is the cardinality of the set $S_i$. As the sum of the elements of these vectors is one, the set of all possible vectors can be represented as a *simplex*, the simplexes for $m_i = 2$ and $m_i = 3$ are shown in figure 2.3.



$m_i = 2$ $m_i = 3$

Figure 2.3: Unit simplex for $m_i = 2$ and $m_1 = 3$.

Clearly pure strategies are also part of the simplex. A pure strategy $s_k$ is a particular case where $\sigma(s_k) = 1$ and $\sigma(s_j) = 0 \ \forall j \neq k$, i.e. a vertex of the simplex. A *strategy profile* is a set of strategies containing one strategy for every player in the game. A strategy profile is a complete specification of all action of the game. When the strategies are mixed, this profile $\sigma$ is called *randomized strategy profile.*

$$\sigma = (\sigma_1, ..., \sigma_n), \sigma \in \mathop{\times}_{i \in N}, \Delta(S_i)$$

is a vector of mixed strategies for all players $i \in N$ where $n$ is the size of $N$.

### 2.2.2 Nash Equilibrium

The Nash Equilibrium (NE) is the most famous and widely used equilibrium concept. This solution concept has been proposed by John Forbes Nash in [72].
In order to explain the NE a few more concepts have to be introduced. Let's define as $u_i(\sigma)$ the expected payoff of player $i$ when all players choose their strategies independently according to a strategy profile $\sigma$. Let's also denote with $(\sigma_{-i}, \tau_i)$ the randomized strategy equal to $\sigma$ except for player $i$ that will use strategy $\tau_i$.

A Nash Equilibrium is a randomized strategy profile $\sigma^*$ that verifies:

$$u_i(\sigma^*) \geq u_i(\sigma_{-i}, \tau_i), \forall i \in N, \forall \tau_i \in \Delta(S_i)$$

Which means that $\sigma^*$ is a Nash Equilibria iff no player could expect an higher payoff with an unilateral deviation from this profile. J. Nash proved that such an equilibrium exist for every finite game $\Gamma$. The proof of this theorem can be found in [71], page 138, and in [122], page 51.

If we go back to the two variants of the game presented in section 2.1.1, in the first version, without complete information, the NE is $(T, L)$ with payoff $(2, 2)$. Any deviation from this will lower the payoff of the the player that changes strategy. In the second version of the game the NE is $(B, Lr)$ with a payoff of $(3, 1)$. In this case player 2 excludes strategies $Rl$, $Rr$ and also $Ll$ because they are less interesting. Knowing that player 2 strategy should be $Lr$ player 1 will play B to get an higher payoff.

### 2.2.3 Pareto efficiency

The Pareto efficiency (or Pareto optimality) is an economic concept that finds applications in other fields. When different possibilities exist for allocating benefits, a change in the allocations that improves the benefit of one individual without worsening the situation for the others, is called a Pareto-optimal move, or a Pareto improvement. An allocation is defined as Pareto efficient when no Pareto improvement is possible anymore, i.e. it is not possible to improve the payoff of an individual without putting another in a less desirable position. The concept of Pareto optimality does not take into account the desirability of an allocation.

## 2.3 Some games

Let's now illustrate some simple games that have been used in this work. All these two-person, two strategies games are symmetric, which means that each player has the same strategy set. The payoff matrix for these games is shown in table 2.3.

### 2.3.1 Coordination and cooperation

In the games presented in the following sections two concepts will be used: *Cooperation* and *Coordination*. Cooperation is used to identify the action of an agent that is taking some risks

Table 2.3: The normal form of a generic two-person, two-strategies, symmetric game.

| | $s_1$ | $s_2$ |
|---|---|---|
| $s_1$ | $(u_1, u_1)$ | $(u_2, u_3)$ |
| $s_2$ | $(u_3, u_2)$ | $(u_4, u_4)$ |

and sacrificing the possibility of an higher (or guaranteed) benefit to help his partner(s). Coordination is merely the results of two agents applying the same action at the same type. In some games, coordination is preferred, as it provides the highest benefits to all agents involved, but this is not always the case.

In the case of Prisoner's dilemma, cooperation can be seen, as said previously, as the action of a altruist player trying to help his opponent and exposing himself to the worse possible outcome. Coordinating on the cooperative strategy is unstable because both players are tempted by the highest payoff that could be obtained by defecting and are also "scared" to be betrayed by the opponent. Coordination on the $(s_2, s_2)$ is the only stable equilibria because in this way the player are protected from the betrayal.

In the Stag hunt game, cooperation also means being exposed to the betrayal of the partner, but in this case the act of cooperation is also enforced by the attempt to obtain the highest possible benefit. In this game coordination is stable in both cases, however coordination on the cooperative strategy is Pareto-dominant and thus preferred.

In the Hawk-doves game, cooperation simply means sharing resources with the opponent. By coordinating on the cooperative strategy both players would obtain a good payoff, however the possibility of higher gains tends to push the players to bully the opponent.

Finally, on the pure coordination case, the only way for players to obtain a benefit is to coordinate. There is no possibility of higher gain in deviating from any of the coordination points and cooperation does have almost any relevance in this setting.

Although coordination and cooperation are close but distinct concepts, in the literature often the term cooperation is used to mean both; for example in the Stag Hunt game. This is also the case of the articles appearing in this thesis.

### 2.3.2 Prisoner's Dilemma

The prisoner's dilemma is one of the fundamental problems in game theory and demonstrate why two personas may not be interested in cooperation even if it's in their best interest to do so. A classic example of the prisoner's dilemma (hereafter PD) is the following:

Two suspects are arrested by the police. The police have insufficient evidence for a conviction, and, having separated the prisoners, visit each of them to offer the same deal. If one testifies for the prosecution against the other (*defects*) and the other remains silent (*cooperates*), the defector goes free and the silent accomplice receives the full 10-year sentence. If both remain silent, both prisoners are sentenced to only six months in jail for a minor charge. If each betrays the other, each receives a five-year sentence. Each prisoner must choose to betray the other or to remain silent. Each one is assured that the other would not know about the betrayal before the end of the investigation. How should the prisoners act?

The payoff matrix for this game is in table 2.4.

Table 2.4: A payoff matrix for the prisoner's dilemma. The payoff represents the years of jail-time, C stands for "Cooperate" and D stands for "Defect".

| | $C$ | $D$ |
|---|---|---|
| $C$ | $(-0.5, -0.5)$ | $(-10, 0)$ |
| $D$ | $(0, -10)$ | $(-5, -5)$ |

It is in the best interest of both players to cooperate, and get only 6 months of jail-time. However both player will be tempted to betray the partner and try to go free. The only NE is exactly $(D, D)$. Each player defects, and they both obtain a bad payoff. A game where the payoff respect the same ordering $u_3 > u_1 > u_4 > u_2$ is also an example of PD and shares the same properties.

Many examples of prisoner's dilemma can be found in real-life. Its wide applicability is the reason of the interest in this game.

**Sports:** This dilemma applies to the use or not of performance enhancing drugs. Each athlete will obtain approximately the same benefit from drugs. So it's an advantage to everyone that no athlete take drugs, because drugs have side effects. However if an athlete takes the drugs, he will gain an advantage over the others, unless all the athletes do the same. When all take the drugs the advantage is cancelled, but everyone is affected by the side-effects.

**Climate change:** All countries would benefit from a stable climate, but any country is often hesitant in enforcing the reduction of the emissions of $CO_2$. The benefit to one country of maintaining the current behavior is greater than the benefit to everyone if all countries comply and modify their behavior.

**Politics:** The arms race between two states can be seen as an example of prisoner's dilemma. Both states can agree in reducing weapons or invest in the army. Both states will profit from increasing the military expenses, no matter what the other does. So both states rationally decide to invest more in the army, even if the result is apparently irrational.

### 2.3.3 Chicken

This game is also often called “Hawks and Doves” or “Snowdrift”. It is another well known metaphor for conflict that captures some important features of social and geopolitical interaction. The principle of this game, is that if a player yields to the other he gets an inferior payoff. However, if neither player yield the outcome is worse for everyone. The game can be described as follows: two drivers, in two cars, both headed towards a bridge. The first drivers who stops or swerves leaves the bridge to the other and loses, he is the “chicken”. If neither one yields, they will both potentially end up in a fatal head-on collision. Theoretically the best thing for both players is to wait for the other player to yield. The crash is the worse possible outcome for both players, not yielding when the opponent does is the best outcome. This leads to a situation where each players try to reach the best possible outcome while risking the worst. This game has also been compared with nuclear brinkmanship [98]. Two parties engage in a showdown with nothing to gain, while an uncontrollable risk can lead to a potential disaster. For this reason this game has been used to describe the mutual assured destruction involved in the Cuban Missile Crisis [94]. In the Hawks and Doves version it represents two animals, competing for some resource. If they agree in sharing the good (doves) they obtain a good outcome. However the best possible payoff is obtained by being a bully (hawk) and scaring the other (dove). The worst case scenario happens when both animals decide to compete for the resource and they end up injuring them selves for more than the value of the good itself.

The payoff matrix for this game is in table 2.5.

Table 2.5: A payoff matrix for the Hawks Doves Game.

| | dove | hawk |
|---|---|---|
| dove | $(2, 2)$ | $(0, 4)$ |
| hawk | $(4, 0)$ | $(-1, -1)$ |

When we formally analyze this game we find three equilibria: (dove, hawk) and (hawk,dove) are two pure strategy NE, the third equilibrium is a mixed strategy. The mixed strategy equilibrium can be found in the following way. Let’s suppose that player 2 will play *dove* with probability $p$ and *hawk* with probability $1 - p$. In this case the expected payoff for player 1 if he plays *dove* will be:

$$E_1[dove] = 2p$$

and if he plays *hawk* his payoff will be:

$$E_1[hawk] = 4p - (1 - p)$$

Player one will be indifferent in playing *dove* or *hawk* when $E_1[dove] = E_1[hawk]$. Which means $2p = 4p - (1-p) \Rightarrow 3p = 1 \Rightarrow p = 1/3$. Assuming that also player 1 also plays *dove* with probability $q$, player 2 will go through the same reasoning and conclude that $q = p = 1/3$. Thus the NE for this game is $\sigma^* = (1/3, 2/3)$ for both players. This kind of game is also called an *anti-coordination game* as it is mutually beneficial for both players to play different strategies.

### 2.3.4 Stag Hunt

This game describes the conflict between social cooperation and safety. It's origin is probably J.J.Rousseau 'Discourse on inequality'. The story is about two hunters, they can try to hunt a stag (strategy $S$), which could feed them and their families for several days, but they have to cooperate to catch one. Or they can hunt for rabbits (strategy $R$), which is an easier pray and can be caught by one hunter alone, but provides food for only one day. Clearly if one hunter decides to go for rabbits and one hunts for a stag the first one will catch something and the second one will be back empty handed.
The payoff matrix for this game showed is in table 2.6.

Table 2.6: A payoff matrix for the Stag Hunt Game.

| | S | R |
|---|---|---|
| S | $(3, 3)$ | $(0, 2)$ |
| R | $(2, 0)$ | $(1, 1)$ |

In this game mutual cooperation $(S, S)$ is the best outcome and it's a NE. This equilibrium is also *Pareto-efficient* because the payoff is optimal for both players. $(R, R)$ is also a NE of the game, this equilibrium is inferior to the other one but it's *risk-dominant* as a player is not exposed to the risk of the other player playing the other strategy. In between these two pure strategy equilibria, there is a third one in mixed strategy. With the current payoff matrix the probability to play $S$ is $p_S = 1/2$ for both players. The dilemma lays here in the "fear" that may lead the players to miss the optimal equilibrium to protect themselves from the other player selfish behavior.

### 2.3.5 Pure Coordination

This class of games model many common situations in society. The idea is that cooperating on a problem will help all parties in realizing a mutual gain. But only making mutually consistent decisions. A common example is the choice of technological standards.
A common example of the payoff matrix for this class of games, for two-players and two-strategies is shown in table **??**. This table can easily be generalized for the case with $n-$strategies for each player by using an $n \times n$ payoff matrix with positive values only on its main diagonal.

Table 2.7: A payoff matrix for a pure coordination game.

| | 1 | 2 |
|---|---|---|
| 1 | $(u_1, u_1)$ | $(0, 0)$ |
| 2 | $(0, 0)$ | $(u_2, u_2)$ |

A simple example of pure coordination game is the *driving game*. Roads have two sides, in some parts of the world people drive on the left side, on some other places they drive on the right side. Both situations are stable and perfectly fine. There is no reason to prefer one or the other and some countries got accustomed to drive on one side of the road, and some others on the other side.

Usually such conventions are stabilized over times, but sometimes are introduced overnight for some other reason. For example recently, in september 2009, the Samoa government decided to change to driving on the left side of the road to reduce the price of the cars. As left-hand drives had to be imported from the Americas while right-hand drives could be cheaply imported from the closer countries. Moreover the government suggested that samoan expatriates in Australia and New Zealand could send used and cheapest cars to their relatives in Samoa.

Another famous example is the overnight change to right side driving imposted by the Swedish government in 1963, against the decision of a 1955 referendum where 83% of the swedish voted to keep the driving on the left.

It is easily seen that apart from the two pure strategies equilibria, a third NE exist in mixed strategy. The probabilities depends on the actual payoff matrix, for example $p = 1/2$ for the driving game.

Bargaining problems like selling and buying goods are also a coordination game. There certainly are conflicting interest in play but in final there should be coordination to conclude the sale.

## 2.4 Evolutionary game theory

Evolutionary game theory (EGT) introduces a dynamical concept in game theory. In real world agents can observe their behavior and sometimes also have a perception of the behavior of other players, and can adapt their strategy to improve their payoffs. Real agents will adapt to their increasing knowledge of the game and the environment. EGT has originated from the work of Maynard Smith and Price in 1973 [65, 64], as an application to the mathematical framework of game theory to biological problems. EGT now has cought the interest of sociologists, anthropologists, philosofers, and economists. Biological evolution, the inspiration for EGT, submit a *population of individuals* to a *source of variation* that should provide diversity and applies a *selection mechanism* to favor fitter variant over individuals who are less adapted to the current environment. EGT translates these concept in the following elements:

- It uses a large population of players, each player uses a fixed strategy when engaged in a two-person game.
- Pair of anonymous players are randomly extracted from the population to play the game and receive the corresponding payoff.
- A selection mechanism ensures that the diffusion on the population of best performing strategies is increased.
- Players are not rational in the sense proposed in the previous sections. Here they only need to apply their built-in strategy.

These elements have been formalized in a selection mechanism called *replicator dynamics*.

### 2.4.1 Replicator Dynamics

For mathematical convenience replicator dynamics (RD) is applied to an infinite population. At each time $t$ the population is in a state that corresponds to a vector of $n$ pure strategies $x(t) = (x_1(t), ..., x_n(t))$ where $x_i(t)$ is the fraction of the population using strategy $i$. This *polymorphic population* state can also be interpreted as a whole population of players playing the corresponding mixed strategy. Pairs of agents are randomly selected from the population to play the game. $u(e_i, x)$ is the payoff of the pure strategy $i$ in state $x$, and $u(x, x)$ is the average payoff of the population defined as : $u(x, x) = \sum_{i=1}^{n} x_i u(e_i, x)$.

If the payoffs are considered like fitness in biology, i.e. they reflect the amount of offspring that will inherit the same trait, the frequency of change of a strategy in the population is proportional to the difference between the average payoff of that strategy and the average payoff of the whole population. We can write the following equation:

$$\frac{dx_i}{dt} = \dot{x}_i = x_i[u(e_i, x) - u(x, x)] = x_i(u_i - \bar{u}), i = 1, ..., n$$

where $\bar{u}$ is the average payoff of the population and $u_i = u(e_i, x)$. This system of differential equations represents the replicator dynamics. From these equations it's easily seen that a strategy that does better than average will end up reinforcing its presence in the population. It's also clear however that a strategy that is absent in the population, cannot appear as basic replicator dynamics does not include the concept of *mutation*.

The next step will be to find if this selection mechanism allows the emergence of stable states in the population state and the differences between these states and the NE found by means of classic game theory. The general result is that, among the stationary states of the dynamics one usually finds the Nash equilibria of the corresponding static game. For details the reader is referred to [122].

### 2.4.2 Evolutionary stable strategy

Evolutionary stable strategies (ESS) have been introduces by Maynard Smith and Price in 1973 [65, 64]. A strategy $x$ is an ESS in a population if a small number of individuals playing an alternative strategy $y$ are unable to invade the population and eventually replace $x$. This notion is simple yet elegant and it is also very close to the concept of the NE. More formally, let's say that all the individuals of a population act according to strategy $x$ and a second strategy $y \in \Delta(S_i)$ is used by the few invaders. The payoff for strategy $x$ when played against $y$ as $u(x, y)$. Let's also define $\epsilon \in (0, 1)$ as the share of mutants in the population. As the pairs of players are drawn uniformly from the population the probability that an $y$ player will be engaged in a match is $\epsilon$, the corresponding probability for an $x$ player is $1-\epsilon$. This is equivalent in playing the mixed strategy $w = \epsilon y + (1-\epsilon)x$. The payoff of the "old" strategy $x$ against the new strategy $w$ is thus $u(x, w)$ and the payoff of the mutant strategy $y$ is $u(y, w)$. Strategy $x$ is an ESS if:

$$u[x, \epsilon y + (1-\epsilon)x] > u[y, \epsilon y + (1-\epsilon)x], \forall y \in (S_i), x \neq y,$$

and the share $\epsilon$ of mutants is small enough.

A second formulation for this same concept can highlight its relationship with the NE; strategy $x$ is an ESS if:

$$u(x, x) \geq u(y, x) \ \forall y,$$

or if

$$u(x, x) = u(y, x) \Rightarrow u(x, y) > u(y, y) \ \forall y \neq x$$

The first condition is equivalent to the NE for the underlying game. The second condition refines this formulation by stating that an ESS is not only at least as good against itself then any other invading strategy, but that the original strategy performs better against the invader than the mutant strategy performs against other mutants. Finally the ESS concept can be seen as a refinement of the NE and $\Delta^{ESS} \subset \Delta^{NE}$, which means that some NE are not ESS, this concept can thus help reduce the number of solutions of a given game. For further details the reader is again referred to [122].

As an example we can look at the Hawks-Dove game presented in section 2.3.3. Let's assume a population of only $Hawks$. If a few $Doves$ appear in the population, as the result of a mutation or an error, the invaders will perform well against the original players, and in the rare encounters with other doves. Given that the ratio $\epsilon$ of doves is small, encounters between doves are really rare. However as $u(D, H) \geq u(H, H)$ the ratio of doves will increase in the population.

consequently $(H, H)$ is not an ESS. The same reasoning can be applied to a population of only doves with a few mutant haws. The outcome will be that also $(D, D)$ is evolutionary unstable. With this process we discarded two NE as not evolutionary stable. The third mixed strategy equilibrium is thus the only NE that can be "translated" in a ESS. The stable state is reached when the proportion $p$ of doves in the population is equal to the probability to play D in the mixed strategy NE for the corresponding payoff matrix.

## 2.5 Structured populations

Standard game theory is about a limited number of players, engaged in an instance of a game. It predicts what a completely rational individual will do to maximize his benefit at the end of the game.

Evolutionary game theory extends this to an infinite number of players, randomly matched to play infinite matches of a game always against a new random opponent. In this case we apply the biological concept of natural selection to refine the solutions found using standard GT and remove those that are unstable. Despite it's advantages, EGT fails to explain the levels of cooperation found on populations of living creatures.

Let's look at the assumptions made by these two theories and compare them to real societies: more specifically, in reality, individuals are not completely rational, their knowledge is limited, population size is limited, and encounters are not random. The first two observations have been addressed with EGT as, in that case agents act on the basis of their traits and not their rationality.

To have a more realistic population, in term of size and possible interactions, in 1992 Nowak and May [84] introduced the use of networks as the underlying structure for the population. In their pioneering work they empirically showed that in a population of PD players placed in the nodes of a regular two-dimensional lattice and only interacting with their Moore neighborhood (the eight closest neighbors, see figure 2.4) cooperation can be maintained.

Although this structure is a quite simple one, the structure and the absence of random mixing seems to enable the formation of clusters of cooperators. The presence of such clusters help the cooperators to improve their wellness as encounters with exploiting defectors are less likely, and also help the cooperative strategy to be spread trough imitation to the agents close to the cluster as they are able to perceive the better performance of that strategy and they are thus pushed by selection to change strategy. These first results on structured population were really promising, however to acquire a better understanding of the role of the population structure, more complex structures such as complex networks have to be studied. As networks will be used to represent and analyze the structure of the populations used in this work, the next chapter will introduce some aspect of networks and their characteristics.

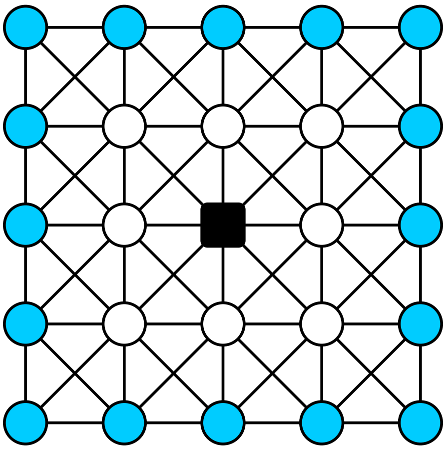


Figure 2.4: In a two-dimentional lattice sigh as the one pictured here, the Moore neighborhood of the central black node, correspond to the eight white nodes around him.

# Chapter 3

# Networks

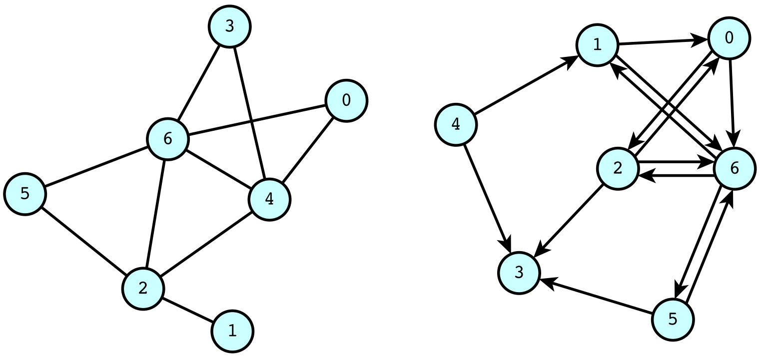
> i have a friend request from some stranger on facebook and i delete it without looking at the profile because that doesn't seem natural. 'cause friendship should not be as easy as that. it's like people believe all you need to do is like the same bands in order to be soulmates. or books. omg... U like the outsiders 2... it's like we're the same person! no we're not. it's like we have the same english teacher. there's a difference.
>
> David Levithan

Networks can be found everywhere. Some networks are well apparent, like roads or power lines. In some other cases networks are just an abstract representation of relationships or dependancies: for example we can build the network of the commercial exchanges between nations or the network of the interaction between a community of chimpanzees. Many objects in many fields like social sciences, physics, and biology can be represented using a network. These networks can provide useful information about the system they represent and allow the study of the system itself and its dynamics.

In mathematics, a *graph* is an abstract representation of a set of *vertices* or *nodes* that are connected by *edges* or *links* (Figure 3.1(a)). A network is special case of a graph where links represents relationships and nodes represents "personas". In this work both terms network and graph will be used to indicate the same concept of *interconnected system of things*. The links in a network can be *undirected* when the relationship is reciprocal or *directed*, to represent one-way relations (Figure 3.1(b)). A network composed by directed edges is a *directed network*, one with undirected links is an *undirected network*.

When a graph is used to represent a real system, this representation generally require a great simplification as many details that characterize the real system will be lost in the process. In the end, even if so much information disappears, what's left often allows the discovery or a better understanding of several phenomena.

As a relatively new science, *Network Science*, is recently gathering much interest. This field is aimed to the discovery of the common phenomena than characterize the behavior of networks and is defined by the National Research Council as "the study of network representations of physical, biological, and social phenomena leading to predictive models of these phenomena." The first known study about networks is "Seven Bridges of Königsberg" written by Leonhard Euler in 1736; in this writing Euler present his solution for crossing every bridge that links the islands of the city of Königsberg representing the island as nodes and the bridges as edges. The definitions and analysis of the swiss mathematician are the foundation of graph theory and lead to the introduction of the concept of topology. In this work the concepts of *structure* and *topology* referred to networks are used with the same meaning, however in their mathematical sense they represent two different concepts.



(a) (b)

Figure 3.1: Example of undirected (a) and directed (b) networks.

As a example of a network of social interactions we can look at Facebook. In this case each link represents a friendship relationship, both users have to agree to create one link, therefore Facebook is an undirected network. This network represents a notable simplification in regard to a real acquaintance network. It's obvious that, in real-life, someone cannot have an interaction with another person if this second person does not also have an interaction with the first one. However there directed interactions can have a different intensity. This difference can be included in the model as a numerical value attached to the link, in which case we will say that the edge is *weighted*. This weight (or *strength*) can be an abstract representation of the importance of a friendship or the number of times this relation has been exploited. If we want to add another degree of complexity we can use inhomogeneous networks. A network is labeled as inhomogeneous when links and/or nodes represent different kind of relationship respectively personas. For example a model for geographic localization of acquaintances will include two kind of nodes: locations and people; and several kind of relationships between these entities like: "friendship" undirected, between people; "knowing someone" directed between people; "home address" between people and locations; and "road" between two locations, etc.

## 3.1 Definitions

In the previous section we briefly presented the basic concepts that define a network. Now we will introduce a few mathematical tools that will allow a characterization of such networks and a better understanding of the properties, the similarities, and the differences of different topologies. A more extensive presentation of graph properties can be found in the book of Newman [78].

### 3.1.1 Graph

A graph $G$ is an ordered pair $G(V, E)$ where $V$ is a set of $N$ vertices and $E$ is a set of binary relationships between two elements of $V$. If $G$ is an *undirected*, respectively *directed*, graph any pair of vertex $i$ and $j$ elements of $E$ is an unordered, respectively ordered, pair. When an edge $(i, j)$ exist in the graph, the vertices connected by this links are called *neighbors*.

A *path* between two vertices $i$ and $j$ is a sequence of links that should be traversed to reach $j$ when starting from $i$ with no edge traversed more than once. The length of such path is the number of edges in it. The shortest path between $i$ and $j$ is the shortest possible sequence and the *distance* between $i$ and $j$ is the number of edges in the shortest path between $i$ and $j$. The *characteristic path length* $L$ of a network is the average of the shortest distances between the nodes of the network.

A graph is *connected* when a path exist between each pair of nodes $i, j \in V$. An undirected graph is *completely connected* when $\forall i, j \in V, \exists (i, j) \in E$. The total number of nodes of a complete graph is $|E| = N(N-1)/2$. A *sparse graph* is a graph where $|E| \ll N(N-1)/2$. A completely connected subgraph with $M < N$ vertices is called an *$M$-clique.*

The *neighborhood* of a node $i$ in graph $G$ is normally denoted as $\mathcal{N}_G(i)$ (or simply $\mathcal{N}(i)$) and represents all the nodes that are connected to $i$ with a link: $\mathcal{N}_G(i) = \{j \in V | (i, j) \in E\}$. That is also the set of nodes that are at distance one from $i$.

The *degree* $k$ of a node $i$ is the number of edges departing from this node; it's equivalent to the number of neighbors of $i$. In the case of a directed graph every node has two distinct degrees: $k_{in}$ and $k_{out}$ that represent the number of links arriving, respectively departing, from that node. The *average degree* $\bar{k}$ (or $\langle k \rangle$) is the mean value of all the vertex degree of a graph $G$. In the case of a directed graph note that $\bar{k} = \bar{k}_{in} = \bar{k}_{out}$

A *regular graph* is a graph where all the nodes share the same degree $k$. In this case the average degree $\bar{k}$ of the graph is equal to the degree $k$ of every node and is often simply noted degree of the graph $K = \bar{k}$.

The *degree distribution* of a graph is the probability distribution of the degrees $P(k)$ of the whole network. It represents the probability for a vertex $i$ to have a particular degree $k$. The *cumulative degree distribution* represent instead the probability for a node $k$ to have *at least* a degree $k$.

The *assortativity* represents the preference for the nodes of a networks to be attached to other nodes that are similar (or different) in some way. Usually, assortativity in networks is mesured through neighbors degree, in this case the assortativity will represent the correlation between the degree of the nodes. A networks is *assortative* when the highly connected nodes tend to be connected to other high degree nodes or *disassortative* when highly connected nodes are mainly linked low degree nodes [74, 78].

### 3.1.2 Clustering Coefficient

The clustering coefficient is a measure of how much the nodes of the graph are clustered together. In other words it represent the probability of two neighbors of a node $u$ to be connected. The clustering coefficient $\mathcal{C}_i$ of a single node $i$ is defined as the ratio of existing edges $E_i$ between the neighbors of $i$ and the total number of edges that could exist:

$$\mathcal{C}_i = \frac{2E_i}{k(k-1)}$$

because in an undirected graph, $k(k-1)/2$ links can exist between the neighbors of a node with degree $k$.

The clustering coefficient of a graph, often simply called clustering coefficient, is the average value of $\mathcal{C}_i$ for all the nodes $i \in V$:

$$\mathcal{C} = \frac{\sum_{i=1}^{N} \mathcal{C}_i}{N}$$

The value of the clustering coefficient is in general quite high in real world network and especially social networks because "the friends of someone tend to also be friends".

### 3.1.3 Communities

Another characteristic of a network can be the existence of a community structure. A community is a subgraph $G'$ within $G$ with an high density of links between the nodes of $G'$ and only few links between the elements of $G'$ and the rest of the graph. It's not possible to find a community structure in all networks, but many real-world network have one.

The interest in communities stems from the fact that knowing the community structure of a given network can provide useful insight about the functioning of the network and how the topology influences it. For example, the community structure that can be found on Zachary's "karate club" network [130](Fig. 3.2), predicts almost exactly the communities formed by the members of the club after part of them left the original club to found a second one.

Finding communities within an arbitrary network is a quite difficult task because the number and size of the communities cannot be know in advance. Moreover, the size of the communities of a network is often very different and these communities present a different density.

*Modularity* is a measure of the quality of a division of a network into communities [77], the value of the modularity $Q \in [-1, 1]$. The higher this value the better is the division. A value of $Q > 0$ qualifies a structure where the number of edges between the nodes of a group is greater

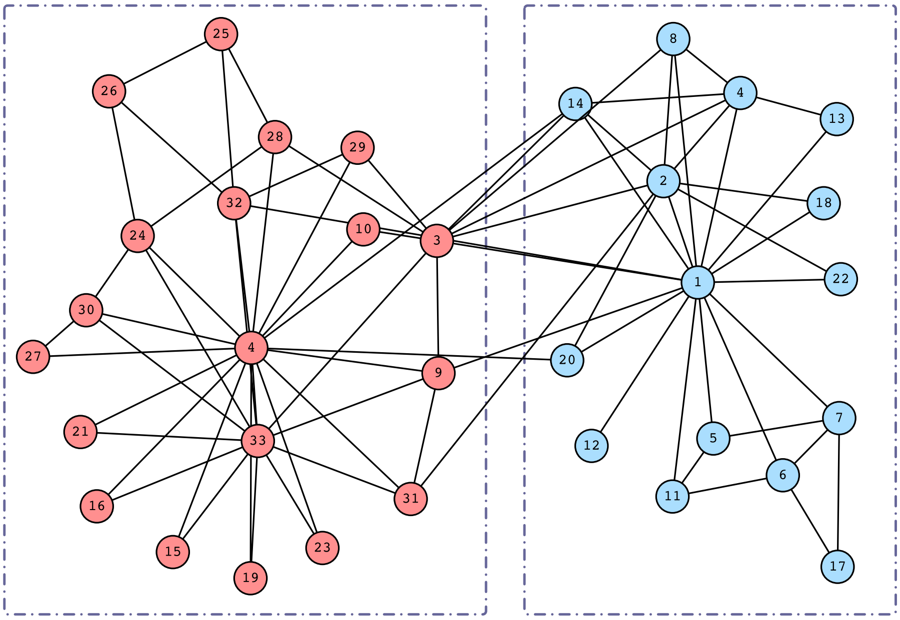


Figure 3.2: The communities on Zachary's Karate Club graph.

than the number expected on the basis of chance.

## 3.2 Network Topologies

### 3.2.1 Regular networks

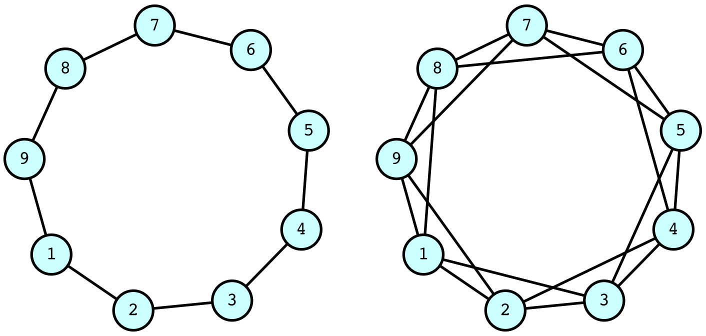
Regular networks have a constant degree $K$ for all the nodes, and, in the case of lattices, an high characteristic path length. Community structure is absent but the clustering coefficient can be high. The most common regular networks are probably one-dimensional and two-dimensional lattices, commonly known as rings and grids respectively. A ring can have different values of connectivity, in figure 3.3 an example with of $K = 2$ and $K = 4$ is shown. Grids can also have different degrees $K$ the most common are 4 and 8.

In order to have a completely regular grid, the structure should be wrapped around on itself to form a three-dimensional ring, called torus. In practice the nodes on the first row have to be connected to the nodes on the last row, and the nodes on the first column should be connected to the nodes on the last column (see figure 3.4).

This kind of network is commonly used in mathematical models because of their regular properties. However real world networks often present structures that are not of a regular type.

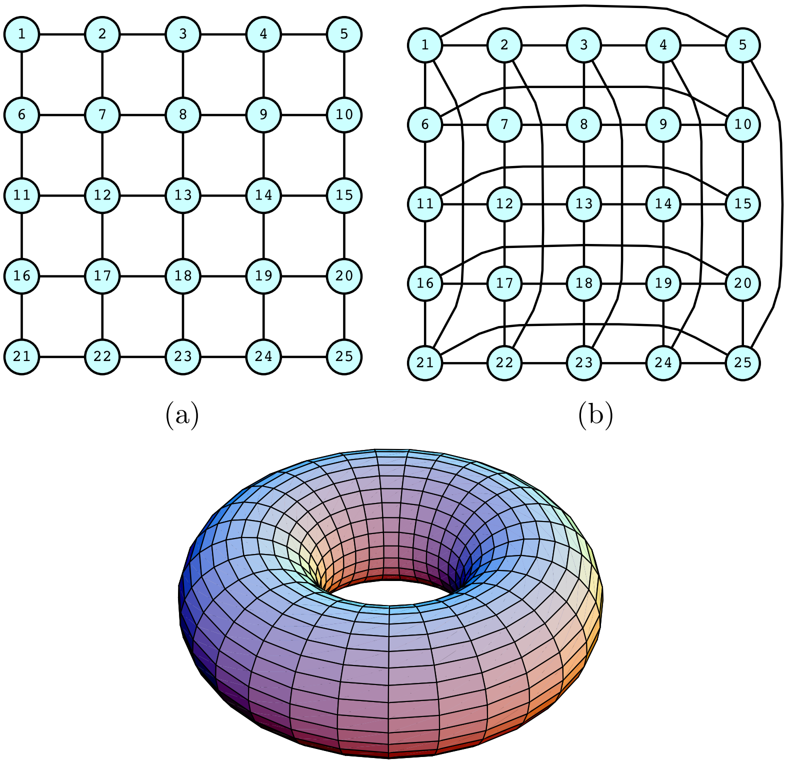
### 3.2.2 Random graph

Another common structure is the *Random graph*. In this networks pairs of edges are connected at random. The most common model for these networks is the one created by Erdös-Rényi [27].



(a) (b)

Figure 3.3: One-dimensional lattices with $K = 2$ (a), and $K = 4$ (b) respectively.

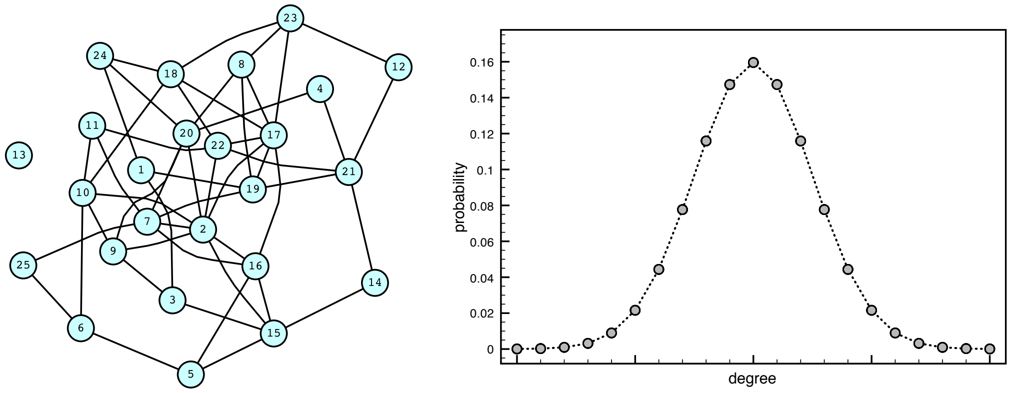


(a) (b)

(c)

Figure 3.4: Two-dimensional lattices with $K = 4$. In (a) the degree is not constant. To have a homogeneous degree the links in (b) should be added. A three-dimensional representation of a two-dimensional lattice is a torus like the one in figure (c).

In this model the probability $p$ of creating a link between each possible pair of node is given. A family of graphs $G$ with $N$ nodes and probability $p$ is defined as $G(N, p)$; this graph has $|E| = p(N(N-1)/2)$ links. A second variation of this model also exist. In this second version instead of defining the probability of a link we decide how many links in total will be present in the network. This model is defined as $G(N, m)$, where $m$ is the total number of links. As $m = |E|$ it's easy do calculate the probability $p$ for this graph. This second variation is especially useful when generating small graphs when the stochastic of the process generate a variable number of links at each try. The clustering coefficient of a random network is close to $p$. The average path length is short compared to that of regular networks.



(a) (b)

Figure 3.5: (a) A random graph and an example of binomial distribution (b).

The degree distribution of a random graph is a *binomial* distribution with the average degree $\bar{k} = p(N-1)$ of the graph as the center of the characteristic bell.

Another variant of random graph are *regular random graphs*. These graphs have a constant degree $K$ but the the neighborhood is random. These structures maintain the clustering coefficient $\mathcal{C} = p$, and a short average path length like the other random graphs, but the degree of the nodes is now constant like in more regular networks.

### 3.2.3 Complex networks

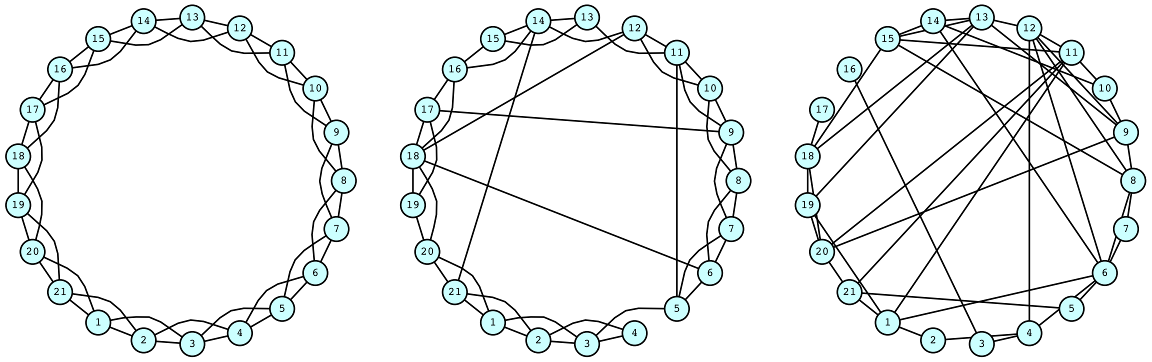
When looking at real networks in several fields like economy, biology, engineering, sociology, etc. one can see that these networks are not regular nor random. The study of complex networks is still young and its largely inspired by the increasing amount of empirical data on technological and social networks.

Complex networks display non-trivial properties that are not found in regular or in random graphs. These properties are for example, a community structure, high clustering coefficient, fat-tailed degree distribution, (dis)assortativity, etc.

**It's a Small-World:** In a short story published in the 1929, Frigyes Karinthy, an hungarian author, proposed the concept of the six-degrees of separation. The idea is that everyone can reach any other person in the world through a chain of friends of friends of maximal length 6. This concept was at the base of an experiment conducted by the american social psychologist Stanley Milgram, an published in 1967 [67]. Milgram's experiment consisted in confiding to randomly chosen persons, a letter addressed to someone in a distant social and geographical situation. The letter could only be given to a known person in the aim of approaching the selected target. Most of the chains where interrupted but the average length of the chains that arrived at destination was 5.5. This result was a probably the first demonstration of the *small world problem*, i.e. that even in a large network such as the US citizenes, short path existed between complete and distant strangers.

Small-world networks are networks that mimic this property: the average path length is relatively short. Watts and Strogatz proposed a simple procedure to a particular class of small-world networks in [127]. These networks are constructed starting with a regular one-dimensional lattice and then replacing with a probability $p_r$ a current link with a new one between one of the two old nodes and a new random one.

This algorithm allows a transition between a regular lattice with $p_r = 0$ and a random graph when $p_r = 1$. When $0 < p_r < 1$, or more precisely for a small interval of values of $p_r$ between 0 and 1, the rewiring process allows the creation of graphs that have at the same time a small average path length and an high clustering coefficient (see figure 3.6).



$p_r = 0$
(a)

$0 < p_r < 1$
(b)

$p_r = 1$
(c)

Figure 3.6: The regular lattice (a) and the random graph (c) are the extreme cases for a Watts-Strogatz small-world graph. A regular small-world graph with the typical short-paths (b).

**Scale-free networks:** A scale-free network is characterized by it's degree distribution. The networks of this class have a non-negligible number of highly connected nodes, called *hubs* and many nodes with a low degree. The resulting degree distribution $P(k)$ has a heavy tail and follows a power-law distribution: $P(k) \sim k^{-\gamma}$, normally the constant $\gamma \in [2, 3]$ but it is sometimes outside these bounds. This class of complex networks is important because many real world networks have a fat-tailed distribution including protein networks, citation networks, neural networks, and some social networks. The interest in scale-free networks emerged in 1999 when Barabási et al. [2] mapped a portion of the Web and showed that a small portion of the pages, the hubs, have many more connections than the rest and the degree distribution follows a power-law. Barabási et al. [3] proposed an interesting mechanism to explain the emergence of networks with a power-law distribution called *preferential attachment*. The idea is inspired by the evolution of the World Wide Web and lays on the idea that when new pages is created this pages will, with high probability, have links to well know, and thus well connected, pages.

The *Barabási-Albert's algorithm* progressively builds a graph starting from a clique with $m \ll N$ nodes, and then adding one-by-one the other $N - m$ nodes. When a node is inserted in the networks $m$ new links are created starting from this new nodes and are attached to $m$ existing nodes selected with a probability proportional to the degree they already have. This model allows the creation of networks with an exponent $\gamma \approx 3$. An example of a small network build using this algorithm can be seen in figure 3.7.

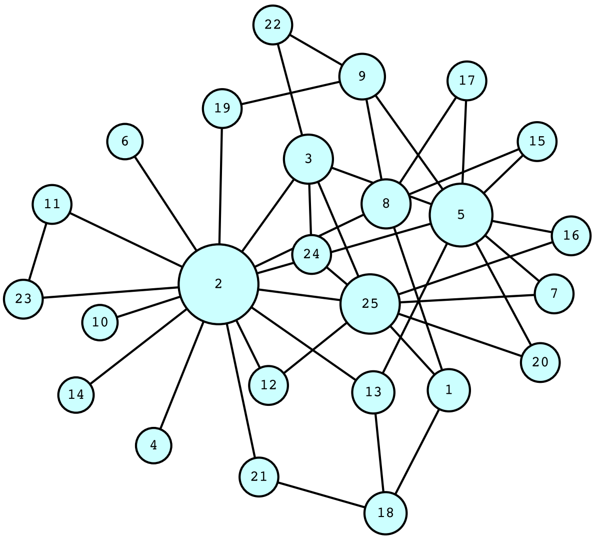


Figure 3.7: An example of a small scale-free graph. The size of the nodes is proportional to the degree: the big vertices are the hubs of this network.

Another popular way to build scale-free networks is the *configuration model*. In this case the distribution is decided and the degree assigned to the node before creating any link. Then the algorithm matches the nodes in order to create the links, while avoiding self-loops, and multiple-links between the same nodes. With this method one can create scale-free graphs with exponent $\gamma \neq 3$.The configuration model has the advantage of avoiding the correlations between the nodes degrees introduced by the Barabási–Albert model, however establishing the links and avoid the deadlock is often too time consuming.

**Social networks:** Social network are not characterized by a precise mathematical property or a precise set of properties. They are simply structures where the nodes are individuals and the ties represents a relationship such as kinship, friendship, common interest, sexual relationship, economic exchange, scientific co-authorship, and so on. The first example of such networks have been constructed "on the field" by direct observation, like the karate club network in figure 3.2; by mean of interviews and questionnaires; or using archives or third-party records, like the co-autorship network shown in figure 3.8.

Social networks are indeed very varied: both by simple networks characteristics like size, average degree, etc. but also by the kind of social structure they represent. Despite all their differences these networks have also many similarities from a statistical point of vie: like the small-world property; a fat-tailed degree distribution, even if often this distribution is not power-law but rather an exponentially truncated or stretched exponential; they also present a high clustering coefficient and a strong community structure. Moreover they often show positive assortativity, which means that highly connected agents tend to associate with other highly connected individuals. A famous example of social network is the Zachary's “karate club” [130], this network is pictured in figure 3.2. This network was constructed by direct observation and represents the pattern of friendship between the members of a karate club in an american university. It's a small structure with 34 agents but gathering the data by direct observation took about two years of work. The network in figure 3.8 [117] represents the scientific collaboration in the relatively young community of the scientists working on Genetic Programming. The image represents the giant component of the network and include $\sim$ 1000 authors. Nowadays online services like Facebook can provide data on huge networks with thousands of agents.

**Model social networks:** Social networks are, as said in the previous paragraph, representation of the social interaction of individuals. In this thesis, we also make use of “model social networks”, the nature of which we explain as follows. Model social networks are networks generated by means of an algorithm, that display characteristics close to social networks collected by empirical study. To differentiate these two classes of networks we often use the terms “real social networks” and “model social networks”. An example of a model for creating social networks is presented in the next chapter, in section 4.5.

### 3.2.4 Dynamic networks

Networks exist all around us, however these networks are not frozen, they are in constant evolution. Agents may join or leave a networks, new bonds can be established and old ones can disappear, the strength of a link may change, etc. When we analyze a real network we in fact take a picture of this network and use this static image. This kind of procedure will lead to accurate results especially if the network evolution is rather slow, as is the case for a network of scientific co-authorship or for the air transportation network. For an example we can look at the evolution over 20 years of the GP co-authorship network [115] (in figure 3.9). The changes are pretty obvious, and in particular in the time frame between 1991 and 1997 the network undergoes profound transformations due to the increasing interest in the topic and a snapshot of the network in this specific time frame will not be representative of the previous and successive state of the network. However the current characteristics of the network are evolving slowly and for this reason working with a snap shot can provide useful insight on the properties of this graph.

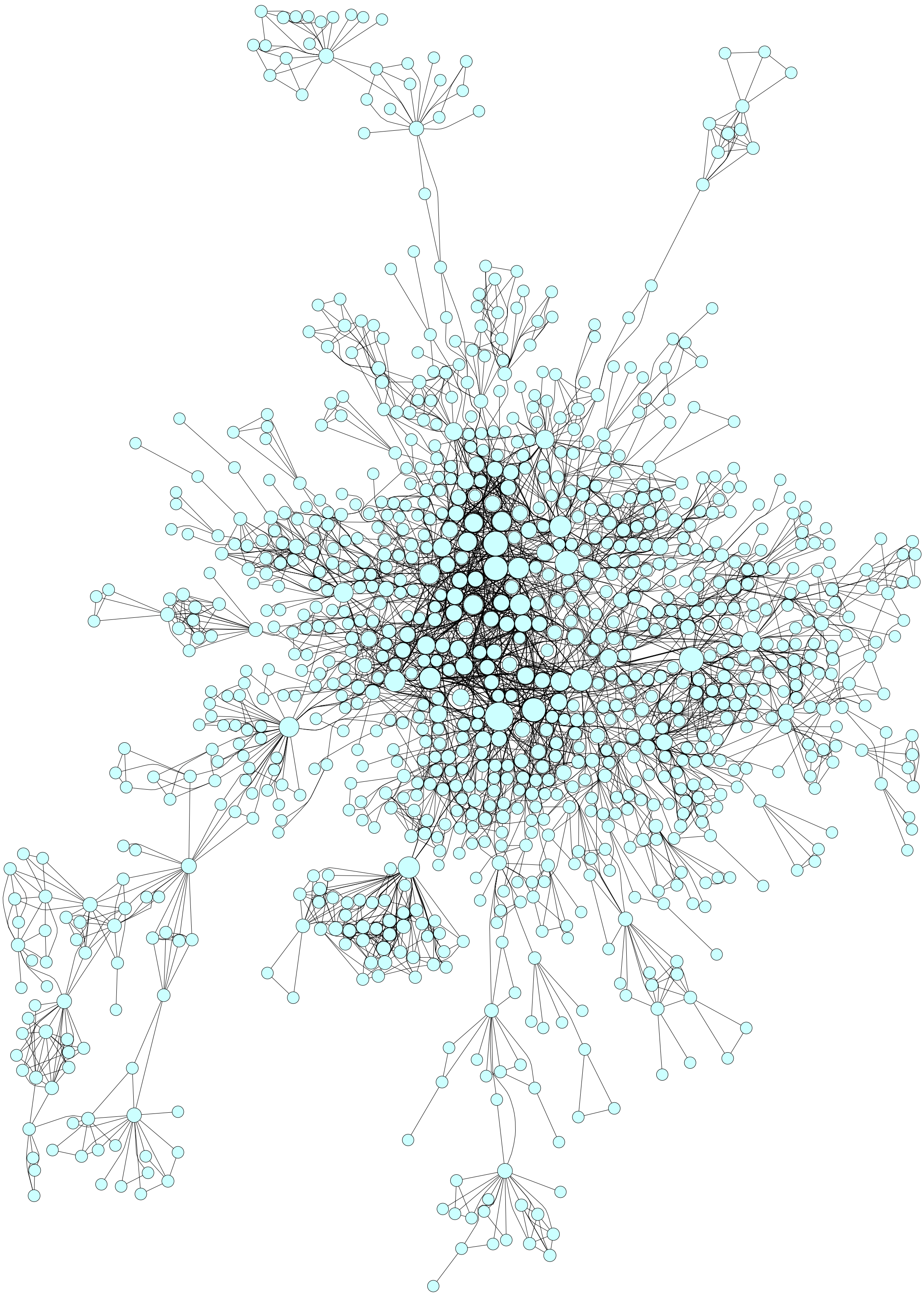
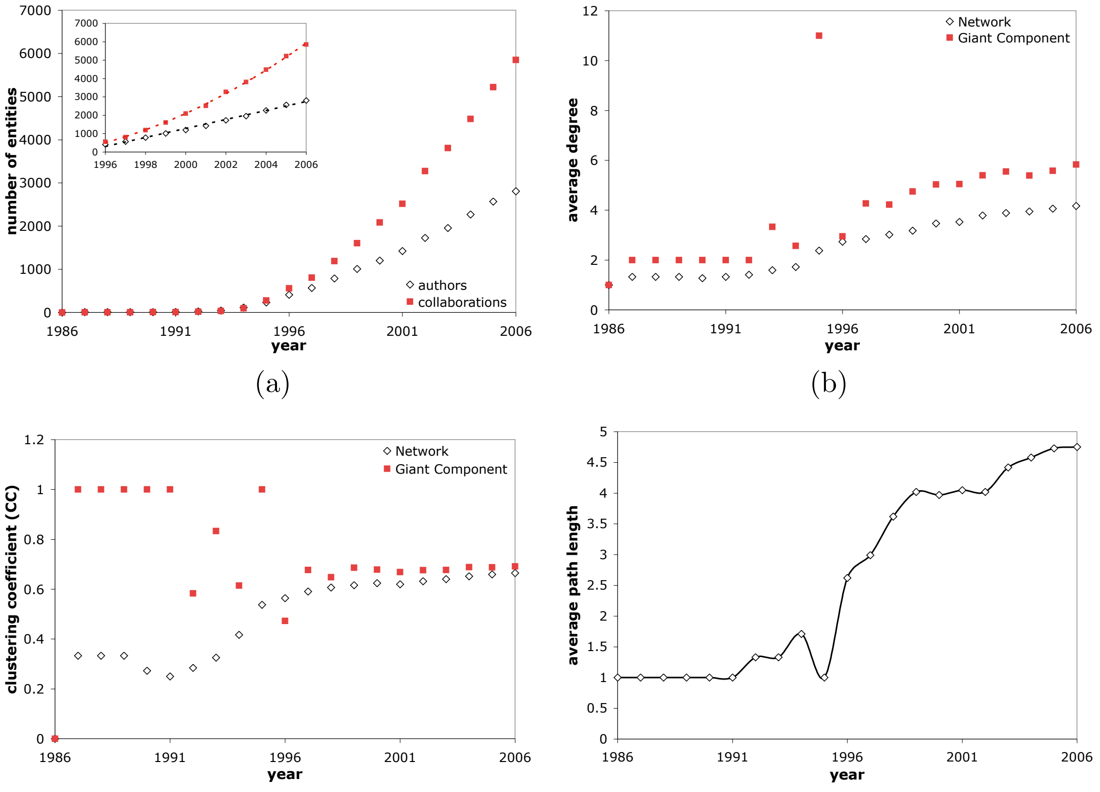


Figure 3.8: An example of a social network: the co-authorship network of the Genetic Programming community in 2007 (W.B.Langdon, http://www.cs.bham.ac.uk/∼wbl/biblio/).



(a) (b)

(c) (d)

Figure 3.9: The evolution of the network of co-authorship of the Genetic Programming community (W.B.Langdon, http://www.cs.bham.ac.uk/~wbl/biblio/). (a) shows the number of authors and collaborations within the community, (b) represent the evolution of the average degree of the nodes of the network. (c) reports the values for the average clustering coefficient and (d) shows the average path length of the network.

On the other hand, networks for other phenomena show a faster evolution. For example the analysis by Kossinets and Watts in [54] of the network of e-mail exchanges within the $\sim 40000$ members of the community of a large US university, shows that the properties of this kind of structure are highly variable.

Because of the importance of the dynamics of the network, we included this aspect in the second part of this work.

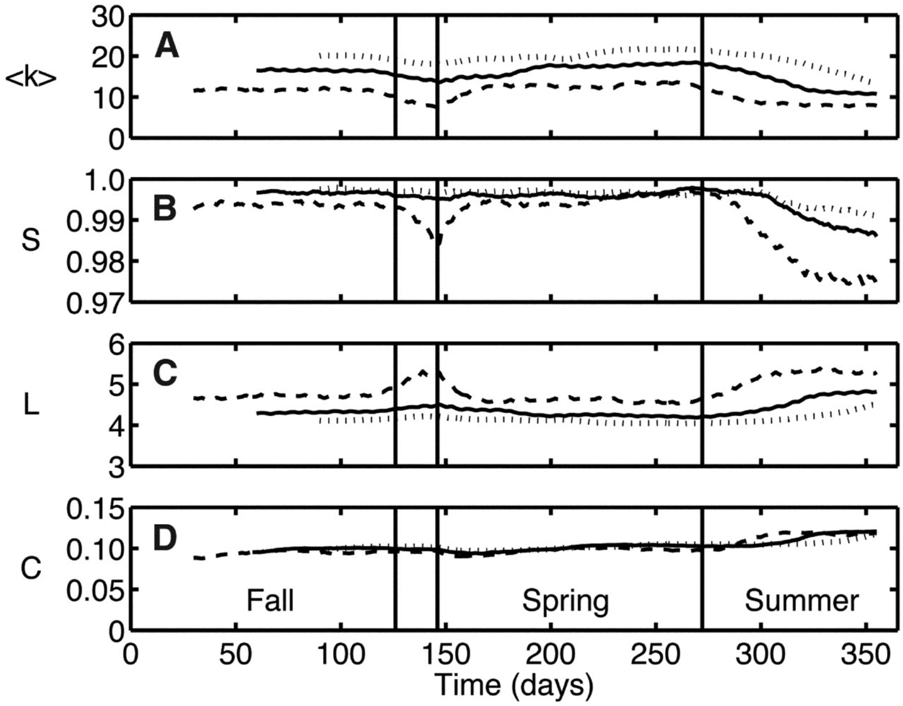


Figure 3.10: The evolving characteristics of the networks of email exchanges analyzed in [54]. The dashed line represent a smoothing window of $\tau = 30$ days, for the continuos line $\tau = 60$ days and for the dotted line $\tau = 90$ days. Image (A) represent the average degree of the network, (B) the fractional size of the largest component, (C) the average path length in the giant component, and (D) the clustering coefficient. This image has been redrawn from [54].

# Chapter 4

# Games on networks

> Today the network of relationships linking the human race to itself and to the rest of the biosphere is so complex that all aspects affect all others to an extraordinary degree. Someone should be studying the whole system, however crudely that has to be done, because no gluing together of partial studies of a complex nonlinear system can give a good idea of the behavior of the whole.
>
> Murray Gell-Mann

In part II, III, and IV of this manuscript we will present part of the results we obtained in the last few years in the form of the articles published in some specialized international journals.

In the three articles included in part II we will deal with several kinds of static networks and see how the topology of these networks and some different update mechanism effects the cooperation between the agents.

In the two articles included in part III we will look at the behavior of the agents when they can not only adapt their strategy to their neighbors, but also modify their neighborhood.

In the last two articles, in part IV we will deal with a particular family of games: coordination games, both from the static and the dynamic points of view, in this case there is no social pressure to prefer one strategy to the other but where only coordination leads to a positive benefit.

## 4.1 How simulations work

It can be useful to introduce some additional nomenclature and definitions and to explain how the simulations used to collect the empirical datas in this works are made. To simulate the evolution of a strategy in a population we run a computer program that requires a few elements. First of all a *population* of players is represented by an undirected graph $G(V, E)$, where the set $V$ of vertices represents the agents, and the set $E$ of edges, represent their undirected interaction

between pair of agents $\in V$. We also recall that the neighbors of a player $i \in V$ is a set $\mathcal{N}(i) = \{j \in V, \exists\ (i,j) \in E\}$ (see Section 3.1).
The payoff of an agent $i$ is noted $\Pi_i$ and it is obtained by combining the individual utilities $u$ obtained by $i$ when interacting with his neighbors $\mathcal{N}(i)$.

**An algorithm to generate the network topology:** this can generate a model network, like a Barábasi-Albert scale-free network using a few parameters, or simply read a file containing a pre-defined, precise topology.

**A game:** which is defined by its payoff matrix. In this work we use two-person, two-strategies symmetric games, thus to define a game we have to define four payoff values.

**An update rule:** that will be used by the agents to decide how to update their strategy when a decision to update has been taken.

The initial strategy of every player in the network is also to be defined before starting a simulation. Generally half of the agents are randomly selected to use strategy 1 and the other half will use strategy 2, but to analyze the performance of a strategy in a particular situation sometimes the initial ratio is different.

The timing of the interactions between players is also an important aspect in these simulations. In section 4.4 we explain this factor in more detail.

Once that these elements have been defined, the simulation proceeds to generate a network, initialize the strategy of the players in the network, and then allow the agents to interact (*play the game*) with their neighbors and update their strategy using the update rule. This process goes on until the population reaches a pseudo-equilibrium. We use the notion of *pseudo-equilibria* to indicate a situation where the population is not completely frozen but small fluctuations continue to occur with no effect on the global trend of the population in the long-term.

To obtain usable statistical measures of the behavior of the agents these simulations are repeated many times using the same or different parameters and generally for several different payoff values.

In the following sections there is a brief description of the update rules used in this part of the manuscript.

### 4.1.1 Limitations

It is clear that this kind of model is not able to completely capture the complexity of a real social system. On the other hand, a simplification may help the isolation of important factors and key mechanisms that allow cooperation to be sustained and also to grow in certain situations.

For example, the assumption that an individual is using only one strategy for all his interactions in a given time step, represents a great limitation when compared to reality, where for every interaction an agent can use a different strategy in order to improve to a maximum his benefit. By using this simplification we of course loose some details, but we simplify the model, which is important both from an analytical and from a computational point of view. Moreover, we are interested in phenomena that allows cooperation to be sustained in the whole population, and not to maximizing the benefit of single players.

Another limitation of the present model is the fact that the global behavior is seen as the sum of pairwise interactions between players. This is a common approximation and most work in the field uses it [113, 101, 96, 32]. However, there is also a lot of interest in $n-$person games, especially in studies about the provision of public good games [122, 40, 18]. In this thesis we have limited ourselves to two-person games only.

## 4.2 Update rules

An update rule allows an agent to select her new strategy by looking at some feature of the current population. Most of the update rules used here are based on the utility of the agents.

**Local replicator dynamics:** This update rule has already been explained in section 2.4.1, but the algorithm used in the local version is slightly different. In a networked population the agent $i$ will randomly choose one of her neighbors $j$ and compare her fitness $\Pi_i$ with the fitness of this neighbor $\Pi_j$. Player $i$ will keep her current strategy if $\Pi_i \geq \Pi_j$ otherwise she will adopt the strategy of player $j$ with a probability $p_{j \to i} = \Phi(\Pi_j - \Pi_i)/\mathcal{N}$, where $\mathcal{N}$ is a normalization factor and $\Phi$ is a monotone increasing function. In practice $i$ can only imitate the strategy of a fittest player, moreover the better this player is, the more probable it will be for $i$ to adopt her strategy.
The normalization factor $\mathcal{N}$ depends on the function used to calculate the utility of a player, and is needed in order to obtain a probability $p_{j \to i} \in [0, 1]$.

**Imitate the best:** This update rule is simpler. When it is used an agent $i$ will select her fittest neighbor $j$ and adopt $j$'s strategy if $\Pi_j > \Pi_i$. When this update rule is applied the decisions taken by an agent are completely deterministic. In case of a tie, eighter the agent retains her strategy or will switch with probability equal to 0.5.

**Conformism:** This update rule has been used in the third paper of this collection (article C) and does not take into account the utility of players. In this case a player $i$ simply takes one of her's neighbors as a cultural model and adopts the strategy of this neighbor if this is the same strategy used by the majority of $i$ neighbors. This update rule has been introduced because humans do not only tend to imitate the fittest individual, but they also have a tendency to comply to the behavior of the majority [16].

**Myopic best-response:** When this rule is applied, a player is able to decide wether her current strategy is optimal or not. If the strategy is not optimal the player, with a probability $p$, will decide the optimal strategy for her next move on the basis of how her neighbors have

acted on the last interaction. This rule has been used in the last two articles of this collection. If the strategy has to be updated, agent $i$ choses the strategy $s_i$ that will maximize her payoff $\Pi_i$, under the assumption the the neighbors will not change their strategy. This rules assumes that players are able to decide if their strategy is optimal and also that they have the cognitive abilities needed to select the best strategy amongst the possible ones. This rule also allows the emergence of strategies that are possible but not present in the population as the evolution does not rely on imitation.
The rule is *myopic* as the agent do not consider the effect of her eventual change of strategy on the evolution of the game.

**Other update rules:** Other updates rule are for example the *Fermi rule* [113] and the *Death-Birth rule* [81].
Under the Death-Birth rule a player imitate the strategy of one of her neighbors (or her strategy) with a probability proportional to the payoff; when this rule is used a player $i$ can imitate an agent that is currently obtaining a payoff lower then $i$ herself.
The Fermi rule is based on the Fermi distribution. With this rule, a neighbor $j$ is randomly selected and player $i$ will imitate his with a probability depending on the difference between $j$ and $i$ payoffs and on a fixed parameter $\beta$ that controls the intensity of selection. The probability for player $i$ to imitate the strategy of $j$ on the next activation is:

$$p_{j\to i} = \frac{1}{1 + exp(-\beta(\Pi_j - \Pi_i))}.$$

These last rules have not been analyzed in the works presented here.

## 4.3 Utility calculation

The utility calculation also has an important role in the success of a strategy. Two particular examples are *accumulated payoff* and *average payoff*. The former consists in summing all the payoffs obtained by agent $i$ in the encounters with all her neighbors, in this case the payoff of a player $i$ would be:

$$\Pi_i^{ACC} = \sum_{j\in\mathcal{N}_i} \pi_{ij},$$

where $j$ is a neighbor of $i$ and $\pi_{ij}$ is the payoff obtained by $i$ when playing with $j$. The average payoff instead is the average value of the payoffs obtained by $i$ when interacting with her neighbors:

$$\Pi_i^{AV} = \frac{\sum_{j\in\mathcal{N}_i} \pi_{ij}}{k_i},$$

where $k_i$ is the degree of $i$, i.e. the number of neighbors $|\mathcal{N}_i|$.

Using a different way to calculate the utility of an agent may introduce several changes in the behavior of the players. For an example, when considering degree heterogenous networks, a lousy player with many neighbors may have a greater accumulated payoff than a more efficient player with only a few neighbors. In the opposite case, when using average payoff, there is almost no difference in having few or many neighbors; on the contrary, having few good neighbors is probably better than to have many average neighbors. As some strategy update rules rely on the utility of the players to decide who to imitate and whether to switch to another strategy, the way this utility is calculated can introduce great differences in the system.

In article A we look at the different results obtained when these two utility calculation function are used and we also try to combine them. Moreover we show that when using the accumulated payoff, the behavior of the agents is no longer only dependent on the order of the payoff values since an affine transformation of the payoff matrix can completely change the quasi-stable states of a game (when using replicator dynamics).

In [61] we fully analyze this behavior and propose a different utility calculation function that preserve the invariance of a game under affine transformation of its payoff matrix and retain part of the advantages of the accumulated payoff (when replicator dynamics is used).

## 4.4 Timing

Basically, two different timing policies can be used. In the *synchronous* case, the interaction and the update of the behavior of the players are synchronized, and all (virtually) happens at the same time. This assumption insures that every agent is treated equally in term of number of times she is able to revise her strategy.

The opposite update timing is the fully *asynchronous* update. In this case, at each time step an agent is randomly chosen with uniform probability, she will then interact with her neighbors, collect a utility, apply her strategy update and decide if she wants to revise her strategy or not.

A third update mode, called *semi-synchronous*, is used in the articles of part III and consist in randomly choosing $m$ agents in the population, where $m \ll N$, and updating this agents in a synchronous way. Obviously when $m = N$ we recover the fully synchronous case, while when $m = 1$ we are back to the fully asynchronous one.

## 4.5 Network topologies

In these simulations we used several network topologies. Random graphs, regular lattices, and Barábasi-Albert scale-free networks have already been presented in chapter 3, we also used the real social network of the GP coauthors, also presented in chapter 3.

However to better compare the results obtained on these populations with what could be expected from a real society we also used a model to generate social-like networks proposed by Toivonen et al. in 2006 [114].

### Toivonen model

This model has been conceived to generate a graph with most of the features of a real social network. The generated networks are assortative, have an high clustering coefficient and also have community structure. The degree distribution is also long-tailed and presents a finite cutoff.

This algorithm incrementally grows a network, starting from a seed of $m_0$ randomly connected nodes. This is how the topology evolves at each successive time step:

- On average $m_r \geq 1$ are choosed to be initial contacts
- On average $m_s \geq 0$ of each initial contact are selected to be secondary contacts.
- A new vertex $v$ is added to the network and connected to all the initial and secondary contacts determined in the previous steps.

This process is repeated until the network reaches the desired size.
The network growing process is completed before the simulation begins, thus the network used in the actual simulation is static.

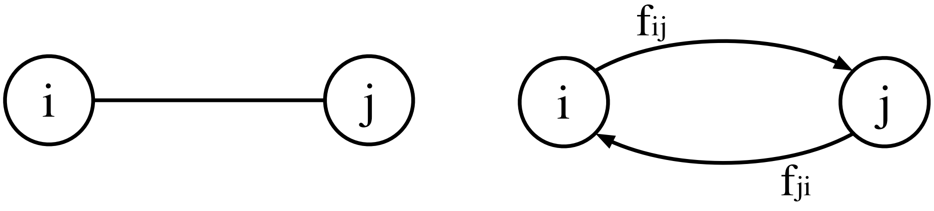
## 4.6 Dynamic networks

In part III and in the second article of part IV, we introduce and use dynamic networks to try to better understand how the profit and the behavior of agents may influence the evolution of the structure of the population.
This new aspect has been introduced because in reality, networks structures are not static but evolve over time and are indeed influenced by their environment, as explained in Section 3.2.4.

### 4.6.1 Our dynamic network model

This model used has been developed with an eye on how social interactions work. Here we report a brief description of this model, a more detailed explanation can be found in section D.3 of the first article in part III.

**Underground network structure:** In this model the original undirected graph $G'$ that defines who interacts with whom, is coupled with a directed graph $G$. All undirected links in $G'$ (figure 4.1(a)) are projected in $G$'s into pairs of directed and weighted links (figure 4.1(b)). In this new graph the weights or forces $f_{ij}$ and $f_{ji}$ represent respectively the “trust” player $i$ has in $j$ and the trust player $j$ has in $i$. These forces are represented by numerical values $\in [0, 1]$, they have a value of 0.5 at the beginning of the simulation, and they evolve independently based on the quality on the interactions between $i$ and $j$, each agent here modifies the trust of her outgoing links based only on her perceptions, thus the forces model her subjective view of her neighbors.



(a) (b)

Figure 4.1: The link between $i$ and $j$ in the undirected graph $G'$ (a) and the same link once projected in the directed graph $G$ (b).

When the players interact and update their strategy they only use the information embedded in the undirected network $G'$. Each encounter has the same value for an agent and each neighbor has the same influence when a player is looking to update her strategy.

**Updating the trusts:** The interactions influence the trust the players have in each other. The trusts between two agents $i$ and $j$, represented by $f_{ij}$ and $f_{ji}$ are modified to represent the perception of each agent of the interaction that just happened. To update this trust a player considers the difference between the payoff $\pi_{ij}$ he just obtained and the payoff $\bar{\pi}_{ij}$ she could have obtained if player $j$ had played the opposite strategy, this value is normalized by taking in to account the maximal and minimal utility an agent cam obtain from a single interaction and the number of neighbors agent $i$ has. The formula for thus trust update is the following:

$$f_{ij}(t+1) = f_{ij}(t) + \frac{\pi_{ij} - \bar{\pi}_{ij}}{k_i(\pi_{max} - \pi_{min})}.$$

Given that $f_{ij}$ have to be $\in [0, 1]$ if the updated value is outside this range, the final updated trust is set at the closest bound of the interval. The number of neighbors of an agent has an impact on the speed at which the trust varies: the higher the number of neighbors and the slower is the trust variation. We chose this option to represent the attention an agent is capable to pay to her relationships. When there is only few neighbors, it's of great importance to be more selective, maintain the good partners and try to replace the bad ones as soon as possible. On the other hand, when an agent has many ties, it's more difficult to invest the necessary energy in every relationship, for this reason they are condemned to evolve more slowly.

**Updating agent behavior:** When a player is called to update her behavior she can either decide to update her strategy or to try to replace a bad relationship with a better one. The choice between these two possibilities is regulated by a parameter $q \in [0, 1]$ that represents the probability for an agent to update his neighborhood. The case $q = 0$ represents a static network, and the case where $q = 1$ is that of a society where agents cannot update their strategy but only theirs relationships.

**Network evolution:** If an agent $i$ decides to update her neighborhood, she will first pick one of his neighbors $j$ with a probability proportional to $1 - f_{ij}$, thus an untrusted neighbor has a higher probability to be selected. Depending on the game played, an interaction can be bad for a player and good for the other one, because of this, agent $j$ can try to maintain her relationship with $i$. The probability of finally removing the link is represented by a number equal to $1 - (f_{ij} + f_{ji})/2$ where both trusts are taken into account.
If the link is cut, both directed links $(i, j)$ and $(j, i)$ disappear. Now player $i$ will create a new link to replace the lost one. To do so she will ask a neighbor $k$ chosen this time proportionally to $f_{ik}$ to introduce her to one of his neighbors. Player $k$ will then select a neighbor $l$ also proportionally to $f_{kl}$ and $i$ will try to create a new link $(i, l)$ (and the sibling link $(l, i)$) with $l$. If this link already exist, $i$ will continue and ask $l$ for a possible match. If also this second try fails, $i$ will select a random agent in the network and establish a relationship with her (see figure 4.2). The forces on this new links are initialized with the value 0.5.

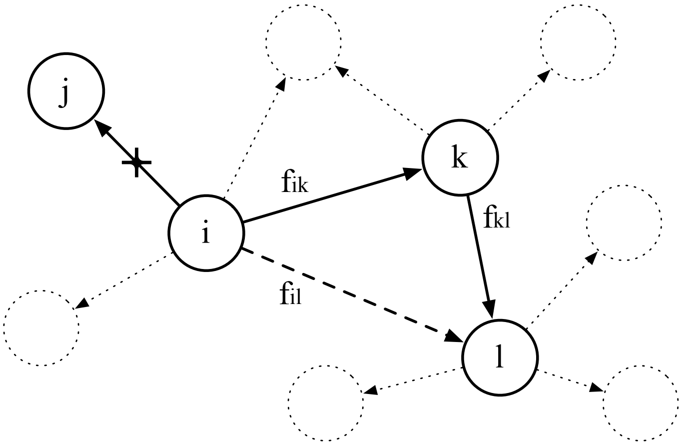


Figure 4.2: Illustration of the rewiring of link $(i, j)$ to $(i, l)$. Agent $k$ is chosen by $i$ and introduces player $i$ to $l$ (see text). To avoid cluttering the figure, only one of the two directed links between the agents is represented in the picture.

At the end of this process $j$ generally ends up loosing a link and will not have the opportunity to create a new link to replace this one. The only exception to this is the case where $j$ ends up without any connection. In this case, the next time she will be selected for update, she will simply establish a relationship with a random agent selected among the whole population.

### 4.6.2 Model Justifications

In this brief section we will try to explain and justify some choices we had to make when the model was conceived. We will also try to address some question that may arise after reading the description of the model.

**Trust, memory and repeated games:** The attentive reader would have noticed that the trust that evolves between two agents contains some sort of memory of their past encounters.

This is indeed the case, as the trust is dictated by the quality and thus strategies used in these encounters. However, this value only vaguely represent the quality of the interaction as an agent is unable to extract precise strategic information, such as the pattern of the moves of the opponent, a possible mutation in the strategy of the partner or even the number of past encounters. This value can for example lead to the termination of the interaction with an agent that recently switched to a more useful strategy or to the protection of a link with agent that has suddenly became a bad influence.

For these reasons the game played by our agent is not a repeated game in the sense of game theory. Moreover, this trust is only used by an heuristic that tries to remove the bad links, and never help a decision concerning a strategy update.

**Rupture and creation of links:** The link rupture is a bilateral, two-ways process, we apply this method to represent the embedded cost for an agent to dismiss one of her relationships. In particular, by using the average value of the trusts between the agents, we intend to capture the simplicity of breaking a relationship that is bilaterally identified as bad, but also the more difficult task that is the rupture of an interaction that is profitable by the other partner.

The link creation process is clearly biased and tends to create triangles if the quality of the interactions is good. However, this is based on reality, since interaction are often established with “friends of friends”. This is a unilateral, one-way process but the social interpretation of the algorithm relies on a common neighbor of the two “newly connected” agents that is charged to introduce the agents and to facilitate the creation of the new link.

It’s also important to say that the link rupture and creation process is not a strategic move aimed to improve without doubt the benefit of an agent, but merely an heuristic that an individual is able to apply in the attempt to improve his wellness. In this way, the present heuristic fundamentally differs from rigorous models of strategic, i.e. game-theoretic, network formation processes found in economics (e.g. see[123, 49]).

## 4.7 Community structure

Introduced in Section 3.1.3, communities represent an important characteristic of social networks. The presence of communities in a networked population can affect the speed of diffusion of strategies. As an example, a sub-optimal strategy $\sigma$ can coexist with a more efficient one if the agents using $\sigma$ represent the majority of the agents in a community. The high connectivity within the community and the low connectivity toward the rest of the network can protect this minority and allow the existence of some diversity in the population.

To detect communities in networks in this work we used the algorithm based on edge betweenness proposed by Girvan and Newman in [31].

*Betweenness* is a centrality measure for the nodes of a network. The betweenness of a node $i$ represents how many shortest paths between pair of nodes run through $i$. *Edge betweenness* is an extension to this definition that characterizes an edge with a number corresponding to how many shortest path between pair of nodes run through it.

We analyzed the community structure of static networks in B and in F to understand how the underlying structure of the network can affect the quasi-stable state reached with simulations.

In dynamic networks we analyzed the communities in paper G at several points during the simulations to monitor the evolution of the networks and understand how the behavior of players affects the evolution of the topology.

# Part II

# Games on Static Networks

# Article A

# Social Dilemmas and Cooperation in Complex Networks

| | |
|---|---|
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### Abstract

In this paper we extend the investigation of cooperation in some classical evolutionary games on populations were the network of interactions among individuals is of the scale-free type. We show that the update rule, the payoff computation and, to some extent the timing of the operations, have a marked influence on the transient dynamics and on the amount of cooperation that can be established at equilibrium. We also study the dynamical behavior of the populations and their evolutionary stability.

## A.1 Introduction and Previous Work

The object of game theory is the analysis of situations where the different social actors have conflicting requirements and individual decisions will have a mutual influence on each other[71]. In this framework, and due to their importance as simplified models of many common important socio-economic situations, the *Prisoner's Dilemma* (PD) and the *Snowdrift* (SD) games have received much attention in the literature. According to game theory, the PD and the SD are paradigmatic examples of games in which cooperative attitude should vanish in the PD, and should be limited to a given fraction in the SD. This is also the case when large populations of individuals play the game pairwise in a random manner and anonimously, as prescribed by evolutionary game theory [128]. In spite of this, numerical simulations of the PD have convincingly shown that, when the population of players possesses a spatial structure, a certain amount of cooperation can emerge and remain stable. Nowak and May [84] were the first to empirically show this using a population structured as a square lattice where each site is a player. Standard evolutionary game theory is based on an infinite (or very large) population model, and on the random pairing of two players at each time step. This amounts to a *mean-field* spatially homogeneous model. The square grid is also spatially homogeneous but the absence of random mixing enables the formation of clusters of cooperators, which allows for more frequent positive encounters between cooperators than would be possible in the mean-field case. More recently, it has become apparent that many real networks are neither regular nor random graphs; instead, they have short diameters, like random graphs, but much higher clustering coefficients than the latter, i.e. they have more local structure. These networks are collectively called *small-world* networks (see [75] for a recent review). Many technological, social, and biological networks are now known to be of this kind. Social networks, in addition, show recognizable community structure [79, 35]. Since evolutionary PD or SD games are metaphors for conflicting social interactions, the research attention has recently shifted from random graphs and regular lattices towards better models of social interaction structures [1, 47, 101, 116].

Recently, Santos and Pacheco [101] presented a numerical study of the evolution of cooperation on (static) scale-free (SF) networks for the PD and the SD games. Their main result was that, in contrast with what one observes in mixing populations or on regular lattices, much higher levels of cooperation are sustainable on this kind of graphs, both for the PD as well as the SD. These results are obviously interesting and encouraging for cooperation but they prompt a number of questions. First of all, Barábasi–Albert or correlationless configuration SF graphs [3] that were used in [101] are not faithful representations of most typical social networks. In fact, although social interaction networks where the degree distribution can be well described by a power-law have been found [56, 34], several recent studies show that social networks in general do not have a pure power-law degree distribution function, as they often show signs of exponential decay of the tail of the distribution [4, 73]. In addition, they usually have more clustering than pure scale-free graphs [75]. Nevertheless, model SF networks are a useful bounding case to study as they are closer to typical social networks than other more artificial kind of graphs, such as Watts–Strogatz small worlds [127]. A second aspect of social networks that is not captured by fixed graph structures is that they are not static; rather, the number of vertices and the links between them continuously evolve as social actors come and go, and relationships are created or abandoned. Dynamical features such as these have been introduced in evolutionary games, among others, in [132, 131, 13, 59, 103]. However, in this paper we only focus on the static aspects of the interaction networks. In other words, we make the hypothesis that the network is at equilibrium and that network dynamics are either absent, or their time scale is longer (slower) with respect to the strategy-change dynamics. This proves to be a useful approach, especially for social acquaintance networks.

In the following we present a brief introduction to the games studied. This is followed by a discussion of the population model and of individual's payoff calculation scheme for the players in a complex network. Next we describe the numerical simulations and their results, including a study of evolutionary stability. We finally present our conclusions.

## A.2 Two Social Dilemmas

Let us first recall a few elementary notions on the PD and the SD. These are two-person, symmetric games in which each player has two possible strategies: cooperate (C) or defect (D). In strategic form, also known as normal form, these games have the following payoff bi-matrix:

|   | C | D |
|---|---|---|
| C | (R,R) | (S,T) |
| D | (T,S) | (P,P) |

In this matrix, R stands for the *reward* the two players receive if they both cooperate, P is the *punishment* for bilateral defection, and T is the *temptation*, i.e. the payoff that a player receives if it defects, while the other cooperates. In this latter case, the cooperator gets the *sucker's* payoff S. For the PD, the payoff values are ordered numerically in the following way: $T > R > P > S$, while in the SD game $T > R > S > P$. Defection is always the best rational individual choice in the PD – (D,D) is the unique Nash equilibrium and also an evolutionary stable strategy (ESS). Mutual cooperation would be preferable but it is a strongly dominated strategy. Thus the dilemma is caused by the "selfishness" of the actors.

In the SD, when both players defect they each get the lowest payoff; (C,D) and (D,C) are Nash equilibria of the game in pure strategies, and there is a third equilibrium in mixed strategies where strategy D is played with probability $1/(2\beta - 1)$, and strategy C with probability $1 - 1/(2\beta - 1)$, where $\beta$ is another name for the temptation $T$, used in biological circles. The dilemma in this game is caused by "greed", i.e. players have a strong incentive to "bully" their opponent by playing D, which is harmful for both parties if the outcome produced is (D,D).

## A.3 Numerical Simulations

The two games were simulated in [101] on Barabási-Albert (BA) [3] and configuration model [75] scale-free networks of size $10^4$ over $10^4$ time steps, using a discrete analogue of *replicator dynamics* equations [128, 41]. The customary rescaling of the payoff values was used such that there is only one independent parameter. For the PD, setting $R = 1$, $P = S = 0$, leaves $T = b > 1$ to be the only parameter (temptation). For the SD, $T$ is set equal to $\beta > 1$, $R = \beta - 1/2$, $S = \beta - 1$, and $P = 0$, which makes the cost-to-benefit ratio of mutual cooperation $r = 1/(2\beta - 1)$ the only parameter. For the sake of comparison, our simulations were done under the same conditions as in [101] ($10^4$ players and $10^4$ time steps).

However, replicator dynamics is not the only possibility for updating the agents' strategies in discrete, finite populations of players using hard-wired strategies. Moreover, in small non degree-homogeneous populations, the mathematical requirements behind the replicator dynamics, strictly speaking, are not satisfied [85]. Thus, we extended the investigation by simulating an *imitate the best* evolution rule according to which an individual $i$ will adopt the strategy of the player with the highest payoff among its neighbors and itself. If a tie occurs, the winner is chosen uniformly at random between the best. This rule is deterministic and was the original rule used in [84].

Concerning the calculation of an individual's payoff, there are several possibilities. A player's payoff may be defined as the sum (*accumulated payoff*) of all pair interactions with its nearest neighbors, which is the form used for instance in [101]. Another possibility consists in using *average payoff*, which is the accumulated payoff divided by the number of interactions. Accumulated and average payoff give the same results when considering degree-homogenous networks such as lattices. Accumulated payoff seems more logical in degree-heterogeneous networks such as scale-free graphs since it reflects the very fact that players may have different numbers of neighbors in the network. Average payoff, on the other hand, smooths out the possible differences although it might be justified in terms of number of interactions that a player may sustain in a given time. For instance, an individual with many connections is likely to interact less often with each of its neighbors than another that has a lower number of connections. Also, if there is a cost to maintain a relationship, average payoff will roughly capture this fact, while it will be hidden if one uses accumulated payoff. For the sake of comparing the two extreme views, here we use both accumulated and average payoff.

Under discrete replicator dynamics rule with accumulated payoff, and using *synchronous update*, Santos and Pacheco [101] found that, when compared to regular lattices, SF networks lead to high levels of cooperation for all values of the parameters $b$ (for PD) and $r$ (for SD). These results have been reproduced by us and are shown in the upper half of figure A.1. Cooperation is also much higher in SF graphs than what has been obtained for Watts–Strogatz small-world graphs [1, 116]. When using the "imitation of the best" strategy-switching rule with synchronous update and accumulated payoff the results are similar, as one can see in the lower part of figure A.1, although there is a marked fall in the high-b and high-r region with respect to replicator dynamics. However, when one lingers on the standard deviations (represented as error bars in the figure), one sees that the results for the imitate the best rule are noisy, with quite large fluctuations. Deviations are smaller for the replicator dynamics, see figure A.1. The reason for the instability and the large fluctuations can be traced to the step function nature of the update rule, as can be seen in figure A.3 (a), in which 40 individual PD runs are plotted, all with $b = 1.8$. In all runs cooperation falls at the beginning, the cooperators then often recover but not always, as there are several runs (about 1/5 for the data used here) in which cooperation never recovers. On the other hand, when using replicator dynamics, there is still a systematic drop of cooperation at the beginning (figure A.3 (c)), nevertheless it tends to rise again in the long run, although this may happen very late in the simulation (see figure A.3 (b)). To better observe this phenomenon, we have doubled the number of time steps ($2 \times 10^4$).

We thus see that the results on BA SF graphs depend on the update rule, although the level of cooperation is still higher than what is found on regular, Watts–Strogatz, and random graphs [41, 116]. However, we wish to point out that if we use an *asynchronous update* policy[1] with the "imitate the best" rule instead of the usual synchronous one, the result is a higher level of cooperation with far less fluctuations than the synchronous case (compare lower parts of figures A.1 and A.2). One might reason that the combination of synchronous update and of "imitate the best" is fully deterministic, which implies that particular chains of events, such as cascades of defection, will be amplified. Introducing stochasticity through asynchrony in the update sequence strongly mitigates the likelihood of such series of events. On the other hand, when using replicator dynamics, the lack of stochasticity in synchronous update is somehow compensated for by the probabilistic strategy change rule, which could explain the similarity of the results in this latter case (compare the upper parts of figures A.1 and A.2 respectively).

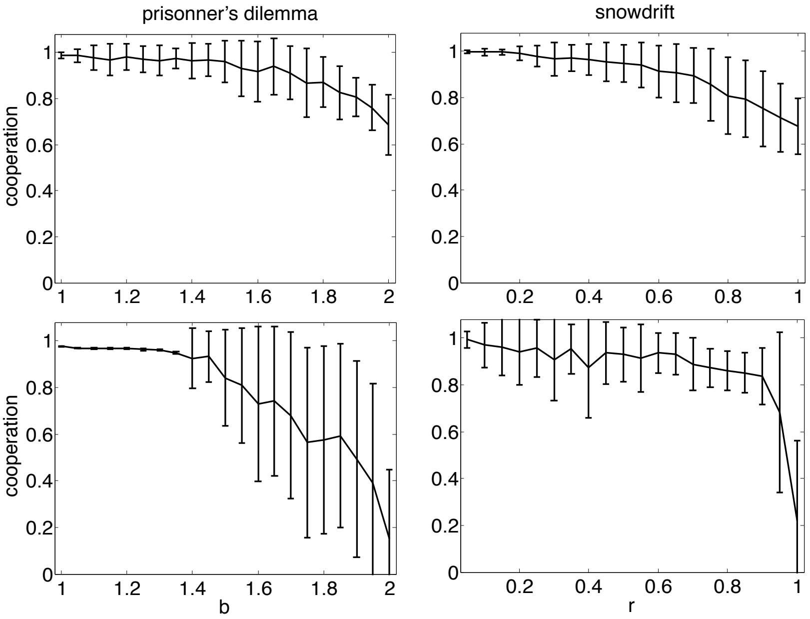


Figure A.1: Fraction of cooperators on SF BA networks of size $10^4$ and average degree $\bar{k} = 4$ with accumulated payoff and synchronous dynamics. Mean values over 50 runs. Upper figures: replicator dynamics; lower figures: imitation of the best.

To illustrate the influence of timing when "imitate the best" is the rule used for strategy update, suppose that a defector occupies the most highly connected node in the graph and that it is surrounded by cooperators exclusively. Then, at the next time step in synchronous update, all those cooperators will turn into defectors. From there, a wave of defection could quickly propagate through the network, leading to a state whereby cooperation cannot be recovered. On the other hand, when players are updated in random order, only a fraction of the neighbors

---

[1] We use the standard uniform random choice (with replacement) of players in the population, which is a discrete approximation of a Poisson process.

will imitate the defector, at the same time lowering the payoff of the central defector, and thus making it less attractive to be imitated in future encounters. This kind of process limits the propagation of defection and allows cooperation to establish itself and be stable. This highlights some shortcomings of synchronous dynamics, which is unrealistic and may give rise to spurious effects [48]. Our conclusion is that, although there is often no significant difference between synchronous and asynchronous update in evolutionary games, as it is the case here under replicator dynamics, the latter is to be preferred for reasons of generality and reliability. However, for the sake of comparison with previous results, in the rest of the paper we use synchronous update.

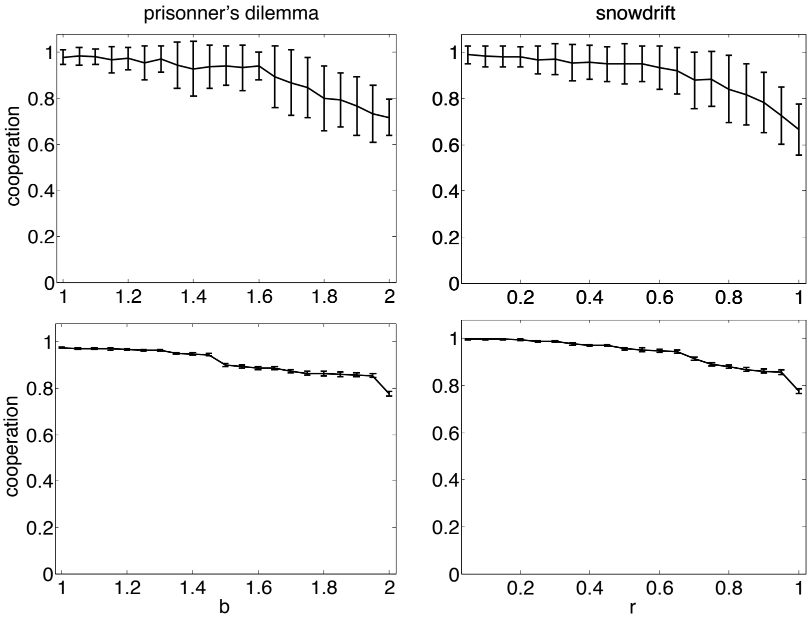
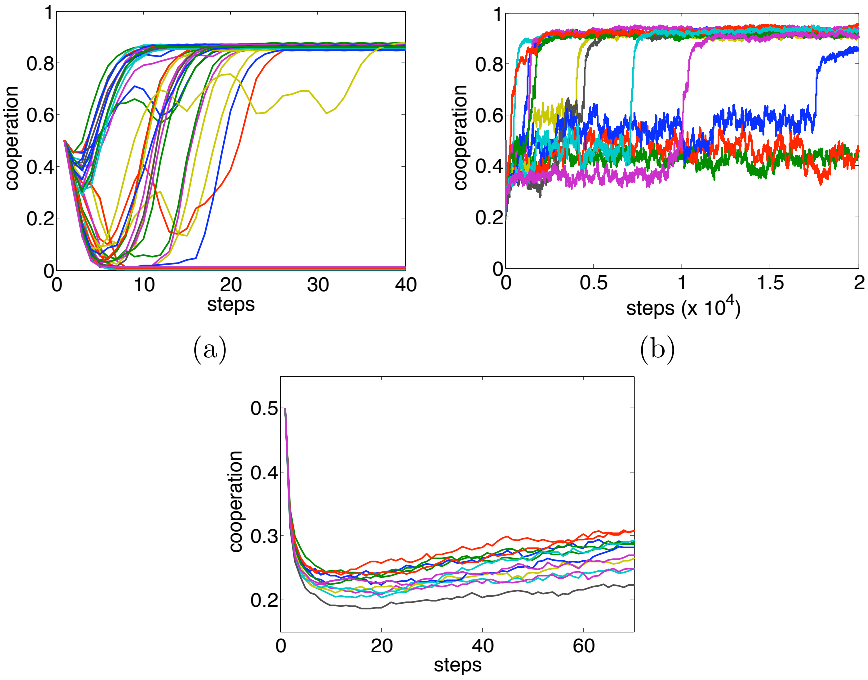


Figure A.2: Fraction of cooperators on SF BA networks of size $10^4$ and average degree $\bar{k} = 4$ with accumulated payoff and asynchronous dynamics. Mean values over 50 runs. Upper figures: replicator dynamics; lower figures: imitation of the best.

Now we turn our attention to the assumption that a player's utility is the sum, i.e. the accumulated payoff of all pair interactions with its nearest neighbors. Although this appears to be a logical step to follow, we shall show that it may cause both conceptual and technical problems. Obviously, one would assume that if an individual has more links to cooperators, and that the payoffs are positive quantities, she should earn more than another player with fewer cooperating neighbors. However, this begs the question of how the network got there in the first place. BA SF graphs are incrementally built by using linear preferential attachment [3]. In this model there is no cost associated to the formation of a new link. However, although this model may be adequate for citation networks or, to some extent, the Web, it is well known that this cannot be the case in most other instances. Thus, other models have been proposed that take into account cost and other factors in network formation [75]. In our case, it is as if the population would be "injected" on an already full-grown, topology-favorable network, while the



(a) (b)

(c)

Figure A.3: PD time series with $b = 1.8$; (a) imitation of the best; (b) replicator dynamics; (c) replicator dynamics (first 70 steps).

rules of the game and other considerations necessarily should play a role in the network formation and dynamics. The same remarks also hold for the "configuration" SF graphs, although these networks are built starting from the degree distribution and a fixed number of nodes, rather than incrementally. Furthermore, a technical problem arises when combining replicator dynamics with accumulated payoff. In infinite mixing populations, classical evolutionary game theory states that replicator dynamics is invariant under positive affine transformations of payoffs with merely a possible change of time scale [128]. This invariance still holds in finite degree-homogenous populations. However, when different individuals start having different degrees, things are not quite the same. Let $\Pi_i$ denote a player $i$'s aggregated payoff. Furthermore, let $\phi(\Pi_j - \Pi_i) = (\Pi_j - \Pi_i)/(Dk_>)$ be the probability function according to which $i$ adopts neighbor $j$'s strategy, with $D = max\{T, R, P, S\} - min\{T, R, P, S\}$ and $k_> = max\{k_i, k_j\}$, where $k_x$ represents the degree of player $x$ [101]. If we now apply a positive affine transformation of the payoff matrix, this leads to the new aggregated payoff $\Pi'_i = \alpha\Pi_i + \beta k_i$ and hence $\phi(\Pi'_j - \Pi'_i) = (\alpha\Pi_j + \beta k_j - \alpha\Pi_i - \beta k_i)/(\alpha D k_>) = \phi(\Pi_j - \Pi_i) + (k_j - k_i)/(\alpha D k_>)$. One can clearly see that using accumulated payoff does not lead to an invariance of the replicator dynamics under shifts of the payoff matrix. As an illustration of the violation of this invariance, figure A.4 shows cooperation curves for the PD when applying such payoff transformations.

This has several implications such as limiting the results obtained in [101] strictly to the studied values of $b$ and $r$, and to an impossibility to rescale the payoff matrix. In a more recent

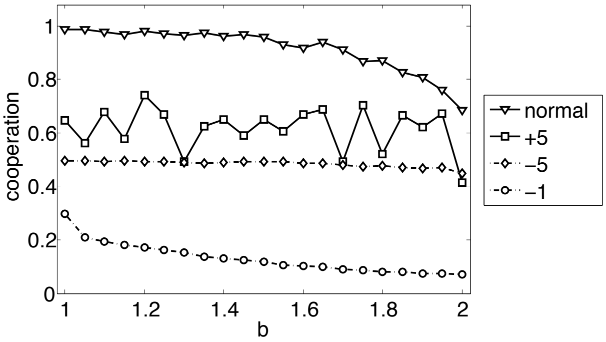


Figure A.4: Fraction of cooperation for the PD game using replicator dynamics and accumulated payoff. A translation of the payoff matrix can produce a fall in cooperation (shift of $-1$) as well as unpredictable behaviors (shift of $+5$) with some runs containing high levels of cooperation and others ending up with massive defection. Standard deviations are not plotted here to improve readability.

study [104] Santos et al. investigated the same games in a wider parameter space, but still using accumulated payoff, which again makes the results non-invariant with respect to a positive affine transformation. Therefore, we repeated the numerical simulations with *average payoff*, i.e. the

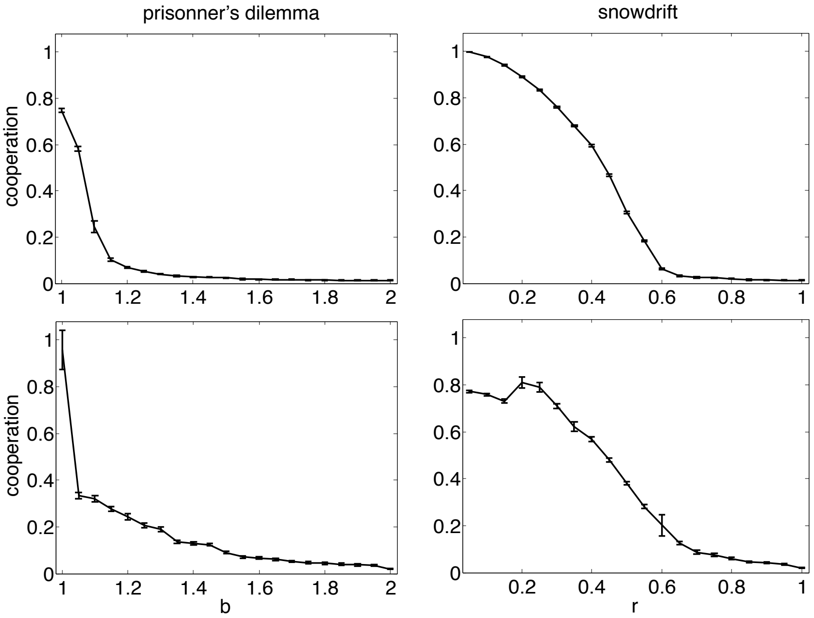


Figure A.5: Fraction of cooperators on SF BA networks of size $10^4$ with average degree $\bar{k} = 4$ using average payoff and synchronous dynamics. Mean values over 50 runs. Upper figures: replicator dynamics; lower figures: imitation of the best.

aggregated payoff obtained by one player divided by the number of links the player has to nearest neighbors, which, along with the shortcomings described above, has the advantage of leaving the replicator dynamics invariant under positive affine transformations.

In figure A.5 we report results for the PD and SD games using average payoff with synchronous updating dynamics, and the same parameter set as in [101]. Looking at the figures, and comparing them with the results of [101] (replicated here for $\bar{k} = 4$ in figure A.1), one immediately sees that the cooperation level reached after the transient equilibration period is much lower, and comparable with the results found for regular and random graphs. This is reasonable, given that now it is as if each individual had the same average number of neighbors as far as its payoff is concerned.

To reach a better understanding of the difference between accumulated and average payoff, we interpolated between the two extreme cases according to the formula

$$\Pi_i = \frac{1}{k^d} \sum_j \pi_{i,j}, \tag{A.1}$$

where $d \in [0, 1]$, $\Pi_i$ is the net payoff of player $i$, and $\pi_{i,j}$ is the payoff player $i$ obtains when interacting with neighbor $j$. One can see that, when $d = 0$ we recover the accumulated payoff value, while $d = 1$ corresponds to the average payoff case. Figure A.6 clearly shows that, as $d$ varies from 0 to 1, and thus the ratio varies from 1 to $1/k$, cooperation levels steadily decrease for all values of the temptation on the y-axis. So, the way in which individual payoff is computed has a large influence on cooperation levels that can be reached, in the average, on a given network topology.

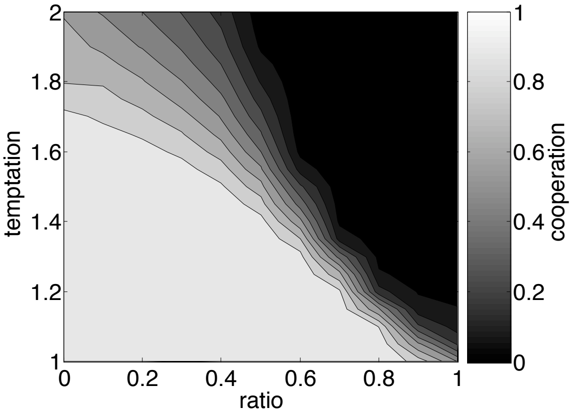


Figure A.6: Cooperation level as a function of the parameter $d$ of equation A.1 in the PD for temptation values between 1 and 2. Cooperation prevails in light areas; Darker areas mean more defection. Results are the average of 50 runs.

## A.4 Evolutionary Stability

Evolutionary stability, i.e. the resistance to invasion by mutant strategies, is an important issue when dealing with evolutionary games [128]. The effect of switching the strategy of the hub with largest connectivity in a totally cooperating population has been studied in [102]. Here we

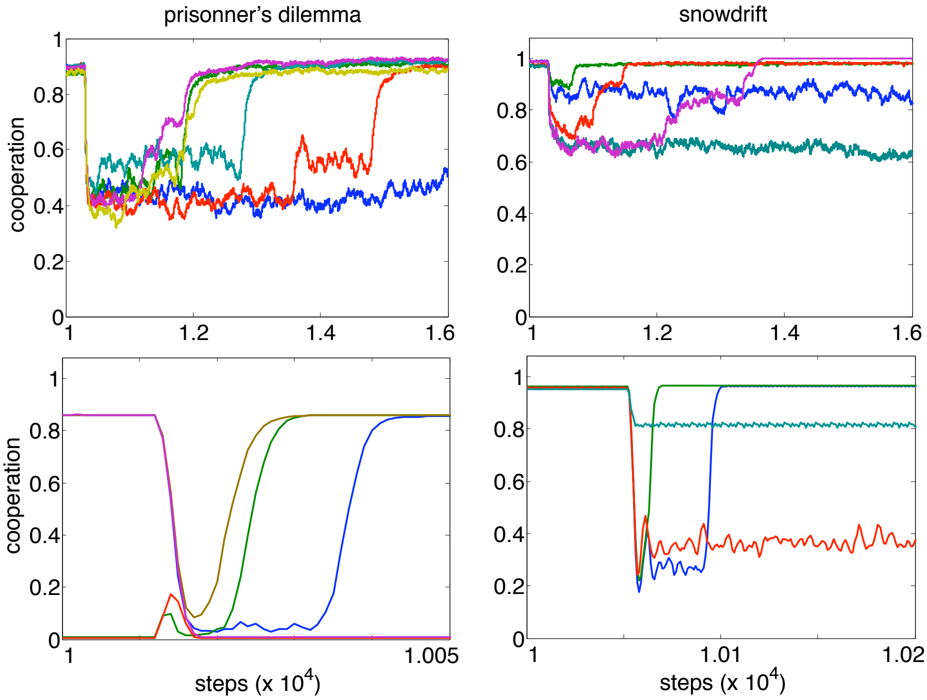


Figure A.7: System stability when using accumulated payoff. For each parameter set, 100 runs have been perturbed, but only a few individual runs are plotted here to expose the behaviors encountered. Upper figures: replicator dynamics; lower figures: imitation of the best. Left-hand figures: $b = 1.8$; right-hand figures: $r = 0.5$.

use a different approach to perturb the population after it has reached a quasi-stable state by switching the strategy of a few players having the strategy of the greater number. This was done for values of $b \in \{1.2, 1.5, 1.8\}$ and $r \in \{0.2, 0.5, 0.8\}$. We then give the system 6000 time steps to attempt to reattain its initial stable state. For reasons of space, we only plot the results obtained for $b = 1.8$ and $r = 0.5$ (see figure A.7). Given the scale-free nature of the interaction network, introducing a small amount of random noise does not have any effect on the population stability. On the other hand, when cooperator hubs switch strategy (one to five in our study), avalanches of defection can form and propagate through the population. Under replicator dynamics and when using accumulated payoff, about 1/6 of the PD runs do not recover the state previously attained at time step $10^4$. This fraction rises to 1/3 for the SD game. With the imitation of the best rule, 1/10 of the PD and SD runs fail to recover from the perturbations. In contrast to accumulated payoff, average payoff does not allow perturbations to generate any noticeable effect, i.e. the system remains quite stable.

## A.5 Conclusions

In conclusion, we have deepened and extended the study presented in [101] clarifying the role of the updating rule and the type of payoff attributed to players. We have shown that the games are not invariant under linear affine tranformations when using accumulated payoff, while average payoff does not have this problem, although it may artificially reduce the impact of scale-free degree networks. We have also seen that asynchronous update dynamics, being more likely in a system of independently interacting agents, by eliminating artificial effects due to the nature of synchronous update, may give rise to steadier quasi-equilibrium states. Moreover, we have studied several dynamical aspects of the evolution of the populations such as their transients before attaining the steady-state, and their evolutionary stability, showing that scale-free networks of interactions provide a quite stable environment for the emergence of cooperation when using accumulated payoff, except when hubs are targeted by the mutations, in which case a sizable number of runs do not recover the original state, at least within the simulation times allowed in our numerical experiments.

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# Article B

# Cooperation and Community Structure in Social Networks

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### Abstract

Situations of conflict giving rise to social dilemmas are widespread in society. One way of studying these important phenomena is by using simplified models of individual behavior under conflicting situations such as evolutionary game theory. Starting from the observation that individuals interact through networks of acquaintances, we study the evolution of cooperation on model and real social networks through well known paradigmatic games. Using a new payoff scheme which leaves replicator dynamics invariant, we find that cooperation is sustainable in such networks, even in the difficult case of the prisoner's dilemma. The evolution and stability of cooperation implies the condensation of game strategies into the existing community structures of the social network in which clusters of cooperators survive thanks to their higher connectivity towards other fellow cooperators.

## B.1 Introduction

Game theory [122, 39] deals with social situations of conflict where two or more individuals take decisions that will mutually influence each other. It is thus a view of collective systems in which global social outcomes emerge as a result of the interaction of the individual decisions made by each agent. The theory makes simplifying assumptions about the behavior of the agents to be able to cast results into a coherent mathematical framework. However, in spite of their abstractness, some extremely simple games lead to puzzles and dilemmas that have a deep social meaning. The Prisoner's Dilemma, a universal metaphor for the tension that exists between social welfare and individual selfishness, is the most famous game of this type. It stipulates that, in situations where individuals may either cooperate or behave selfishly and thus defect, they will rationally choose the latter. However, cooperation would be the preferred outcome when global welfare is considered. And indeed, cooperation actually emerges when the game is played by humans and in many other cases. For instance, in experiments where people play the Prisoner's Dilemma for money without repetition and anonymously, the fraction of cooperation is observed to be about 40 to 50 per cent [99]. Other simple games that give rise to social dilemmas are the Hawk-Dove and the Stag-Hunt games, to be described below.

Several mechanisms have been invoked to explain the emergence of cooperative behavior. Among them, repeated interaction, reputation, and belonging to a recognizable group have often been mentioned [6, 57]. However, the work of Nowak and May [84] showed that the simple fact that players are arranged according to a spatial structure and only interact with neighbors is sufficient to sustain a certain amount of cooperation even when the game is played anonymously and without repetition. Nowak and May's study and much of the following work was based on regular structures such as two-dimensional grids (see also [41] for the Hawk-Dove case, and [86] for a recent review). Axelrod et al. [7] showed that by randomizing the choice of neighbors, i.e. by actually giving up a strictly local geographical structure, cooperation can still emerge, provided that the interaction patterns remain stable in time, which is a first step toward a social network structure. However, all these topologies can only be considered as approximations, as it has now become clear that many actual networks, social or otherwise, usually have a topological structure that is neither regular nor random but rather of the *small-world* type. Roughly speaking, small-world networks are graphs in which any node is relatively close to any other node. In this sense, they are similar to random graphs but unlike regular lattices. However, in contrast with random graphs, they also have a certain amount of local structure, as measured, for instance, by a quantity called the *clustering coefficient* which essentially represents the probability that two neighbors of a given node are themselves connected (an excellent review of the subject appears in [75]). Thus, most real conflicting situations in economy and sociology are not well described neither by a fixed geographical position of the players in a regular lattice, nor by a mixing population or a random graph, and it becomes relevant to study these dilemmas on other, more realistic social structures. Some previous work has been done in this direction. In particular we mention Santos and Pacheco's work on scale-free networks [101, 104] and work on Watts–Strogatz small-world graphs [1, 53, 116, 126]. These network types do have the right global "statistical" properties, but we emphasize that they are only an approximation of the actual local topological properties of measured networks of interactions. Thus, we introduce more socially relevant topologies, including an actual coauthorship network, and we emphasize the relationships between community structure and cooperation. Some recent works close to the present one in spirit are Holme et al. [47], Santos et al. [101, 104], and [58]. However, the authors of [47] study only the prisoner's dilemma on particular social networks using a different, noisy, strategy update rule, while [101, 104] deal with scale-free graphs exclusively. Reference [58] has similar goals as our work and it deals only with the Prisoner's Dilemma on a couple of empirically determined social networks. It uses a different strategy update rule and a restricted parameter space. We shall discuss these and other differences with our work in the model and results sections.

## B.2 Social Dilemmas

The three representative games studied here are the Prisoner's Dilemma (PD), the Hawk-Dove (HD), and the Stag-Hunt (SH) of which we briefly summarize the significance and the main results. More detailed accounts can be found in many places, for instance [122, 39, 6]. In their simplest form, they are all two-person, two-strategies, symmetric games with the following payoff bi-matrix:

| | C | D |
|---|---|---|
| C | (R,R) | (S,T) |
| D | (T,S) | (P,P) |

In this matrix, R stands for the *reward* the two players receive if they both cooperate (C), P is the *punishment* for bilateral defection (D), and T is the *temptation*, i.e. the payoff that a player receives if it defects, while the other cooperates. In this case, the cooperator gets the *sucker's payoff* S. In the three games the condition $2R > T + S$ is imposed so that mutual cooperation is preferred over an equal probability of unilateral cooperation and defection. For the PD, the payoff values are ordered numerically in the following way: $T > R > P > S$. Defection is always the best rational individual choice in the PD; (D,D) is the unique *Nash equilibrium* (NE) and also an *evolutionarily stable strategy* (ESS) [122, 128]. Mutual cooperation would be preferable but it is a strongly dominated strategy.

In the HD, the order of $P$ and $S$ is reversed yielding $T > R > S > P$. Thus, in the HD when both players defect they each get the lowest payoff. (C,D) and (D,C) are Nash equilibria of the game in pure strategies, so the game is *antagonistic*, and there is a third equilibrium in mixed strategies where strategy D is played with probability $p$, and strategy C with probability $1 - p$, where $p$ depends on the actual payoff values. The only ESS of the game is the mixed strategy, while the two pure NE are not ESSs [128]. The dilemma in this game is caused by "greed", i.e. players have a strong incentive to "bully" their opponent by playing D, which is harmful for both parties if the outcome produced happens to be (D,D).

In the SH, the ordering is $R > T > P > S$, which means that mutual cooperation (C,C) is the best outcome, Pareto-superior, and a Nash equilibrium. However, there is a second equilibrium in which both players defect (D,D) which is somewhat "inferior" to the previous one, although perfectly equivalent from a NE point of view. Here the dilemma is represented by the fact that the socially preferable coordinated equilibrium (C,C) might be missed for “fear” that the other player will play D instead. There is a third mixed-strategy NE in the game, but it is commonly dismissed because of its inefficiency and also because it is not an ESS [128].

## B.3 Nature of Social Networks

From the evolutionary game theory perspective [122, 128], these dilemmas have been classically studied by modeling the behavior of a large population in which randomly paired individuals play the game in an anonymous manner using the corresponding payoff matrix. Non-rational players are “hard-wired” to play a given strategy, and those better than average increase their share in the population. Among the fixed points of these *replicator dynamics* one finds evolutionarily stable strategies, i.e. strategies that cannot be invaded by a mutant strategy [128].

If we take a network view in describing the structure of the population, the previous “mixing” population would be represented by a complete graph, i.e. any individual may interact with any other player. The advantage of the mixing model and also of random graphs is that they admit an approach by *mean-field* methods, which treat the system as being homogeneous, ignoring space dependences and correlations. For instance, the replicator dynamics leads to a system of differential equations which, given an initial distribution of strategies among the agents, describe the evolution of the vector of population frequencies [128]. However, we do know that real social networks do not conform to these simple models. Instead, they are of finite size, have heterogeneous connectivity, and often display small-world properties, in the sense that any individual is only a few steps away from any other, and individuals cluster together in neighborhoods [75, 126, 4, 12]. Therefore, the previous evolutionary games should be studied on this type of networks, to understand the limitations of the theory, and to extend it as far as possible to structures encountered in real-life.

We simulate the games on three main types of networks: scale-free graphs, a theoretical social network model, and an existing coauthorship network. These network types go from least realistic, the scale-free, to real, the coauthorship network. We construct scale-free networks according to the classical Barabási-Albert (BA) model [3]. The social network model follows [114]. Details of their construction are given in section B.4. As a typical example of a true social network, we use a coauthorship network among researchers in the genetic programming (GP) community. This network displays small-world properties, with a connected giant component of 942 scientists and it has recently been analyzed [117]. It has clusters and communities and it should be representative of other similar human acquaintance networks. Its degree distribution function is not a pure power-law; rather, it can be fitted by an exponentially truncated power-law.

Scale-free networks are characterized by a skewed degree distribution function $P(k)$, i.e. the probability that a given node has exactly $k$ neighbors is a slow-decaying function of $k$. However, except perhaps for sexual contact networks [56, 34], and some collaboration networks[8], most social networks studied to date are not of the pure scale-free type, and show a faster decay of the tail of the degree distribution [75, 4]. Intuitively, there must be a cutoff in the number of acquaintances a given agent can have, and in many cases also a typical number of acquaintances, which gives a scale to the network. Besides, it has been observed that social networks have a higher clustering coefficient than the typical values reached in scale-free graphs, another manifestation of the complex neighborhood structure of the network. Furthermore, the appearance of communities – sets of densely connected vertices with sparse connections between the sets – is yet another typical feature found in social structures [79]. Communities can highly influence the way information is propagated throughout the network or opinion formation is processed. Finally, another interesting aspect of real-life social networks is the correlation between the degrees of neighbouring nodes, called *degree assortativity.* Technological and biological networks typically have a negative correlation, i.e. high-degree vertices are preferentially connected to low-degree vertices, whereas measured social networks are assortative, meaning highly connected nodes tend to be connected with other highly connected nodes [75, 80].

We note that real social, communication, and technological networks are dynamical, i.e. new nodes may join the network forming new links, and old nodes may leave it as social actors come and go. As a first approximation we model only static networks, thus ignoring fluctuations and non-equilibrium phenomena. In other words, we make the hypothesis that the network is at equilibrium and that network dynamics are either absent, or their time scale is longer with respect to the strategy-change dynamics. This proves to be a useful approach, especially for social acquaintance networks. Some recent work has targeted the dynamical aspects of network evolution [132, 25, 103, 38].

## B.4 Model Description

### B.4.1 Population Structure

We consider a population of players of size $N$. Each individual $i$ in the population is represented as a vertex $v_i$ of a graph $G(V, E)$, with $v_i \in V$. An interaction between two players $i$ and $j$ is represented by the undirected edge $e_{ij} \in E$. The number of neighbors of player $i$ is the degree $k_i$ of vertex $v_i$. The average degree of the network will be called $\bar{k}$. The terms vertex, node, individual, or player shall be used interchangeably in the sequel; likewise for edge, link, interaction, and acquaintance. In the next two paragraphs we give details on the construction of our population graphs.

**Scale-Free Graphs Construction.** We use the model proposed by Barabási and Albert [3]. Networks are grown incrementally starting with a small clique of $m_0$ nodes. At each successive time step a new node is added such that its $m \leq m_0$ edges link it to $m$ nodes already present in the graph. It is assumed that the probability $p$ that a new node will be connected to node $i$ depends on the current degree $k_i$ of the latter. This is called the *preferential attachment* rule. The probability $p(k_i)$ of node $i$ to be chosen is given by $p(k_i) = k_i / \sum_j k_j$, where the sum is over all nodes already in the graph. The model evolves into a stationary network with power-law probability distribution for the vertex degree $P(k) \sim k^{-\gamma}$, with $\gamma \sim 3$. For the simulations, we constructed Barabási-Albert scale-free networks (henceforth BA SF) of size $N = 10000$ and with an average degree $\overline{k} = 4$, starting with a clique of $m_0 = 2$ nodes and at each time step the new incoming node has $m = 2$ links.

**Social Network Construction.** This model, called here the TSN model and presented in detail in [114], was conceived to construct a graph with most of the desired features of real-life social networks i.e, assortative, highly clustered, showing community structures, having an adjustable decay rate of the degree distribution, and a finite cutoff. The network is incrementally grown starting from a seed of $m_0$ randomly connected vertices. At each successive time step, the following algorithm is applied:

1. On average $m_r \geq 1$ random vertices are picked to be initial contacts.
2. On average $m_s \geq 0$ neighbors of the $m_r$ initial contacts are chosen to be secondary contacts.
3. A newly added vertex $v$ is connected to all the initial and secondary contacts determined in the two previous steps.

The above is iterated until the network reaches the desired size. Notice that the process responsible for the appearance of high clustering, assortativity and community structure is step 2. In the numerical experiments, we used graphs of size $N = 10000$ with $m_0 = 30$ initial nodes. Every time a new node is added, its number of initial contacts $m_r$ is distributed as $p(\#\text{ of initial contacts} = 1) = 0.95$ and $p(\#\text{ of initial contacts} = 2) = 0.05$. The number of its secondary contacts $m_s$ is uniformly distributed between 0 and 3. The resulting degree distribution falls below a power-law for high values of $k$ [114].

### B.4.2 Strategy Update Rules

To update the strategies of the individuals given an initial strategy distribution in the population, we use a discrete analogue of replicator dynamics as discussed in [46] and used in lattices by [41]. The replicator dynamics assumes that the share of the population playing a particular strategy grows in proportion to how well this strategy is doing relative to the average population payoff. Replicator dynamics is usually defined for very large populations [128] where correlations are absent and mean-field approximations can be used. In finite populations the behavior may be different, as studied by Nowak et al. [85].

Let $\Pi_x$ be a player $x$'s aggregated payoff and $k_x$ the number of neighbors $x$ has. We define the replicator dynamics function $\phi(\Pi_j - \Pi_i)$ as being the probability function according to which player $i$ adopts neighbor $j$'s strategy, namely

$$\phi(\Pi_j - \Pi_i) = \begin{cases} \dfrac{\Pi_j - \Pi_i}{\Pi_{j,\max} - \Pi_{i,\min}} & \text{if } \Pi_j - \Pi_i > 0 \\ \\ 0 & \text{otherwise,} \end{cases} \tag{B.1}$$

where $\Pi_{x,\max}$ (resp. $\Pi_{x,\min}$) is the maximum (resp. minimum) payoff a player $x$ can get. Note that Santos et al. [101, 104] make use of a very similar expression which only differs from ours by the choice of the normalization factor. The authors of [58] use another common strategy update rule which consists in imitating the strategy of the most successful neighbor. This is a deterministic rule while ours is stochastic. This, together with the use of accumulated payoff (see next section) makes the results difficult to compare directly.

### B.4.3 Utility Calculation

There exist several possibilities for determining a player's utility or payoff. Some previous work [84, 86, 101, 104, 58, 132] defined a player's payoff as the sum (*accumulated payoff*) of all pair interactions with its nearest neighbors. Other studies [53, 116] use the *average payoff*, i.e. the accumulated payoff divided by the number of interactions. Accumulated and average payoff give the same results when considering degree-homogenous networks such as lattices [84, 86]. Accumulated payoff seems more logical to use in degree-heterogeneous networks since it reflects the very fact that players may have different numbers of neighbors. However, accumulated payoff may lead to a technical problem when players have a different number of interactions. Evolutionary game theory states that replicator dynamics is invariant under positive affine transformations of payoffs save for a possible change of time scale [128]. However, on degree-heterogenous networks, this assumption is not satisfied when combining accumulated payoff with the replicator dynamics, as shown in [118]. This is essentially due to the translation component of the affine transformation and can be verified by considering a player $x$'s new accumulated payoff after transformation of the payoff matrix which is $\Pi'_x = \alpha\Pi_x + \beta k_x$, with $\alpha \in \mathbb{R}_+, \beta \in \mathbb{R}$.

Average payoff respects the replicator dynamics invariance but it prevents nodes with many edges to potentially have higher payoffs than those with only a few links, although it might be justified in terms of the number of interactions that a player may sustain in a given time, i.e. an individual with many connections is likely to interact less often with each of its neighbors than another that has a lower number of connections. Also, if there is a cost to maintain a relationship, average payoff will roughly capture this fact, while it will be hidden if one uses accumulated payoff. Moreover, nodes are very vulnerable to defecting neighbors who have just one link.

We propose here a third definition for a player's payoff that retains the advantages of the accumulated and average payoffs without their drawbacks. Before proceeding, let us first make a few notations. Let $\pi_{ij}$ denote the payoff player $i$ receives when interacting with player $j$, and let $V_i$ be the set of $i$'s neighbors. Finally, let $\pi_{i,\gamma}$ denote the guaranteed minimum payoff player $i$ can obtain in a one-shot two-person game. This is what $i$ would receive were he to attempt to maximize his minimum payoff. Note that for symmetric games such as the ones studied here, $\pi_{i,\gamma} = \pi_\gamma$ for $i = 1, \ldots, N$, where $N$ is the size of the population. For example in the PD, a player could choose to play $C$ with the risk of obtaining the lowest payoff $S$ were its opponent to play $D$. However, by opting for strategy $D$ a player would maximize its minimum payoff thus guaranteeing itself at least $\pi_\gamma = P > S$ no matter what its opponent's strategy might be. In the HD we have $\pi_\gamma = S$, for this time the payoff ordering is $T > R > S > P$ and a player needs only to play $C$ to receive at least payoff $S$. Finally, in the SH, $\pi_\gamma = P$. We can now define a player $i$'s aggregated payoff as being $\Pi_i = \sum_{j \in V_i} (\pi_{ij} - \pi_{i,\gamma})$. Intuitively, it can be viewed as the difference between the payoff an individual collects and the payoff it would get by "playing it safe". Our modified payoff has the advantage of leaving the replicator dynamics invariant with respect to a positive affine transformation of the payoff matrix both on degree-homogeneous and heterogeneous graphs while still allowing the degree distribution of the network to have a strong impact on the dynamics of the game. Indeed, players placed on highly connected nodes of a graph can benefit from their numerous interactions which enable them to potentially collect a high payoff. However, these same players run a risk of totaling a much lower score than a player with only a few links. One can notice that on degree-homogeneous graphs such as lattices or complete graphs, using the adjusted accumulated payoff yields the same results as using accumulated or average payoff. The proof of the invariance under positive affine transformation of this new payoff definition is straightforward; simply note that in this case $\Pi_{x,\max} = k_x(\pi_{x,\max} - \pi_{x,\gamma})$ and $\Pi_{x,\min} = k_x(\pi_{x,\min} - \pi_{x,\gamma})$, where $\pi_{x,\max} = max\{T, R, P, S\}$ and $\pi_{x,\min} = min\{T, R, P, S\}$ for the symmetric games studied here.

### B.4.4 Population Dynamics

Calling $C(t) = (s_1(t), s_2(t), \ldots, s_N(t))$ a *configuration* of the population strategies $s_i \in \{C, D\}$ at time step $t$, the global *synchronous* system dynamics leads to $C(t+1)$ by simultaneously updating all the player's strategies according to the chosen rule which is here a discrete analogue of replicator dynamics. Synchronous update, with its idealization of a global clock, is customary in spatial evolutionary games, and most results have been obtained using this model [84, 86, 52]. However, perfect synchronicity is only an abstraction as agents normally act at different and possibly uncorrelated times [48]. In spite of this, it has been shown that the update mode does not fundamentally alter the results for replicator dynamics [41, 83]. We have also checked that asynchronous update dynamics does not influence the system evolution in a significant way and so, all results presented here refer to synchronous systems.

## B.5 Results

### B.5.1 Simulation Parameters

We simulate on our networks the three games previously mentioned in section B.2. As stated above, we study three types of networks: BA SF, the TSN model, and the real acquaintance GP network. Let us point out that the BA SF graphs are only used as a benchmark for comparison with more realistic networks. Indeed, they have already been studied in [101, 104], although here we use a different payoff scheme.

For each game, we can explore the entire game space by limiting our study to the variation of only two parameters per game. This is possible without loss of generality owing to the invariance of Nash equilibria and replicator dynamics under positive affine transformations of the payoff matrix and using our payoff scheme [128]. In the case of the PD, we set $R = 1$ and $S = 0$, and vary $1 \leq T \leq 2$ and $0 \leq P \leq 1$. For the HD, we set $R = 1$ and $P = 0$ and the two parameters are $1 \leq T \leq 2$ and $0 \leq S \leq 1$. Finally, in the SH, we decide to fix $R = 1$ and $S = 0$ and vary $0 \leq T \leq 1$ and $0 \leq P \leq T$.

We deliberately choose not to vary the same two parameters in all three games. The reason we choose to set $T$ and $S$ in both the PD and the SH is to simply provide natural bounds on the values to explore of the remaining two parameters. In the PD case, $P$ is limited between $R = 1$ and $S = 0$ in order to respect the ordering of the payoffs ($T > R > P > S$) and $T$'s upper bound is equal to 2 due to the $2R > T + S$ constraint. Had we fixed $R = 1$ and $P = 0$ instead, $T$ could be as big as desired, provided $S \leq 0$ is small enough. In the HD, setting $R = 1$ and $P = 0$ determines the range of $S$ (since this time $T > R > S > P$) and gives an upper bound of 2 for $T$, again due to the $2R > T + S$ constraint. Note however, that the only valid value pairs of $(T, S)$ are those that satisfy the latter constraint.

We used networks of size $N = 10000$, except for the GP network case, whose size is $N = 942$. Each network is randomly initialized with exactly 50% cooperators and 50% defectors. In all cases, the parameters are varied between their two bounds by steps of 0.1. For each set of values, we carry out 50 runs of 16000 time steps each, using a fresh graph realization in each run (the GP network is constant). Cooperation level is averaged over the last 1000 time steps, well after the transient equilibration period.

### B.5.2 Evolution of Cooperation

In this section we present global results as they pertain to the whole network of agents; discussion of local structures, such as clusters and communities is deferred to the next section. In Figure B.1, we report average cooperation levels for the three games on three different types of networks, for systems having attained a steady-state. As expected, the region in which cooperation is possible is much more restricted in the PD than for the other two games. Cooperation is more widespread for the HD, as mutual defection is the worst outcome in this game. For the PD and the SH, cooperation is sensitive to the "punishment" level P, for a given T, with the PD

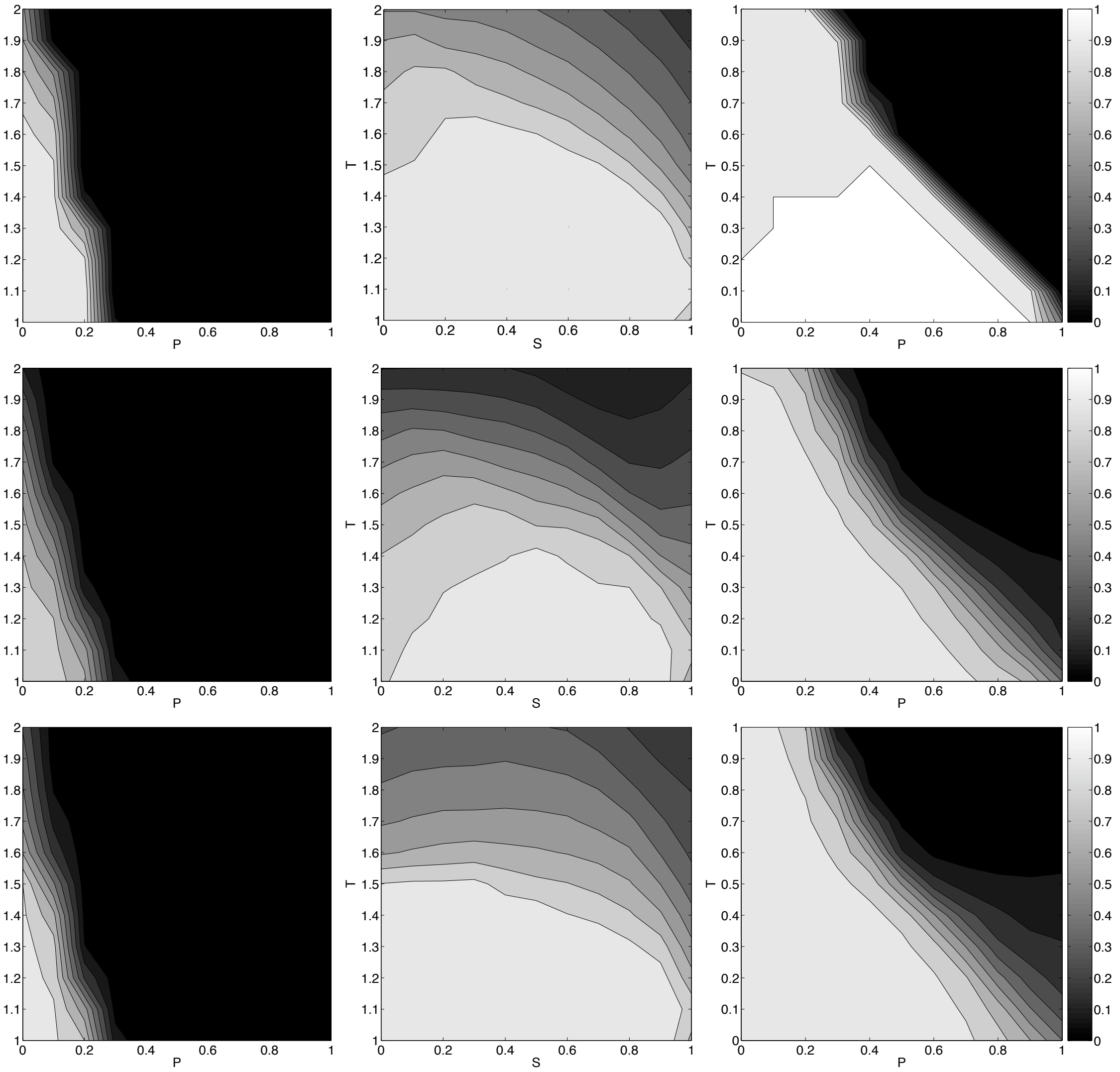
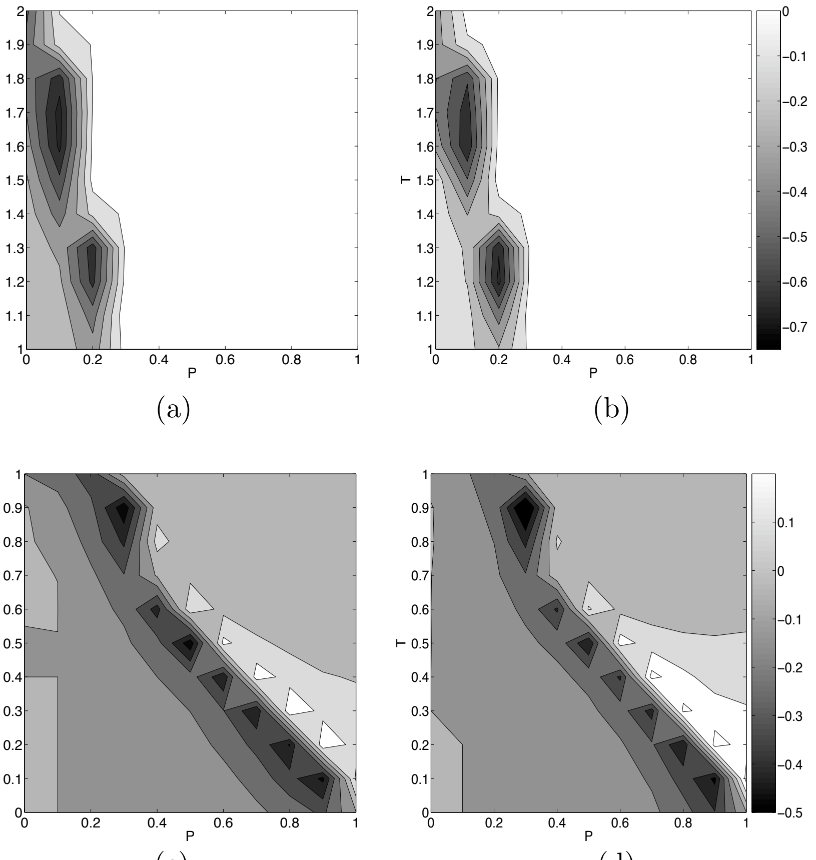


Figure B.1: Level of cooperation at the end of the simulation. From left to right: PD, HD, SH; from top to bottom: BA SF, TSN model, GP network. For the HD, the meaningful phase space is the lower left triangle; for the SH it is the upper left triangle. Results are averaged over 50 runs for each game, each network structure, and parameter set.

being influenced in a higher degree. Concerning the HD, one can see that the S parameter has moderate influence on cooperation for a given T. We also notice that the transition from cooperation to defection is much steeper in the PD and SH cases than for the HD.

Now, interpreting the results in terms of the three different topologies, BA SF networks are the structures that yield the highest cooperation levels for the three games (compare the top row with the second and third rows of Figure B.1), which qualitatively confirms the findings of [101, 104]. However, note that Santos et al.'s results differ quantitatively from ours due to the different payoff scheme used. Moreover, while our results are valid in the whole game's space, the results of [101, 104] are not because of the non-invariance of their dynamics using accumulated



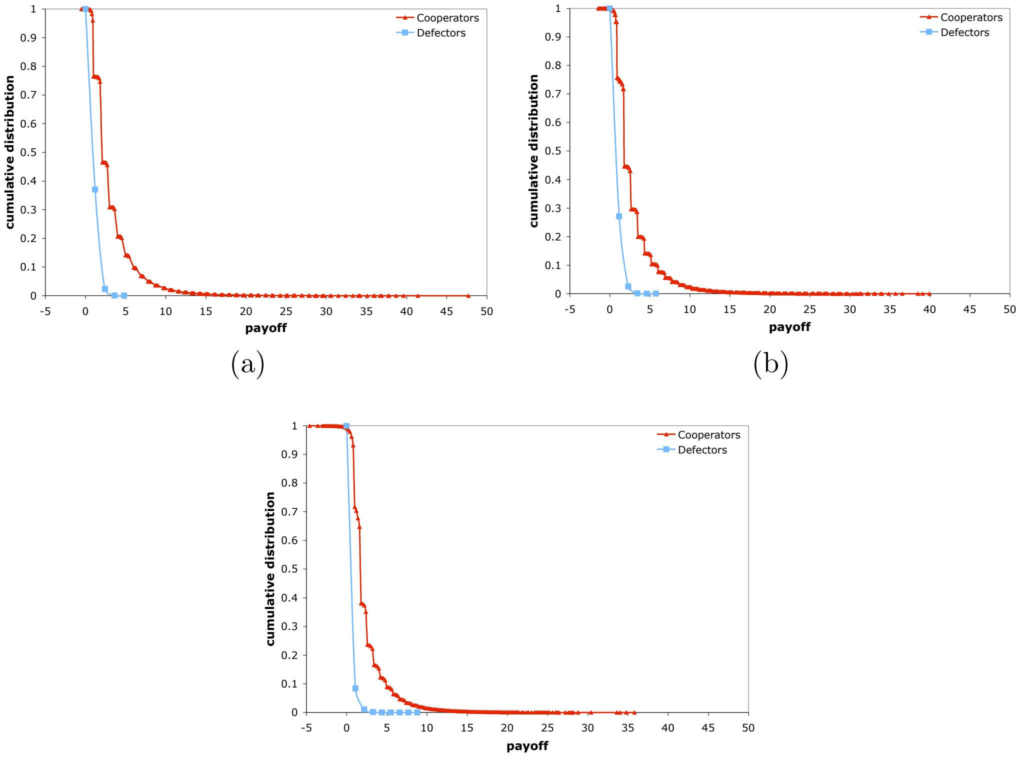
(a) (b)

(c) (d)

Figure B.2: Differences in cooperation between (a) TSN model and BA SF and (b) GP network and BA SF for the PD. (c) and (d) show the same differences for the SH.

payoff on degree-heterogeneous networks. To highlight the effects of the network topology, in Figure B.2 we show the cooperation difference plots between, respectively, the BA SF and the TSN model, and the BA SF and the GP scientific coauthorship network. The left part refers to the PD, while in the right two figures, differences are reported for the SH. Differences for the HD are not shown as they are less marked. The difference plot between the TSN model and the GP network is almost flat, as the differences are very small (not shown here). This can also be inferred from the similarity between Figures B.2 (a), (b) and (c), (d) respectively. We can thus say that the TSN model seems to be a good approximation of a real social network, at least as far as the cooperation distribution in the games spaces is concerned.

Another important global quantity characterizing a population (or subpopulation) of players as a whole is the total payoff at the end of the simulated games, sometimes called the *wealth*. The cumulated wealth of defectors and cooperators is plotted in Figure B.3 for the PD in the case of the TSN model. This is done for $T = 1.3$, for three values of the punishment P, giving rise to different cooperation regimes: one in which cooperation prevails, a second one with approximately an equal amount of cooperators and defectors, and a third case where defection predominates. We found that, while defectors' wealth curves are rather well fitted by an exponential function, the cooperators' wealth is larger and has a broader distribution which can be fitted by a stretched exponential. This hints at a clustering of cooperators, as this is the only way for them to increase their payoff. In the next section we shall provide topological evidence of this phenomenon.



(a) (b)

(c)

Figure B.3: Cumulative wealth distribution in the PD game averaged over 50 runs for the social network model; (a) $T = 1.3, P = 0.1$ yielding ∼73% of cooperation. Average C-wealth=2.92, average D-wealth=0.47; (b) $T = 1.3, P = 0.15$ yielding ∼52% of cooperation. Average C-wealth=2.74, average D-wealth=0.34; (c) $T = 1.3, P = 0.2$ yielding ∼15% of cooperation. Average C-wealth=2.30, average D-wealth=0.11.

### B.5.3 Community Structure and Cooperation

As mentioned in a previous section, real social networks show community structure. A community can be seen as a set of highly connected vertices having few connections with vertices belonging to other communities. It is a difficult task to distinguish the different communities composing the network and to determine whether a given vertex belongs to only one or several of them. There exist several algorithms to split a network into communities [79, 17, 76], each one with its pros and cons. In our case, due to the computational burden of the simulations, we chose one of the algorithms proposed by Newman [76].

When the community algorithm is run on the TSN model, the important observation is

Table B.1: Comparison of the clustering coefficients and degree assortativity of the three network types studied. For the TSN and BA SF, the values are averaged over 100 realizations. Assortativity coefficient is computed using the expression that appears in [75].

| | TSN | GP | BA SF |
|---|---|---|---|
| Clustering coefficient | $0.442 \pm 0.004$ | 0.665 | $0.026 \pm 0.005$ |
| assortativity coefficient | $0.133 \pm 0.014$ | 0.131 | $-0.089 \pm 0.027$ |

that, independent of the cooperation level, in most communities either cooperators or defectors predominate. This is best seen in the GP coauthorship network, as its relatively small size makes visualization easier. We also notice that the TSN model and the GP network both have a high clustering coefficient, very similar degree assortativity, and also qualitatively similar community structure. The corresponding features for the BA SF networks are very different (see Table B.1).

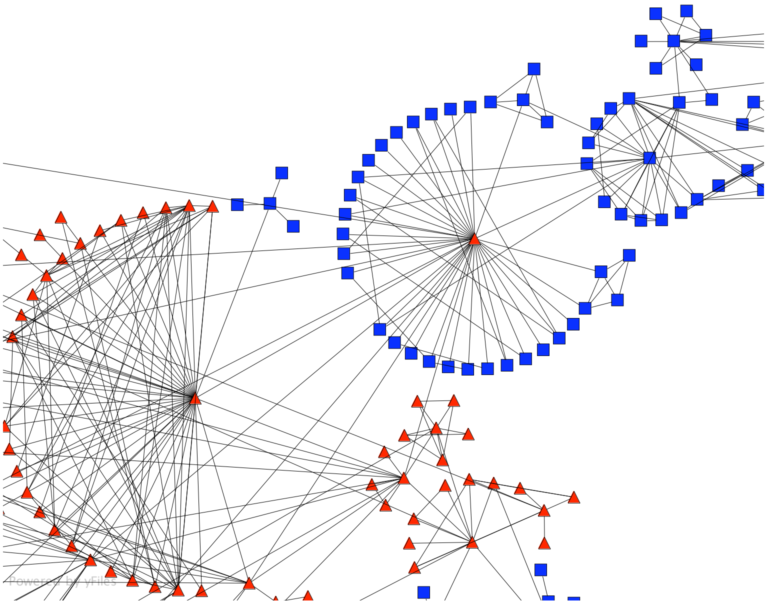
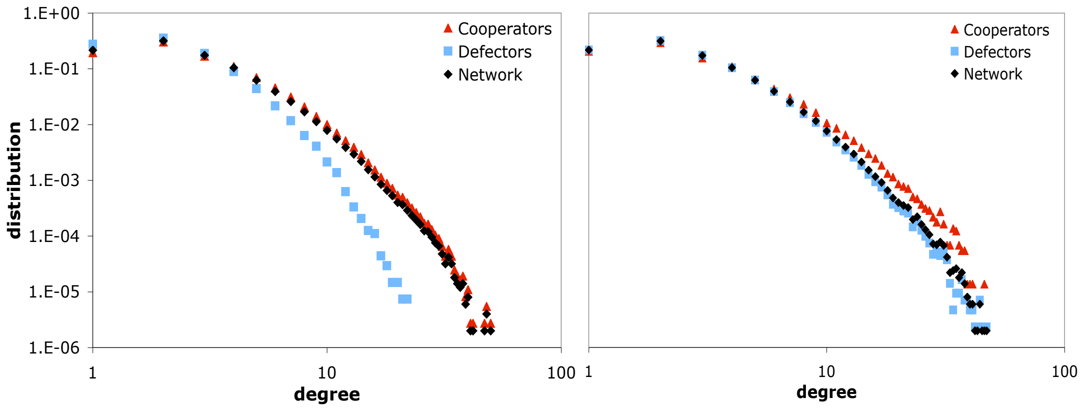


Figure B.4: Communities: cooperators are represented by triangles and defectors by squares.

For reasons of space, in the following we mainly show results for the PD, giving occasional comments for the other two games. In Figure B.4 we depict a portion of the GP graph, distinguishing between cooperators and defectors for the PD. As noted above, tightly-bound communities are mostly composed of players with the same strategy. Although we only show a small portion of the whole network for reasons of clarity, we could have chosen many other places as the phenomenon is widespread. Cooperators tend to "protect" themselves by occupying sites with many links toward other cooperators. On the other hand, a cooperator like the central one in the largest defecting community will have a tendency to become a defector since its neighbors are nearly all defectors; but when its highly connected "wealthy" cooperator neighbor on the left of the figure is probabilistically selected to be imitated, then it will certainly become a cooperator again. So, the rare cooperators that are not tightly clustered with other cooperators will tend to oscillate between strategies. Table B.2 illustrates the strategy clustering phenomenon in a quantitative way by giving the average percentage of cooperating neighbors of a player as a function of its strategy and of the global cooperation regime (high cooperation, medium cooperation, and low cooperation). We can see that in all cases a cooperator is surrounded by a large majority of cooperators, whereas a defector mainly interacts with other defectors. However, the amount of strategy segregation is less extreme when considering the HD with respect to PD and SH. In the HD case the two strategies are slightly more intermingled, confirming analogous findings for grid-structured populations [41]. These results are in qualitative agreement with the well known fact that for large mixing populations, the only evolutionary stable state in the HD is one in which neither hawks nor doves completely disappear [46]. The community structure of cooperators, together with the mutual payoff advantage of cooperating also explains the previous observation that the average cooperators' wealth exceeds the average wealth of defectors. While we focused on the way cooperators and defectors self-organize themselves within network communities and how these community structures tend to naturally segregate the two strategies, the authors of [58] concentrated on the study of the influence of the intra and intercommunity structure on global cooperation levels for the PD only.

Table B.2: The proportion of cooperators in a player's neighborhood depending on the game played, the regime at the quasi-stable state, and the player's strategy. The values were obtained by averaging over 50 runs.

| | PD | | HD | | SH | |
|---|---|---|---|---|---|---|
| | *C* | *D* | *C* | *D* | *C* | *D* |
| C regime | $0.98 \pm 0.01$ | $0.18 \pm 0.04$ | $0.93 \pm 0.02$ | $0.28 \pm 0.03$ | $0.98 \pm 0.02$ | $0.11 \pm 0.05$ |
| 50-50 regime | $0.96 \pm 0.03$ | $0.10 \pm 0.06$ | $0.91 \pm 0.01$ | $0.19 \pm 0.03$ | $0.96 \pm 0.03$ | $0.06 \pm 0.05$ |
| D regime | $0.92 \pm 0.03$ | $0.02 \pm 0.02$ | $0.61 \pm 0.02$ | $0.36 \pm 0.02$ | $0.93 \pm 0.07$ | $0.02 \pm 0.04$ |



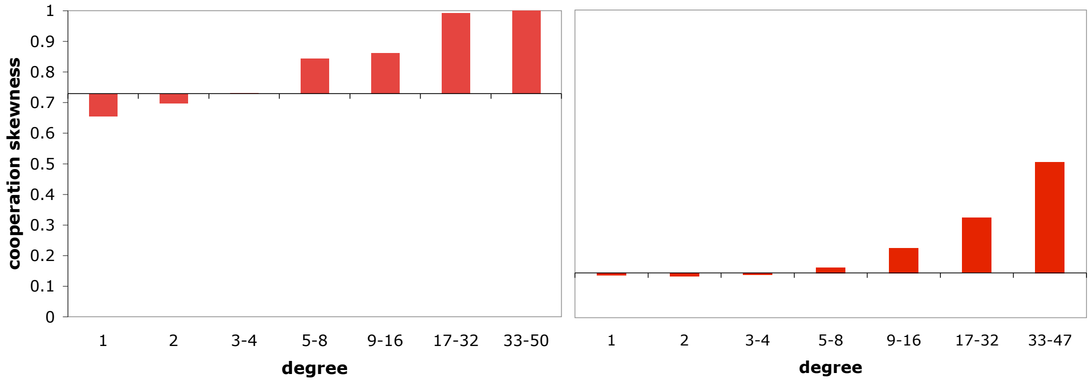
(a) (b)

Figure B.5: Degree distribution functions for the TSN model; Prisoner's dilemma. Data are averaged over 50 runs; log-log scale. (a) $T = 1.3, R = 1, P = 0.1, S = 0$; 73% cooperation. (b) $T = 1.3, R = 1, P = 0.2, S = 0$; 15% cooperation.

If we now focus on the degree distributions for the TSN model as a whole, and for cooperators and defectors separately as shown in Figure B.5, we can make the following observations. Firstly, although the distributions are fat-tailed, they are not well fitted by pure single power-laws for the number of acquaintances an individual may establish is limited, similar to what happens in real societies. Looking at the cooperators' and defectors' degree distributions, when cooperation prevails, cooperators tend to occupy the highest degree nodes (Figure B.5 (a)). When defectors predominate, as in Figure B.5 (b), the two curves tend to be closer, although the cooperators still monopolize higher degree nodes. The 50/50 case falls just in between these two extremes (not shown here to save space). As a matter of fact, for high cooperation, the defectors' curve are well fitted by an exponential distribution $P(k) = 0.473 \exp(-0.453k)$. Another way of seeing this is depicted in Figure B.6, where we plot the distribution of cooperators according to node degree, relative to the average for a given level of cooperation. Both for high and low cooperation fractions, cooperators' degrees are skewed toward higher values. For the HD and the SH the results are similar, namely, degree distributions for defectors fall off more rapidly than those for cooperators. However, when defection prevails, the effect is more marked for the HD, while for the SH the skewness is less pronounced. Finally, the same measures on the GP graph give qualitatively similar trends.

## B.6 Conclusions

Our results have implications for evolutionary games, and they may also serve to illuminate how the structure of social networks influences the game dynamics and the emergence of cooperation. For that purpose, we have chosen to use a model for the construction of networks producing features that are typical of social networks, and an actual social network that is precisely known. The standard BA SF model has also been used but only as a benchmark against which to



(a) (b)

Figure B.6: Distribution of cooperators as a function of the players' degree relative to the population average. Prisoner's dilemma. Data are averaged over 50 runs and logarithmically binned. Left: 73% cooperation. Right: 15% cooperation.

compare the results obtained on social networks. It is encouraging that the results on the two different social network structures are quite similar, and have much more in common between themselves than with BA SF graphs. This is shown quite clearly by the values of the clustering coefficient and by the degree assortativity, which is assortative for the social networks and slightly disassortative for the BA SF graphs.

For evolutionary games, the study of graph structures that extend the usual regular lattices and the more complex, but still socially unrealistic Watts–Strogatz and BA SF graphs, should be a useful one. As these games are supposedly metaphorically played by people and entities in many kinds of social interactions, it becomes important to take into account in as precise a manner as possible the actual ties that exist. When this is done, one discovers that the community structures that are so common in society play an extremely important role in the evolution of cooperation. Starting with an equal amount of randomly distributed cooperators and defectors, all simulations end up with the majority of cooperators separated from the defectors according to the underlying community structure of the network. This phenomenon is quite independent of the final global level of cooperation, at least for the simulations performed.

When one takes the structure of the social interactions into account, there are serious challenges from the theoretical point of view. However, this is a very promising direction to advance our understanding of social processes. Simulations like those presented here can help to make progress and can pave the way for more formal models.

# Article C

# Conformity Hinders the Evolution of Cooperation on Scale-Free Networks

| | |
|---|---|
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### Abstract

We study the effects of conformity, the tendency of humans to imitate locally common behaviors, in the evolution of cooperation when individuals occupy the vertices of a graph and engage in the one-shot Prisoner's Dilemma or the Snowdrift game with their neighbors. Two different graphs are studied: rings (one-dimensional lattices with cyclic boundary conditions) and scale-free networks of the Barabási-Albert type. The proposed evolutionary-graph model is studied both by means of Monte Carlo simulations and an extended pair-approximation technique. We find improved levels of cooperation when evolution is carried on rings and individuals imitate according to both the traditional pay-off bias and a conformist bias. More important, we show that scale-free networks are no longer powerful amplifiers of cooperation when fair amounts of conformity are introduced in the imitation rules of the players. Such weakening of the cooperation-promoting abilities of scale-free networks is the result of a less biased flow of information in scale-free topologies, making hubs more susceptible of being influenced by less-connected neighbors.

## C.1 Introduction

Understanding the emergence and stability of cooperation is a central problem in many fields of both natural and social sciences. Researchers have traditionally adopted evolutionary game theory [128] as common formal framework for studying the dynamics of strategy change, and games like the Prisoner's Dilemma (PD) and the Snowdrift Game (SG) as metaphors for the tension between group welfare and individual selfishness. The PD and the SG (also known as Chicken or Hawks-Doves) are two-person, symmetric games in which a given player can be, at each time step, either a Cooperator (C) or a Defector (D). Cs are willing to engage in cooperative tasks, while Ds prefer not to, thus exploiting Cs. If two individuals of the same type interact, they both get the reward for mutual cooperation $R$ if they cooperate or the punishment for mutual defection $P$ if they defect. If a D and a C interact, the D receives the temptation to defect $T$ and the C receives the sucker's pay-off $S$. In the PD, the pay-offs are ordered such that $T > R > P > S$ with $2R > T + S$. Since $T > R$ and $P > S$, the only Nash equilibrium of the game is the pure strategy (D,D). In this case, the dilemma is caused both by "greed" (or the temptation to cheat) and "fear" that the other player cheats. In the SG, the order of $P$ and $S$ is reversed, yielding $T > R > S > P$. Thus, when both players defect they get the lowest possible pay-off. The pairs of pure strategies (C,D) and (D,C) are Nash equilibria of the game. There is also a third equilibrium in mixed strategies in which strategy D is played with probability $p$ and strategy C with probability $1 - p$, where $p$ depends on the actual pay-off values. The dilemma in this game is caused only by "greed", i.e. players have a strong incentive to threat their opponent by playing D, which is harmful for both parties if the outcome happens to be (D,D).

Conventional evolutionary game theoretical models assume an infinite population in which pairs of randomly drawn individuals interact according to a given game. Selection is strictly pay-off biased, which implies that fitter individuals reproduce more (genetic evolution) or successful individuals tend to be imitated more frequently (cultural evolution). In both genetic and cultural evolution, the evolutionary process can be analytically described by a set of equations called the replicator dynamics [128]. In the SG, the only stable equilibrium of such equations is an internal one, corresponding to the mixed strategy of classical game theory, while the two pure equilibria are unstable. In the PD, the only stable rest point occurs when the population is entirely composed of Ds: Cs are doomed to extinction in this game.

Given these unfavorable predictions for the evolution of cooperation, several mechanisms have been invoked in order to explain why altruism can actually emerge, such as kin selection, group selection, direct reciprocity, indirect reciprocity and network reciprocity [82]. Network reciprocity [84, 101, 55, 112] arises when individuals occupy the vertices of a graph (modeling spatially subdivided populations or social networks) such that interactions are constrained to direct neighbors. When the population of players possesses such a structure, Cs can survive in clusters of related individuals for certain ranges of the game parameters, as it has been known since the pioneering work by Nowak and May [84]. Among the different conceivable population topologies, scale-free networks have received particular attention since they have been found to promote cooperation to a point that Cs dominate Ds in a significant portion of the parameters' space [101].

In addition to positing infinite well-mixed populations, the replicator dynamics relies on the assumption that selection is entirely pay-off biased. Such a premise, although natural to posit in genetic evolution, is less straightforward to postulate in cultural evolution where information is transmitted by means of imitation. Humans not only have a bias for imitating more successful people, but also to conform, or to show a disproportionate tendency to copy the behavior of the majority [16]. Recent empirical research has shown that conformity is an important bias in our social learning psychology [20, 24], and that it can partially account for the results obtained in laboratory experiments on social dilemmas [18, 9]. Theoretical research has also shown that conformity can promote cooperation in the PD. In the standard case of a large, well-mixed population, the dynamics can lead either to full defection or to bi-stability, depending on the amount of conformity [45, 89, 44]. In [89] the case of square lattices was studied by simulation, with the result that conformity stabilizes cooperation in such population topologies, a result confirmed for rings in [66] and, in a more detailed way, in the work presented here.

In this paper we investigate the evolution of cooperation when individuals imitate with a given amount of conformity and both interaction and imitation are constrained to nearest neighbors in a network. In order to extend previous work [84, 41, 23, 101, 116, 89] and to study the influence of the network topology, we use rings and Bárabasi-Albert scale-free networks as examples of, respectively, simple degree-homogeneous (i.e. regular) and highly degree-heterogeneous graphs. It will be shown that, while conformity reinforces the cooperation-promoting advantages of network reciprocity in rings, the very same mechanism may strongly hinder the evolution of cooperation when the network topology is scale-free. Indeed, when Cs are not initially in the majority and imitation is partly conformist, scale-free networks are no longer the powerful amplifiers of cooperation expected from the results of previous studies. There is thus an interesting interplay between conformity and network reciprocity so that the cooperation-promoting effects of conformity depend on the particular type of networks on which evolutionary dynamics are played.

## C.2 Model

We consider a population of size $N$ where the $i$-th individual is represented by the vertex $v_i$ of an undirected, simple graph $G(V,E)$. The neighborhood of $i$, $\Gamma(i)$, is the set of all individuals $j$ such that there is an edge $e_{ij} \in E$. The number of neighbors of $i$ is thus the degree $k_i$ of vertex $v_i$.

At each time step, each individual is either a C or a D. The system evolves by the successive application of interaction and imitation phases. During the interaction phase, individuals simultaneously engage in a single round of the game with their neighbors. As a result, individual $i$ collects an accumulated payoff $\Pi_i = \sum_{l \in \Gamma(i)} \pi_{il}$, where $\pi_{il}$ is the pay-off player $i$ receives when

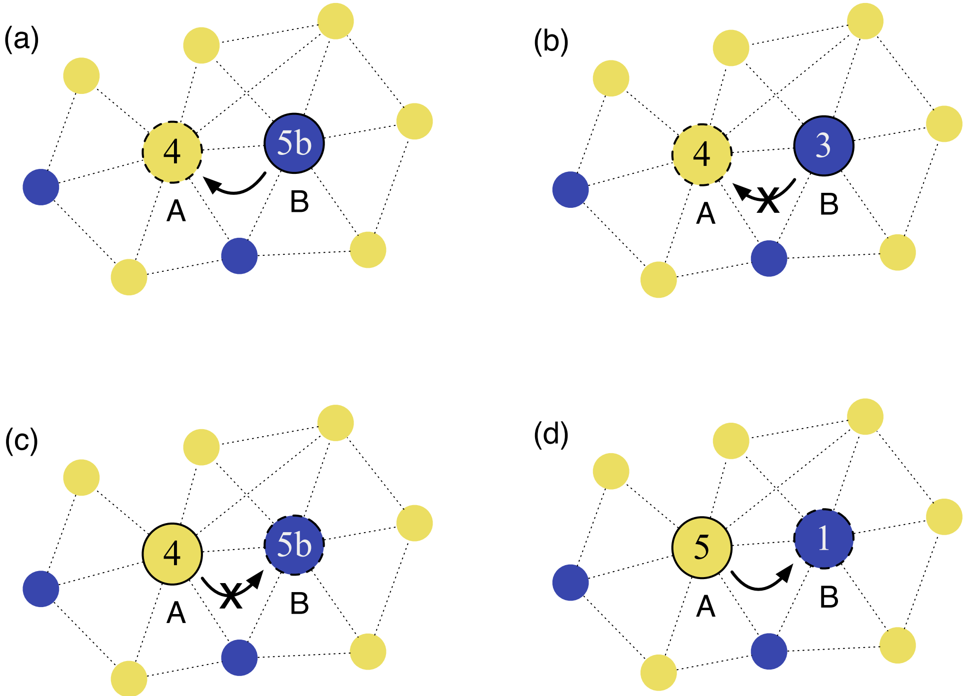

Figure C.1: (Color online) Individuals imitate following two different update rules, each reflecting a different bias of our social learning psychology: pay-off based imitation and conformist imitation. Cooperators are shown in light yellow; defectors in dark blue. Social interaction is modeled by a rescaled Prisoner's Dilemma with $T = b > 1$, $R = 1$, $P = S = 0$. (a) Successful pay-off biased transmission. When applying a pay-off biased rule of imitation, $A$ can copy $B$'s strategy and become a defector, since $B$'s pay-off is greater than $A$'s. (b) Unsuccessful conformist transmission. If $A$ were to imitate $B$ according to conformity, no transmission would take place, since defectors are in the minority of $A$'s neighborhood (3 defectors vs. 4 cooperators). (c) Unsuccessful pay-off biased transmission. $B$ will not copy $A$'s strategy under a pay-off biased rule since $A$'s pay-off is smaller than $B$'s. (d) Successful conformist transmission. Conformist transmission from $A$ to $B$ can take place because cooperators constitute the majority in $B$'s neighborhood (5 cooperators vs. 1 defector).

interacting with player $l$ (e.g. $T$, $R$, $P$ or $S$). During the imitation phase, each individual randomly chooses one of its neighbors as its *cultural model*. Let us denote $i$'s cultural model by $j$. We consider two update rules for the cultural evolutionary dynamics: pay-off biased imitation and conformist imitation. (i) For pay-off biased imitation, $i$ copies $j$'s strategy with a probability given by $f\left((\Pi_j - \Pi_i)/(\theta k_>)\right)$, where $f(x)$ is equal to $x$ if $x > 0$ and 0 otherwise, $k_> = \max\{k_i, k_j\}$, $\theta = T - S$ in the PD and $\theta = T - P$ in the SG. This update rule is a local, finite population analogue of the replicator dynamics, commonly used in the literature [41, 101]. (ii) For conformist imitation the probability that $i$ copies $j$'s strategy is given by $f\left((n_{j|i} - n_{i|i})/k_i\right)$ where $n_{l|i}$ is the number of $i$'s neighbors with strategy $l$. This update rule is related to the majority rule and to the voter model, commonly used in interdisciplinary physics studies [21]. In our model individuals imitate according to a pay-off bias with probability $1 - \alpha$, and according to a conformist bias with probability $\alpha$. Thus, the parameter $\alpha$ represents the amount of

conformity in the individuals' behavior and gives the average proportion of players imitating according to the conformity rule at each time step. When $\alpha = 0$ our local dynamics reduce to the strictly pay-off biased imitation rule used in previous studies [101, 41]. Figure C.1 gives some illustrative examples of the imitation dynamics of the proposed model.
In order to allow comparison with previous studies, we focus on the commonly used rescaled version of the PD [84, 101], for which $T = b$, $1 \leq b \leq 2$, $R = 1$ and $P = S = 0$. The parameter $b$ represents the advantage of defectors over cooperators. For the SG we make, as in [101], $T = \beta > 1$, $R = \beta - 1/2$, $S = \beta - 1$, and $P = 0$, such that the cost-to-benefit ratio of mutual cooperation is given by $r = 1/(2\beta - 1)$. It is worthy of note that, in degree-inhomogeneous networks, the local replicator dynamics using accumulated payoff is not invariant with respect to affine transformations of the payoff matrix [118, 61]. Although this fact invalidate generalizations of the obtained results to the extended parameter space, it allows us to compare our results with relevant previous work.

Before studying our model with actual network models (rings and scale-free networks) by means of numerical simulation, we briefly present analytical results obtained using the mean-field method and the pair approximation. Such analytical results are important in order to identify the dynamical regions of the system and to serve as starting point for comparisons with the dynamics on actual networks studied in Section C.4.

## C.3 Analytical Results

### C.3.1 Mean-Field Approach

Within the framework of the traditional mean-field approach [112] network locality is ignored and the system is assumed to have an infinite size, leading to an infinite, well-mixed population. In this case, it is easy to show that the time evolution of the fraction of Cs $\rho$ is ruled by the following equation:

$$\dot{\rho} = \rho(1-\rho)\left\{\gamma\left[\pi_C - \pi_D\right] + \alpha(2\rho - 1)\right\}, \tag{C.1}$$

where $\pi_C = \rho R + (1-\rho)S$ and $\pi_D = \rho T + (1-\rho)P$ are the average pay-offs to Cs and Ds, and $\gamma = (1-\alpha)/\theta$. Equation C.1 (or a similar formula) has been derived in related work on cultural transmission processes including both pay-off biased imitation and conformist imitation [44, 45, 18, 89, 110]. The dynamics has the two trivial fixed points $\rho_0^* = 0$ and $\rho_1^* = 1$, as well as (possibly) one internal non-trivial equilibrium given by

$$\rho^* = \frac{\gamma(P-S) + \alpha}{\gamma\left\{R - T + P - S\right\} + 2\alpha}.$$

For $\alpha = 0$ (pure pay-off biased transmission) Eq. C.1 recovers the standard replicator dynamics of the original game, whereas for $\alpha = 1$ (pure conformist transmission), Eq. C.1 is equivalent to the replicator dynamics of a pure coordination game with internal (unstable) equilibrium

$\rho^* = 1/2$. For $0 < \alpha < 1$, variations in the amount of conformity and the entries of the pay-off matrix can change the evolutionary dynamics of the social dilemma. In particular, the global behavior of the system depends on the two critical values $\alpha_D = (S - P)/(\theta + S - P)$ and $\alpha_C = (T - R)/(\theta + T - R)$ so that the system is in one of the following four dynamical regions:

1. Dominant defection ($\alpha > \alpha_D \wedge \alpha < \alpha_C$): $\rho_0^* = 0$ is the only stable equilibrium. In this case, Cs are doomed to extinction regardless of their initial frequency in the population.

2. Co-existence ($\alpha < \alpha_D \wedge \alpha < \alpha_C$): only the internal equilibrium $\rho^*$ is stable. Cs and Ds coexist in equilibrium at proportions given by $\rho^*$ and $1 - \rho^*$, respectively.

3. Bi-stability ($\alpha > \alpha_D \wedge \alpha > \alpha_C$): both $\rho_0^* = 0$ and $\rho_1^* = 1$ are stable whereas the internal fixed point $\rho^*$ is unstable. In this case, the evolutionary dynamics depends on the initial frequency of Cs, $\rho(0)$. For $\rho(0) > \rho^*$ cooperation prevails, whereas it vanishes for $\rho(0) < \rho^*$.

4. Dominant cooperation ($\alpha < \alpha_D \wedge \alpha > \alpha_C$): $\rho_1^* = 1$ is the only stable equilibrium; Cs get fixed regardless of their initial frequency in the population.

These regimes can be seen in Fig. C.2, which shows the phase diagrams of the two rescaled games. In the PD with conformity, $S < P \Rightarrow \alpha_D < 0$, so that only dominant defection and bi-stability are possible. In particular, for the rescaled version of the game, conformity can make the system bi-stable if $\alpha > (b-1)/(2b-1)$. However, for all values of $b$ in the bi-stability region, the basin of attraction of $\rho_0^*$ is greater than the basin of attraction of $\rho_1^*$, i.e. Cs initially in the minority are doomed to extinction regardless of their initial proportion and the values of $b$ and $\alpha$. In the SG with conformity, the four dynamical regions above described are possible, with $\alpha_D = (1 - r)/2$ and $\alpha_C = r/(1 + 2r)$. In the co-existence region, the equilibrium proportion of Cs is larger than what is expected in the $\alpha = 0$ case when $r < 1/2$ and smaller when $r > 1/2$. In the bi-stability region, the basin of attraction of $\rho_1^*$ is greater than the basin of attraction of $\rho_0^*$ for $r < 1/2$.

In sum, conformity can promote cooperation in the PD to a certain degree in the mean-field limit. If in the majority (and if conformity is strong enough) Cs now have a chance of surviving invasion from Ds, and eventually take over the whole population [89]. In the SG, whether conformity helps or hinders the evolution of cooperation actually depends on the cost-to-benefit ratio $r$. Cs are favored for $r < 1/2$ and disfavored for $r > 1/2$.

### C.3.2 Pair Approximation

Pair approximation [63, 121] improves over traditional mean-field approach for structured populations by considering the frequency of strategy pairs (i.e. C-C, C-D and D-D). Since the technique assumes regular graphs without loops, it only applies to Bethe lattices in a strict sense [42]. However, pair approximation has been used to predict evolutionary dynamics on more general regular graphs with considerable success [41, 42]. We extended the pair-dynamics

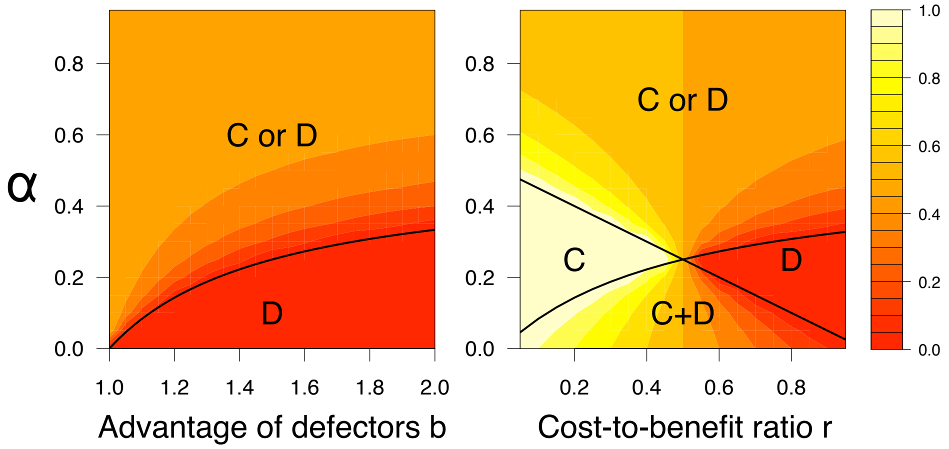

Figure C.2: (Color online) Mean-field solutions of the phase diagrams for the PD with conformity on the $b$-$\alpha$ plane (left) and for the SG with conformity on the $r$-$\alpha$ plane (right). For the PD, the system can be in the dominant defection ($D$) or the bi-stability ($C$ *or* $D$) regions. For the SG, dominant cooperation ($C$) and co-existence ($C$+$D$) are also possible outcomes. Darker colors indicate more defection in the average. In the $C$ *or* $D$ region, colors indicate the size of the basin of attraction for the cooperative equilibrium. In the $C$+$D$ region, colors indicate the equilibrium proportion of Cs.

model presented in the Supplementary Information of Ref. [41] to investigate the cultural evolutionary dynamics of social dilemmas on graphs. The pair approximation of our model leads to a system of ordinary differential equations tracking changes in the proportions $p_{c,c}$, $p_{c,d}$ and $p_{d,d}$ of, respectively, the C-C, C-D and D-D links in the population graph. The resulting system, although impossible to solve analytically due to the nonlinearity of the equations, can be solved numerically after specifying suitable initial conditions.

Figure C.3 shows the phase diagrams for the pair approximation of our model, for regular graphs with degree $k = 4$ and $k = 8$. The figures were constructed by numerically integrating the equations under different initial proportions of Cs ($\rho(0) = \{0.1, 0.2, \ldots, 0.9\}$) and averaging over all initial conditions. Pure spatial effects can be seen when $\alpha = 0$. For the PD, the dynamical regime of the game is no longer of dominant defection, but of co-existence. Locality of interactions thus favors Cs by allowing them to survive extinction. In addition to this classical result, for $k = 4$ conformity is largely favorable to Cs. Indeed, augmenting $\alpha$ increases the proportion of Cs in the co-existence region and, depending on the value of $b$, can shift the system to the region of dominant cooperation. In the SG with $k = 4$ conformity has similar effects, resulting in an analogous dynamic picture. The fact that the SG represents a less stringent dilemma makes larger the area of dominant cooperation. For $k = 8$, phase diagrams get closer to those predicted by the mean-field method (see Fig. C.2) but important levels of cooperation are still sustained. In the PD, for instance, the basins of attraction of the cooperative equilibrium in the

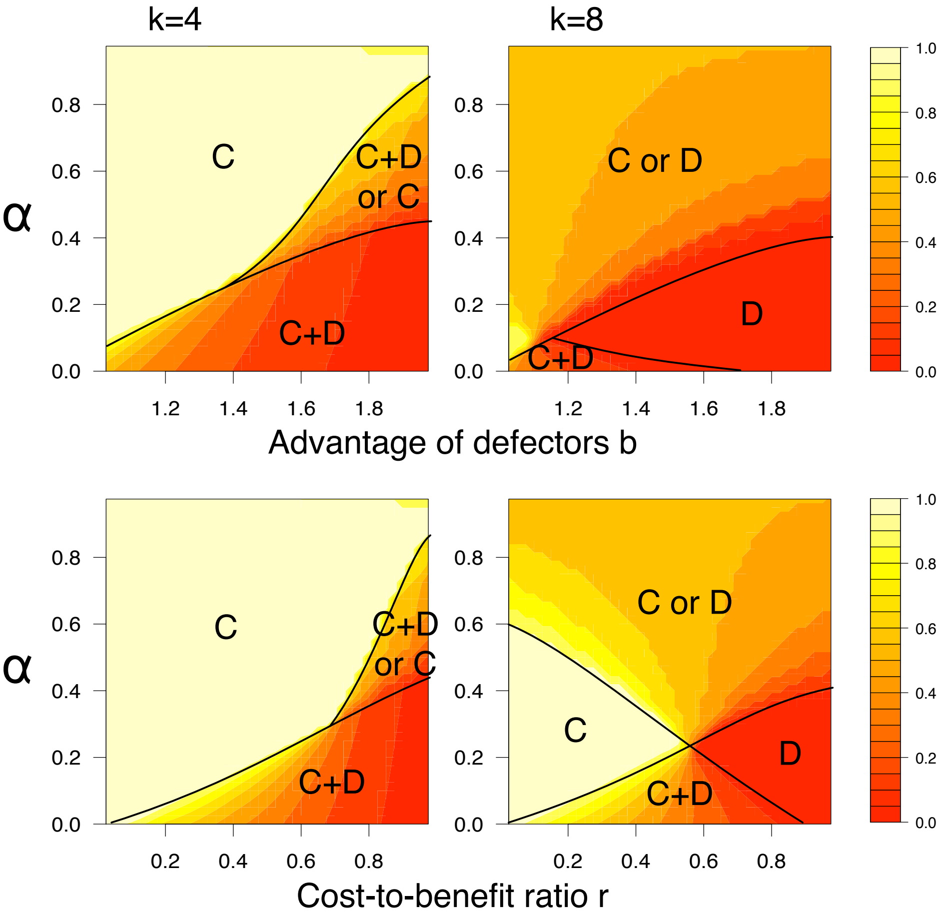

Figure C.3: (Color online) Pair approximations of the phase diagrams for the PD with conformity on the $b$-$\alpha$ plane (top row) and the SG with conformity on the $r$-$\alpha$ plane (bottom row). The first column shows the results for $k = 4$, the second column for $k = 8$. The system exhibits different dynamical regimes depending on the game: dominant cooperation ($C$), dominant defection ($D$), coexistence ($C$+$D$), and bi-stability ($C$ *or* $D$ and $C$+$D$ *or* $C$). In the $C$+$D$ *or* $C$ region, the system can stabilize in a mixed state or in pure cooperation. Darker colors indicate more defection in the average.

bi-stability region are larger than those expected in a well-mixed population (compare the top right panel of Fig. C.3 with the left panel of Fig. C.2).

In a nutshell, when the population of players possesses local structure, a given amount of conformity in the imitation rules of the players is able to foster cooperation, at least for low values of the mean degree $k$. The reason for this is the easier formation of clusters of individuals playing the same strategy induced by conformist imitation.

## C.4 Simulation Results

We now turn our attention to actual networks as population topologies, in particular (i) rings (regular 1D-lattices with cyclic boundary conditions) with degrees $k = 4$, $k = 8$ and $k = 16$, and (ii) Barabási–Albert scale-free networks [3] with average degrees $\bar{k} = 4$, $\bar{k} = 8$, and $\bar{k} = 16$. For both types of networks we generated graphs of size $N = 10^4$. In the case of rings, graphs are constructed by arranging the nodes on a circle and connecting each node to the $k$ most-neighboring nodes.

We study the model by Monte Carlo simulations in populations randomly initialized with 50% Cs and 50% Ds (but see Section C.4.3 for other initial conditions). The probability $\alpha$ of conformist transmission was set to $\alpha \in [0, 0.5]$ in steps of 0.1. We privilege values of $\alpha \leq 0.5$ so that dynamics are primarily driven by pay-off differences in the competing strategies. However, we also study the limiting case $\alpha = 1$ in Section C.4.4 and the case $0 \leq \alpha \leq 1$ in Section C.4.3. The advantage of defectors $b$ (PD) and the cost-to-benefit ratio $r$ (SG) were varied in steps of 0.05. We carried out 50 runs for each couple of values of $\alpha$ and the game parameter. For the scale-free networks, we used a fresh graph realization in each run. The average final frequency of Cs $\hat{\rho}$ was obtained by averaging over $10^3$ time steps after a relaxation time of $10^4$ time steps.

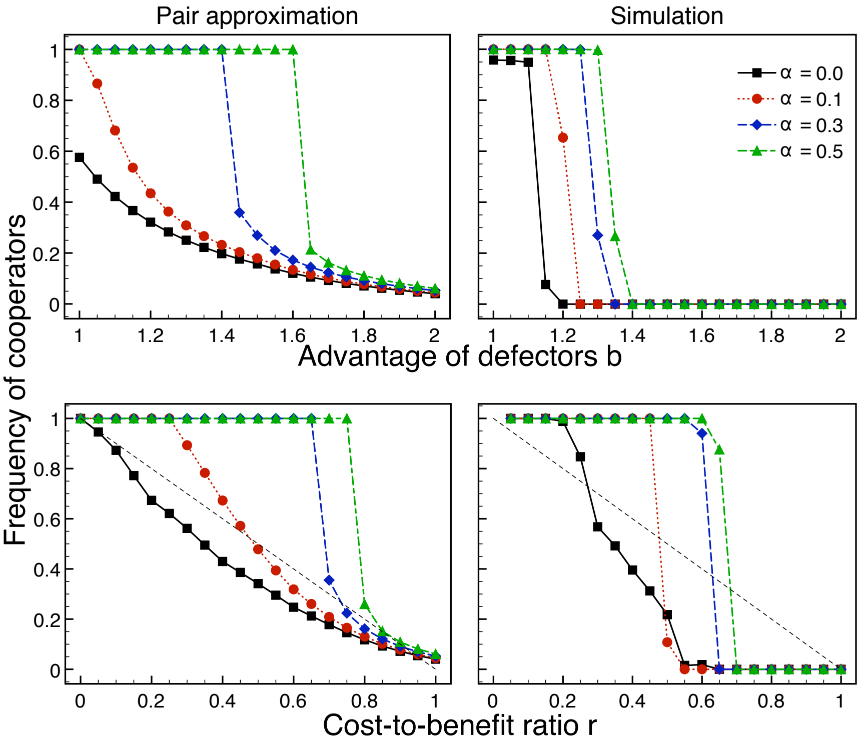

Figure C.4: (Color online) Final average frequency of Cs on rings with $k = 4$ for the PD (upper panels) and the SG (lower panels) as a function of $b$ or $r$ for different values of the propensity to conform $\alpha$. Results by Monte Carlo simulations are shown in the right panels while predictions by pair approximation are shown in the left panels. Mean-field approximations for the SG and $\alpha = 0$ are shown with dotted lines.

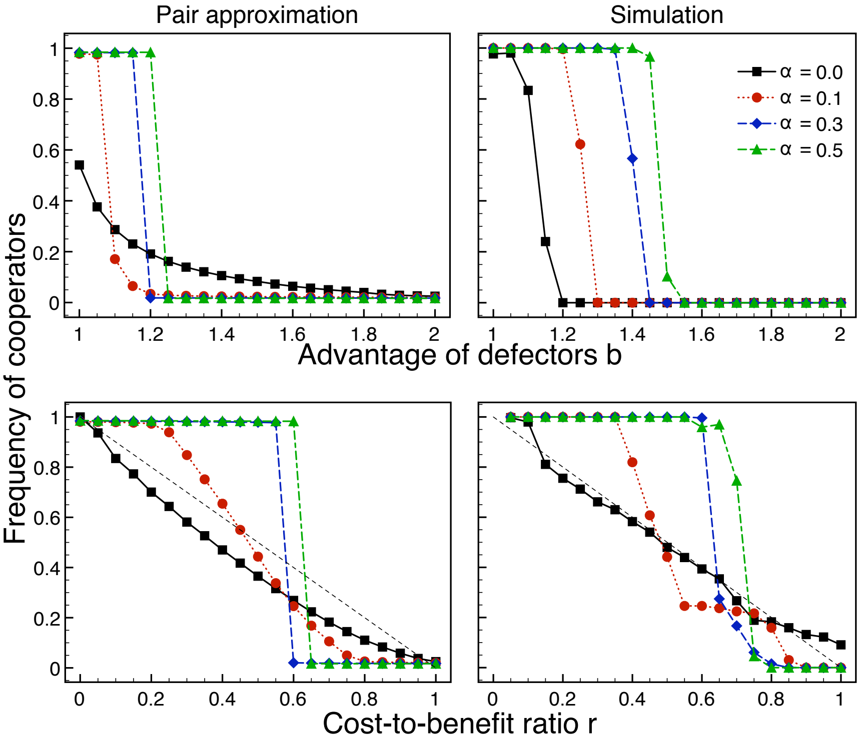

Figure C.5: (Color online) Final average frequency of Cs on rings with $k = 8$ for the PD (upper panels) and the SG (lower panels) as a function of $b$ or $r$ for different values of the propensity to conform $\alpha$. Results by Monte Carlo simulations are shown in the right panels while predictions by pair approximation are shown in the left panels. Mean-field approximations for the SG and $\alpha = 0$ are shown with dotted lines.

## C.4.1 Results for Rings

Figure C.4 summarizes the results obtained for the PD and the SG on rings with $k = 4$. These plots confirm the results previously obtained for the standard $\alpha = 0$ case on these population topologies [101, 116], which in turn are qualitatively similar to those obtained for square lattices [84, 41]. In the PD, Cs are able to survive for low values of $b$ by forming clusters wherein they interact more often with their own strategy than what is expected in well-mixed populations. Cs can thus benefit from mutual cooperation and counterbalance the exploitation of Ds at the borders of the clusters [23]. In the SG, spatial structure hinders the evolution of cooperation [41], such that only for small values of $r$ (i.e. $r < 0.3$) the final fraction of Cs is higher than what is expected in a well-mixed population. As it is evident from our results, conformity enhances cooperation in rings, moving rightward the critical value $b^*$ for which $\hat{\rho} = 0$ in the PD, and the value $r^*$ for which the $\hat{\rho}$ becomes smaller than the corresponding proportion in a well-mixed population in the SG. Furthermore, the different curves are ordered in a way that the higher $\alpha$, the higher $\hat{\rho}$ for all values of $b$ and $r$ (except for the SG, $r = 0.5$, $\alpha = 0.1$) and the larger the critical values $b^*$ and $r^*$.

Figure C.5 plots the results for rings with $k = 8$. In the PD, conformity enhances cooperation even more pronouncedly than in the $k = 4$ case. Indeed, the threshold $b^*$ has moved rightward for every value of $\alpha$. Such trend is still present in the results obtained for rings with $k = 16$ (not shown here to avoid cluttering the figures). In the SG, the increase in the degree of the graph makes conformity cooperation-enhancing up to a threshold value $\hat{r}$ (where a curve with $\alpha > 0$ crosses the curve with $\alpha = 0$) but detrimental afterwards. As $b^*$ in the PD, also $\hat{r}$ moves rightward as $\alpha$ increases.

With respect to simulation results, pair approximation tends to underestimate cooperation for low values of $\alpha$ and $b$ or $r$ and to overestimate it for medium to large values of these parameters. For the PD with conformity, results for $k = 8$ are rather pessimistic and are much closer to what we have obtained for random graphs (data not shown here). This is not surprising since random graphs are locally similar to Bethe lattices [14]. Notice, however, that pair approximation predicts reasonably well the cooperation-enhancing effects of conformity in the PD and the ordering of the curves for different values of $\alpha$. Also, for the SG, pair approximation accurately predicts the fact that the curves with conformity ($\alpha > 0$) are above the curve without conformity ($\alpha = 0$) when $k = 4$ (Fig. C.4, lower panels), but that they cross it when $k = 8$ (Fig. C.5, lower panels). This means that pair approximation correctly predicts the fact that, for $k = 8$, there is a point up to which conformity helps Cs but beyond which Ds are favored with respect to the standard case without conformity.

### C.4.2 Results for Scale-Free Graphs

Let us now turn our attention to the results obtained for scale-free networks (Fig. C.6). When imitation is strictly pay-off biased ($\alpha = 0$) these degree-heterogeneous graphs importantly foster cooperation in both the PD and the SG with respect to what is obtained in rings and other degree-homogeneous graphs [101]. As an aside, we note that the higher the average degree $\bar{k}$, the lower the gains in cooperation [1]. The addition of conformity has important consequences in the evolution of cooperation on scale-free graphs. In the PD, conformity improves $\hat{\rho}$ for all values of $b$ only for a scale-free topology with $\bar{k} = 4$ and $\alpha < 0.3$. For the other cases, conformity does not hamper cooperation for small values of $b$ but is detrimental for medium to large values of the game parameter. Furthermore, the threshold value $\hat{b}$ above which $\hat{\rho}$ is higher than in the case without conformity is a monotonically decreasing function of both $\alpha$ and $\bar{k}$, such that the higher the amount of conformity and the average connectivity of the graph, the smaller the value of $\hat{b}$. Particularly, for scale-free networks with $\bar{k} = 8$ and $\alpha \geq 0.2$, conformity weakens the advantage of these graphs in promoting cooperation to a point that $\hat{\rho}$ becomes comparable to the corresponding fraction obtained in rings (compare the right upper panels of Fig. C.6 and Fig. C.5).

Results for the SG on scale-free networks (lower panels of Fig. C.6) are qualitatively similar to those obtained for the PD. Again, conformity is beneficial for cooperation for all values of the game parameter $r$ only for $\bar{k} = 4$ and $\alpha < 0.3$. For the remaining cases, there is a threshold

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[1] When comparing our results with those of [101], note that the curves are in the wrong order in [101] as cooperation should decrease with increasing mean degree for scale-free networks.

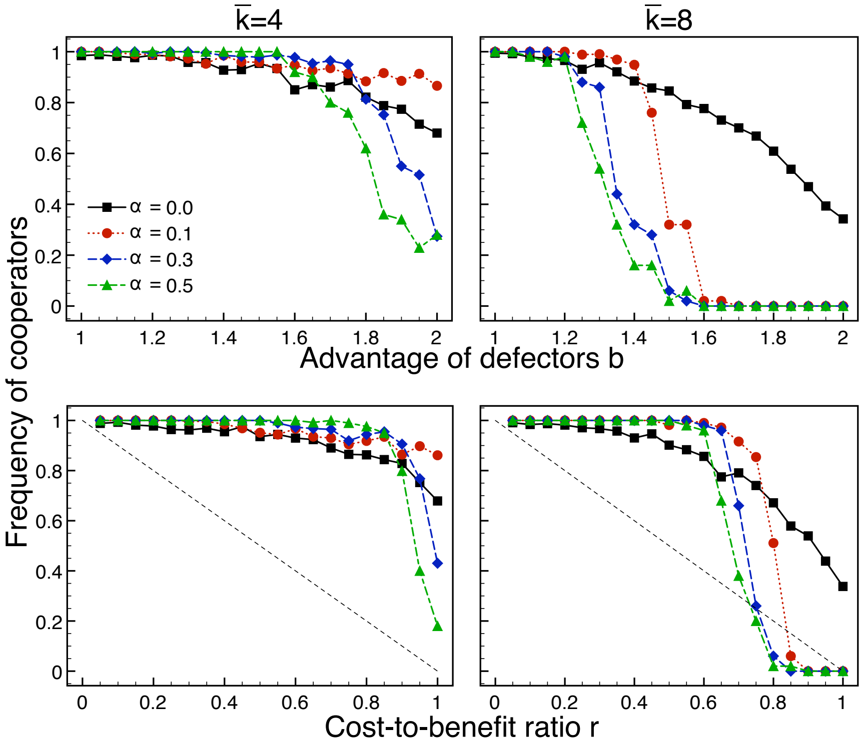

Figure C.6: (Color online) Final average frequency of Cs on scale-free networks for the PD (upper panels) and the SG (lower panels) as a function of $b$ (PD) or $r$ (SG) for different values of the propensity to conform $\alpha$. Results are shown for $\bar{k} = 4$ (left panels) and $\bar{k} = 8$ (right panels). Mean-field approximations for the SG and $\alpha = 0$ are shown with dotted lines.

value $\hat{r}$ of the cost-to-benefit ratio above which $\hat{\rho}$ is smaller than the corresponding frequency of Cs in the $\alpha = 0$ case. We note again the fact that the higher the value of $\alpha$, the lower the value of $\hat{r}$. Finally, and as in the PD, for $\bar{k} = 8$ and $\alpha \geq 0.2$ there are no important quantitative differences in $\hat{\rho}$ between rings and scale-free networks: scale-free networks have again lost the cooperation-enhancing capabilities they feature when imitation is strictly pay-off biased. For $\bar{k} = 8$ and high values of $r$, the addition of conformity can even make Cs go extinct, which would not happen in the non-conformist case.

### C.4.3 Dependence on the initial conditions

In order to investigate the robustness of cooperation and to study the influence of the initial fraction of Cs $\rho(0)$ we have also run simulations for the PD on rings and scale-free graphs for $\bar{k} = 8$ starting from values of $\rho(0)$ other than 0.5, and on an extended range of values of $\alpha$ going from 0 to 1. Results are shown in Fig. C.7 in the form of phase diagrams for each initial condition. In contrast to the notion of bi-stability in a system of ordinary differential equations (such as those resulting from the mean-field approach and the pair approximation), here we define bi-stability as the ability of the system to reach either full cooperation or full defection starting from the same *global* initial conditions, due to its stochastic dynamics and finite size.

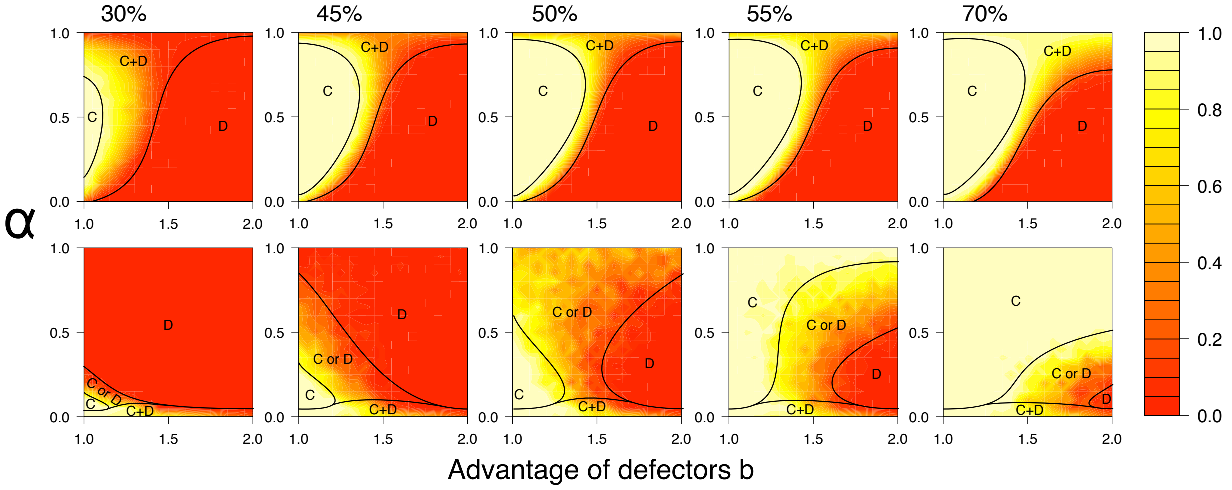

Figure C.7: (Color online) Phase diagrams for the PD game on rings with $k = 8$ (top row) and for scale-free graphs with $\bar{k} = 8$ (bottom row) as a function of $b$ and $\alpha$. The images are for increasing initial fractions of cooperation $\rho(0)$ from left to right.

Not unexpectedly, initial conditions influence the final outcomes of the simulations, so that the strategy initially in the majority is always favored with respect to the case when $\rho(0) = 0.5$. Notice, however, that the effects of conformity are still qualitatively different for each of the two types of networks considered in this study. On these phase diagrams the transition from the region of dominant cooperation ($C$) to dominant defection ($D$) is steeper on rings, where the two zones with monomorphic populations are divided by a narrow region of co-existence ($C$+$D$). On scale-free networks a large region of bi-stability ($C$ *or* $D$) tends to be formed in the middle of the parameter's space, being the largest for $\rho(0)$ close to 50%. Indeed, the cultural evolutionary dynamics are much more sensitive to the initial conditions when applied on top of scale-free networks than when they are played on top of rings. For rings, conformity favor Cs even if they are initially in the minority, such that, in general, the higher the value of $\alpha$ the higher the final fraction of Cs in the population. For scale-free networks, conformity can be favorable to cooperation when Cs are initially in the majority, but decidedly detrimental if they are in the minority. The remarkable observation is that in scale-free networks even a small change in the initial fraction of Cs can drastically change the final outcome (see the second and fourth images in the lower row of Fig. C.7 for $\rho(0) = 0.45$ and $\rho(0) = 0.55$). It would be tempting to compare the numerical results for scale-free graphs with those obtained analytically in the mean-field case and with the pair approximation (Figs. C.2 and C.3). However, this cannot be done as both the mean-field and pair approximation approaches give poor results in highly degree-inhomogeneous networks.

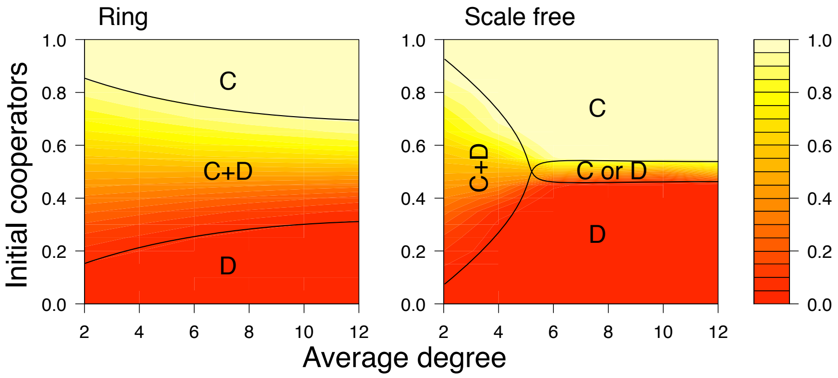

Figure C.8: (Color online) Final population composition as a function of the average degree $\bar{k}$ and the initial proportion of cooperators when imitation is purely conformist ($\alpha = 1$).

### C.4.4 Pure conformist dynamics

We briefly comment on the case with $\alpha = 1$ which is special as the dynamics is completely driven by the majority rule and games' payoffs play no role. Figure C.8 shows what happens in this case as a function of the network's average degree $\bar{k}$ and the initial proportion of Cs. For $\bar{k} = 2$ there is a large co-existence region for both graphs, and the pure equilibria have relatively small basins of attraction. With increasing $\bar{k}$, the co-existence region decreases so that a greater connectivity favors fixation in a monomorphic population. Whereas in rings co-existence is still reached for $\bar{k}$ as large as 12, for scale-free networks such regime disappears for $\bar{k} > 5$. For these networks, only in the narrow central strip around $\rho(0) = 0.5$ may bi-stability arise. Note that in this case the C and D labels indicating cooperators or defectors are purely conventional as payoffs (and so, the behavioral strategies of the individuals) are completely ignored.

## C.5 Discussion

Conformity and network reciprocity are able to act together and foster cooperation in degree-homogeneous graphs for social dilemmas such as the PD and the SG. The basic principle behind network reciprocity is the formation of clusters of related individuals leading to assortative interactions that favor Cs. Conformity further helps such cluster formation thus improving the efficiency of cooperative behavior in a network of interacting individuals.

More interestingly, conformity may hinder the evolution of cooperation on the otherwise co-operation-promoting scale-free networks. The different dynamical organization of cooperation in degree-heterogeneous graphs with conformity can explain the reason of such phenomenon. When individuals imitate exclusively according to a pay-off bias, Cs and Ds coexist in quasi-equilibrium, with some nodes fixed in cooperative or defective behavior and others where there is no fixation and cycles of invasion follow indefinitely [32]. Thus, the gradual drop in cooperation seen in Fig. C.6 for the case $\alpha = 0$ is mostly due to fluctuating individuals spending less and

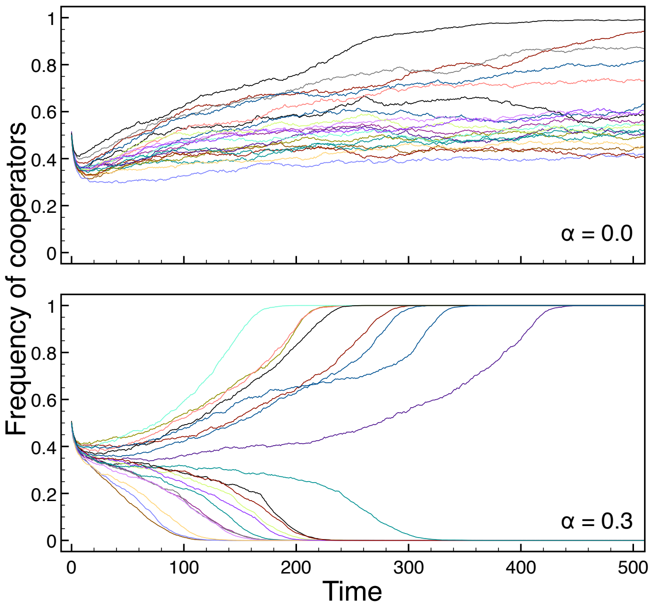

Figure C.9: (Color online) Evolution of the frequency of Cs on scale-free networks ($\bar{k} = 8$) during the first 500 time steps for the PD, without conformity (upper panel) and with conformity (lower panel). In both figures, $b = 1.35$. 20 distinct curves are shown.

less time engaging in cooperative behavior. This dynamical picture changes when individuals imitate not only according to a pay-off bias, but also to conformity. In this case, for $\bar{k} = 8$, the population always reaches one of the two absorbing states, so that in the limit only one strategy gets fixed: Cs for low values of $b$, Cs or Ds (with a certain probability) for intermediate values of $b$, and Ds for large values of $b$ (see also the bottom row of Fig. C.7). In general, and contrary to what happens without conformity, intermediate levels of cooperation for $\alpha > 0$ (when averaging over several runs) are not the result of the co-existence or fluctuation of different strategies but of the fact that, for an interval of values of $b$, whose length increases with $\alpha$, the system sometimes converges to the cooperative equilibrium and some others to the defective equilibrium (see Fig. C.7 bottom row, central image). Additionally, evolutionary dynamics develop much faster in the presence of conformity. Figure C.9 illustrates these observations for the case of scale-free networks with $\bar{k} = 8$ and $b = 1.35$. Without conformity (upper panel of Fig. C.9) the fraction of Cs for each run slowly increases during the initial part of the simulation until, eventually, it stabilizes around 0.9. Conversely, with conformity (lower panel of Fig. C.9), very early in the evolutionary process the population goes either to full cooperation or to full defection.

We can gain an insight into the interplay between network reciprocity and conformity by making use of the notion of the temperature of players [55, 62]. Hot players are those who play more since they have a large number of neighbors, whereas cold players are those who have few neighbors and, consequently, play less games. By playing more often, and provided that pay-offs are positively biased (i.e. $S \geq 0$ in the PD), hot players get higher accumulated payoffs than cold players. Under pure pay-off biased imitation ($\alpha = 0$) this implies that hot players are also

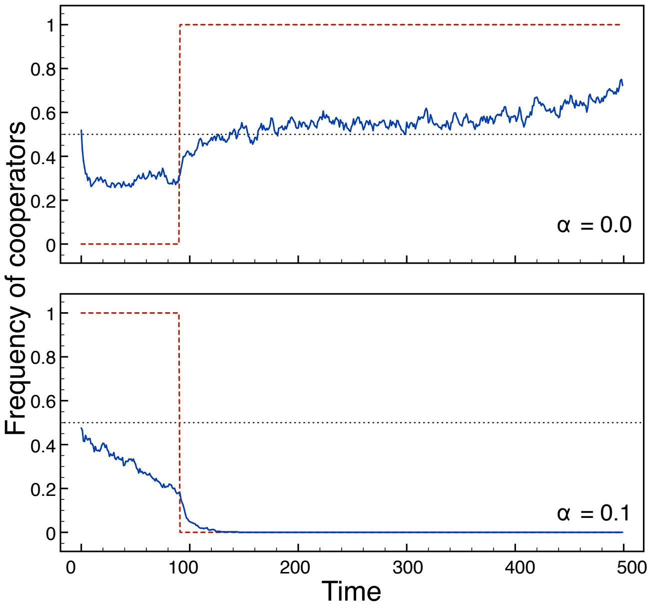

Figure C.10: (Color online) Evolution of cooperation around the most connected hub of a scale-free network with $\bar{k} = 8$. The game is a rescaled PD with $b = 1.35$ for $\alpha = 0$ (upper panel) and $\alpha = 0.1$ (lower panel). The fraction of C neighbors is shown in solid lines and the strategy of the hub in dashed lines (D corresponds to 0; C to 1). As a reference, the level of 50% cooperation is depicted in dotted lines. The most connected hub is initially set to D (upper panel) or to C (lower panel). The rest of the population is initialized to around 50% Cs.

more successful in being imitated and in disseminating their strategies [62].

Both Cs and Ds do better when they are surrounded by Cs. By spreading defective behavior, hot Ds become less and less successful, since the number of their C neighbors decreases. Hot Cs, on the contrary, see their pay-off increased by spreading their own strategy. The more hot Cs are imitated the more they earn and the more difficult it is for a surrounding D to invade. A typical example of such "hub dynamics" is illustrated in Fig. C.10 (upper panel) for the most connected hub of a scale-free network. The hub is D at the beginning of the simulation, while the rest of the population is initialized to around 50% Cs. Many C neighbors imitate the defective hub (or other surrounding Ds) during the first steps of simulation, so that the proportion of C neighbors is reduced to approximately 30%. As a consequence, the total pay-off of the hub is reduced, and the hub becomes vulnerable to invasion from a neighboring C. When the hub becomes a C, more and more of its D neighbors also switch their strategies. Consequently, the proportion of C neighbors (and the total pay-off to the hub) increases and is maintained at a high level afterwards. The presence of such positive feedback mechanism, and the fact that it only works for Cs, greatly enhances cooperation in degree-heterogeneous graphs and, particularly, in scale-free networks [100].

The introduction of conformity decreases the bias in the flow of information in degree-heterogeneous graphs, making hubs vulnerable to invasion from their cold neighbors. While hubs are unlikely to imitate their low connected neighbors when using a pay-off biased rule, nothing prevents them from imitating a cold surrounding player if it holds the strategy of the local majority (see Fig. C.1(d)). Since the fraction of Cs generally decreases at the outset of the simulation (see the first time steps of the curves shown in Fig. C.9), conformity further favors Ds, which become predominant in the population. An example of this dynamics is shown in Fig. C.10 (lower panel). Initially, the hub is a C. Many of the hub's neighbors turn to defection during the first time steps, making cooperation the less common strategy in the hub's neighborhood. Around the 100th time step, the hub imitates by conformity one of its defector neighbors, leading to a quicker decrease in the proportion of Cs in its neighborhood. Shortly after, Cs completely vanish around the most connected hub. During those first time steps, hubs imitating according to a conformist bias will have many chances of becoming Ds. When Cs are not initially in the large majority, such initial asymmetry in the strategies of the hubs can account for the negative effects of conformity in the evolution of cooperation in scale-free networks. Conformity partly reverses the flow of information on degree-heterogeneous networks so that hubs no longer conduct the dynamics and instead quickly conform to the general trend of the whole population.

## C.6 Conclusions

To sum up, we have investigated the effects of conformity in the evolution of cooperation on regular one-dimensional lattices (rings) and scale-free networks. This was done by proposing an updating rule that is a stochastic average of the traditional local replicator dynamics, which models pay-off biased imitation, and a conformist biased rule of transmission favoring the most common variants around focal individuals. We explored rings and scale-free networks with different average degrees, as well as different values of the propensity to conform $\alpha$. Two games representing social dilemmas were studied: the rescaled versions of the PD, and the SG. In addition to Monte Carlo simulations, we also used an extended pair-dynamics model to predict the average fraction of cooperators in equilibrium, and compare them with the results obtained from our simulations.

The results presented in this paper show that whether conformity strengthens or weakens the evolution of cooperation depends on the intrinsic characteristics of the underlying graph. In the PD, conformity favors cooperation on rings by allowing clusters of Cs forming more easily. Conversely, it can hinder cooperation in scale-free networks for medium to large values of $b$, due to the exposure of hubs to the opinions of the local majority in their neighborhoods. In particular, and already for small amounts of conformity in the imitation rules of the players, scale-free networks do not show the great improvement over regular structures that has been previously reported in the literature. In the SG, conformity fosters cooperation on rings in the case $k = 4$ for all values of the cost-to-benefit ratio $r$, and for low to medium values of $r$ in the case $k = 8$. In scale-free networks, conformity is rather detrimental for large values of $r$. Thus, for both the PD and the SG, conformity often hinders the evolution of cooperation on scale-free networks for the cultural evolutionary dynamics described in this paper.

It is worth pointing out that other factors dismissing the advantage of scale-free networks in the evolution of cooperation have been identified, such as participation costs [62], other positive affine transformations of the pay-off matrix [118, 61], and the use of average instead of accumulated pay-offs [102]. While these factors are extrinsic to the imitation rules of the agents, conformity is a simple mechanism undoubtedly present in our social learning psychology and central to better understand cultural dynamics and the way cooperation evolves on real social networks.

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## C.7 Appendix: Pair approximation

An analytical approximation of the dynamics of evolutionary games on graphs can be obtained by means of pair approximation [63, 121]. For detailed surveys of this technique, and its applications to games on graphs, we refer the interested reader to Refs. [41, 42, 112]. We limit ourselves to briefly introduce the pair approximation and to explain how we have extended it for taking into account conformity in the imitation rules of the players.

Pair approximation is a method for constructing a system of ordinary differential equations for the global frequencies of strategies by tracking the changes in the frequencies of strategy pairs. In our case, we are interested in determining the global frequency $\rho$ of Cs by tracking the fluctuations in $p_{c,c}$, $p_{c,d}$, $p_{d,c}$ and $p_{d,d}$, where $p_{s,s'}$ is the probability of having an individual playing strategy $s$ connected to an individual playing strategy $s'$. For pair approximation to be consistent with the mean-field approach, it is assumed that $p_s = \sum_{s'} p_{s,s'}$. Furthermore, and in order to "close" the set of equations, configurations of triplets and more complicated configurations are approximated by the configuration probabilities of strategy pairs. For example, the configuration probability of the triplet $s, s', s''$ is approximated by $p_{s,s',s''} = p_{s,s'} p_{s',s''} / p_{s'}$. It is important to note that pair approximation (i) requires regular graphs and (ii) corrections arising from loops are ignored. Finally, note that the predictions of the pair approximation for any two regular graphs with the same degree $k$ are exactly the same. This allows us to compare our results to those of [41] when $\alpha = 0$.

Let us consider individuals sitting on the vertices of a graph of degree $k$. Whenever a randomly chosen site $A$ updates its strategy, a random neighbor $B$ is selected as $A$'s cultural model. Common neighbors of any pair of vertices are considered to be independent by pair approximation (i.e. loops are neglected). Thus, let us denote by $a_1, \ldots, a_{k-1}$ (resp. $b_1, \ldots, b_{k-1}$) the $k-1$ the neighbors of $A$ (resp. $B$) other than $B$ (resp. $A$). The probability of a generic configuration

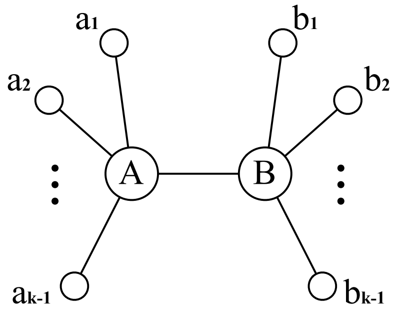

Figure C.11: A generic configuration for pair approximation. $A$ is the focal individual, $B$ is $A$'s cultural model, $a_1, a_2, \ldots, a_{k-1}$ are $A$'s neighbors other than $B$, and $a_1, a_2, \ldots, a_{k-1}$ are $B$'s neighbors other than $A$. $A$ and $B$ are assumed to have no common neighbors, i.e. triangles and loops are neglected.

(see Fig. C.11) is given by:

$$p_{A,B}\frac{\prod_{i=1}^{k-1} p_{a_i,A}p_{b_i,B}}{p_A^{k-1}p_B^{k-1}}.$$

The probability that the pair $A, B$ becomes $B, B$ is calculated by multiplying the transition probability $\sigma_{A\to B}$ by the configuration probability and summing over all possible configurations, so that:

$$p_{A,B\to B.B} = \sum_{a_1,\ldots,a_{k-1}} \sum_{b_1,\ldots,b_{k-1}} \sigma_{A\to B} \times p_{A,B}\frac{\prod_{i=1}^{k-1} p_{a_i,A}p_{b_i,B}}{p_A^{k-1}p_B^{k-1}}.$$

In our model, the transition probability $\sigma_{A\to B}$ depends not only on the payoffs of $A$ and $B$ but also on $\alpha$ (the probability to imitate according to a conformist bias) and on the number of players among $a_1, \ldots, a_{k-1}$ playing the same strategy of $A$ and $B$. The transition probability is given by:

$$\begin{aligned}\sigma_{A\to B} &= (1-\alpha)f\left(\frac{\Pi_B(b_1,\ldots,b_{k-1}) - \Pi_A(a_1,\ldots,a_{k-1})}{k\theta}\right) + \\ &\quad \alpha f\left(\frac{n_B(a_1,\ldots,a_{k-1},B) - n_A(a_1,\ldots,a_{k-1},B)}{k}\right),\end{aligned}$$

where $\Pi_B(x_1,\ldots,x_{k-1})$, $\Pi_A(x_1,\ldots,x_{k-1})$ denote the payoffs of $B$ ($A$) interacting with $x_1,\ldots,x_{k-1}$ plus $A$ ($B$), and $n_B(a_1,\ldots,a_{k-1},B)$, $n_A(a_1,\ldots,a_{k-1},B)$ specify the number of players with strategy $B$ ($A$) among $a_1,\ldots,a_{k-1}$ and $B$. The definitions of the parameter $\theta$ and the function $f$ are given in Section C.2.

Whenever $A$ imitates $B$, the pair configuration probabilities change so that $p_{B,B}$, $p_{B,a_i},\ldots,p_{B,a_{k-1}}$ increase, while $p_{A,B}$, $p_{A,a_i},\ldots,p_{A,a_{k-1}}$ decrease. All these changes lead to a set of ordinary differential equations governing the dynamics of the system:

$$\begin{aligned}
\dot{p}_{c,c} = & \sum_{a_1,\dots,a_{k-1}} (n_c(a_1,\dots,a_{k-1})+1) \prod_{i=1}^{k-1} p_{d,a_i} \sum_{b_1,\dots,b_{k-1}} \prod_{j=1}^{k-1} p_{c,b_j} \times \\
& \left\{(1-\alpha) f\left(\frac{\Pi_c(b_1,\dots,b_{k-1}) - \Pi_d(a_1,\dots,a_{k-1})}{k\theta}\right) + \right. \\
& \left. \alpha f\left(\frac{2n_c(a_1,\dots,a_{k-1})+2-k}{k}\right)\right\} - \\
& \sum_{a_1,\dots,a_{k-1}} n_c(a_1,\dots,a_{k-1}) \prod_{i=1}^{k-1} p_{c,a_i} \sum_{b_1,\dots,b_{k-1}} \prod_{j=1}^{k-1} p_{d,b_j} \times \\
& \left\{(1-\alpha) f\left(\frac{\Pi_d(b_1,\dots,b_{k-1}) - \Pi_c(a_1,\dots,a_{k-1})}{k\theta}\right) + \right. \\
& \left. \alpha f\left(\frac{k - n_c(a_1,\dots,a_{k-1})}{k}\right)\right\}
\end{aligned}$$

$$\begin{aligned}
\dot{p}_{c,d} = & \sum_{a_1,\dots,a_{k-1}} \left(\frac{k}{2} - 1 - n_c(a_1,\dots,a_{k-1})\right) \prod_{i=1}^{k-1} p_{d,a_i} \sum_{b_1,\dots,b_{k-1}} \prod_{j=1}^{k-1} p_{c,b_j} \times \\
& \left\{(1-\alpha) f\left(\frac{\Pi_c(b_1,\dots,b_{k-1}) - \Pi_d(a_1,\dots,a_{k-1})}{k\theta}\right) + \right. \\
& \left. \alpha f\left(\frac{2n_c(a_1,\dots,a_{k-1})+2-k}{k}\right)\right\} - \\
& \sum_{a_1,\dots,a_{k-1}} \left(\frac{k}{2} - n_c(a_1,\dots,a_{k-1})\right) \prod_{i=1}^{k-1} p_{c,a_i} \sum_{b_1,\dots,b_{k-1}} \prod_{j=1}^{k-1} p_{d,b_j} \times \\
& \left\{(1-\alpha) f\left(\frac{\Pi_d(b_1,\dots,b_{k-1}) - \Pi_c(a_1,\dots,a_{k-1})}{k\theta}\right) + \right. \\
& \left. \alpha f\left(\frac{k - n_c(a_1,\dots,a_{k-1})}{k}\right)\right\},
\end{aligned}$$

where $n_c(a_1,\dots,a_{k-1})$ gives the number of Cs among $a_1,\dots,a_{k-1}$ and $\Pi_c(x_1,\dots,x_{k-1})$, $\Pi_d(x_1,\dots,x_{k-1})$ denote the payoffs of a C (D) interacting with $x_1,\dots,x_{k-1}$ plus a D (C). Because of the symmetry condition $p_{c,d} = p_{d,c}$ and the constraint $p_{c,c}+p_{c,d}+p_{d,c}+p_{d,d} = 1$ these two differential equations are sufficient to describe the system. Note that whenever $\alpha = 0$ the system of equations is equivalent to that derived in the supplementary information of Ref. [41] and the appendix of Ref. [42]. Following those works, the above equations also omit the common factor $2p_{c,d}/(\rho^{k-1}p_d^{k-1})$, which has no influence in the equilibria of the system. The equilibrium values $\hat{p}_{c,c}$, $\hat{p}_{c,d}$, were obtained by numerically integrating the equations after specifying initial conditions for $10^{10}$ time steps. In all cases, $p_{c,c}(0) = (\rho(0))^2$, $p_{c,d}(0) = \rho(0)(1-\rho(0))$. The equilibrium frequency of Cs was then approximated by $\hat{p}_c = \hat{p}_{c,c} + \hat{p}_{c,d}$.

# Part III

# Games on Dynamic Networks

# Article D

# Evolution of Cooperation and Coordination in a Dynamically Networked Society

| | |
|---|---|
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### Abstract

Situations of conflict giving rise to social dilemmas are widespread in society and game theory is one major way in which they can be investigated. Starting from the observation that individuals in society interact through networks of acquaintances, we model the co-evolution of the agents' strategies and of the social network itself using two prototypical games, the Prisoner's Dilemma and the Stag Hunt. Allowing agents to dismiss ties and establish new ones, we find that cooperation and coordination can be achieved through the self-organization of the social network, a result that is non-trivial, especially in the Prisoner's Dilemma case. The evolution and stability of cooperation implies the condensation of agents exploiting particular game strategies into strong and stable clusters which are more densely connected, even in the more difficult case of the Prisoner's Dilemma.

## D.1 Introduction

In this paper we study the behavior of a population of agents playing some simple two-person, one-shot non-cooperative game. Game theory [71] deals with social interactions where two or more individuals take decisions that will mutually influence each other. It is thus a view of collective systems in which global social outcomes emerge as a result of the interaction of the individual decisions made by each agent. Some extremely simple games lead to puzzles and dilemmas that have a deep social meaning. The most widely known among these games is the Prisoner's Dilemma (PD), a universal metaphor for the tension that exists between social welfare and individual selfishness. It stipulates that, in situations where individuals may either cooperate or defect, they will rationally choose the latter. However, cooperation would be the preferred outcome when global welfare is considered. Other simple games that give rise to social dilemmas are the Hawk-Dove and the Stag-Hunt (SH) games.

In practice, however, cooperation and coordination on common objectives is often seen in human and animal societies [6, 109]. Coordinated behavior, such as having both players cooperating in the SH, is a bit less problematic as this outcome, being a Nash equilibrium, is not ruled out by theory. For the PD, in which cooperation is theoretically doomed between rational agents, several mechanisms have been invoked to explain the emergence of cooperative behavior. Among them, repeated interaction, reputation, and belonging to a recognizable group have often been mentioned [6]. Yet, the work of Nowak and May [84] showed that the simple fact that players are arranged according to a spatial structure and only interact with neighbors is sufficient to sustain a certain amount of cooperation even when the game is played anonymously and without repetition. Nowak and May's study and much of the following work were based on regular structures such as two-dimensional grids (see [86] for a recent review). Nevertheless, many actual social networks usually have a topological structure that is neither regular nor random but rather of the *small-world* type. Roughly speaking, small-world networks are graphs in which any node is relatively close to any other node. In this sense, they are similar to random graphs but unlike regular lattices. However, in contrast with random graphs, they also have a certain amount of local structure, as measured, for instance, by a quantity called the *clustering coefficient* which essentially represents the probability that two neighbors of a given node are themselves connected (an excellent review of the subject appears in [75]). Some work has been done in recent years in the direction of using those more realistic kind of networks, including actual social networks. In particular we mention Santos and Pacheco's work on scale-free networks [101], work on Watts–Strogatz small-world graphs [1, 116], and on model and real social networks [60]. A recent contribution focuses on repeated games and learning [125] and Szabó and Fáth have published an excellent and very complete review of work done up to 2006 [112]. These investigations have convincingly shown that a realistic structure of the society, with interactions mainly limited to neighbors in the network, is well sufficient in allowing cooperative and coordinated behavior to emerge without making any particular assumption about the rationality of the actors or their computational and forecasting capabilities.

Most of the above mentioned studies have assumed a fixed population size and structure, which amounts to dealing with a closed system and ignoring any fluctuations in the system's size and internal interactions. However, real social networks, such as friendship or collaboration networks, are not in an equilibrium state, but are open systems that continually evolve with new agents joining or leaving the network, and relationships (i.e. links in network terms) being made or dismissed by agents already in the network [8, 54, 115]. Thus, the motivation of the present work is to re-introduce these coupled dynamics into our model and to investigate under which conditions, if any, cooperative and coordinated behavior may emerge and be stable. In this paper, we shall deal with networked populations in which the number of players remains constant but the interaction structure, i.e. who interacts with whom, does not stay fixed; on the contrary, it changes in time and its variation is dictated by the very games that are being played by the agents. A related goal of the present work is to study the topological structures of the emergent networks and their relationships with the strategic choices of the agents.

Some previous work has been done on evolutionary games on dynamic networks [59, 103, 111, 131]. Skyrms and Pemantle [111] was recently brought to our attention by a reviewer. It is one of the first important attempts to study the kind of networks that form under a given game and, as such, is closely related to the work we describe here. The main ideas are similar to ours: agents start interacting at random according to some game's payoff matrix and, as they evolve their game strategy according to their observed payoffs, they also have a chance of breaking ties and forming new ones, thus giving rise to a social network. The main differences with the present work is that the number of agents used is low, of the order of 10 instead of the $10^3$ used here. This allows us to study the topological and statistical nature of the evolving networks in a way that is not possible with a few agents, while Skyrms' and Pemantle's work is more quantitative in the study of the effects of the stochastic dynamics on the strategy and network evolution process. The work of Zimmermann and Eguíluz [131] is based on similar considerations too. There is a rather large population which has initially a random structure. Agents in the population play the one-shot two-person Prisoner's Dilemma game against each other and change their strategy by copying the strategy of the more successful agent in their neighborhood. They also have the possibility of dismissing interactions between defectors and of rewiring them randomly in the population. The main differences with the present work are the following. Instead of just considering symmetrical undirected links, we have a concept of two directed, weighted links between pairs of agents. In our model there is a finite probability of breaking any link, not only links between defectors, although defector-defector and cooperator-defector links are much more likely to be dismissed than cooperator-cooperator links. When a link is broken it is rewired randomly in [131] while we use a link redirection process which favors neighbors with respect to more relationally distant agents. In [131] only the Prisoner's Dilemma is studied and using a reduced parameter space. We study both the Prisoner's Dilemma and the Stag Hunt games covering a much larger parameter space. Concerning timing of events, we use an asynchronous update policy for the agents' strategies, while update is synchronous in [131]. Finally, instead of a best-takes-over discrete rule, we use a smoother strategy update rule which changes an agent's strategy with a probability proportional to the payoffs difference. Santos et al. [103] is a more recent paper also dealing with similar issues. However, they use a different algorithm for severing an undirected link between two agents which, again, does not include the concept of a link weight. Furthermore, the Stag Hunt game is only mentioned in passing, and their strategy update rule is different. In particular, they do not analyze in detail the statistical structure of the emerging networks, as we do here. Other differences with the above mentioned related works will be described in the discussion and analysis of results. Finally, our own previous work [59] also deals with the co-evolution of strategy and structure in an initially random network. However, it is very different from the one presented here since we used a semi-rational threshold decision rule for a family of games similar, but not identical to the Prisoner's Dilemma in [59]. Furthermore, the idea of a bidirectional weighted link between agents was absent, and link rewiring was random.

This article is structured as follows. In sect. D.2, we give a brief description of the games used in our study. This part is intended to make the article self-contained. In sect. D.3, we present a detailed description of our model of co-evolving dynamical networks. In sect. D.4, we present and discuss the simulation results and their significance for the social networks. Finally, in sect. D.5, we give our conclusions and discuss possible extensions and future work.

## D.2 Social Dilemmas

The two representative games studied here are the Prisoner's Dilemma (PD) and the Stag-Hunt (SH) of which we briefly summarize the significance and the main results. More detailed accounts can be found elsewhere, for instance in [6, 109]. In their simplest form, they are two-person, two-strategies, symmetric games with the following payoff bi-matrix:

$$
\begin{array}{c|cc}
 & \text{C} & \text{D} \\
\hline
\text{C} & (\text{R},\text{R}) & (\text{S},\text{T}) \\
\text{D} & (\text{T},\text{S}) & (\text{P},\text{P})
\end{array}
$$

In this matrix, R stands for the *reward* the two players receive if they both cooperate (C), P is the *punishment* for bilateral defection (D), and T is the *temptation*, i.e. the payoff that a player receives if it defects, while the other cooperates. In this case, the cooperator gets the *sucker's payoff* S. In both games, the condition $2R > T + S$ is imposed so that mutual cooperation is preferred over an equal probability of unilateral cooperation and defection. For the PD, the payoff values are ordered numerically in the following way: $T > R > P > S$. Defection is always the best rational individual choice in the PD; (D,D) is the unique *Nash equilibrium* (NE) and also an *evolutionarily stable strategy* (ESS) [71, 128]. Mutual cooperation would be preferable but it is a strongly dominated strategy.

In the SH, the ordering is $R > T > P > S$, which means that mutual cooperation (C,C) is the best outcome, Pareto-superior, and a Nash equilibrium. However, there is a second equilibrium in which both players defect (D,D) and which is somewhat "inferior" to the previous one, although perfectly equivalent from a NE point of view. The (D,D) equilibrium is less satisfactory yet "risk-dominant" since playing it "safe" by choosing strategy D guarantees at least a payoff of P, while playing C might expose a player to a D response by her opponent, with the ensuing minimum payoff S. Here the dilemma is represented by the fact that the socially preferable coordinated equilibrium (C,C) might be missed for "fear" that the other player will play D instead. There is a third mixed-strategy NE in the game, but it is commonly dismissed because of its inefficiency and also because it is not an ESS [128]. Although the PD has received much more attention in the literature than the SH, the latter is also very useful, especially as a metaphor of coordinated social behavior for mutual benefit. These aspects are nicely explained in [109].

## D.3 Model Description

Our model is strictly local as no player uses information other than the one concerning the player itself and the players it is directly connected to. In particular, each agent knows its own current strategy and payoff, and the current strategies and payoffs of its immediate neighbors. Moreover, as the model is an evolutionary one, no rationality, in the sense of game theory, is needed [128]. Players just adapt their behavior such that they copy more successful strategies in their environment with higher probability, a process commonly called *imitation* in the literature [46]. Furthermore, they are able to locally assess the worth of an interaction and possibly dismiss a relationship that does not pay off enough. The model and its dynamics are described in detail in the following sections.

### D.3.1 Network and Interaction Structure

The network of agents will be represented as an undirected graph $G(V, E)$, where the set of vertices $V$ represents the agents, while the set of edges (or links) $E$ represents their symmetric interactions. The population size $N$ is the cardinality of $V$. A neighbor of an agent $i$ is any other agent $j$ such that there is an edge $\{ij\} \in E$. The set of neighbors of $i$ is called $V_i$ and its cardinality is the degree $k_i$ of vertex $i \in V$. The average degree of the network will be called $\bar{k}$.

Although from the network structure point of view there is a single undirected link between a player $i$ and another player $j \in V_i$, we shall maintain two links: one going from $i$ to $j$ and another one in the reverse direction (see fig. D.1). Each link has a weight or "force" $f_{ij}$ (respectively $f_{ji}$). This weight, say $f_{ij}$, represents in an indirect way an abstract quality that could be related to the "trust" player $i$ attributes to player $j$, it may take any value in $[0, 1]$ and its variation is dictated by the payoff earned by $i$ in each encounter with $j$, as explained below.

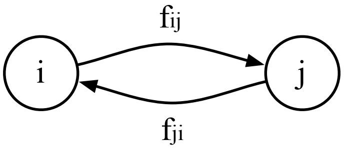


Figure D.1: Schematic representation of mutual trust between two agents through the strengths of their links.

We point out that we do not believe that this model could represent, however roughly, a situation of genetic relatedness in a human or animal society. In this case, at the very least, one should have at the outset that link strengths between close relatives should be higher than the average forces in the whole network and such groups should form cliques of completely connected agents. In contrast, we start our simulations from random relationships and a constant average link strength (see below). Thus, our simplified model is closer to one in which relationships between agents are only of socio-economic nature.

The idea behind the introduction of the forces $f_{ij}$ is loosely inspired by the potentiation/depotentiation of connections between neural networks, an effect known as the *Hebb rule* [43]. In our context, it can be seen as a kind of "memory" of previous encounters. However, it must be distinguished from the memory used in iterated games, in which players "remember" a certain amount of previous moves and can thus conform their future strategy on the analysis of those past encounters [71]. Our interactions are strictly one-shot, i.e. players "forget" the results of previous rounds and cannot recognize previous partners and their possible playing patterns. However, a certain amount of past history is implicitly contained in the numbers $f_{ij}$ and this information may be used by an agent when it will come to decide whether or not an interaction should be dismissed (see below)[1]. This bilateral view of a relationship is, to our knowledge, new in evolutionary game models on graphs.

We also define a quantity $s_i$ called *satisfaction* of an agent $i$ which is the sum of all the weights of the links between $i$ and its neighbors $V_i$ divided by the total number of links $k_i$:

$$s_i = \frac{\sum_{j \in V_i} f_{ij}}{k_i}.$$

We clearly have $0 \leq s_i \leq 1$.

### D.3.2 Initialization

The constant size of the network during the simulations is $N = 1000$. The initial graph is generated randomly with a mean degree comprised between $\bar{k} = 5$ and $\bar{k} = 20$. These values of $\bar{k}$ are of the order of those actually found in many social networks (see, for instance, [8, 54, 73, 117]). Players are distributed uniformly at random over the graph vertices with 50% cooperators. Forces between any pair of neighboring players are initialized at 0.5. With $\bar{k} > 1$ a random graph finds itself past the percolation phase transition [15] and thus it has a giant connected component of size $O(N)$ while all the other components are of size $O(log(N))$. We do not assure that the whole graph is connected, as isolated nodes will draw a random link during the dynamics (see below).

Before starting the simulations, there is another parameter $q$ that has to be set. This is akin to a "temperature" or noise level; $q$ is a real number in $[0, 1]$ and it represents the frequency with which an agent wishes to dismiss a link with one of its neighbors. The higher $q$, the faster the link reorganization in the network. This parameter has a role analogous to the "plasticity"

---

[1] A further refinement of the concept could take obsolescence phenomena into account. For instance, in the same way that pheromone trails laid down by ants evaporate with time, we could introduce a progressive loss of strength of the links proportional to the time during which there is no interaction between the concerned agents. For the sake of simplicity, we prefer to stick with the basic model in this work

of [131] and it controls the speed at which topological changes occur in the network. As social networks may structurally evolve at widely different speeds, depending on the kind of interaction between agents, this factor might play a role in the model. For example, e-mail networks change their structure at a faster pace than, say, scientific collaboration networks [54, 115]. A similar coupling of time scales between strategy update and topological update also occurs in [111, 103].

### D.3.3 Timing of Events

Usually, agents systems such as the present one, are updated synchronously, especially in evolutionary game theory simulations [60, 84, 101, 131]. However, there are doubts about the physical signification of simultaneous update [48]. For one thing, it is strictly speaking physically unfeasible as it would require a global clock, while real extended systems in biology and society in general have to take into account finite signal propagation speed. Furthermore, simultaneity may cause some artificial effects in the dynamics which are not observed in real systems [48, 59]. Fully asynchronous update, i.e. updating a randomly chosen agent at a time with or without replacement also seems a rather arbitrary extreme case that is not likely to represent reality very accurately. In view of these considerations, we have chosen to update our population in a partially synchronous manner. In practice, we define a fraction $f = n/N$ (with $N = an, a \in \mathbb{N}$) and, at each simulated discrete time step, we update only $n \leq N$ agents randomly chosen with replacement. This is called a *microstep*. After $N/n$ microsteps, called a *macrostep*, $N$ agents will have been updated, i.e. the whole population will have been updated in the average. With $n = N$ we recover the fully synchronous update, while $n = 1$ gives the extreme case of the fully asynchronous update. Varying $f$ thus allows one to investigate the role of the update policy on the dynamics. We study several different values of $f$, but we mainly focus on $f = 0.01$.

### D.3.4 Strategy and Link Dynamics

Here we describe in detail how individual strategies, links, and link weights are updated. Once a given node $i$ is chosen to be activated, i.e. it belongs to the fraction $f$ of nodes that are to be updated in a given microstep, $i$ goes through the following steps:

- if the degree of agent $i$, $k_i = 0$ then player $i$ is an isolated node. In this case a link with strength 0.5 is created from $i$ to a player $j$ chosen uniformly at random among the other $N - 1$ players in the network.
- otherwise,
    - either agent $i$ updates its strategy according to a local *replicator dynamics* rule with probability $1-q$ or, with probability $q$, agent $i$ may delete a link with a given neighbor $j$ and creates a new 0.5 force link with another node $k$ ;
    - the forces between $i$ and its neighbors $V_i$ are updated

Let us now describe each step in more detail.

**Strategy Evolution.** We use a local version of replicator dynamics (RD) as described in [41] and further modified in [60] to take into account the fact that the number of neighbors in a degree-inhomogeneous network can be different for different agents. The local dynamics of a player $i$ only depends on its own strategy and on the strategies of the $k_i$ players in its neighborhood $V_i$. Let us call $\pi_{ij}$ the payoff player $i$ receives when interacting with neighbor $j$. This payoff is defined as

$$\pi_{ij} = \sigma_i(t)\ M\ \sigma_j^T(t),$$

where $M$ is the payoff matrix of the game (see sect. D.2) and $\sigma_i(t)$ and $\sigma_j(t)$ are the strategies played by $i$ and $j$ at time $t$. The quantity

$$\widehat{\Pi}_i(t) = \sum_{j \in V_i} \pi_{ij}(t)$$

is the *accumulated payoff* collected by player $i$ at time step $t$. The rule according to which agents update their strategies is the conventional RD in which strategies that do better than the average increase their share in the population, while those that fare worse than average decrease. To update the strategy of player $i$, another player $j$ is drawn at random from the neighborhood $V_i$. It is assumed that the probability of switching strategy is a function $\phi$ of the payoff difference, where $\phi$ is a monotonically increasing function [46]. Strategy $\sigma_i$ is replaced by $\sigma_j$ with probability

$$p_i = \phi(\widehat{\Pi}_j - \widehat{\Pi}_i).$$

The major differences with standard RD is that two-person encounters between players are only possible among neighbors, instead of being drawn from the whole population, and the latter is finite in our case. Other commonly used strategy update rules include imitating the best in the neighborhood [84, 131], or replicating in proportion to the payoff [41, 116]. Although, these rules are acceptable alternatives, they do not lead to replicator dynamics and will not be dealt with here. We note also that the straight accumulated payoff $\widehat{\Pi}_i$ has a technical problem when used on degree-inhomogeneous systems such as those studied here, where agents (i.e. nodes) in the network may have different numbers of neighbors. In fact, in this case $\widehat{\Pi}_i$ does not induce invariance of the RD with respect to affine transformations of the game's payoff matrix as it should [128], and makes the results depend on the particular payoff values. Thus, we shall use a modified accumulated payoff $\Pi$ instead as defined in [60]. This payoff, which is the standard accumulated payoff corrected with a factor that takes into account the variable number of neighbors an agent may have, does not suffer from the standard accumulated payoff limitations.

**Link Evolution.** The active agent $i$, which has $k_i \neq 0$ neighbors will, with probability $q$, attempt to dismiss an interaction with one of its neighbors. This is done in the following way. Player $i$ will look at its satisfaction $s_i$. The higher $s_i$, the more satisfied the player, since a high satisfaction is a consequence of successful strategic interactions with the neighbors. Thus, there should be a natural tendency to try to dismiss a link when $s_i$ is low. This is simulated by drawing a uniform pseudo-random number $r \in [0, 1]$ and breaking a link when $r \geq s_i$. Assuming that the decision is taken to cut a link, which one, among the possible $k_i$, should be chosen? Our solution again relies on the strength of the relevant links. First a neighbor $j$ is chosen with probability proportional to $1 - f_{ij}$, i.e. the stronger the link, the less likely it will be chosen. This intuitively corresponds to $i$'s observation that it is preferable to dismiss an interaction with a neighbor $j$ that has contributed little to $i$'s payoff over several rounds of play. However, in our system dismissing a link is not free: $j$ may "object" to the decision. The intuitive idea is that, in real social situations, it is seldom possible to take unilateral decisions: often there is a cost associated, and we represent this hidden cost by a probability $1 - (f_{ij} + f_{ji})/2$ with which $j$ may refuse to be cut away. In other words, the link is less likely to be deleted if $j$ appreciates $i$, i.e. when $f_{ji}$ is high. A simpler solution would be to try to cut the weakest link, which is what happens most of the time anyway. However, with a finite probability of cutting any link, our model introduces a small amount of noise in the process which can be considered like "trembles" or errors in game theory [71] and which roughly reproduces decisions under uncertainty in the real world.

Assuming that the $\{ij\}$ link is finally cut, how is a new link to be formed? The solution adopted here is inspired by the observation that, in social networks, links are usually created more easily between people who have a mutual acquaintance than those who do not. First, a neighbor $k$ is chosen in $V_i \setminus \{j\}$ with probability proportional to $f_{ik}$, thus favoring neighbors $i$ trusts. Next, $k$ in turn chooses player $l$ in his neighborhood $V_k$ using the same principle, i.e. with probability proportional to $f_{kl}$. If $i$ and $l$ are not connected, a link $\{il\}$ is created, otherwise the process is repeated in $V_l$. Again, if the selected node, say $m$, is not connected to $i$, a new link $\{im\}$ is established. If this also fails, a new link between $i$ and a randomly chosen node is created. In all cases the new link is initialized with a strength of 0.5 in both directions. This rewiring process is schematically depicted in fig. D.2 for the case in which a link can be successfully established between players $i$ and $l$ thanks to their mutual acquaintance $k$.

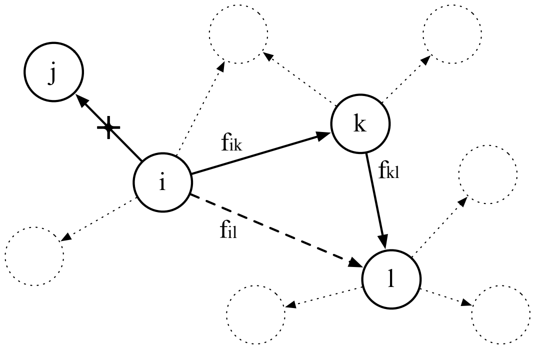


Figure D.2: Illustration of the rewiring of link $\{ij\}$ to $\{il\}$. Agent $k$ is chosen to introduce player $l$ to $i$ (see text).

At this point, we would like to stress several important differences with previous work in which links can be dismissed in evolutionary games on networks [59, 103, 131]. In [131], only links between defectors are allowed to be cut unilaterally and the study is restricted to the PD. Instead, in our case, any link has a finite probability to be abandoned, even a profitable link between cooperators if it is recent, although links that are more stable, i.e. have high strengths, are less likely to be rewired. This smoother situation is made possible thanks to our bilateral view of a link which is completely different from the undirected choice made in [131].

In [103], links can be cut by an unsatisfied player, where the concept of satisfaction is different from ours, and simply means that a cooperator or a defector will wish to break a link with a defector. The cut will be done with a certain probability that depends on the strategies of the two agents involved and their respective payoffs. Once a link between $i$ and $j$ is actually cut and, among the two players, $i$ is the one selected to maintain the link, the link is rewired to a random neighbor of $j$. If both $i$ and $j$ wish to cease their interaction, the link is attributed to $i$ or $j$ probabilistically, as a function of the respective payoffs of $i$ and $j$, and rewiring takes place from there. Thus, although both $i$'s and $j$'s payoffs are taken into consideration in the latter case, there is no analogous of our "negotiation" process as the concept of link strength is absent. In [59] links are cut according to a threshold decision rule and are rewired randomly anywhere in the network.

A final observation concerns the evolution of $\bar{k}$ in the network. While in [103, 131] the initial mean degree is strictly maintained during network evolution through the rewiring process, here it may increase slightly owing to the existence of isolated agents which, when chosen to be updated, will create a new link with another random agent. While this effect is of minor importance and only causes small fluctuations of $\bar{k}$, we point out that in real evolving networks the mean connectivity fluctuates too [8, 54, 115].

**Updating the Link Strengths.** Once the chosen agents have gone through their strategy or link update steps, the strengths of the links are updated accordingly in the following way:

$$f_{ij}(t+1) = f_{ij}(t) + \frac{\pi_{ij} - \bar{\pi}_{ij}}{k_i(\pi_{max} - \pi_{min})},$$

where $\pi_{ij}$ is the payoff of $i$ when interacting with $j$, $\bar{\pi}_{ij}$ is the payoff earned by $i$ playing with $j$, if $j$ were to play his other strategy, and $\pi_{max}$ ($\pi_{min}$) is the maximal (minimal) possible payoff obtainable in a single interaction. This update is performed in both directions, i.e. both $f_{ij}$ and $f_{ji}$ are updated $\forall j \in V_i$ because both $i$ and $j$ get a payoff out of their encounter.

The following algorithms schematically describe the whole co-evolution process for one microstep:

**Algorithm 1:** Simulation of a microstep of the population evolution

```
Data: $V$ is a set of players; $I$ is a subset of $n$ randomly selected players in $V$; $V_i$ is the
      set of first neighbors of player $i$; $f_{ij}$ is the strength of the oriented arc $\{\vec{ij}\}$; $r$ is a
      pseudo-random number $\in [0, 1)$
forall the players $i \in I$ do
    foreach $j \in V_i$ do
        $i$ plays with $j$ and updates its payoff
        $f_{ij}(t+1) = f_{ij}(t) + \Delta f_{ij}(t)$
        $f_{ji}(t+1) = f_{ji}(t) + \Delta f_{ji}(t)$
    end
    if $k_i = 0$ then
        $i$ is isolated and creates a new link to a randomly selected $j \in V$
    else
        if $r < q$ then
            Link Evolution($i$)
        else
            Strategy Evolution($i$)
        end
    end
end
```

**Algorithm 2:** Link Evolution($i$)

```
Data: $r_1$ and $r_2$ are pseudo-random numbers $\in [0, 1)$; $s_i$ is the satisfaction of player $i$
if $r_1 > s_i$ then
    $j \in V_i$ selected proportionally to $1 - f_{ij}$
    if $r_2 > (f_{ij} + f_{ji})/2$ then
        remove the link $\{ij\}$
        $k \in V_i$ selected proportionally to $f_{ik}$
        $l \in V_k$ selected proportionally to $f_{kl}$
        if $\{il\}$ link doesn't exist then
            create the link$\{il\}$
        else
            $m \in V_l$ selected proportionally to $f_{lm}$
            if $\{im\}$ link doesn't exist then
                create the link$\{im\}$
            else
                $i$ creates a new link to a randomly selected $j \in V \setminus V_i$
            end
        end
    end
end
```

**Algorithm 3:** Strategy Evolution($i$)

```
Data: $r$ is a pseudo-random number $\in [0, 1)$; $\Pi_k$ is the aggregated payoff of player $k$; $\phi$
      is a monotonically increasing function
$j \in V_i$ randomly selected
$p_i = \phi(\Pi_j - \Pi_i)$
if $r > p_i$ then
    $i$ imitate the strategy of $j$
end
```

## D.4 Simulation Results

### D.4.1 Simulation Parameters

We simulate on our networks the two games previously mentioned in sect. D.2. For each game, we can explore the entire game space by limiting our study to the variation of only two parameters per game. This is possible without loss of generality owing to the invariance of Nash equilibria and replicator dynamics under positive affine transformations of the payoff matrix using our payoff scheme [128]. In the case of the PD, we set $R = 1$ and $S = 0$, and vary $1 \leq T \leq 2$ and $0 \leq P \leq 1$. For the SH, we decided to fix $R = 1$ and $S = 0$ and vary $0 \leq T \leq 1$ and $0 \leq P \leq T$. The reason we choose to set $T$ and $S$ in both the PD and the SH is to simply provide natural bounds on the values to explore of the remaining two parameters. In the PD case, $P$ is limited between $R = 1$ and $S = 0$ in order to respect the ordering of the payoffs ($T > R > P > S$) and $T$'s upper bound is equal to 2 due to the $2R > T + S$ constraint. Had we fixed $R = 1$ and $P = 0$ instead, $T$ could be as big as desired, provided $S \leq 0$ is small enough. In the SH, setting $R = 1$ and $S = 0$ determines the range of $T$ and $P$ (since this time $R > T > P > S$). Note however, that for this game the only valid value pairs of $(T, P)$ are those that satisfy the $T > P$ constraint.

As stated in sect. D.3.2, we used networks of size $N = 1000$, randomly generated with an average degree $\bar{k} \in \{5, 10, 20\}$ and randomly initialized with 50% cooperators and 50% defectors. In all cases, the parameters are varied between their two bounds in steps of 0.1. For each set of values, we carry out 50 runs of at most 20000 macrosteps each, using a fresh graph realization in each run. A run is stopped when all agents are using the same strategy, in order to be able to measure statistics for the population and for the structural parameters of the graphs. The system is considered to have reached a pseudo-equilibrium strategy state when the strategy of the agents (C or D) does not change over 150 further macrosteps, which means $15 \times 10^4$ individual updates. We speak of pseudo-equilibria or steady states and not of true evolutionary equilibria because, as we shall see below, the system never quite reaches a totally stable state in the dynamical systems sense in our simulations but only transient states that persist for a long time.

### D.4.2 Cooperation and Stability

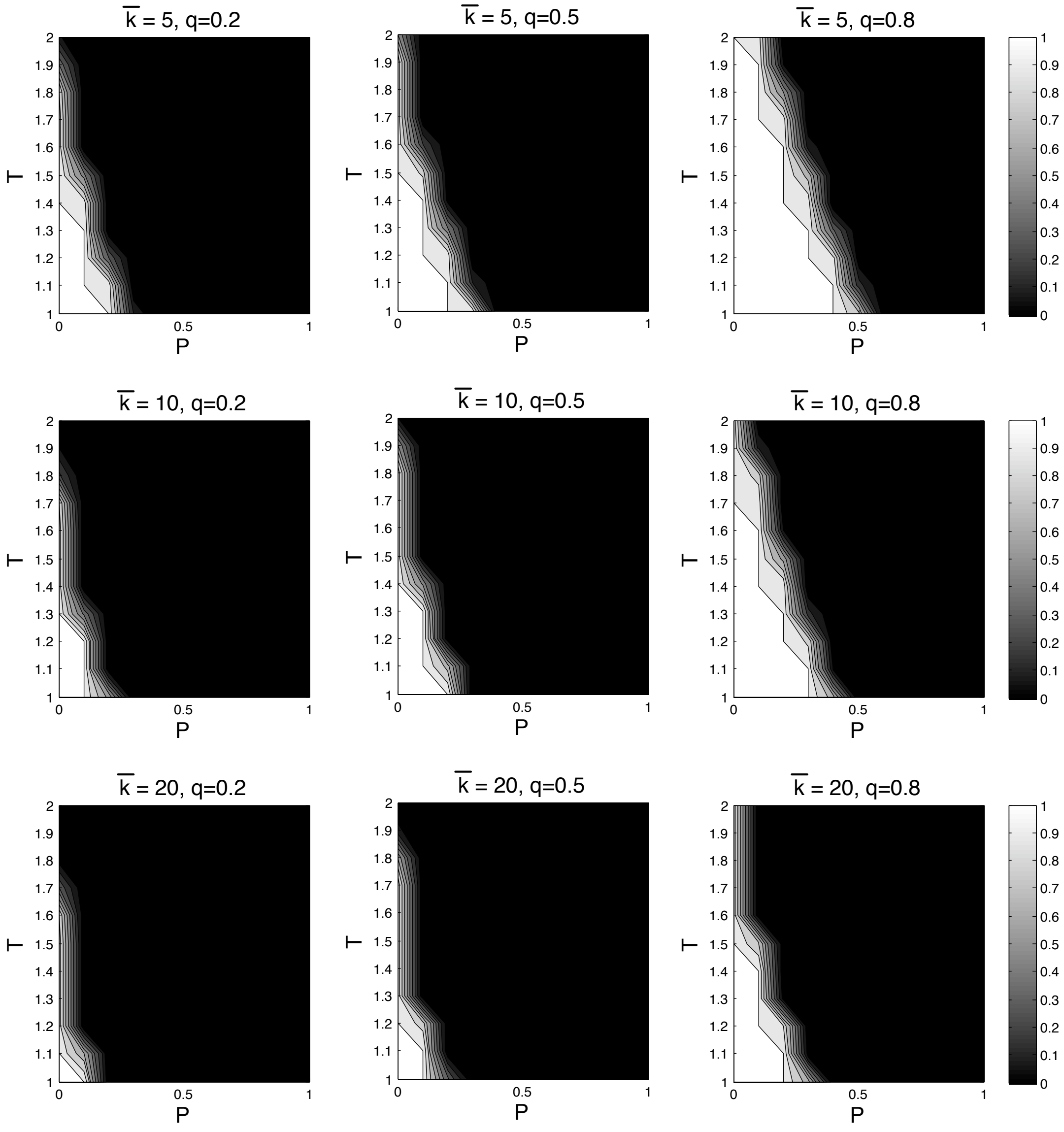


Figure D.3: Cooperation level for the PD in the game's configuration space. Darker gray means more defection.

Cooperation results for the PD in contour plot form are shown in fig. D.3. We remark that, as observed in other structured populations, cooperation may thrive in a small but non-negligible part of the parameter space. Thus, the added degree of freedom represented by the possibility of refusing a partner and choosing a new one does indeed help to find player's arrangements that help cooperation. This finding is in line with the results of [103, 131]. Furthermore, the fact that our artificial society model differs from the latter two in several important ways also shows that the result is a rather robust one. When considering the dependence on the fluidity parameter $q$, one sees in fig. D.3 that the higher $q$, the higher the cooperation level. This was expected since being able to break ties more often clearly gives cooperators more possibilities for finding and keeping fellow cooperators to interact with. This effect has been previously observed also in the works of [103, 131] and, as such, seems to be a robust finding, relatively independent of the other details of the models. The third parameter considered in fig. D.3 is the mean degree $\bar{k}$. For a given value of $q$, cooperation becomes weaker as $\bar{k}$ increases. We believe that, as far as $\bar{k}$ is concerned, a realistic average characterization of actual social networks is represented by $\bar{k} = 10$ (middle row in fig. D.3) as seen, for instance, in [8, 54, 73, 117]. Higher average degrees do exist, but they are found either in web-based pseudo-social networks or in fairly special collaboration networks like the particle physics community, where it is customary to include as coauthors tens or even hundreds of authors [73]. Clearly, there is a limit to the number of real acquaintances a given agent may manage with.

We have also performed many simulations starting from different proportions of randomly distributed cooperators and defectors to investigate the effect of this parameter on the evolution of cooperation. In Fig. D.4 we show five different cases, the central image corresponding to the 50% situation. The images correspond to the lower left quarter of the right image in the middle row of Fig. D.3 with $\bar{k} = 10$, $q = 0.8$, $1 < T < 1.5$, and $0 < P < 0.5$.

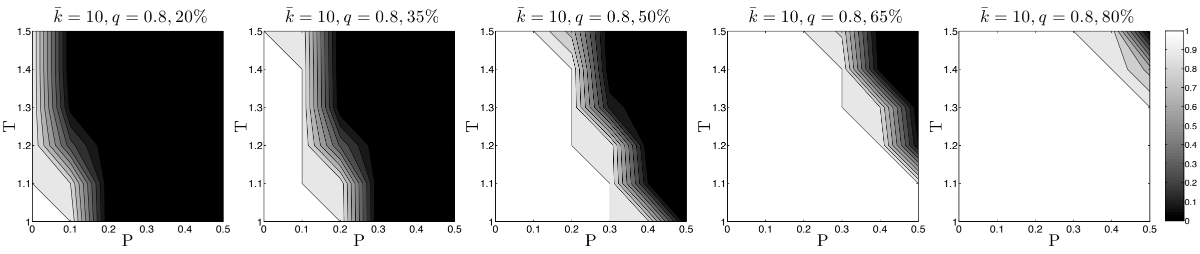


Figure D.4: Cooperation level for the PD starting with different fractions of cooperators increasing from 20% to 80% from left to right. Only the lower left quarter of the parameter space is shown. Results are the average of 50 independent runs.

Compared with the level of cooperation observed in simulations in static networks, we can say that results are consistently better for co-evolving networks. For example, the typical cases with $\bar{k} = 10$ and $q = 0.5, 0.8$ show significantly more cooperation than what was found in model and real social networks in previous work [60]. Even when there is a much lower rewiring frequency, i.e. with $q = 0.2$, the cooperation levels are approximately as good as those observed in our previous study in which exactly the same replicator dynamics scheme was used to update the agents' strategies and the networks were of comparable size. The reason for this behavior is to be found in the added constraints imposed by the invariant network structure. The seemingly contradictory fact that an even higher cooperation level may be reached in static scale-free networks [101], is theoretically interesting but easily dismissed as those graphs are unlikely models for social networks, which often show fat-tailed degree distribution functions but not pure power-laws (see, for instance, [4, 73]). As a further indication of the latter, we shall see in sect. D.4.3 that, indeed, emerging networks do not have a power-law degree distribution.

From the point of view of the evolutionary dynamics, it is interesting to point out that any given simulation run either ends up in full cooperation or full defection. When the full cooperation state of the population is attained, there is no way to switch back to defection by the intrinsic agent dynamics. In fact, all players are satisfied and have strong links with their cooperating neighbors. Even though a small amount of noise may still be present when deciding whether or not to rewire a link, since there are only cooperators around to imitate, there can be no strategy change and only very little link rewiring. On the other hand, well before this stable state is reached and there are still many defectors around, the system may experience some random drift that may drive it to full defection. The converse may also happen, but when the full defection state is reached, the situation is qualitatively different. In this case agents are unsatisfied, they will often try to rewire their links. However, all the other players around being also defectors, there will be constant changes of the local network structure. Thus the system will find itself in a fluctuating state, but this matters little for the bulk statistical properties of the population and of the network. To be assured that this is indeed the case, we have conducted some very long runs with all-defect end states. Global statistics do not change, except that the mean degree tends to increase slightly with time and the degree distribution function continues to evolve (see sect. D.4.3).

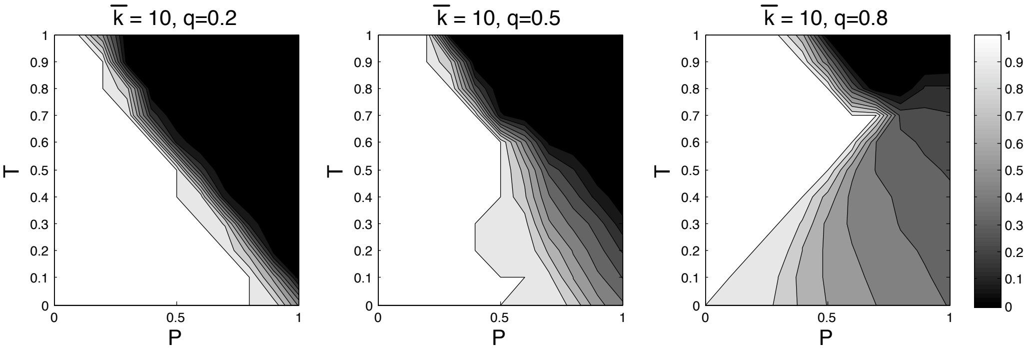


Figure D.5: Cooperation level for the SH game.

Cooperation percentages as a function of the payoff matrix parameters for the SH game are shown in fig. D.5 for $\bar{k} = 10$ and $q = 0.2, 0.5$, and 0.8. Note that in this case only the upper left triangle of the configuration space is meaningful (see sect. D.4.1). The SH is different from the PD since there are two evolutionarily stable strategies which are therefore also NEs: one population state in which everybody defects and the opposite one in which everybody cooperates (see sect. D.2). Therefore, it is expected, and absolutely normal, that some runs will end up with all defect, while others will witness the emergence of full cooperation. In contrast, in the PD the only theoretically stable state is all-defect and cooperating states may emerge and be stable only by exploiting the graph structure and creating more favorable neighborhoods by breaking and forming ties. The value of the SH is in making manifest the tension that exists between the socially desirable state of full cooperation and the socially inferior but less risky state of defection [109]. The final outcome of a given simulation run depends on the size of the basin of attraction of either state, which is in turn a function of the relative values of the payoff matrix entries. To appreciate the usefulness of making and breaking ties in this game we can compare our results with what is prescribed by the standard RD solution. Referring to the payoff table of sect. D.2, let's assume that the column player plays C with probability $\alpha$ and D with probability $1-\alpha$. In this case, the expected payoffs of the row player are:

$$E_r[C] = \alpha R + (1-\alpha)S$$

and

$$E_r[D] = \alpha T + (1-\alpha)P$$

The row player is indifferent to the choice of $\alpha$ when $E_r[C] = E_r[D]$. Solving for $\alpha$ gives:

$$\alpha = \frac{P-S}{R-S-T+P}. \tag{D.1}$$

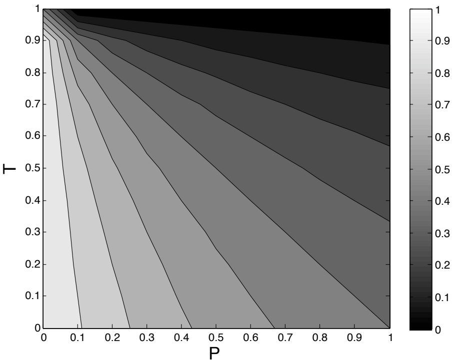


Figure D.6: Probabilities of cooperation for the mixed strategy NE as a function of the game's parameters for the Stag Hunt.

Since the game is symmetric, the result for the column player is the same and $(\alpha C, (1-\alpha)D)$ is a NE in mixed strategies. We have numerically solved the equation for all the sampled points in the game's parameter space, which gives the results shown in fig. D.6. Let us now use the following payoff values in order to bring them within the explored game space (remember that NEs are invariant w.r.t. such a transformation [128]):

| | C | D |
|---|---|---|
| C | $(1,1)$ | $(0,2/3)$ |
| D | $(2/3,0)$ | $(1/3,1/3)$ |

Substituting in(D.1) gives $\alpha = 1/2$, i.e. the (unstable) polymorphic population should be composed by about half cooperators and half defectors. Now, if one looks at fig. D.5 at the points where $P = 1/3$ and $T = 2/3$, one can see that this is approximately the case for the first image, within the limits of the approximations caused by the finite population size, the symmetry-breaking caused by the non-homogeneous graph structure, and the local nature of the RD. On the other hand, in the middle image and, to a greater extent, in the rightmost image, this point in the game space corresponds to pure cooperation. In other words, the non-homogeneity of the network and an increased level of tie rewiring has allowed the cooperation basin to be enhanced with respect to the theoretical predictions of standard RD. Skyrms and Pemantle found the same qualitative result for very small populations of agents when both topology and strategy updates are allowed [111]. It is reassuring that coordination on the payoff-dominant equilibrium can still be achieved in large populations as seen here.

### D.4.3 Structure of the Emerging Networks

In this section we present a statistical analysis of the global and local properties of the networks that emerge when the pseudo-equilibrium states of the dynamics are attained. Let us start by considering the evolution of the average degree $\bar{k}$. Although there is nothing in our model to prevent a change in the initial mean degree, the steady-state average connectivity tends to increase only slightly. For example, in the PD with $q = 0.8$ and $\bar{k}_{init} = 5$ and $\bar{k}_{init} = 10$, the average steady-state (ss) values are $\bar{k}_{ss} \simeq 7$ and $\bar{k}_{ss} \simeq 10.5$ respectively. Thus we see that, without imposing a constant $\bar{k}$ as in [103, 131], $\bar{k}$ nonetheless tends to increase only slightly, which nicely agrees with observations of real social networks [8, 54, 115]. There is a special case when the steady-state is all-defect and the simulation is allowed to run for a very long time ($2 \times 10^4$ macrosteps); in this case the link structure never really settles down, since players are unsatisfied, and $\bar{k}$ may reach a value of about 12 when starting with $\bar{k} = 10$ and $q = 0.8$.

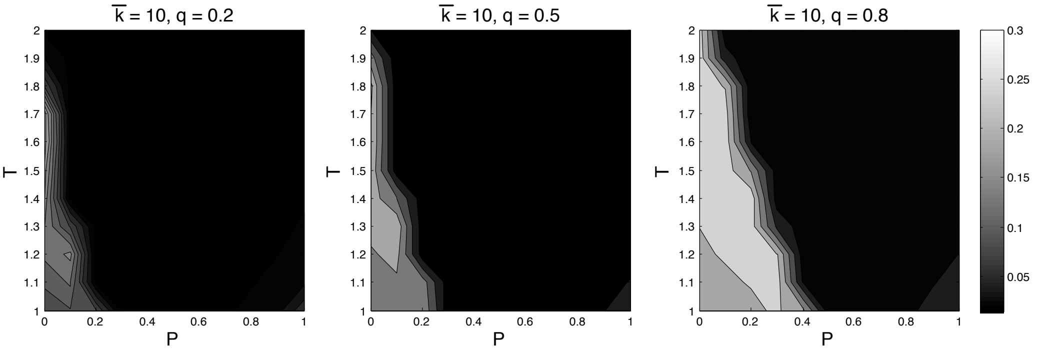


Figure D.7: Clustering coefficient level for the PD game. Lighter gray means more clustering.

Another important global network statistics is the average clustering coefficient $\mathcal{C}$. The clustering coefficient $\mathcal{C}_i$ of a node $i$ is defined as $\mathcal{C}_i = 2E_i/k_i(k_i - 1)$, where $E_i$ is the number of edges in the neighborhood of $i$. Thus $\mathcal{C}_i$ measures the amount of "cliquishness" of the neighborhood of node $i$ and it characterizes the extent to which nodes adjacent to node $i$ are connected to each other. The clustering coefficient of the graph is simply the average over all nodes: $\mathcal{C} = \frac{1}{N}\sum_{i=1}^{N} \mathcal{C}_i$ [75]. Random graphs are locally homogeneous and for them $\mathcal{C}$ is simply equal to the probability of having an edge between any pair of nodes independently. In contrast, real networks have local structures and thus higher values of $\mathcal{C}$. Fig. D.7 gives the average clustering coefficient $\bar{\mathcal{C}} = \frac{1}{50}\sum_{i=1}^{50} \mathcal{C}$ for each sampled point in the PD configuration space, where 50 is the number of network realizations used for each simulation. It is apparent that the networks self-organize and acquire local structure in the interesting, cooperative parts of the parameter's space, since the clustering coefficients there are higher than that of the random graph with the same number of edges and nodes, which is $\bar{k}/N = 10/1000 = 0.01$. Conversely, where defection predominates $\mathcal{C}$ is smaller, witnessing of a lower amount of graph local restructuring. These impressions are confirmed by the study of the degree distribution functions (see below). The correlation between clustering and cooperation also holds through increasing values of $q$: $\mathcal{C}$ tends to increase from left to right in fig. D.7, a trend similar to that observed in the middle row of fig. D.3 for cooperation. This correlation is maintained also for $\bar{k} = 5$ and $\bar{k} = 20$ (not shown).

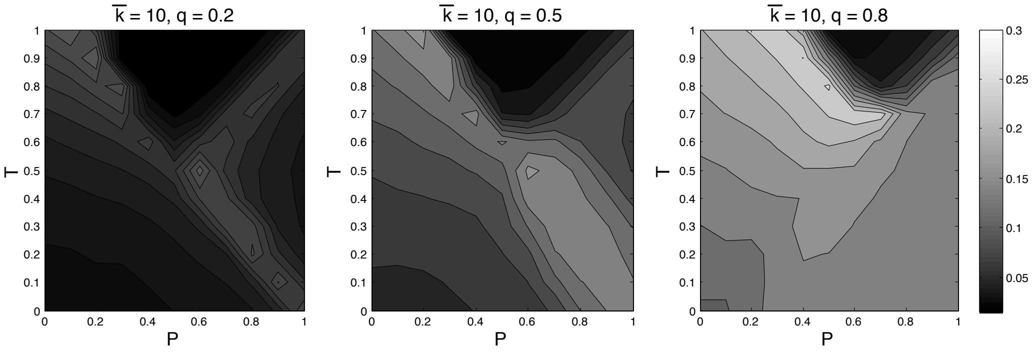
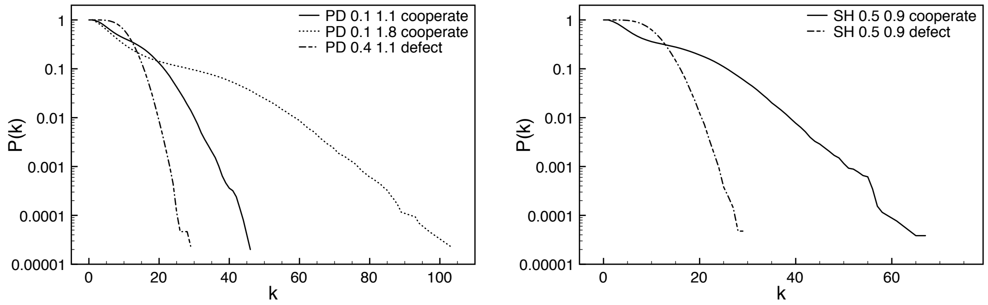


Figure D.8: Clustering coefficient level for the SH game.

As far as the clustering coefficient is concerned, the same qualitative phenomenon is observed for the SH namely, the graph develops local structures and the more so the higher the value of $q$ for a given $\bar{k}$ (see fig. D.8). Thus, it seems that evolution towards cooperation and coordination passes through a rearrangement of the neighborhood of the agents with respect to the homogeneous random initial situation, something that is made possible through the higher probability given to neighbors when rewiring a link, a stylized manifestation of the commonly occurring social choice of partners.

The *degree distribution function* (DDF) $p(k)$ of a of a graph represents the probability that a randomly chosen node has degree $k$ [75]. Random graphs are characterized by DDF of Poissonian form, while social and technological real networks often show long tails to the right, i.e. there are nodes that have an unusually large number of neighbors [75]. In some extreme cases the DDF has a power-law form $p(k) \propto k^{-\gamma}$; the tail is particularly extended and there is no characteristic degree. The *cumulative degree distribution function* (CDDF) is just the probability that the degree is greater than or equal to $k$ and has the advantage of being less noisy for high degrees. Fig. D.9 (a) shows the CDDFs for the PD for three cases of which two are in the cooperative region and the third falls in the defecting region (see fig. D.3). The dotted curve refers to a region of the configuration space in which there is cooperation in the average but it is more difficult to reach, as the temptation parameter is high (T=1.8,P=0.1). The curve has a rather long tail



(a) (b)

Figure D.9: Cumulative degree distributions. Average values over 50 runs. (a): PD, (b): SH. $q = 0.8$, $\bar{k} = 10$. Linear-log scales.

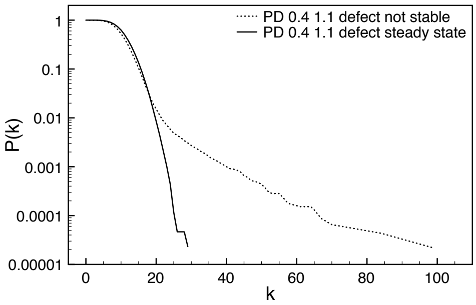


Figure D.10: Cumulative degree distributions for the PD in case of defection before (dotted line) and after (thick line) reaching a steady-state. Linear-log scales.

and is thus broad-scale in the sense that there is no typical degree for the agents. Therefore, in the corresponding network there are cooperators that are linked to many other cooperators. On the other hand, if one considers the dotted-dashed curve, which corresponds to a defecting region (T=1.1,P=0.4), it is clear that the distribution is much closer to normal, with a well-defined typical value of the degree. Finally, the third thick curve, which corresponds to a region where cooperation is more easily attained (T=1.1,P=0.1), also shows a rather faster decay of the tail than the dotted line and a narrower scale for the degree. Nevertheless, it is right-skewed, indicating that the network is no longer a pure random graph. Since we use linear-log scales, the dotted curve has an approximately exponential or slower decay, given that a pure exponential would appear as a straight line in the plot. The tail of the thick curve decays faster than an exponential, while the dashed-dotted curve decays even faster. Almost the same observations also apply to the SH case, shown in fig. D.9 (b). These are quite typical behaviors and we can conclude that, when cooperation is more difficult to reach, agents must better exploit the link-redirection degree of freedom in order for cooperators to stick together in sufficient quantities and protect themselves from exploiting defectors during the co-evolution. When the situation is either more favorable for cooperation, or defection easily prevails, network rearrangement is less radical. In the limit of long simulation times, the defection case leads to networks that have degree distribution close to Poissonian and are thus almost random. Fig. D.10 shows such a case for the PD. The dashed curve is the CDDF at some intermediate time, when full defection has just been reached but the network is still strongly reorganizing itself. Clearly, the distribution has a long tail. However, if the simulation is continued until the topology is quite stable at the mesoscopic level, the distribution becomes close to normal (thick curve).

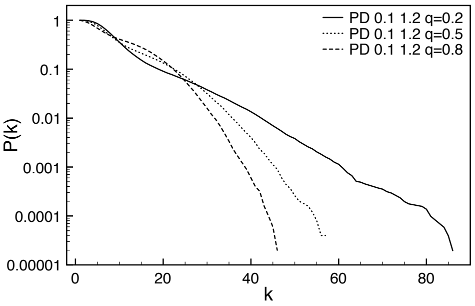


Figure D.11: Cumulative degree distribution functions for three values of $q$, for the same point in the PD configuration space in the cooperating region.

Finally, it is interesting to observe the influence of the $q$ parameter on the shape of the degree distribution functions for cooperating networks. Fig. D.11 reports average curves for three values of $q$. For high $q$, the cooperating steady-state is reached faster, which gives the network less time to rearrange its links. For lower values of $q$ the distributions become broader, despite the fact that rewiring occurs less often, because cooperation in this region is harder to attain and more simulation time is needed.

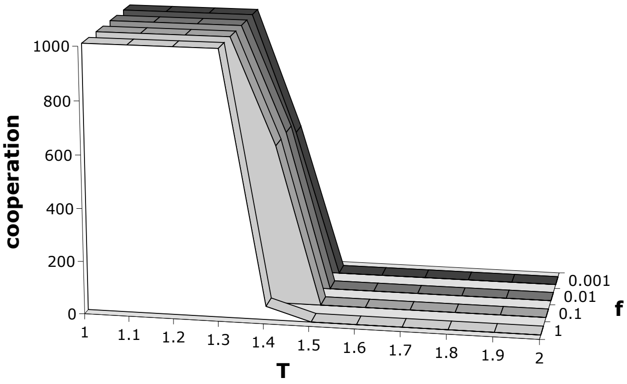


Figure D.12: Cooperation levels in the PD for $P = 0.1$ and $1 \leq T \leq 2$ as a function of the synchronicity parameter $f$.

**Influence of Timing.** Fig. D.12 depicts a particular cut in the configuration space as a function of the synchronicity parameter $f$. The main remark is that asynchronous updates give similar results, in spite of the difference in the number of agents that are activated in a single microstep. In contrast, fully synchronous update ($f = 1$) appears to lead to a slightly less favorable situation for cooperation. Since fully synchronous update is physically unrealistic and can give spurious results due to symmetry, we suggest using fully or partially asynchronous update for agent's simulation of artificial societies.

### D.4.4 Clusters

We have seen in the previous section that, when cooperation is attained in both games as a quasi-equilibrium state, the system remains stable through the formation of clusters of players using the same strategy. In fig. D.13 one such typical cluster corresponding to a situation in which global cooperation has been reached in the PD is shown. Although all links towards the "exterior" have been suppressed for clarity, one can clearly see that the central cooperator is a highly connected node and there are many links also between the other neighbors. Such a tightly packed structure has emerged to protect cooperators from defectors that, at earlier times, were trying to link to cooperators to exploit them. These observations explain why the degree distributions are long-tailed (see previous section), and also the higher values of the clustering coefficient in this case (see sect. D.4.3).

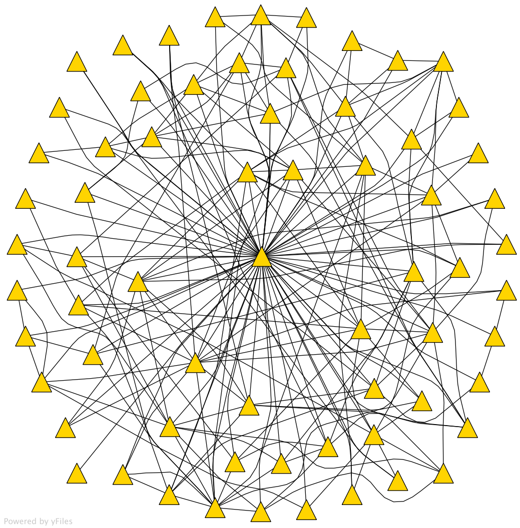
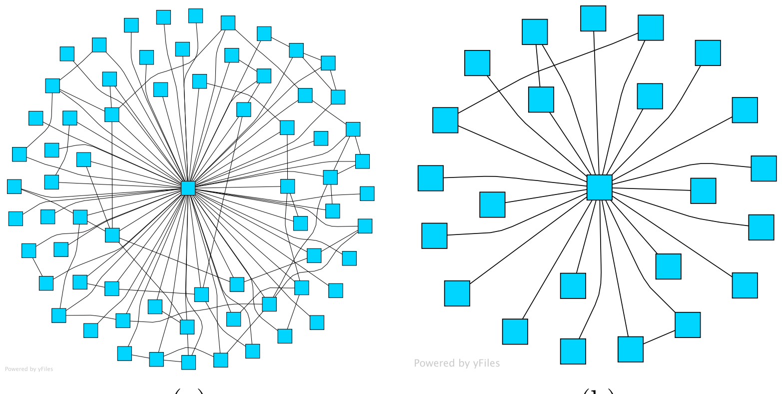


Figure D.13: Example of a tightly packed cluster of cooperators for PD networks. $T = 1.8, P = 0.1$ and $q = 0.8$.

When the history of the stochastic process is such that defection prevails in the end, the situation is totally different. Fig. D.14 (a) and (b) show two typical examples of cluster structures found during a simulation. Fig. D.14 (a) refers to a stage in which the society is composed solely by defectors. However, the forces of the links between them are low, and so many defectors try to dismiss some of their links. This situation lasts for a long simulated time (actually, the system is never at rest, as far as the links are concerned) but the dense clusters tend to dissolve, giving rise to structures such as the one shown in fig. D.14 (b). If one looks at the degree distribution at this stage (fig. D.10) it is easy to see that the whole population graph tends to become random.



(a) (b)

Figure D.14: Example of defector clusters for PD networks, for $T = 1.8, P = 0.3$ and $q = 0.8$. Clusters like (a) exists only just after the all-defect state is reached. When a steady-state is reached only clusters like (b) are present in a network of defectors.

The SH case is very similar, which is a relatively surprising result. In fact, when cooperation finally takes over in regions of the configuration space where defection would have been an almost equally, likely final state, players are highly clustered and there are many highly connected individuals, while in less conflicting situations the clusters are less dense and the degree distribution shows a faster decay of the tail. On the other hand, when defection is the final quasi-stable state, the population graphs looses a large part of its structure. Thus, the same topological mechanisms seem to be responsible for the emergence of cooperation in the PD and in the SH. The only previous study that investigates the structure of the resulting networks in a dynamical setting is, to our knowledge, reference [131], where only the PD is studied. It is difficult to meaningfully compare our results with theirs as the model of Zimmermann et al. differs from ours in many ways. They use a deterministic hard-limit rule for strategy update which is less smooth than our stochastic local replicator dynamics. Moreover, they study the PD in a reduced configuration space, only links between defectors can be broken, and links are rewired at random. They concentrate on the study of the stability of the cooperating steady-states against perturbations, but do not describe the topological structures of the pseudo-equilibrium states in detail. Nevertheless, it is worthy of note that the degree distribution functions for cooperators and defectors follow qualitatively the same trend, i.e. cooperators networks have distributions with fatter tails to the right than defector networks.

## D.5 Conclusions and Future Work

Using two well known games that represent conflicting decision situations commonly found in animal and human societies, we have studied by computer simulation the role of the dynamically networked society's structure in the establishment of global cooperative and coordinated behaviors, which are desirable outcomes for the society's welfare. Starting from randomly connected players which only interact locally in a restricted neighborhood, and allowing agents to probabilistically and bilaterally dismiss unprofitable relations and create new ones, the stochastic dynamics lead to pseudo-equilibria of either cooperating or defecting agents. With respect to standard replicator dynamics results for mixing populations, we find that there is a sizable configuration space region in which cooperation may emerge and be stable for the PD, whereas the classical result predicts total defection. For the SH, where both all-cooperate and all-defect steady-states are theoretically possible, we show that the basin of attraction for cooperation is enhanced. Thus, the possibility of dismissing a relationship and creating a new one does indeed increase the potential for cooperation and coordination in our artificial society. The self-organizing mechanism consists in both games in forming dense clusters of cooperators which are more difficult to dissolve by exploiting defectors. While the beneficial effect of relational or geographical static population structures on cooperation was already known from previous studies, here we have shown that more realistic dynamic social networks may also allow cooperation to thrive. Future work will deal with the stability of the cooperating states against stronger perturbations than merely the implicit noise of the stochastic dynamics. We also intend to study more fully the structure of the emerging clusters and their relationships, and we plan to extend the model to other important paradigmatic games such as Hawks-Doves and coordination games.

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# Article E

# Mutual Trust and Cooperation in the Evolutionary Hawks-Doves Game

| | |
|---|---|
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### Abstract

Using a new dynamical network model of society in which pairwise interactions are weighted according to mutual satisfaction, we show that cooperation is the norm in the Hawks-Doves game when individuals are allowed to break ties with undesirable neighbors and to make new acquaintances in their extended neighborhood. Moreover, cooperation is robust with respect to rather strong strategy perturbations. We also discuss the empirical structure of the emerging networks, and the reasons that allow cooperators to thrive in the population. Given the metaphorical importance of this game for social interaction, this is an encouraging positive result as standard theory for large mixing populations prescribes that a certain fraction of defectors must always exist at equilibrium.

## E.1 Introduction and Previous Work

Game Theory [122] is the study of how social or economical agents take decisions in situations of conflict. Some games such as the celebrated Prisoner's Dilemma have a high metaphorical value for society in spite of their simplicity and abstractness. Hawks-Doves, also known as Chicken, is one such socially significant game. Hawks-Doves is a two-person, symmetric game with the generic payoff bi-matrix of Table E.1. In this matrix, D stands for the defecting strategy "hawk", and C stands for the cooperating strategy "dove". The "row" strategies correspond to player 1 and the "column" strategies to player 2. An entry of the table such as (T,S) means that if player 1 chooses strategy D and player 2 chooses strategy C, then the payoff or utility to player 1 is T, while the payoff of player 2 is S. Metaphorically, a hawkish behavior means a strategy of fighting, while a dove, when facing a confrontation, will always yield. R is the *reward* the two players receive if they both cooperate, P is the *punishment* for bilateral defection, and T is the *temptation*, i.e. the payoff that a player receives if it defects, while the other cooperates. In this case, the cooperator gets the *sucker's payoff* S. The game has a structure similar to that of the *Prisoner's Dilemma* [6]. However, the ordering of payoffs for the Prisoner's Dilemma is $T > R > P > S$ rendering defection the best rational individual choice, while in the Hawks-Doves game studied here the ordering is $T > R > S > P$ thus making mutual defection, i.e. result (D,D), the worst possible outcome. Note that in game theory, as long as the above orderings are respected, the actual numerical payoff values do not change the nature and number of equilibria [122].

| | C | D |
|---|---|---|
| C | (R,R) | (S,T) |
| D | (T,S) | (P,P) |

Table E.1: Payoff matrix for a symmetric two person game.

In contrast to the Prisoner's Dilemma which has a unique Nash equilibrium that corresponds to both players defecting, the strategy pairs (C,D) and (D,C) are both Nash equilibria of the Hawks-Doves game in pure strategies, and there is a third equilibrium in mixed strategies in which strategy D is played with probability $p$, and strategy C with probability $1 - p$, where $0 < p < 1$ depends on the actual payoff values. We recall that a Nash equilibrium is a combination of strategies (pure or mixed) of the different players such that any unilateral deviation by any agent from this combination can only decrease her expected payoff [122].

As it is the case for the Prisoner's Dilemma (see for example [6, 57] for the iterated case, among a vast literature), Hawks-Doves, for all its simplicity, appears to capture some important features of social interactions. In this sense, it applies in many situations in which "parading", "retreating", and "escalating" are common. One striking example of a situation that has been thought to lead to a Hawks-Doves dilemma is the Cuban missile crisis in 1962 [94]. Territorial threats at the border between nations are another case in point as well as bullying in teenage gangs. Other well known applications are found in the animal kingdom during ritualized fights [64].

In this article, we shall present our methods and results in the framework of *evolutionary game theory* [46]. In evolutionary game theory a very large mixing population of players is considered, and randomly chosen pairs of individuals play a sequence of one-shot two-person games. In the

Hawks-Doves game, the theory prescribes that the only *Evolutionary Stable Strategy* (ESS) of the population is the mixed strategy, giving rise, at equilibrium, to a polymorphic population composed of hawks and doves in which the frequency of hawks equals $p$, the probability with which strategy hawk would be played in the NE mixed strategy.

In the case of the Prisoner's Dilemma, one finds a unique ESS with all the individuals defecting. However, Nowak and May [84] showed that cooperation in the population is sustainable under certain conditions, provided that the network of the interactions between players has a lattice spatial structure. Killingback and Doebeli [52] extended the spatial approach to the Hawks-Doves game and found that a planar lattice structure with only nearest-neighbor interactions may favor cooperation, i.e. the fraction of doves in the population is often higher than what is predicted by evolutionary game theory. In a more recent work however, Hauert and Doebeli [41] were led to a different conclusion, namely that spatial structure does not seem to favor cooperation in the Hawks-Doves game.

Further studies extended the structured population approach to other graph structures representing small worlds (for an excellent review, see [112]). Small-world networks are produced by randomly rewiring a few links in an otherwise regular lattice such as a ring or a grid [127]. These "shortcuts", as they are called, give rise to graphs that have short path lengths between any two nodes in the average as in random graphs, but in contrast to the latter, also have a great deal of local structure as conventionally measured by the *clustering coefficient*[1]. These structures are much more typical of the networks that have been analyzed in technology, society, and biology than regular lattices or random graphs [75]. In [116] it was found that cooperation in Hawks-Doves may be either enhanced or inhibited in small-world networks depending on the gain-to-cost ratio $r = R/(R - P)$, and on the strategy update rule using standard local evolutionary dynamics with one-shot bilateral encounters. However, Watts–Strogatz small-world networks, although more realistic than lattices or random graphs, are not good representations of typical social networks. Santos and Pacheco [101] extended the study of the Hawks-Doves game to scale-free networks, i.e. to networks having a power-law distribution of the connectivity degree [75]. They found that cooperation is remarkably enhanced in them with respect to previously described population structures through the existence of highly connected cooperator hubs. Scale-free networks are much closer than Watts–Strogatz ones to the typical socio-economic networks that have been investigated, but they are relatively uncommon in their "pure" form due to finite cutoffs and other real-world effects (for example, see [75, 4, 73, 50]), with the notable exception of sexual contact networks [56]. Using real and model static social networks, Luthi et al. [60] also found that cooperation is enhanced in Hawks-Doves, although to a lesser degree than in the scale-free case, thanks to the existence of tight clusters of cooperators that reinforce each other.

Static networks resulting from the analysis of actual social networks or good models of the latter

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[1] The clustering coefficient $\mathcal{C}_i$ of a node $i$ is defined as $\mathcal{C}_i = 2E_i/k_i(k_i - 1)$, where $E_i$ is the number of edges in the neighborhood of $i$. Thus $\mathcal{C}_i$ measures the amount of "cliquishness" of the neighborhood of node $i$ and it characterizes the extent to which nodes adjacent to node $i$ are connected to each other. The clustering coefficient of the graph is simply the average over all nodes: $\mathcal{C} = \frac{1}{N}\sum_{i=1}^{N}\mathcal{C}_i$ [75].

are a good starting point; however, the static approach ignores fluctuations and non-equilibrium phenomena. As a matter of fact, in many real networks nodes may join the network forming new links, and old nodes may leave it as social actors come and go. Furthermore, new links between agents already in the network may also form or be dismissed. Often the speed of these network changes is comparable to that of the agent's behavioral adaptation, thus making it necessary to study how they interact. Examples of slowly-changing social networks are scientific collaborations, friendships, firm networks among others. A static network appears to be a good approximation in these cases. On the other hand, in our Internet times, there exist many social or pseudo-social networks in which topology changes are faster. For example, e-mail networks [54], web-based networks for friendship and entertainment, such as Facebook, or professional purposes such as LinkedIn, and many others. Furthermore, as it is not socially credible that people will keep for a long time unsatisfying relationships, addition and dismissal of partners are an extremely common phenomenon, also due to natural causes such as moving, changing fields, or interests. We note at this point that some previous work has focused on the possibility of allowing players to choose or refuse social partners in game interactions [10, 108], which has been shown to potentially promote cooperation. However, this work does not consider an explicit underlying interaction network of agents, nor does it use the social link strengths as indicators of partner's suitability as we do here.

In light of what has been said above, the motivation of the present work is to study the coevolution of strategy and network structure and to investigate under which conditions cooperative behavior may emerge and be stable in the Hawks-Doves game. A related goal is to study the topological structures of the emergent networks and their relationships with the strategic choices of the agents. Some previous work has been done on evolutionary games on dynamic networks [111, 25, 131, 59, 103] almost all of them dealing with the Prisoner's Dilemma. The only one briefly describing results for the Hawks-Doves game is [103] but our model differs in several important respects and we obtain new results on the structure of the cooperating clusters. The main novelty is the use of pairwise interactions that are dynamically weighted according to mutual satisfaction. The new contributions and the differences with previous work will be described at the appropriate points in the article. An early preliminary version of this study has been presented at the conference [91].

The paper is organized as follows. In the next section we present our coevolutionary model. This is followed by an exhaustive numerical study of the game's parameter space. After that we present our results on cooperation and we describe and discuss the structure of the emerging networks. Finally we give our conclusions and suggestions for possible future work.

## E.2 The Model and its Dynamics

The model is strictly local as no player uses information other than the one concerning the player itself and the players it is directly connected to. In particular, each agent knows its own current strategy and payoff. Moreover, as the model is an evolutionary one, no rationality, in the sense of game theory, is needed [122]. Players just adapt their behavior such that they imitate more successful strategies in their environment with higher probability. Furthermore, they are able to locally assess the worthiness of an interaction and possibly dismiss a relationship that does not pay off enough. The model has been introduced and fully explained in [92], where we study the Prisoner's Dilemma and the Stag-Hunt games; it is reported here in some detail in order to make the paper self-contained.

### E.2.1 Agent-Agent and Network Interaction Structure

The network of agents is represented by a directed graph $G(V, E)$, where the set of vertices $V$ represents the agents, while the set of oriented edges (or links) $E$ represents their unsymmetric interactions. The population size $N$ is the cardinality of $V$. A neighbor of an agent $i$ is any other agent $j$ such that there is a pair of oriented edges $\vec{ij}$ and $\vec{ji} \in E$. The set of neighbors of $i$ is called $V_i$. For network structure description purposes, we shall also use an unoriented version $G^{'}$ of $G$ having exactly the same set of vertices $V$ but only a single unoriented edge $ij$ between any pair of connected vertices $i$ and $j$ of $G$. For $G'$ we shall define the degree $k_i$ of vertex $i \in V$ as the number of neighbors of $i$. The average degree of the network $G^{'}$ will be called $\bar{k}$.
A pair of directed links between vertices $i$ and $j$ in $G$ is schematically depicted in Fig. E.1. Each link has a weight or "force" $f_{ij}$ (respectively $f_{ji}$). This weight, say $f_{ij}$, represents in an indirect way the "trust" player $i$ attributes to player $j$. This weight may take any value in $[0, 1]$ and its variation is dictated by the payoff earned by $i$ in each encounter with $j$, as explained below.

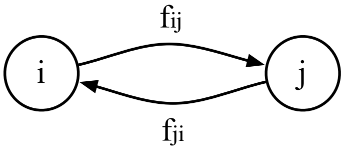

Figure E.1: Schematic representation of mutual trust between two agents through the strengths of their links.

The idea behind the introduction of the forces $f_{ij}$ is loosely inspired by the potentiation/depotentiation of connections between neurons in neural networks, an effect known as the *Hebb rule* [43]. In our context, it can be seen as a kind of "memory" of previous encounters. However, it must be distinguished from the memory used in iterated games, in which players "remember" a certain number of previous moves and can thus conform their future strategy on the analysis of those past encounters [122]. Our interactions are strictly one-shot, i.e. players "forget" the results of previous rounds and cannot recognize previous partners and their possible playing patterns. However, a certain amount of past history is implicitly contained in the numbers $f_{ij}$ and this information may be used by an agent when it will come to decide whether or not an interaction should be dismissed (see below).
We also define a quantity $s_i$ called *satisfaction* of an agent $i$ which is the sum of all the weights of the links between $i$ and its neighbors $V_i$ divided by the total number of links $k_i$:

$$s_i = \frac{\sum_{j \in V_i} f_{ij}}{k_i}.$$

We clearly have $0 \leq s_i \leq 1$. Note that the term satisfaction is sometimes used in game-theoretical work to mean the amount of utility gained by a given player. Instead, here satisfaction is related to the average willingness of a player to maintain the current relationships in the player's neighborhood.

### E.2.2 Initialization

The network is of constant size $N = 1000$; this allows a simpler yet significant model of network dynamics in which social links may be broken and formed but agents do not disappear and new agents may not join the network. The initial graph is generated randomly with a mean degree $\bar{k} = 10$ which is of the order of those actually found in many social networks such as collaboration, association, or friendship networks in which relations are generally rather long-lived and there is a cost to maintain a large number; see, for instance, [73, 75, 69, 117]. Players are distributed uniformly at random over the graph vertices with 50% cooperators. Forces of links between any pair of neighboring players are initialized at 0.5.

We use a parameter $q$ which is akin to a "temperature" or noise level; $q$ is a real number in $[0, 1]$ and it represents the frequency with which an agent wishes to dismiss a link with one of its neighbors. The higher $q$, the faster the link reorganization in the network. This parameter has been first introduced in [131] and it controls the speed at which topological changes occur in the network, i.e. the time scale of the strategy-topology co-evolution. It is an important consideration, as social networks may structurally evolve at widely different speeds, depending on the kind of interaction between agents. For example, e-mail networks change their structure at a faster pace than, say, scientific collaboration networks.

### E.2.3 Strategy and Link Dynamics

Here we describe in detail how individual strategies, links, and link weights are updated. The node update sequence is chosen at random with replacement as in many previous works [48, 41, 59]. Once a given node $i$ of $G$ is chosen to be activated, it goes through the following steps:

- if the degree of agent $i$, $k_i = 0$ then player $i$ is an isolated node. In this case a link with strength 0.5 is created from $i$ to a player $j$ chosen uniformly at random among the other $N - 1$ players in the network.
- otherwise,
  - either agent $i$ updates its strategy according to a local *replicator dynamics* rule with probability $1-q$ or, with probability $q$, agent $i$ may delete a link with a given neighbor $j$ and creates a new 0.5 force link with another node $k$ ;
  - the forces between $i$ and its neighbors $V_i$ are updated

Let us now describe each step in more detail.

### E.2.4 Strategy Evolution

We use a local version of replicator dynamics (RD) for regular graphs [41] but modified as described in [61] to take into account the fact that the number of neighbors in a degree-inhomogeneous network can be different for different agents. Indeed, it has been analytically shown that using straight accumulated payoff in degree-inhomogeneous networks leads to a loss of invariance with respect to affine transformations of the payoff matrix under RD [61]. The local dynamics of a player $i$ only depends on its own strategy and on the strategies of the $k_i$ players in its neighborhood $V_i \in G'$. Let us call $\pi_{ij}$ the payoff player $i$ receives when interacting with neighbor $j$. This payoff is defined as

$$\pi_{ij} = \sigma_i(t)\ M\ \sigma_j^T(t),$$

where $M$ is the payoff matrix of the game and $\sigma_i(t)$ and $\sigma_j(t)$ are the strategies played by $i$ and $j$ at time $t$. The quantity

$$\widehat{\Pi}_i(t) = \sum_{j \in V_i} \pi_{ij}(t)$$

is the weighted accumulated payoff defined in [61] collected by player $i$ at time step $t$. The rule according to which agents update their strategies is the conventional RD in which strategies that do better than the average increase their share in the population, while those that fare worse than average decrease. To update the strategy of player $i$, another player $j$ is drawn at random from the neighborhood $V_i$. It is assumed that the probability of switching strategy is a function $\phi$ of the payoff difference; $\phi$ is required to be monotonic increasing; here it has been taken linear [46]. Strategy $\sigma_i$ is replaced by $\sigma_j$ with probability

$$p_i = \phi(\widehat{\Pi}_j - \widehat{\Pi}_i),$$

where

$$\phi(\widehat{\Pi}_j - \widehat{\Pi}_i) = \begin{cases} \dfrac{\widehat{\Pi}_j - \widehat{\Pi}_i}{\widehat{\Pi}_{j,\max} - \widehat{\Pi}_{i,\min}} & \text{if } \widehat{\Pi}_j - \widehat{\Pi}_i > 0 \\ \\ 0 & \text{otherwise.} \end{cases}$$

In the last expression, $\widehat{\Pi}_{x,\max}$ (resp. $\widehat{\Pi}_{x,\min}$) is the maximum (resp. minimum) payoff a player $x$ can get (see ref. [61] for more details).

The major differences with standard RD is that two-person encounters between players are only possible among neighbors, instead of being drawn from the whole population, and the latter is of finite size in our case. Other commonly used strategy update rules include imitating the best in the neighborhood [84, 131], or replicating in proportion to the payoff [41, 116].

### E.2.5 Link Evolution

The active agent $i$, which has $k_i \neq 0$ neighbors will, with probability $q$, attempt to dismiss an interaction with one of its neighbors in the following way. In the description we focus on the outgoing links from $i$ in $G$, the incoming links play a subsidiary role. Player $i$ first looks at its satisfaction $s_i$. The higher $s_i$, the more satisfied the player, since a high satisfaction is a consequence of successful strategic interactions with the neighbors. Thus, the natural tendency is to try to dismiss a link when $s_i$ is low. This is simulated by drawing a uniform pseudo-random number $r \in [0, 1]$ and breaking a link when $r \geq s_i$. Assuming that the decision is taken to cut a link, which one, among the possible $k_i$, should be chosen? Our solution is based on the strength of the relevant links. First a neighbor $j$ is chosen with probability proportional to $1 - f_{ij}$, i.e. the stronger the link, the less likely it is that it will be selected. This intuitively corresponds to $i$'s observation that it is preferable to dismiss an interaction with a neighbor $j$ that has contributed little to $i$'s payoff over several rounds of play. However, dismissing a link is not free: $j$ may "object" to the decision. The intuitive idea is that, in real social situations, it is seldom possible to take unilateral decisions: often there is a cost associated, and we represent this hidden cost by a probability $1 - (f_{ij} + f_{ji})/2$ with which $j$ may refuse to be cut away. In other words, the link is less likely to be deleted if $j$ appreciates $i$, i.e. when $f_{ji}$ is high.

Assuming that the $\vec{ij}$ and $\vec{ji}$ links are finally cut, how is a new interaction to be formed? The solution adopted here is inspired by the observation that, in social settings, links are usually created more easily between people who have a mutual acquaintance than those who do not. First, a neighbor $k$ is chosen in $V_i \setminus \{j\}$ with probability proportional to $f_{ik}$, thus favoring neighbors $i$ trusts. Next, $k$ in turn chooses player $l$ in his neighborhood $V_k$ using the same principle, i.e. with probability proportional to $f_{kl}$. If $i$ and $l$ are not connected, two links $\vec{il}$ and $\vec{li}$ are created, otherwise the process is repeated in $V_l$. Again, if the selected node, say $m$, is not connected to $i$, an interaction between $i$ and $m$ is established by creating two new links $\vec{im}$ and $\vec{mi}$. If this also fails, new links between $i$ and a randomly chosen node are created. In all cases the new links are initialized with a strength of 0.5 in each direction. This rewiring process is schematically depicted in Fig. E.2 for the case in which a link can be successfully established between players $i$ and $l$ thanks to their mutual acquaintance $k$.

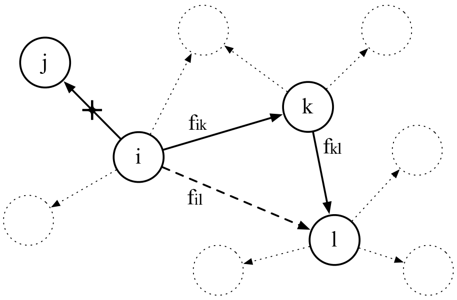

Figure E.2: Illustration of the rewiring of link $\{ij\}$ to $\{il\}$. Agent $k$ is chosen to introduce player $l$ to $i$ (see text). Only outgoing links are shown for clarity.

At this point, we would like to stress several important differences with previous work in which links can be dismissed and rewired in a constant-size network in evolutionary games. First of all, in all these works the interaction graph is undirected with a single link between any pair of agents. In [131], only links between defectors are allowed to be cut unilaterally and the study is restricted to the Prisoner's Dilemma. Instead, in our case, any interaction has a finite probability to be abandoned, even a profitable one between cooperators if it is recent, although links that are more stable, i.e. have high strengths, are less likely to be rewired. This smoother situation is made possible thanks to our bilateral view of a link. It also allows for a moderate amount of "noise", which could reflect to some extent the uncertainties in the system. The present link rewiring process is also different from the one adopted in [103], where the Fermi function is used to decide whether to cut a link or not and also from their new version of it which has appeared in [107]. Finally, in [59] links are cut according to a threshold decision rule and are rewired randomly anywhere in the network.

### E.2.6 Updating the Link Strengths

Once the chosen agents have gone through their strategy or link update steps, the strengths of the links are updated accordingly in the following way:

$$f_{ij}(t+1) = f_{ij}(t) + \frac{\pi_{ij} - \bar{\pi}_{ij}}{k_i(\pi_{max} - \pi_{min})},$$

where $\pi_{ij}$ is the payoff of $i$ when interacting with $j$, $\bar{\pi}_{ij}$ is the payoff earned by $i$ playing with $j$, if $j$ were to play his other strategy, and $\pi_{max}$ ($\pi_{min}$) is the maximal (minimal) possible payoff obtainable in a single interaction. If $f_{ij}(t+1)$ falls outside the $[0,1]$ interval then it is reset to 0 if it is negative, and to 1 if it is larger than 1. This update is performed in both directions, i.e. both $f_{ij}$ and $f_{ji}$ are updated $\forall j \in V_i$ because both $i$ and $j$ get a payoff out of their encounter.

## E.3 Numerical Simulations and Discussion

### E.3.1 Simulation Parameters

We simulated the Hawks-Doves game with the dynamics described above exploring the game space by limiting our study to the variation of only two game parameters. We set $R = 1$ and $P = 0$ and the two parameters are $1 \leq T \leq 2$ and $0 \leq S \leq 1$. Setting $R = 1$ and $P = 0$ determines the range of $S$ (since $T > R > S > P$) and gives an upper bound of 2 for $T$, due to the $2R > T + S$ constraint, which ensures that mutual cooperation is preferred over an equal probability of unilateral cooperation and defection. Note however, that the only valid value pairs of $(T, S)$ are those that satisfy the latter constraint.

We simulated networks of size $N = 1000$, randomly generated with an average degree $\bar{k} = 10$ and randomly initialized with 50% cooperators and 50% defectors. In all cases, parameters $T$ and $S$ are varied between their two bounds in steps of 0.1. For each set of values, we carry out 50

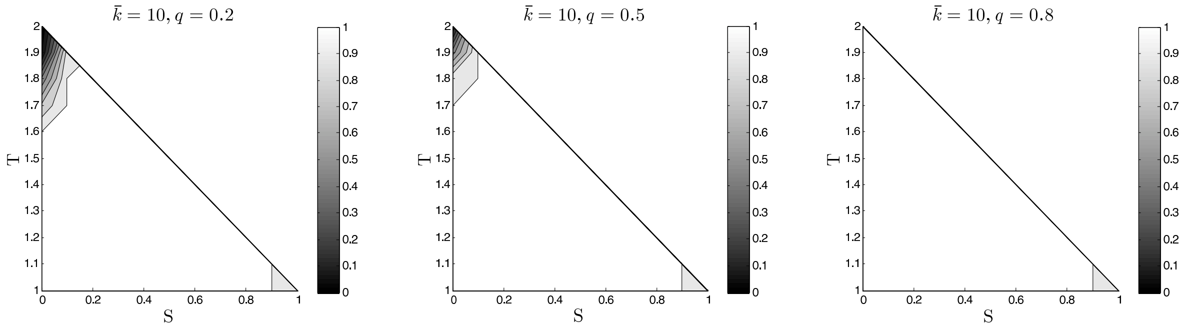

Figure E.3: Average cooperation values for the Hawks-Doves game for three values of $q$ at steady-state. Results are the average of 50 runs.

runs of at most 10000 steps each, using a fresh graph realization in each run. Each step consists in the update of a full population. A run is stopped when all agents are using the same strategy, in order to be able to measure statistics for the population and for the structural parameters of the graphs. After an initial transient period, the system is considered to have reached a pseudo-equilibrium strategy state when the strategy of the agents (C or D) does not change over 150 further time steps, which means $15 \times 10^4$ individual updates. It is worth mentioning that equilibrium is always attained well before the allowed 10000 time steps, in most cases, less than $1'000$ steps are enough. We speak of pseudo-equilibria or steady states and not of true evolutionary equilibria because there is no analog of equilibrium conditions in the dynamical systems sense.

To check whether scalability is an issue for the system, we have run several simulations with larger graphs namely, $N = 3000$ and $N = 10000$. The overall result is that, although the simulations take a little longer and transient times are also slightly longer, at quasi-equilibrium all the measures explored in the next sections follow the same trend and the dynamics give rise to comparable topologies and strategy relative abundance.

### E.3.2 Emergence of Cooperation

Cooperation results in contour plot form are shown in Fig. E.3. We remark that, as observed in other structured populations, cooperation is achieved in almost the whole configuration space. Thus, the added degree of freedom represented by the possibility of refusing a partner and choosing a new one does indeed help to find player's arrangements that help cooperation. When considering the dependence on the parameter $q$, one sees in Fig. E.3 that the higher $q$, the higher the cooperation level, although the differences are small, since full cooperation prevails already at $q = 0.2$. This is a somewhat expected result, since being able to break ties more often clearly gives cooperators more possibilities for finding and keeping fellow cooperators to interact with. The results reported in the figures are for populations starting with 50% cooperators randomly distributed. We have also tried other proportions with less cooperators, starting at 30%. The results, not reported here for reasons of space, are very similar, the only difference being that it takes more simulation time to reach the final quasi-stable state. Finally, one could ask whether cooperation would still spread starting with very few cooperators. Numerical simulations show that cooperation could indeed prevail even starting from as low as 1% cooperators, except on the far left border of the configuration space where cooperation is severely disadvantaged.

Compared with the level of cooperation observed in simulations in static networks, we can say that results are consistently better for co-evolving networks. For all values of $q$ (Fig. E.3) there is significantly more cooperation than what was found in model and real social networks [60] where the same local replicator dynamics was used but with the constraints imposed by the invariant network structure. A comparable high cooperation level has only been found in static scale-free networks [101, 104] which are not as realistic as a social network structures.

The above considerations are all the more interesting when one observes that the standard RD result is that the only asymptotically stable state for the game is a polymorphic population in which there is a fraction $\alpha$ of doves and a fraction $1 - \alpha$ of hawks, with $\alpha$ depending on the actual numerical payoff matrix values. To see the positive influence of making and breaking ties we can compare our results with what is prescribed by the standard RD solution. Referring to the payoff table E.1, let's assume that the column player plays C with probability $\alpha$ and D with probability $1 - \alpha$. In this case, the expected payoffs of the row player are:

$$E_r[C] = \alpha R + (1 - \alpha)S$$

and

$$E_r[D] = \alpha T + (1 - \alpha)P$$

The row player is indifferent to the choice of $\alpha$ when $E_r[C] = E_r[D]$. Solving for $\alpha$ gives:

$$\alpha = \frac{P - S}{R - S - T + P}. \tag{E.1}$$

Since the game is symmetric, the result for the column player is the same and $(\alpha C, (1-\alpha)D)$ is a NE in mixed strategies. We have numerically solved the equation for all the sampled points in the game's parameter space. Let us now use the following payoff values in order to bring them within the explored game space (remember that NEs are invariant w.r.t. such an affine transformation):

| | C | D |
|---|---|---|
| C | $(1, 1)$ | $(2/3, 4/3)$ |
| D | $(4/3, 2/3)$ | $(0, 0)$ |

Substituting in equation E.1 gives $\alpha = 2/3$, i.e. the dynamically stable polymorphic population should be composed by about 2/3 cooperators and 1/3 defectors. Now, if one looks at Fig. E.3 at the points where $S = 2/3$ and $T = 4/3$, one can see that the point, and the region around it, is one of full cooperation instead. Even within the limits of the approximations caused by the finite population size and the local dynamics, the non-homogeneous graph structure and an increased level of tie rewiring has allowed cooperation to be greatly enhanced with respect to the theoretical predictions of standard RD.

### E.3.3 Evolution of Agents' Satisfaction

According to the model, unsatisfied agents are more likely to try to cut links in an attempt to improve their satisfaction level, which could be simply described as an average value of the strengths of their links with neighbors. Satisfaction should thus tend to increase during evolution. In effect, this is what happens, as can be seen in Fig. E.4. The figure refers to a particular run that ends in all agents cooperating, but it is absolutely typical. One can remark the "spike" at time 0. This is clearly due to the fact that all links are initialized with a weight of 0.5. As the simulation advances, the satisfaction increases steadily and for the case of the figure, in which all agents cooperate at the end, it reaches its maximum value of 1 for almost all players.

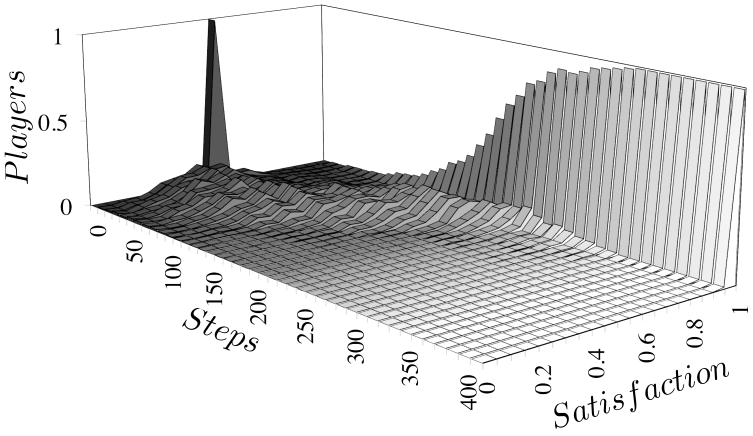

Figure E.4: Fraction of agents having a given satisfaction level as a function of evolution time.

### E.3.4 Stability of Cooperation

Evolutionary game theory provides a dynamical view of conflicting decision-making in populations. Therefore, it is important to assess the *stability* of the equilibrium configurations. This is even more important in the case of numerical simulation where the steady-state finite population configurations are not really equilibria in the mathematical sense. In other words, one has to be reasonably confident that the steady-states are not easily destabilized by perturbations. To this end, we have performed a numerical study of the robustness of final cooperators' configurations by introducing a variable amount of random noise into the system. A strategy is said to be *evolutionarily stable* when it cannot be invaded by a small amount of players using another strategy [46]. We have chosen to switch the strategy of an increasing number of highly connected cooperators to defection, and to observe whether the perturbation propagates in the population, leading to total defection, or if it stays localized and disappears after a transient time. Figs. E.5 and E.6 show how the system recovers when the most highly connected 30% of the cooperators are suddenly and simultaneously switched to defection. In Fig. E.5 the value chosen in the game's configuration space is $T = 1.6$ and $S = 0.4$. This point lies approximately on the diagonal in Fig. E.3 and corresponds to an all-cooperate situation. As one can see, after the perturbation is applied, there is a sizable loss of cooperation but, after a while, the system recovers full cooperation in all cases (only 10 curves are shown in each figure for clarity, but the phenomenon is qualitatively identical in all the 50 independent runs tried). From left to right, three values of $q = 0.2, 0.5, 0.8$ are used. It is seen that, as the rewiring frequency $q$ increases, recovering from the perturbation becomes easier as defection has less time to spread around before cooperators are able to dismiss links toward defectors. Switching the strategy of the 30 % most connected nodes is rather extreme since they include most cooperator clusters but, nonetheless, cooperation is rather stable in the whole cooperating region. In Fig. E.6 we have done the same this time with $T = 1.9$ and $S = 0,1$. This point is in a frontier region in which defection may often prevail, at least for low $q$ (see Fig. E.3) and thus it represents one of the hardest cases for cooperation to remain stable. Nevertheless, except in the leftmost case

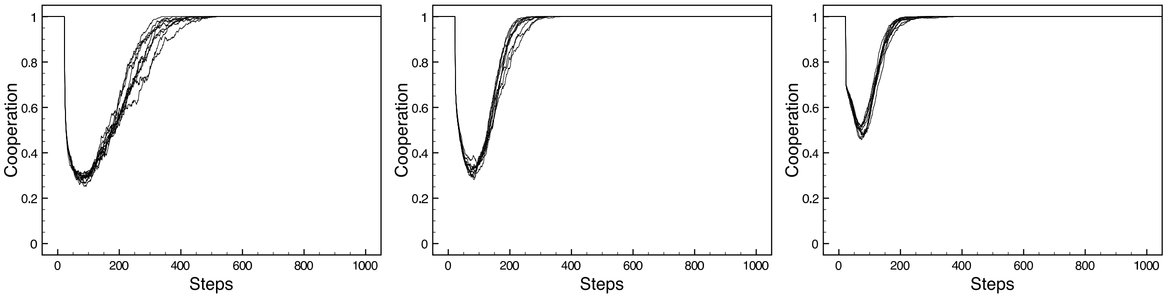

Figure E.5: Cooperation percentage as a function of simulated time when the strategy of the 30% most connected nodes is switched from cooperation to defection. $T = 1.6$, $S = 0.4$ and, from left to right, $q = 0.2, 0.5, 0.8$.

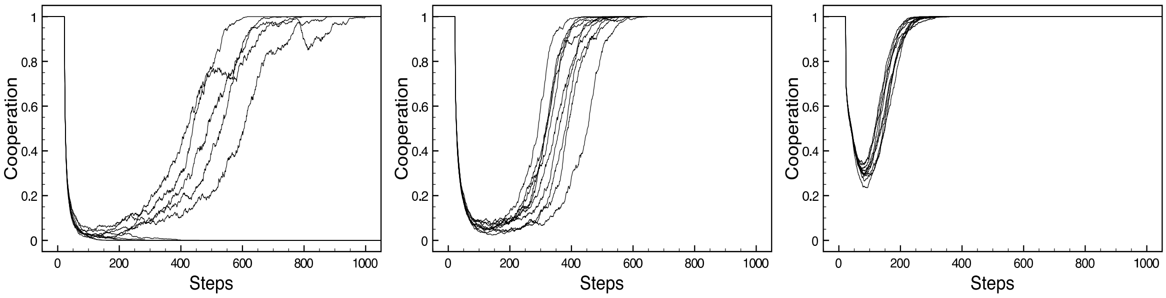

Figure E.6: Cooperation percentage when the strategy of the 30% most connected nodes is switched from cooperation to defection. $T = 1.9$, $S = 0.1$ and, from left to right, $q = 0.2, 0.5, 0.8$.

($q = 0.2$) where half of the runs permanently switch to all-defect, in all the other cases the population is seen to recover after cooperation has fallen down to less than 10%. Note that the opposite case is also possible in this region that is, in a full defect situation, switching of 30% highly connected defectors to cooperation can lead the system to one of full cooperation. In conclusion, the above numerical experiments have empirically shown that cooperation is extremely stable after cooperator networks have emerged. Although we are using here an artificial society of agents, this can hopefully be seen as an encouraging result for cooperation in real societies.

### E.3.5 Structure of the Emerging Networks

In this section we present a statistical analysis of the global and local properties of the networks that emerge when the pseudo-equilibrium states of the dynamics are attained. Note that in the following sections the graph we refer to is the unoriented, unweighted one that we called $G^{'}$ in Sect. E.2.1. In other words, for the structural properties of interest, we only take into account the fact that two agents interact and not the weights of their directed interactions.

#### Small-World Nature

Small-world networks are characterized by a small mean path length and by a high clustering coefficient [127]. Our graphs start random, and thus have short path lengths by construction since their mean path length $\bar{l} = O(logN)$ scales logarithmically with the number of vertices $N$ [75]. It is interesting to notice that they maintain short diameters at equilibrium too, after rewiring has taken place. We took the average $\bar{L} = \sum_{k=1}^{660} \bar{l}$ of the mean path length of 660 evolved graphs, which represent ten graphs for each $T, S$ pair. This average is 3.18, which is of the order of $log(1000)$, while its initial random graph average value is 3.25. This fact, together with the remarkable increase of the clustering coefficients with respect to the random graph (see below), shows that the evolved networks have the small-world property. Of course, this behavior was expected, since the rewiring mechanism favors close partners in the network and thus tends to increase the clustering and to shorten the distances.

#### Average Degree

In contrast to other models [131, 103], the mean degree $\bar{k}$ can vary during the course of the simulation. We found that $\bar{k}$ increases only slightly and tends to stabilize around $\bar{k} = 11$. This is in qualitative agreement with observations made on real dynamical social networks [54, 8, 115] with the only difference that the network does not grow in our model.

#### Clustering Coefficients

The clustering coefficient $\mathcal{C}$ of a graph has been defined in the Introduction section. Random graphs are locally homogeneous in the average and for them $\mathcal{C}$ is simply equal to the probability of having an edge between any pair of nodes independently. In contrast, real networks have

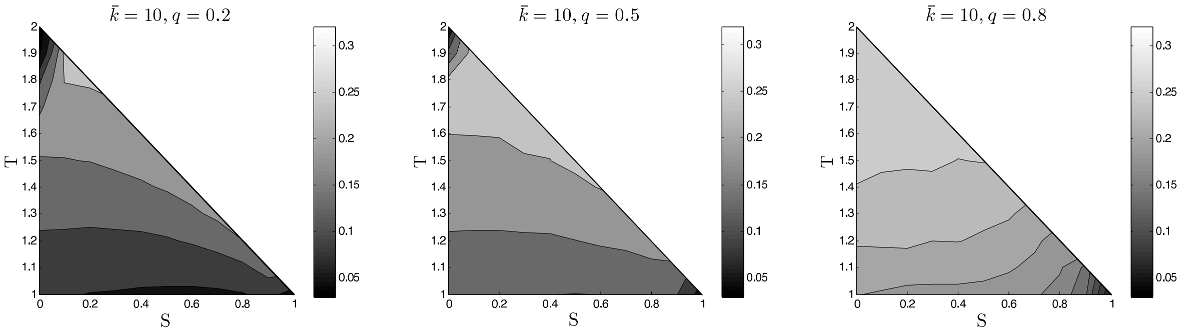

Figure E.7: Average values of the clustering coefficient over 50 runs for three values of $q$.

local structures and thus higher values of $\mathcal{C}$. Fig. E.7 gives the average clustering coefficient $\bar{\mathcal{C}} = \frac{1}{50}\sum_{i=1}^{50}\mathcal{C}$ for each sampled point in the Hawks-Doves configuration space, where 50 is the number of network realizations used for each simulation. The networks self-organize through dismissal of partners and choice of new ones and they acquire local structure, since the clustering coefficients are higher than that of a random graph with the same number of edges and nodes, which is $\bar{k}/N = 10/1000 = 0.01$. The clustering tends to increase with $q$ (i.e. from left to right in Fig. E.7). It is clear that the increase in clustering and the formation of cliques is due to the fact that, when dismissing an unprofitable relation and searching for a new one, individuals that are relationally at a short distance are statistically favored. But this has a close correspondence in the way in which new acquaintances are made in society: they are not random, rather people often get to interact with each other through common acquaintances, or "friends of friends" and so on.

### Degree Distributions

The *degree distribution function* (DDF) $p(k)$ of a graph represents the probability that a randomly chosen node has degree $k$. Random graphs are characterized by DDF of Poissonian form $p(k) = \bar{k}^k e^{-\bar{k}}/k!$, while social and technological real networks often show long tails to the right, i.e. there are nodes that have an unusually large number of neighbors [75]. In some extreme cases the DDF has a power-law form $p(k) \propto k^{-\gamma}$; the tail is particularly extended and there is no characteristic degree. The *cumulative degree distribution function* (CDDF) is just the probability that the degree is greater than or equal to $k$ and has the advantage of being less noisy for high degrees. Fig. E.8 shows the CDDFs for the Hawks-Doves for three values of $T$, $S = 0.2$, and $q = 0.5$ with a logarithmic scale on the y-axis. A Poisson and an exponential distribution are also included for comparison. The Poisson curve actually represents the initial degree distribution of the (random) population graph. The distributions at equilibrium are far from the Poissonian that would apply if the networks would remain essentially random. However, they are also far from the power-law type, which would appear as a straight line in the log-log plot of Fig E.9. Although a reasonable fit with a single law appears to be difficult, these empirical distributions are closer to exponentials, in particular the curve for $T = 1.7$, for which such a fit

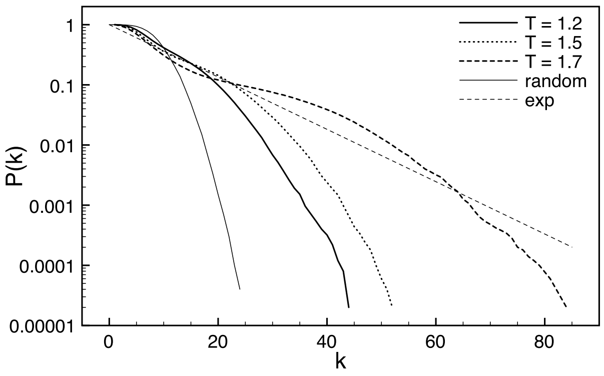

Figure E.8: Empirical cumulative degree distribution functions for three different values of the temptation $T$. A Poissonian and an exponential distribution are also plotted for comparison. Distributions are discrete, the continuous lines are only a guide for the eye. Lin-log scales.

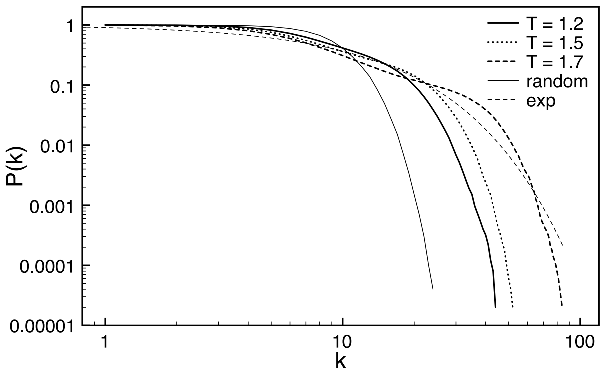

Figure E.9: Empirical cumulative degree distribution functions for three different values of the parameter $T$. Log-log scales.

has been drawn. It can be observed that the distribution is broader the higher $T$ (The higher $T$, the more agents gain by defecting). In fact, although cooperation is attained nearly everywhere in the game's configuration space, higher values of the temptation $T$ mean that agents have to rewire their links more extensively, which results in a higher number of neighbors for some players, and thus it leads to a longer tail in the CDDF. The influence of the $q$ parameter on the shape of the degree distribution functions is shown in Fig. E.10 where average curves for three values of $q$, $T = 1.7$, and $S = 0.2$, are reported. For high $q$, the cooperating steady-state is reached faster, which gives the network less time to rearrange its links. For lower values of $q$ the distributions become broader, despite the fact that rewiring occurs less often, because cooperation in this region is harder to attain and more simulation time is needed. In conclusion, emerging network structures at steady states have DDFs that are similar to those found in actual social networks [75, 4, 73, 50, 117], with tails that are fatter the higher the temptation $T$ and the lower $q$. Topologies closer to scale-free would probably be obtained if the model allowed for growth, since preferential attachment is already present to some extent due to the nature of the

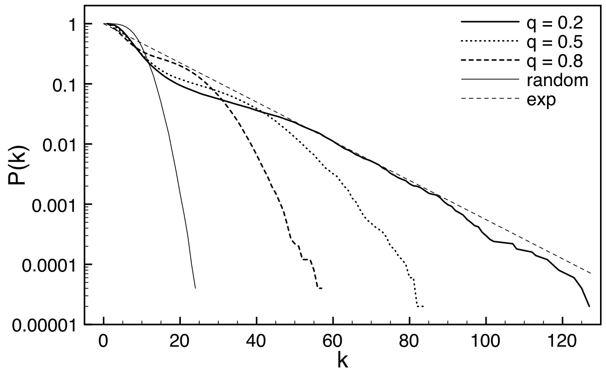

Figure E.10: Empirical cumulative degree distribution functions for three different values of the temptation $q$. Lin-log scales.

rewiring process [93].

### Degree Correlations

Besides the degree distribution function of a network, it is also sometimes useful to investigate the empirical joint degree-degree distribution of neighboring vertices. However, it is difficult to obtain reliable statistics because the data set is usually too small (if a network has $L$ edges, with $L \ll N^2$ where $N$ is the number of vertices for the usually relatively sparse networks we deal with, one then has only $L$ pairs of data to work with). Approximate statistics can readily be obtained by using the average degree of the nearest neighbors of a vertex $i$ as a function of the degree of this vertex, $\bar{k}_{V_i}(k_i)$ [88]. From Fig. E.11 one can see that the correlation is slightly negative, or disassortative. This is at odds with what is reported about real social networks, in which usually this correlation is positive instead, i.e. high-degree nodes tend to connect to high-degree nodes and vice-versa [75]. However, real social networks establish and grow because

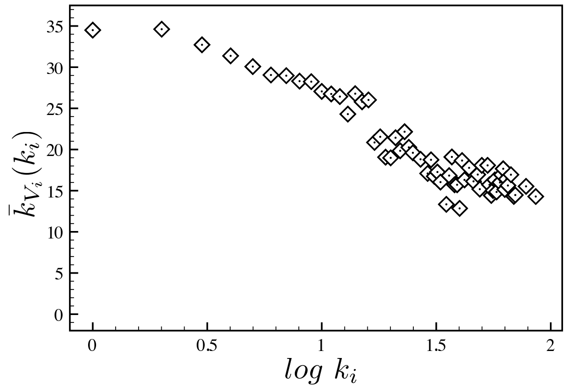

Figure E.11: Average degree of the direct neighbors of a vertex Vs. the vertex degree. The relation is disassortative. Log-lin scales.

of common interests, collaboration work, friendship and so on. Here this is not the case, since the network is not a growing one, and the game played by the agents is antagonistic and causes segregation of highly connected cooperators into clusters in which they are surrounded by less highly connected fellows. This will be seen more pictorially in the following section.

### E.3.6 Cooperator Clusters

From the results of the previous sections, it appears that a much higher amount of cooperation than what is predicted by the standard theory for mixing populations can be reached when ties can be broken and rewired. We have seen that this dynamics causes the graph to acquire local structure, and thus to loose its initial randomness. In other words, the network self-organizes in order to allow players to cooperate as much as possible. At the microscopic, i.e. agent level, this happens through the formation of clusters of players using the same strategy. Fig. E.12 shows one typical cooperator cluster. In the figure one can clearly see that the central cooperator is a highly connected node and there are many links also between the other neighbors. Such tightly packed structures have emerged to protect cooperators from defectors that, at earlier times, were trying to link to cooperators to exploit them. These observations help understand why the degree distributions are long-tailed (see previous section), and also the higher values of the clustering coefficient.

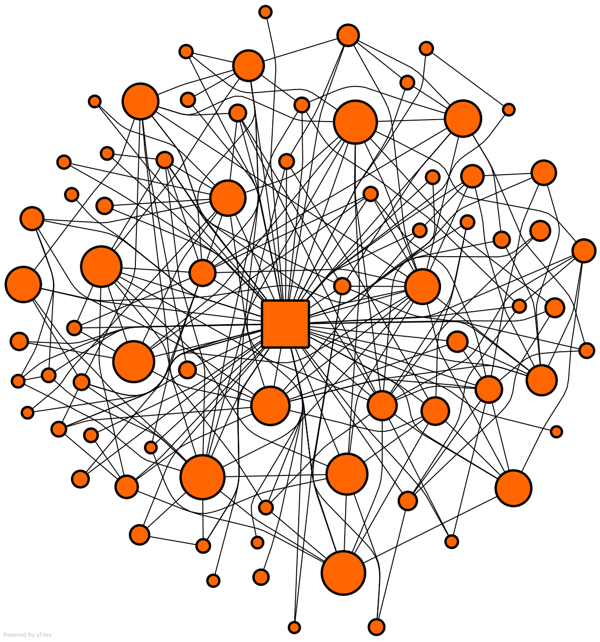

Figure E.12: A typical cooperator cluster. Links to the rest of the network have been suppressed for clarity. The size of a node is proportional to its connectivity in the whole graph. The most connected central cooperator is shown as a square.

Further studies of the emerging networks would imply investigating the communities and the way in which strategies are distributed in them. There are many ways to reveal the modular structure of networks [22] but we leave this study for further work.

## E.4 Conclusions

In this paper we have introduced a new dynamical population structure for agents playing a series of two-person Hawks and Doves game. The most novel feature of the model is the adoption of a variable strength of the bi-directional social ties between pairs of players. These strengths change dynamically and independently as a function of the relative satisfaction of the two end points when playing with their immediate neighbors in the network. A player may wish to break a tie to a neighbor and the probability of cutting the link is higher the weaker the directed link strength is. The ensemble of weighted links implicitly represents a kind of memory of past encounters although, technically speaking, the game is not iterated. While in previous work the rewiring parameters where ad hoc, unspecified probabilities, we have made an effort to relate them to the agent's propensity to gauge the perceived quality of a relationship during time.
The model takes into account recent knowledge coming from the analysis of the structure and of the evolution of social networks and, as such, should be a better approximation of real social conflicting situations than static graphs such as regular grids. In particular, new links are not created at random but rather taking into account the "trust" a player may have on her relationally close social environment as reflected by the current strengths of its links. This, of course, is at the origin of the de-randomization and self-organization of the network, with the formation of stable clusters of cooperators. The main result concerning the nature of the pseudo-equilibrium states of the dynamics is that cooperation is greatly enhanced in such a dynamical artificial society and, furthermore, it is quite robust with respect to large strategy perturbations. Although our model is but a simplified and incomplete representation of social reality, this is encouraging, as the Hawks-Doves game is a paradigm for a number of social and political situations in which aggressivity plays an important role. The standard result is that bold behavior does not disappear at evolutionary equilibrium. However, we have seen here that a certain amount of plasticity of the networked society allows for full cooperation to be consistently attained. Although the model is an extremely abstract one, it shows that there is place for peaceful resolution of conflict. In future work we would like to investigate other stochastic strategy evolution models based on more refined forms of learning than simple imitation and study the global modular structure of the equilibrium networks.

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# Part IV

# Pure Coordination Games

# Article F

# Evolution of Coordination in Social Networks: A Numerical Study

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### Abstract

Coordination games are important to explain efficient and desirable social behavior. Here we study these games by extensive numerical simulation on networked social structures using an evolutionary approach. We show that local network effects may promote selection of efficient equilibria in both pure and general coordination games and may explain social polarization. These results are put into perspective with respect to known theoretical results. The main insight we obtain is that clustering, and especially community structure in social networks has a positive role in promoting socially efficient outcomes.

## F.1 Introduction

Game theory [122] has proved extremely useful in the study of economic, social, and biological situations for describing interactions between agents having possibly different and often conflicting objectives. Paradigmatic games such as the Prisoner's Dilemma [6] have been used in order to represent the tension that appears in society when individual objectives are in conflict with socially desirable outcomes. Most of the vast research literature has focused on conflicting situations in order to uncover the mechanisms that could lead to cooperation instead of socially harmful outcomes (see e.g. [82] for a synthesis). However, there are important situations in society that do not require players to use aggressive strategies. In fact, many frequent social and economic activities require individuals to coordinate their actions on a common goal since in many cases the best course of action is to conform to the standard behavior. For example, if one is used to drive on the right side of the road and travels to a country where the norm is reversed, it pays off to follow the local norm. Bargaining and contracts are also of this type because, even though expectancies may be different between a buyer and a seller, still both would rather trade than not, provided that the respective prices are not too different. For another example, consider a situation in which coordination in working contributions is required in order to produce a good or a service. In a group it might pay off not to contribute, if this behavior goes unnoticed, but the total output will be negatively affected. Games that express this extremely common kind of interactions are called *coordination games*.

Coordination games confront the players with multiple Nash equilibria and the ensuing problem of equilibrium selection. Given that these equilibria are equivalent from the game-theoretical point of view, how to explain how agents make their decisions? This question has important implications in opinion dynamics problems, for example in elections, choice of a new technology and so on [33, 11, 29].

A useful approach has been to use evolutionary and learning ideas which offer a dynamical perspective based on the forces of biological and social evolution. In *evolutionary game theory* (EGT), the concept of a population of players where strategies that score best are more likely to be selected and reproduced provides a justification for the appearance of stable states of the dynamics that represent solutions of the game [122, 128].

For mathematical convenience, standard EGT is based on infinite mixing populations where pairs of individuals are drawn uniformly at random at each step and play the game. Correlations are absent by definition and the population has an homogeneous structure. However, everyday observation tells us that in animal and human societies, individuals usually tend to interact more often with some specified subset of partners; for instance, teenagers tend to adopt the fashions of their close friends group; closely connected groups usually follow the same religion, and so on. Likewise, in the economic world, a group of firms might be directly connected because they share capital, technology, or otherwise interact in some way. In short, social interaction is mediated by networks, in which vertices identify people, firms etc., and edges identify some kind of relation between the concerned vertices such as friendship, collaboration, economic exchange and so on. Thus, locality of interaction plays an important role. This kind of approach was pioneered in EGT by Nowak and May [84] by using simple two-dimensional regular grids. Recently, in the wake of a surge of activity in network research in many fields [75, 19], the dynamical and evolutionary behavior of games on networks that are more likely to represent actual social interactions than regular grids has been investigated (see [112] for a comprehensive recent review). These studies, almost exclusively conducted on games of conflict such as the Prisoner's dilemma or Hawks-Doves, have shown that there are network structures, such as scale-free and actual social networks that may favor the emergence of cooperation with respect to the fully mixing populations used in the theory [104, 60].

In this work we extend this kind of approach to games of the coordination type. We shall use several types of network structures, both networks generated by an algorithm as well as an actual social network to try to unravel the effect of structure on the population behavior. In the present paper, we ignore that social networks are actually dynamical entities that change constantly. Indeed, actors join and leave networks and they may accumulate and abandon ties over time. Using static networks is a useful first approximation however, especially for the cases where the rate of change of the network structure is slow with respect to the rate of change of individual's behaviors which is the approximation that is made here[1]. Comparatively little theoretical work has been done on coordination games on networks, except for some standard types such as rings or complete networks [26] for which rigorous results have been obtained thanks to their regular structure. Although we do mention some known rigorous results as discussed below, our methodology is essentially computer simulation-based. This is because for most network types, inhomogeneity and correlations do not allow standard mean-field methods to be used. Likewise, pair approximation methods [120] provide an acceptable approach for random and regular graphs but not for the other more complex types and thus they are not used here.

The paper is organized as follows. In the next section we first present a brief introduction to the subject of coordination games, in order to make the work self-contained. Then, in Sect. F.3, we enumerate the main theoretical results on coordination games, as well as the necessary definitions for networks of agents and their dynamics. In Sect. F.4 we describe the simulation methodology and the parameters used and, in Sect. F.5 we present and discuss the simulation results on various network classes first for pure coordination games, and then for general coordination ones. Finally, in Sect. F.6 we give our conclusions and ideas for future work.

## F.2 Coordination Games

### F.2.1 General Coordination Games

General two-person, two strategies coordination games have the normal form of Table F.1. With $a > d$ and $b > c$, $(\alpha, \alpha)$ and $(\beta, \beta)$ are both Nash equilibria. Now, if we assume that $a > b$ and $(a-d) \leq (b-c)$ then $(\beta, \beta)$ is the risk-dominant equilibrium, while $(\alpha, \alpha)$ is the Pareto-dominant one. This simply means that players get a higher payoff by coordinating on $(\alpha, \alpha)$ but they risk less by using strategy $\beta$ instead. There is also a third equilibrium in mixed strategies but it is evolutionarily unstable. A well known example of games of this type are the so-called Stag-Hunt

|  | $\alpha$ | $\beta$ |
|---|---|---|
| $\alpha$ | $a, a$ | $c, d$ |
| $\beta$ | $d, c$ | $b, b$ |

Table F.1: A general two-person, two strategies coordination game.

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[1] a companion study on the dynamical network case is in progress.

games [109]. This class of games has been extensively studied analytically in an evolutionary setting [51, 26] and by numerical simulation on several model network types [109, 104, 60, 96]. In the following, we shall first deal with the easier case of pure coordination games which, in spite of their simplicity, already clearly pose the equilibrium selection problem. Then we shall report results on Stag-Hunt games, for which there exist many published studies to compare with, both theoretical and with the use of simulation.

### F.2.2 Pure Coordination Games

Two-person *pure coordination games* have the normal form depicted in Table F.2, with $u_i, u_i > 0$, and $u_i, u_j = 0, 0, i \neq j, \forall i, j \in [1, k]$, where $k$ is the number of strategies available to each player in the strategy set $S = \{s_1, s_2, ..., s_k\}$, and the $u$'s are payoffs. So all the Nash equilibria in pure strategies correspond to diagonal elements in the table where the two players coordinate on the same strategy, while there is a common lower uniform payoff for all other strategy pairs which is set to 0 here. A simple coordination game is the *driving game*. In some countries people drive on the right side of the road, while in others they drive on the left side. This can be represented by the pure coordination game represented in Table F.3. There are two Nash equilibria in pure strategies: (right, right) and (left, left) and obviously there is no reason, in principle, to prefer one over the other, i.e. the two equilibria are equivalent. However, while some countries have got accustomed to drive on the left such as the UK, Australia, and Japan, others have done the opposite such as most European countries and the USA. Such *norms* or *conventions* have stabilized in time and are often the product of social evolution. There is of course a third equilibrium in mixed strategies in the driving game which consists in playing left and right with probability 1/2 each but it would seem rather risky to play the game in this way on a real road. Another well known example of a pure coordination game is the Battle of the Sexes in which the Nash equilibria in pure strategies are those in which players use the same strategy, but the two sides in a two person game prefer a different equilibrium [122].

| | $s_1$ | $s_2$ | ... | $s_k$ |
|---|---|---|---|---|
| $s_1$ | $\mathbf{u_1}, \mathbf{u_1}$ | $0, 0$ | ... | $0, 0$ |
| $s_2$ | $0, 0$ | $\mathbf{u_2}, \mathbf{u_2}$ | ... | $0, 0$ |
| ... | ... | ... | ... | ... |
| $s_k$ | $0, 0$ | ... | ... | $\mathbf{u_k}, \mathbf{u_k}$ |

Table F.2: A general payoff bi-matrix of a two-person pure coordination game. Nash equilibria in pure strategies are marked in bold.

| | right | left |
|---|---|---|
| right | $\mathbf{1}, \mathbf{1}$ | $0, 0$ |
| left | $0, 0$ | $\mathbf{1}, \mathbf{1}$ |

Table F.3: The driving game.

## F.3 Mathematical Setting and Previous Results

In this section, we recall some rigorous results for two-person, two-strategies coordination games on some particular network types. Indeed, network topology has an influence on the stable states of the evolutionary dynamics that will be reached, as it will become clear in what follows. We also give nomenclature and definitions for the graphs representing the population of agents and for the dynamical decision processes implemented by the agents.

Let's thus consider the game's payoff matrix of Table F.4 with $a \geq b > 0$. When $a > b$, strategy $\alpha$ is said to be dominant since a player obtains a higher payoff playing $\alpha$ rather than $\beta$.

| | $\alpha$ | $\beta$ |
|---|---|---|
| $\alpha$ | $a, a$ | $0, 0$ |
| $\beta$ | $0, 0$ | $b, b$ |

Table F.4: A general two-person, two-strategies pure coordination game.

The network of agents will be represented by an undirected graph $G(V, E)$, where the set of vertices $V$ represents the agents, while the set of edges (or links) $E$ represents their symmetric interactions. The population size $N$ is the cardinality of $V$. A neighbor of an agent $i$ is any other agent $j$ at distance one from $i$. The set of neighbors of $i$ is called $V_i$ and its cardinality is the degree $k_i$ of vertex $i \in V$. The average degree of the network is called $\bar{k}$ and $p(k)$ denotes its degree distribution function, i.e. the probability that an arbitrarily chosen node has degree $k$.

Since we shall adopt an evolutionary approach, we must next define the decision rule by which individuals will update their strategy during time. An easy and well known adaptive learning rule is *myopic best-response dynamics*, which embodies a primitive form of bounded rationality and for which rigorous results are known [129, 36]. In the local version of this model, time is discrete i.e. $t = 0, 1, 2, \ldots$ and, at each time step, an agent has the opportunity of revising her current strategy. She does so by considering the current actions of her neighbors and switching to the action that would maximize her payoff if the neighbors would stick to their current choices. The model is thus completely local and an agent only needs to know her own current strategy, the game payoff matrix, who are her neighbors, and their current strategies. This rule is called myopic because the agents only care about immediate payoff, they cannot see far into the future. Given the network structure of the population, the rule is implemented as follows:

- at each time step a player $i$ revises his strategy with probability $p$
- player $i$ will choose the action that maximizes his payoff, given that the strategy profile of his neighbors $V_i$ remains the same as in the previous period
- if there is a tie or $i$ is not given the opportunity of revising his strategy, then $i$ will keep his current strategy

Using the above kind of stochastic evolutionary process, which can be modeled by a Markov chain, the following theoretical results have been proved by several researchers and can be found in Chapter 4 of [36], where references to the original works are given. They are valid for general coordination games, and thus also for the special case of the pure coordination game of Table F.4.

**Theorem.** A strategy profile in which everyone plays the same action is a Nash equilibrium for every graph $G$. If $G$ is complete then these are the only possible equilibria. If $G$ is incomplete, then there may exist polymorphic equilibria as well.

The preceding theorem implies that social diversity may emerge at equilibrium depending on the network structure. Given that complete networks are not socially relevant, this result leaves open the possibility of equilibrium strategy distributions in the population. A second related result states that, starting from any initial strategy profile, the above described stochastic process will converge to a Nash equilibrium of the coordination game with probability 1. To probe for the stability of equilibria, the concept of mutation is introduced. A mutation simply means that a player that is updating its current strategy can make a mistake with some small probability $q$. These small random effects are meant to capture various sources of uncertainty such as deliberate and involuntary decision errors. Deliberate errors might play the role of experimentation in the environment, and involuntary ones might be linked with insufficient familiarity with the game, for example. A state of this adaptive noisy dynamics is called *stochastically stable* if in the long term, the probability of being in that state does not go to tero as the error probability tends to zero (see [129] for a rigorous definition). This idea allows one to discriminate among the possible equilibria according to their stability properties.

From the above considerations, it may be concluded that the network topology plays an important role on the equilibrium states that the population will reach in the long run. However, the graph types for which analytical results are available are far from the complex structures of observed real social networks. Therefore, our aim in the following is to characterize the behavior of such complex networks by using numerical simulations and appropriate statistical analysis.

## F.4 Numerical Simulations Methodology

### F.4.1 Network Types Studied

In the last few years a large amount of knowledge has accumulated about the structure of real social networks and many model networks, both static and growing have been proposed [75, 19, 49]. We are thus in a position that allows us to make use of this recent information in order to study the behavior of coordination games on such realistic networks. In detail, we shall use the following network types: random, Barábasi-Albert scale-free networks, a real social network, and model social networks. We shall now briefly describe each of these network types, directing the reader to the relevant references for more details.

### random graphs

For generating random graphs we use one of the classical models proposed by Erdös and Rényi and described in [15]. Given $N$ indistinguishable vertices, each possible edge has an independent probability $p$ of appearing ($0 \leq p \leq 1$), which gives the $G(N, p)$ ensemble of random graphs. It is worth mentioning that for that type of random graph the average clustering coefficient[2] $\bar{C} = p = \bar{k}/N$. Thus $\bar{C}$ at fixed $\bar{k}$ tends to 0 for increasing $N$. This is one of the reasons that make these random graphs rather unsuitable as model social networks, although they are useful as a known benchmark to evaluate deviations from randomness. Furthermore, $p(k) = e^{-\bar{k}} \, \frac{\bar{k}^k}{k!}$ is Poissonian and thus it allows only small fluctuations around $\bar{k}$, while actual measured networks usually have long-tailed degree distribution functions.

### Scale-Free graphs

Among the several available models for constructing scale-free networks [75], here we use the classical one by Barabási–Albert [3]. Barabási–Albert networks are grown incrementally starting with a small clique of $m_0$ nodes. At each successive time step a new node is added such that its $m \leq m_0$ edges link it to $m$ nodes already present in the graph. It is assumed that the probability $p$ that a new node will be connected to node $i$ depends on the current degree $k_i$ of the latter. This is called the *preferential attachment* rule. The probability $p(k_i)$ of node $i$ to be chosen is given by $p(k_i) = k_i / \sum_j k_j$, where the sum is over all nodes already in the graph. The model evolves into a stationary network with power-law probability distribution for the vertex degree $P(k) \sim k^{-\gamma}$, with $\gamma \sim 3$. For the simulations, we started with a clique of $m_0 = 2$ nodes and at each time step the new incoming node has $m = 2$ links.

Scale-free graphs are rather extreme and are infrequent among social networks (see below), even taking finite degree cutoffs into account. As the random graph, they are rather to be considered as a model network.

### An Actual Social Network

One important reason for introducing true or model social networks is that, as said above, clustering is an important feature in networks of contacts while neither Erdös-Rényi nor Barabási-Albert scale-free graphs show a comparable level of clustering. As a typical example of a true social network, we use a coauthorship network among researchers in the genetic programming (GP) community. This network has a connected giant component of 1024 scientists and it has recently been analyzed [117]. It has clusters and communities and it should be representative of other similar human acquaintance networks. Its degree distribution function $p(k)$, as is usually the case with most measured social networks [4, 75, 49], is not a pure power-law; rather, it can be fitted by an exponentially truncated power-law.

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[2] We use the following common definition. The clustering coefficient $C_i$ of a node $i$ is defined as $C_i = 2E_i/k_i(k_i - 1)$, where $E_i$ is the number of edges in the neighborhood of $i$. Thus $C_i$ measures the amount of "cliquishness" of the neighborhood of node $i$ and it characterizes the extent to which nodes adjacent to node $i$ are connected to each other. The clustering coefficient of the graph is the average over all nodes: $\bar{C} = \frac{1}{N} \sum_{i=1}^{N} C_i$ [75]

### Model Social Networks

Several ways have been proposed for growing artificial networks with properties similar to those of observed social networks. Here we use the model of Toivonen et al. [114], which was conceived to construct a graph with most of the desired features of real-life social networks i.e, assortative, highly clustered, showing community structures, having an adjustable decay rate of the degree distribution, and a finite cutoff. The network is incrementally grown starting from a seed of $m_0$ randomly connected vertices. At each successive time step, the following algorithm is applied:

1. On average $m_r \geq 1$ random vertices are picked to be initial contacts.
2. On average $m_s \geq 0$ neighbors of the $m_r$ initial contacts are chosen to be secondary contacts.
3. A newly added vertex $v$ is connected to all the initial and secondary contacts determined in the two previous steps.

The above is iterated until the network reaches the desired size. Notice that the process responsible for the appearance of high clustering, assortativity and community structure is step 2. In the numerical experiments, we used graphs of size $N = 1000$ with $m_0 = 30$ initial nodes. Every time a new node is added, its number of initial contacts $m_r$ is distributed as $p(\#\text{ of initial contacts} = 1) = 0.95$ and $p(\#\text{ of initial contacts} = 2) = 0.05$. The number of its secondary contacts $m_s$ is uniformly distributed between 0 and 3. The resulting degree distribution falls below a power-law for high values of $k$ [114].

## F.4.2 Simulations Settings

The network used are of size $N = 1000$ except for the GP network, whose giant component has size 1024. The mean degree $\bar{k}$ of the networks generated was 6, except for the GP case which has $\bar{k} \simeq 5.8$.

For pure coordination games the non-zero diagonal payoffs $a$ (see sect. F.2.2) has been varied in the range $[0.5, 1]$ in steps of 0.05 with $b = 1 - a$; the range $[0, 0.5]$ is symmetrically equivalent. For general coordination games (sect. F.2) in which $a > d > b > c$, we have studied a portion of the parameters' space defined by $c \in [-1, 0]$ and $d \in [0, 1]$, $a = 1$, and $b = 0$, as is usually done for the stag-hunt games [104, 96]. The $c - d$ plane has been sampled with a grid step of 0.05. Each value in the phase space reported in the following figures is the average of 50 independent runs. Each run has been performed on a fresh realization of the corresponding graph, except for the GP co-authorship network case which is a unique realization.

As already hinted in sect. F.3, we have used a fully asynchronous update scheme in which a randomly selected agent is chosen for update with replacement at each discrete time step. To detect steady states[3] of the dynamics we first let the system evolve for a transient period of

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[3] True equilibrium states in the sense of stochastic stability are not guaranteed to be reached by the simulated dynamics. For this reason we prefer to use the terms steady states or quasi-equilibrium states which are states that have little or no fluctuation over an extended period of time.

$5000 \times N \simeq 5 \times 10^6$ time steps. After a quasi-equilibrium state is reached past the transient, averages are calculated during $500 \times N$ additional time steps. A steady state has always been reached in all simulations performed within the prescribed amount of time, for most of them well before the limit.

We have experimented with different proportions of uniformly randomly distributed initial strategies $\alpha$ belonging to the set $\{0, 0.05, 0.25, 0.5, 0.75, 0.95, 1\}$ and we have used two different values for the stochastic noise $q$ in the simulations: $q \in \{0, 0.02\}$, i.e. either no noise or a small amount, as prescribed by the most important theoretical stochastic models in order to ensure that the evolutionary process is ergodic [51, 26, 129].

## F.5 Simulation Results

### F.5.1 Results on Pure Coordination Games

Figures F.1 and F.2 show global coordination results for random graphs and scale-free graphs respectively. The plots report on the x-axis the payoff advantage of strategy $\alpha$ with respect to strategy $\beta$, which goes from 0 to 1, and on the y-axis the frequency of $\alpha$-strategists in the population. The curves represent average values over 50 runs for each sampled point. By simple inspection, it is clear that results do not differ by a large extent between the random and the scale-free cases, which means that the degree distribution function has little effect on the outcome. The general trend is for all the populations to converge toward the payoff-dominant Nash equilibrium in pure strategies which is also the case for the standard well-mixed population, as we know from analytical results. Polymorphic populations do exist temporarily but they are unstable and the stochastic dynamics always reaches a monomorphic state. It is also quite obvious that without mutations (Figs. F.1 and F.2 left-hand images), if a strategy is absent at the beginning, it cannot appear later. Instead, with even a small amount of noise ($q = 0.02$ in the figures), the strategy offering the best payoff will take over the population thanks to repeated mutations that will create individuals playing that strategy (Figs. F.1 and F.2 right-hand images) even in case the strategy is absent in the initial population. Furthermore, noise always promotes a quicker transition toward the payoff-dominant steady state.

Figures F.3 and F.4 depict the same quantities as above in the case of the real social network and model social networks respectively. Although the general behavior is the same, i.e. the Pareto-dominant steady state is reached in most situations, some aspects of the dynamics differ from the case of random and scale-free networks. To begin with, one sees on the left-hand images that, without noise, the payoff dominated strategy is able to resist in the population when the payoff differences are small. For example, starting with an equal initial share of strategies $\alpha$ and $\beta$, one sees in Figs. F.3 and F.4 that, up to a difference in payoffs of 0.02 the Pareto-dominated strategy is still present in the population with a sizable fraction. This phenomenon can be explained by looking at the clusters present in the social networks. Results will be presented below.

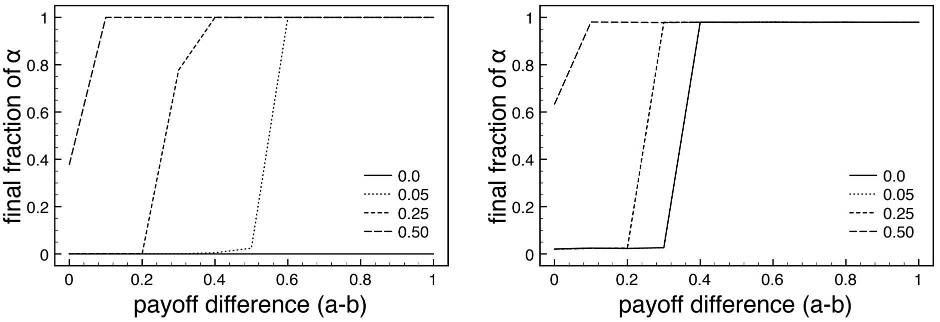

Figure F.1: Random network: $\bar{k} = 6$. Left image refers to noiseless best response dynamics. The right image is for a noisy dynamics with $q = 0.02$. Graphics report the frequency of strategy $\alpha$ in the population as a function of the payoff difference $a - b$. Continuous lines are just a guide for the eye.

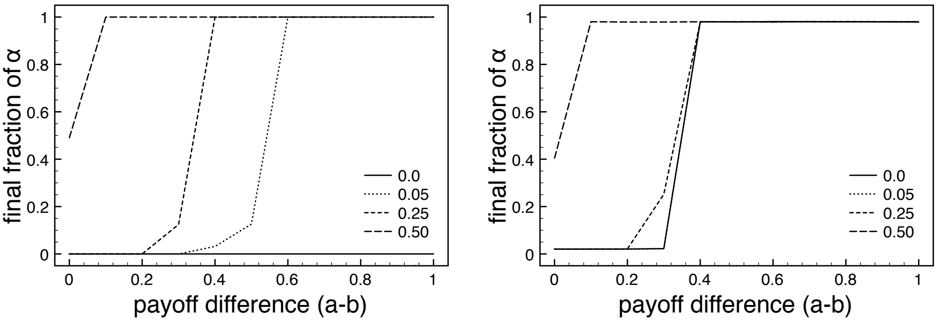

Figure F.2: Scale-free network: Barabasi-Albert Model, $\bar{k} = 6$. Left image refers to noiseless best response dynamics. In the right image the probability of mutation is $q = 0.02$. On the y-axis the frequency of strategy $\alpha$ is plotted against the payoff difference $a - b$.

But the main remark is that, in the presence of noise, the payoff-dominant stable state is reached for smaller differences in payoff (see right-hand images). In other words, a small $a - b$ advantage is enough to quickly steer the dynamics towards the dominant quasi-equilibrium. The behavior is sufficiently different from the previous one to require at least a qualitative explanation, which is presented next by introducing the concept of communities.

### F.5.2 Social Communities and Game Strategies

Communities or clusters in networks can be loosely defined as being groups of nodes that are strongly connected between them and poorly connected with the rest of the graph. These structures are extremely important in social networks and may determine to a large extent the properties of dynamical processes such as diffusion, search, and rumor spreading among others. Several methods have been proposed to uncover the clusters present in a network (for a review see, for instance, [28]). In order to study the effect of community structure on the distribution of behaviors at steady state, here we have used the divisive method of Girvan and Newman [79]

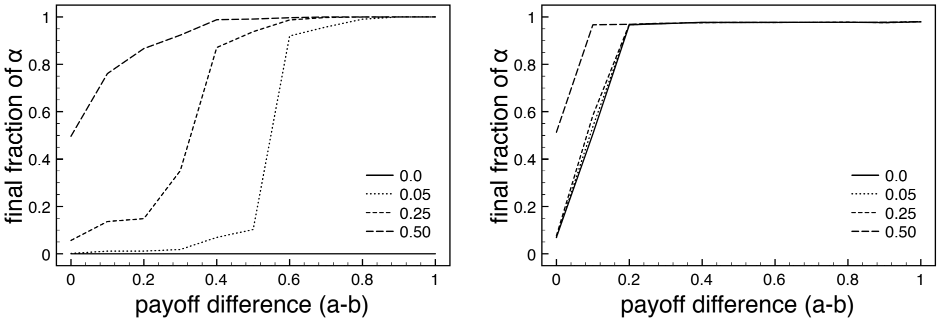

Figure F.3: Coauthorship network in the Genetic Programming community. Left image: no noise. Right image: mutation probability $q = 0.02$. On the y-axis we report the fraction of $\alpha$-strategists in the population as a function of the payoff difference $a - b$.

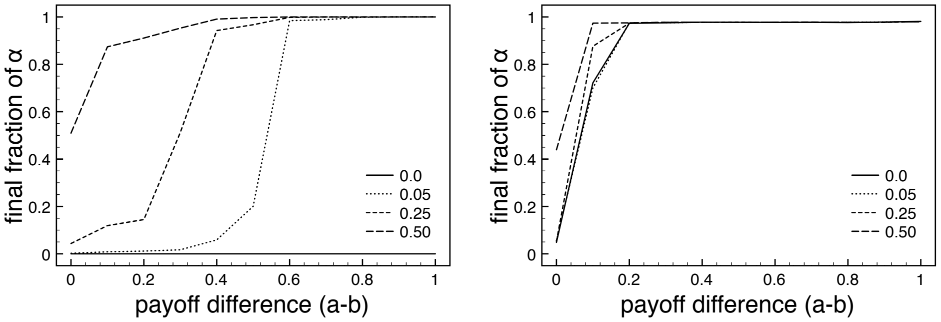

Figure F.4: Model social network. Left: mutation probability $q = 0$; right: $q = 0.02$. On the y-axis we report the fraction of $\alpha$-strategists in the population as a function of the payoff difference $a - b$.

which is based on iteratively removing edges with a high value of edge betweennes.

The presence of communities has a marked effect on the game dynamics. Figure F.5 depicts the community structure of a Barabási–Albert scale-free graph (a) and of a model social network built according to Toivonen et al's model (b). The difference is striking: while clear-cut clusters exist in (b), almost no recognizable communities can be isolated in (a), a fact that is shown by the high number of links between clusters, with a communities graph average degree of $\sim 32$, while $\bar{k}$ is about 6.5 for the communities graphs arising from social networks. A common statistical indicator of the presence of a recognizable community structure is the *modularity* $Q$. According to Newman [77], where quantitative definitions are given, modularity is proportional to the number of edges falling within clusters minus the expected number in an equivalent network with edges placed at random. While modularity is not without flaws [37], it is still a convenient indicator of the presence of clusters. In general, networks with strong community structure tend to have values of $Q$ in the range $0.4 - 0.7$. Indeed, for the networks in Fig. F.5, we have $Q \simeq 0.3$ for the scale-free network, while $Q \simeq 0.6$ for the model social network. Colors in the figure represent frequency of strategies at steady state for a single particular, but representative,

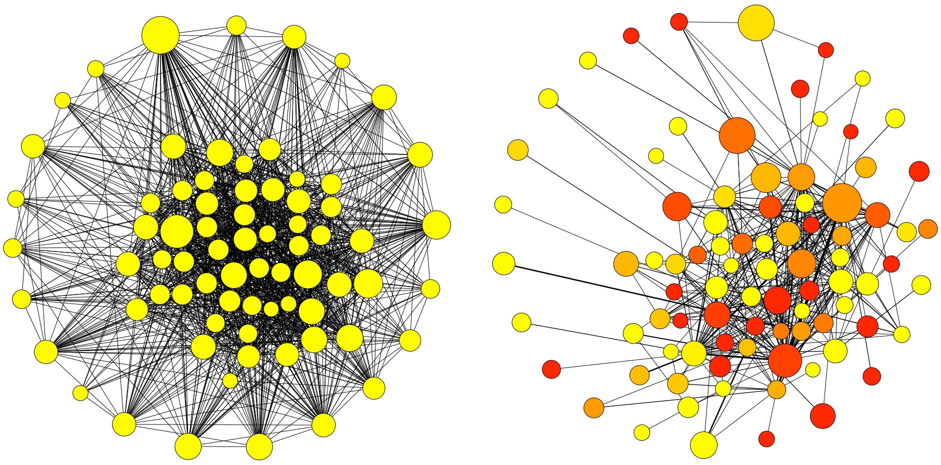
(a) (b)

Figure F.5: Distribution of strategies at steady state in the network communities when both strategies share the same payoff: $a = b = 0.5$. (a) scale-free, frequency of $\alpha = 0.568$. (b) model social network, fraction of $\alpha = 0.585$. Each vertex represents a whole community with size proportional to the size of the community. Links represent inter-community connections and their thickness is proportional to the number of inter-community links. The communities are much less interconnected in the social network and this causes a greater difference in concentration from community to community.

run in each case. In the average over 50 runs, final proportions of strategies $\alpha$ and $\beta$ do not depart much from the initial 50%. However, while in the scale-free case at the steady state the standard deviation is high, meaning that the system converges often to one or the other equilibrium, this is not the case for the social networks. In the latter, at steady state there is always a mix of strategies; in other words, polymorphic equilibria may be stable. This is a remarkable fact that is due to the community structure of social networks, which is almost missing in the scale-free and random network cases. Thanks to this clear-cut cluster structure, as soon as the nodes of a cluster are colonized by a majority of one of the two strategies by statistical fluctuation, it becomes difficult for the other strategy to overtake, which explains why these cluster strategies are robust. The effect of the community structure is even more apparent in Fig. F.6 where strategy $\alpha$ has been given a slight initial advantage. At steady state, in both the co-authorship network (a) as well as the model network (b) strategy $\beta$ is still present in some clusters. If we were to interpret strategies as social norms or conventions, then this would suggest that a realistic social structure may help protect diversity, either political or cultural, for example. The possibility of polymorphic equilibria had been theoretically predicted by Morris [70] for symmetric payoffs in pure coordination games with best response dynamics in the case of infinite populations and making use of a notion of "cohesion" which refers to the relative frequency of ties among groups compared with non-members. Clearly, although it was expressed in a different language that does not make explicit use of networks, this notion is related to the communities we have here and the simulation results nicely confirm the prediction in the case of finite, actual networked systems.

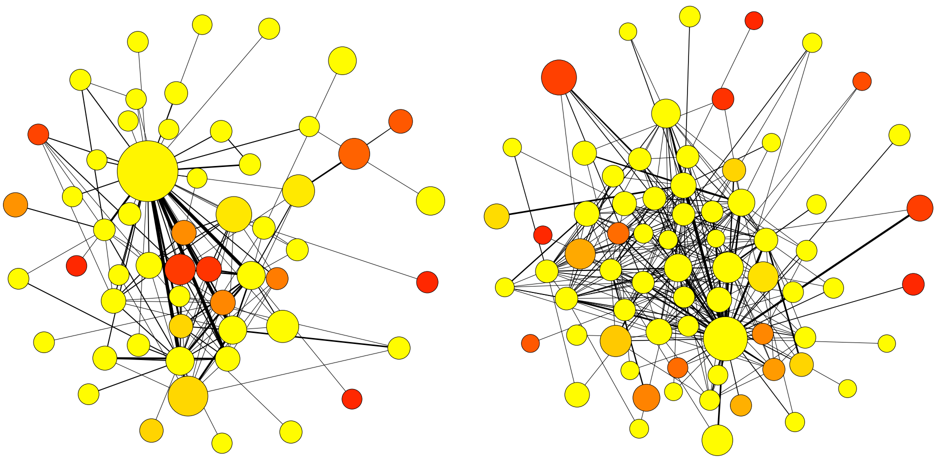
(a) (b)

Figure F.6: Strategy distribution in the network communities when $\alpha$ has a small advantage over $\beta$: $a = 0.55$. (a) Genetic Programming co-authorship network, proportion of $\alpha = 0.839$. (b) model social network, proportion of $\alpha = 0.833$. The cluster structure of these networks allows the preservation of the dominated strategy in some communities.

### F.5.3 Results on the Stag Hunt Games

Figure F.7 shows strategy distribution on the game parameter space for the Stag Hunt class of coordination games for the scale-free case. Results for random graphs are similar to those for scale-free networks and are not shown. The two upper images are for equal initial proportions of each strategy, while the bottom figures refer to an initial proportion of strategy $\alpha = 5\%$. The first image in each row is for best response without noise, while the second image has noise level $q = 0.02$.

For initially equidistributed strategies, although average values are reported in the figures, almost all simulations attain one or the other absorbing state, i.e. all individuals play $\alpha$ or all play $\beta$, and there is almost no difference when noise is present. This is in agreement with previous results on scale-free graphs published by Roca et al. [97] where update was by best response without noise, and also with [60] where replicator dynamics instead of best response dynamics was used as a strategy update rule.

For the more extreme case in which initially the fraction of strategy $\alpha$ is 5% randomly distributed over the graph vertices (bottom row images), a small amount of random noise does not have a large effect: the cooperative strategy emerges in the favorable region of the parameter space, i.e. for low $d$ and high $c$ (upper left corner) in both cases. However, the presence of noise enhances the efficient coordination region. Indeed, even when strategy $\alpha$ is initially absent, once it is created by mutation, it spreads as in the 5% case. It is to be noted that the same phenomenon happens when the minority strategy is $\beta = 0.05$; in this case the images are specularly symmetrical, and

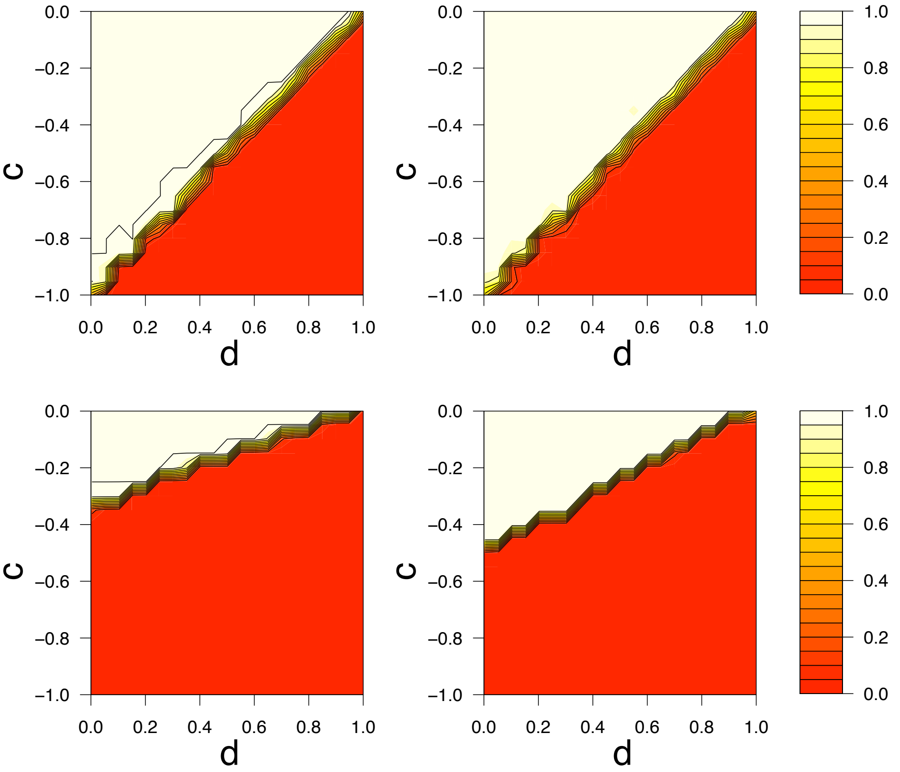

Figure F.7: Distribution of strategies proportions at steady state on the $d-c$ plane for scale-free networks. Each sampled point is the average of 50 independent runs. The upper images are for equal initial proportions of each strategy. In the lower figures the initial proportion of randomly distributed $\alpha$-strategists is 5%. Figures on the left column are for best response dynamics without noise, while those on the right column represent a situation in which the probability of mutation $q = 0.02$. Darker colors mean that risk-dominance prevails; light color design the region where payoff-dominance prevails.

with colors reversed, with respect to the main diagonal, except for sampling differences (not shown to save space).

Figure F.8 depicts average results for the model social network case of Toivonen et al. [114]. Results for the collaboration network are very close to those of model social networks. For this reason, and in order not to clutter the graphics too much, we do not show them. It is immediately apparent that the case in which strategies are initially randomly distributed in equal amounts seems similar to the scale-free results. However, looking more carefully, the average results shown in the figures hide to some extent the fact that now many simulations do not end in one of the monomorphic population states, but rather there is a mix of the two strategies, when noise is absent. This is visible in the upper left figure in the less crisp frontier along the diagonal which is due to a more gradual transition between phase space regions. However, when a small amount of noise is present (upper right image) the transition is again sharp and the dynamics usually leads to a monomorphic population in which one of the two absorbing states is entered. The reason why there can be mixed states in the noiseless case in social networks is related to

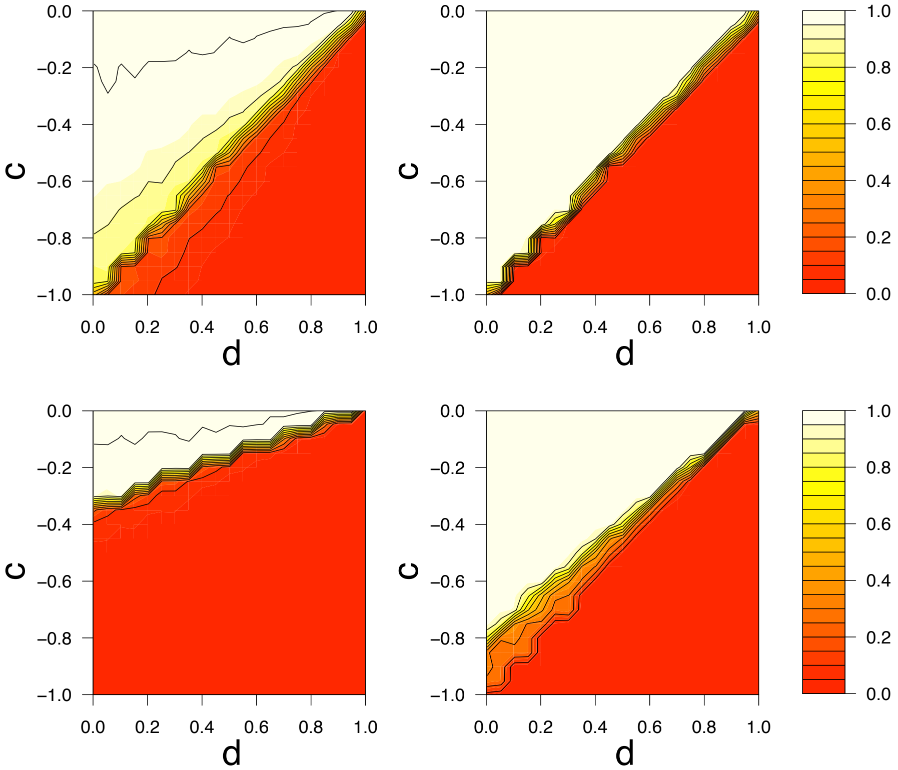

Figure F.8: Distribution of strategies proportions at steady state on the $d-c$ plane for model social networks. Each sampled point is the average of 50 independent runs. The upper images are for equal initial proportions of each strategy. In the lower figures the initial proportion of randomly distributed $\alpha$-strategists is 5%. Figures on the left column are for best response dynamics without noise, while those on the right column represent a situation in which the probability of mutation $q = 0.02$. Darker colors mean that risk-dominance prevails; light color design the region where payoff-dominance prevails.

their mesoscopic structure. As we have seen in sect. F.5.2, model and real social networks can be partitioned into recognizable clusters. Within these clusters strategies may become dominant as in the pure coordination case just by chance. In other words, as soon as a strategy dominates in a given cluster, it is difficult to eradicate it from outside since other communities, being weakly connected, have little influence. This kind of effect in the Stag Hunt game has been observed previously in simulations on grid-structured populations [109, 97]. However, grid structures are not socially realistic; thus, the fact that more likely social structure do support efficient outcomes is an encouraging result. However, when noise is present, there is always the possibility that the other strategy appears in the cluster by statistical fluctuations and, from there, it can takeover the whole community. To end this section, we remark that analogous effects due to the presence of clusters in social networks have been observed and interpreted in the Prisoner's Dilemma game in [58, 60].

We now briefly comment on the relationship between our numerical results and well known theoretical results on Stag-Hunt games. These theoretical models are based on ergodic stochastic processes in large populations and state that, when using best-response dynamics in random two-person encounters, and in the presence of a little amount of noise, both for well-mixed populations as well as for populations structured as rings, the risk-dominant strategy should take over the population in the long run [51, 26, 129]. From our simulation results on all kind of networks this is not the case; in other words, at the steady state there is always either a single strategy, but not necessarily the risk-dominant one, or a mix of both strategies. For scale-free and random graphs, the numerical results of [97] agree with ours. The case of social networks, presented here for the first time, also confirms the above and in addition makes explicit the role played by communities. We may also mention at this point that, for the Stag-Hunt, the presence of a local interaction structure provided by a network has been shown to increase the region of the phase space in which the Pareto-dominant outcome prevails for other strategy update rules, such as imitate the most successful neighbor or reproduce proportionally to fitness (replicator dynamics) [109, 96]. Thus coordination is sensitive to the exact type of underlying dynamics in networks. This is indirectly confirmed by the theoretical study of Robson and Vega-Redondo [95] in which a different matching model is used with respect to Kandori et al [51]. In [95] players are immediately randomly rematched after each encounter and the result is that the Pareto-dominant equilibrium is selected instead.

In summary, it can be said that network effects tend to reinforce cooperation on the Pareto-dominant case, which is a socially appreciable effect. However, these results must be taken with a grain of salt. We are numerically studying finite, network-structured populations during a limited amount of time, while theoretical results have been established for large well mixed populations in the very long run. As for the finite-size population, we have performed many simulations with larger ($N = 2000$) and smaller (down to $N = 100$) networks and the results do not change significantly (not shown to save space). However, in the end, the conditions of the numerical experiments are sufficiently different to conclude that numerical results and theoretical predictions based on different assumptions do not have to agree necessarily.

## F.6 Summary and Conclusions

In this work we have studied pure and general coordination games on complex networks by numerical simulation. Situations described by coordination games are common in society and it is important to understand when and how coordination on socially efficient outcomes can be achieved.

In the case of pure coordination games on model networks using deterministic best response strategy dynamics we have found that network effects are small or non-existent in standard complex networks. On model social networks and a real co-authorship network the behavior is similar, but the transition from one convention to the other is smoother and the cluster structure of the networks plays an important role in protecting payoff-weaker conventions within communities and this leads to a clear polarization of conventions in the network. When a small amount of noise is added in order to simulate errors and trembles in the agent's decisions, the dynamics leads to the payoff-dominant norm for smaller values of the payoff difference between strategies. However, in the case of social networks, even a tiny amount of payoff advantage is enough to drive a minority of $\alpha$-strategists to take over the whole network thanks to the cluster structure and mutations.

In the case of general coordination games of the Stug Hunt type where there is a tension between payoff-dominance and risk-dominance, we have confirmed previous simulation results in the sense that, with deterministic best response dynamics the influence of network structure is very limited [109, 97, 96, 60]. On the other hand, when we consider model and social networks, again their community structure plays an important role which consists in allowing the existence at steady state of dimorphic populations in which both strategies are present and stable. The payoff-dominant strategy is favored in regions where risk-dominance should be the only stable strategy and, conversely, it allows risk-dominant players to survive in clusters when payoff-dominance should prevail.

We have also compared numerical results with theoretical ones when they exist. The latter actually depend on the detailed structure of the stochastic processes generated by the particular theoretical model. In this sense, numerical results are compatible with theoretical predictions when they are applicable, i.e. for well mixed and ring-structured populations [51, 95, 26]. Also, for pure coordination games the predictions of [70] in arbitrary non-homogeneous structures are qualitatively confirmed. However, finite-size and complex network effects are difficult to describe theoretically and thus our results on complex and social networks cannot always be easily compared with theoretical predictions. Our current and future work is to investigate coordination games in a more realistic co-evolutionary scenario in which both the agents' strategies as well as their interactions may vary dynamically.

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# Article G

# Coordination Games on Dynamical Networks

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### Abstract

We propose a model in which agents of a population interacting according to a network of contacts play games of coordination with each other and can also dynamically break and redirect links to neighbors if they are unsatisfied. As a result, there is co-evolution of strategies in the population and of the graph that represents the network of contacts.

We apply the model to the class of pure and general coordination games. For pure coordination games, the networks co-evolve towards the polarization of different strategies. In the case of general coordination games our results show that the possibility of refusing neighbors and choosing different partners increases the success rate of the Pareto-dominant equilibrium.

## G.1 Introduction

The purpose of Game Theory [122] is to describe situations in which two or more agents or entities may pursue different views about what is to be considered best by each of them. In other words, Game Theory, or at least the non-cooperative part of it, strives to describe what the agents' rational decisions should be in such conflicting situations. For example, games such as the well known Prisoner's Dilemma have been heavily used in order to represent the tension that appears in society when individual objectives are in conflict with socially desirable outcomes.

Indeed, a large part of the research literature has focused on conflicting situations in order to uncover the mechanisms that could lead to cooperation instead of socially harmful outcomes (see e.g. [82] for a synthesis). However, there are important situations in society that do not require players to use aggressive strategies. In fact, many frequent social and economic activities require individuals to coordinate their actions on a common goal since in many cases the best course of action is to conform to the standard behavior. For example, if someone's native language is French and she travels to an English-speaking country, it pays off to follow the local norm, i.e. to speak English instead of French. Games that express this extremely common kind of interactions are called *coordination games*.

Coordination games confront the players with multiple Nash equilibria and the consequent problem of how to choose among them. A useful approach has been to turn to evolutionary and learning ideas which offer a dynamical perspective based on the forces of biological and social evolution. In *evolutionary game theory* (EGT), the concept of a population of players where strategies that score best are more likely to be selected and reproduced provides a justification for the appearance of stable states of the dynamics that represent solutions of the game [122, 30]. For mathematical convenience, standard EGT is based on infinite mixing populations where pairs of individuals are drawn uniformly at random at each step and play the game. Correlations are absent by definition and the population has an homogeneous structure. However, everyday observation tells us that in animal and human societies, individuals usually tend to interact more often with some specified subset of partners; for instance, teenagers tend to adopt the fashions of their close friends group; closely connected groups usually follow the same religion, and so on. Likewise, in the economic world, a group of firms might be directly connected because they share capital, technology, or otherwise interact in some way. In short, social interaction is mediated by networks, in which vertices identify people, firms etc., and edges identify some kind of relation between the concerned vertices such as friendship, collaboration, and economic exchange. Thus, locality of interaction plays an important role. This kind of approach was pioneered in EGT by Nowak and May [84] by using simple two-dimensional regular grids. Recently, in the wake of a surge of activity in network research in many fields [75], the dynamical and evolutionary behavior of games on networks that are more likely to represent actual social interactions than regular grids has been investigated (see [112] for a comprehensive recent review). These studies have shown that there are network structures, such as scale-free and actual social networks that may favor the emergence of cooperation with respect to the fully mixing populations used in the theory [101, 60]. Most studies have focused on conflicting games but some work has also been done on games of the coordination type [109, 60, 96].

However, the above approach assumes a static point of view, i.e. it takes the network of contacts as being fixed once and for all and investigates the evolution of the agents' strategies over time. In other words, it is as if we took a snapshot of a given network at a given time and used this situation during all future times. Actually, however, social networks are dynamical entities that change constantly: actors may join and leave networks at unpredictable times and they may accumulate and abandon ties over time. Using static networks is a useful first approximation however, especially for the cases where the rate of change of the network structure is slow with respect to the rate of change of individual's behaviors. This could be the case of long-term collaboration networks, friendship, or of biological networks that are the result of an extremely long and slow evolution. But in many cases this static picture does not fit the reality very well. If we think of social or pseudo-social networks such as e-mail exchanges, Facebook-like networks, rumor-spreading networks and a host of other similar structures, we see that that the evolution of the network of contacts itself can be quite rapid and plays an important role.

In the present work we study the co-evolution of agents' behavior and of the agents' ties in the network over time. For the sake of simplicity, we investigate constant-size systems, i.e. we start with a finite-size network of agents and allow agents to abandon and to create links among themselves but there will be no new agents joining the system, nor agents will be allowed to leave it. This is not what happens in actual social and technological networks, which all tend to grow with time and are actually non-equilibrium systems, but our "closed system" approximation is simpler to simulate and interpret, and will allow us to already draw significant conclusions. Our methodology is essentially computer simulation-based since complex networks inhomogeneity and correlations make standard mean-field methods not adequate.

Some previous work has been done on evolutionary games on dynamic networks essentially dealing with the Prisoner's Dilemma, e.g. [109, 59, 103, 131] and a recent review of these approaches has been written by Perc and Szolnoki [90]. The present study follows our own model described in [92, 119] which differs from other approaches in the way in which links between agents are represented and interpreted, as explained later. In these previous works we studied the antagonistic Hawk Doves game and the Prisoner's Dilemma with replicator dynamics, instead of the best response dynamics used here for coordination games.

We also note that in the last fifteen years economists have put forward a theory of strategic network formation, i.e. formal models of how utility-based link formation moves might be implemented in order to reach a Nash equilibrium for all the members of the network (see e.g. Jackson's book for a synthesis of this work [49]). Our approach is different from the above view of strategic network formation for two reasons. First, we use networks that are at least two orders of magnitude larger and, while the equilibrium predictions resulting from strategic considerations usually lead to very simple topological structures such as small cliques or stars, our large evolving networks show complex structure and behavior. Second, while in strategic network formation the evolution of the network is submitted to utility maximization on the part of the players, our linking moves are based on very simple forms of reinforcement learning. Only the decisions of players concerning their behavioral strategies are based on a formal game payoff matrix.

The paper is organized as follows. In the next section we present a brief introduction to the subject of coordination games, in order to make the work self-contained. Then we describe the dynamical network model in Sect. G.3. Next, in Sect. G.4 we present and discuss the simulation results for pure coordination games. This is followed by the results on general coordination games in Sect. G.5. Finally, in Sect. G.6 we give our conclusions.

## G.2 Coordination Games

### G.2.1 General Coordination Games

General two-person, two strategies coordination games have the normal form of Table G.1. With $a > d$ and $b > c$, $(\alpha, \alpha)$ and $(\beta, \beta)$ are both Nash equilibria. Now, if we assume that $a > b$ and $(a - d) \leq (b - c)$ then $(\beta, \beta)$ is the risk-dominant equilibrium, while $(\alpha, \alpha)$ is the Pareto-dominant one. This simply means that players get a higher payoff by coordinating on $(\alpha, \alpha)$ but they risk less by using strategy $\beta$ instead. There is also a third equilibrium in mixed strategies but it is evolutionarily unstable. A well known example of games of this type are the so-called Stag Hunt games [109]. This class of games has been extensively studied analytically in an evolutionary setting [51, 26] and by numerical simulation on several static model network types [109, 60, 96]. In the following, we shall first deal with the easier case of pure coordination games which, in spite of their simplicity, already clearly pose the equilibrium selection problem. Then we shall report results on Stag Hunt games which are more interesting in social terms as they pose a problem of "trust", since the socially efficient solution is more risky.

| | $\alpha$ | $\beta$ |
|---|---|---|
| $\alpha$ | $a, a$ | $c, d$ |
| $\beta$ | $d, c$ | $b, b$ |

Table G.1: A general two-person, two strategies coordination game.

### G.2.2 Pure Coordination Games

Two-person *pure coordination games* have the normal form depicted in Table G.2, with $u_i, u_i > 0$, and $u_i, u_j = 0, 0, i \neq j, \forall i, j \in [1, k]$, where $k$ is the number of strategies available to each player in the strategy set $S = \{s_1, s_2, ..., s_k\}$, and the $u$'s are payoffs. So all the Nash equilibria in pure strategies correspond to diagonal elements in the table where the two players coordinate on the same strategy, while there is a common lower uniform payoff for all other strategy pairs which is set to 0 here.

| | $s_1$ | $s_2$ | ... | $s_k$ |
|---|---|---|---|---|
| $s_1$ | $\mathbf{u_1}, \mathbf{u_1}$ | $0, 0$ | ... | $0, 0$ |
| $s_2$ | $0, 0$ | $\mathbf{u_2}, \mathbf{u_2}$ | ... | $0, 0$ |
| ... | ... | ... | ... | ... |
| $s_k$ | $0, 0$ | ... | ... | $\mathbf{u_k}, \mathbf{u_k}$ |

Table G.2: A general payoff bi-matrix of a two-person pure coordination game. Nash equilibria in pure strategies are marked in bold.

In this paper we shall consider two-person, two-strategies pure coordination games with the payoff matrix of Table G.3 with $a \geq b > 0$. When $a > b$, strategy $\alpha$ is said to be dominant since a player obtains a higher payoff playing $\alpha$ rather than $\beta$.

| | $\alpha$ | $\beta$ |
|---|---|---|
| $\alpha$ | $a, a$ | $0, 0$ |
| $\beta$ | $0, 0$ | $b, b$ |

Table G.3: A general two-person, two-strategies pure coordination game.

## G.3 Model Description

In this section we provide nomenclature and definitions for the graphs representing the population and for the dynamical decision processes implemented by the agents. The dynamical model has originally appeared in [92, 119]; it is summarized here to make the paper self-contained. The network of players is represented by a directed weighted graph $G(V, E)$, where the set of vertices $V$ represents the agents, while the set of oriented edges (or links) $E$ represents their unsymmetric weighted interactions. The population size $N$ is the cardinality of $V$. A *neighbor* of an agent $i$ is any other agent $j$ such that there is a pair of oriented edges $\bar{ij}$ and $\bar{ji} \in E$. The set of neighbors of $i$ is called $\mathcal{N}_i$. For network structure description purposes, we shall also use an unoriented version $G^{'}$ of $G$ having exactly the same set of vertices $V$ but only a single unoriented unweighted edge $ij$ between any pair of connected vertices $i$ and $j$ of $G$. For $G'$ we shall define the degree $k_i$ of vertex $i \in V$ as the number of neighbors of $i$. The average degree of the network $G^{'}$ will be called $\bar{k}$.

Each link in $G$ has a weight or "force" $f_{ij}$ that represents in an indirect way the "trust" player $i$ places in player $j$. This weight may take any value in $[0, 1]$ and its variation is dictated by the payoff earned by $i$ in each encounter with $j$. The detailed way in which weights evolve dynamically is explained below. We define a quantity $s_i$ called *satisfaction* of an agent $i$ as the mean weight of $i$'s links:

$$s_i = \frac{\sum_{j \in \mathcal{N}_i} f_{ij}}{k_i},$$

with $0 \leq s_i \leq 1$. The link strengths can be seen as a kind of accumulated "memory" of previous encounters. However, it must be distinguished from the memory used in iterated games, in which players "remember" a certain number of previous moves and can thus conform their future strategy on the analysis of those past encounters [122, 30]. Our interactions are strictly one-shot, i.e. players "forget" the strategies used by neighbors in previous rounds and cannot recognize their playing patterns over time. However, they do recognize neighbors in terms of the strengths of the links they maintain with them. It could also be useful to model progressive obsolescence of the $f_{ij}$ over time, i.e. a discount rate of their values but, for the sake of simplicity, we prefer not to consider this effect in a first step.

Since we shall adopt an evolutionary approach, we must next define the decision rule by which individuals update their strategy and their contacts during time. For the strategy update, an easy and well known adaptive learning rule is *myopic best-response dynamics*, which embodies a primitive form of bounded rationality and for which rigorous results for coordination games are known in well mixed populations [51, 95] and in fixed one-dimensional and two-dimensional lattices [26, 70].

In the local version of this model, time is discrete i.e. $t = 0, 1, 2, \ldots$ and, in each time step, an agent has the opportunity of revising her current strategy. She does so by considering the current actions of her neighbors and switching to the action that would maximize her payoff if the neighbors would stick to their current choices. This rule is called myopic because the agents only care about immediate payoff, they cannot see far into the future. The model is thus completely local and an agent only needs to know her own current strategy, the game payoff matrix, who are her neighbors, and their current strategies. Furthermore, the agent must be able to "ask" one of her neighbors to introduce to her one of his neighbors. Given the network structure of the population, the strategy update rule is implemented as follows:

- in each time step a randomly chosen[1] player $i$ revises her strategy with probability $p$
- player $i$ will choose the action that maximizes her payoff, given that the strategy profile of her neighbors $\mathcal{N}_i$ remains the same as in the previous period
- if there is a tie or $i$ is not given the opportunity of revising her strategy, then $i$ will keep her current strategy

Now we describe the dynamics of links. The active agent $i$ will, with probability $q$, attempt to dismiss an interaction with one of her neighbors, say $j$, selected proportionally to $1 - f_{ij}$, i.e. the higher $f_{ij}$, the lower the probability of the link being selected for rewiring. Likewise, the lower the satisfaction $s_i$ of agent $i$, the higher the probability of dismissing the $ij$ link. However, although $i$ may take the lead in the decision to dismiss a link, $j$ has some power in opposing herself. The idea is that, in real social situations, it is seldom possible to take unilateral decisions: often there is a cost associated, and we represent this hidden cost by a probability $1 - (f_{ij} + f_{ji})/2$ with which $j$ may refuse to be cut away. In other words, the link is less likely to be deleted when $j$'s trust in $i$, $f_{ji}$, is high. If the decision is finally taken to cut the link, $i$ attempts to create a new link with a neighbor $l$ of one of her neighbors $k \in \mathcal{N}_i \setminus \{j\}$, such that links $ik$ and $kl$ with high forces are probabilistically favored. Link $ji$ simply disappears, as relations in the weighted graph $G$ are always reciprocal, although the corresponding weights will, in general, be different. This process is schematically depicted in Fig. G.1. This process requires that agent $k$ "introduces" one of its neighbors, say $l$, to $i$. Obviously, this bias will cause the clustering coefficient of the network to increase over time due to the transitive closure it causes, i.e. triangles will be more likely in the evolving graph. The solution adopted here is inspired by the observation that, in social networks, links are usually created more easily between people who have a satisfactory mutual acquaintance than those who do not. If the new link already

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[1] The active agent is chosen with uniform probability and with replacement.

exists, the process is repeated with $l$'s neighbors. If this also fails, a new link between $i$ and a randomly chosen node is created. In all cases the new link is initialized with a strength of 0.5 in both directions.

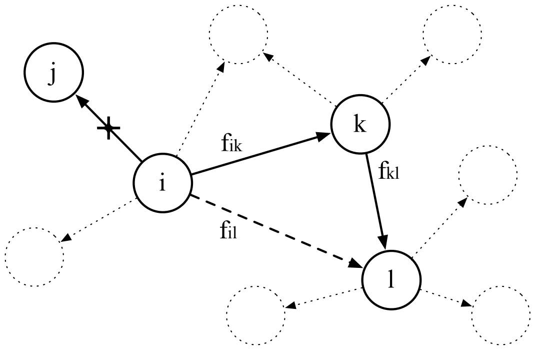

Figure G.1: Illustration of the rewiring of link $\{ij\}$ to $\{il\}$. Agent $k$ is chosen to introduce player $l$ to $i$ (see text). Between any pair of connected agents there are two directed links: only one of them is shown for clarity.

Once the agents have played with their neighbors, and have gone through their strategy or link update steps, the strengths of the links are updated in the following way:

$$f_{ij}(t+1) = f_{ij}(t) + \frac{\pi_{ij} - \bar{\pi}_{ij}}{k_i\left(\pi_{max} - \pi_{min}\right)},$$

where $\pi_{ij}$ is the payoff of $i$ when interacting with $j$, $\bar{\pi}_{ij}$ is the payoff that $i$ would have earned against $j$, if $j$ were to play his other strategy, and $\pi_{max}$ ($\pi_{min}$) is the maximal (minimal) possible payoff obtainable in a single interaction. $k_i = |\mathcal{N}_i|$ is the number of $i$'s neighbors, i.e. its degree. If $f_{ij}(t+1)$ falls outside the $[0,1]$ interval then it is reset to 0 if it is negative, and to 1 if it is larger than 1. This update is performed in both directions, i.e. both $f_{ij}$ and $f_{ji}$ are updated $\forall j \in \mathcal{N}_i$ because both $i$ and $j$ get a payoff out of their encounter.

In summary, calling $G_t$ the population graph at time $t$, where each node is labeled with its present strategy $s(t) \in \{\alpha, \beta\}$ and $G_t = (V, E)$ where $V$ is fixed but $E = E(t)$ evolves, the resulting stochastic process $\{G_0, G_1, \ldots\}$ is a Markov chain since the probability of state $G_t$ only depends on the previous step: $P(G_t|G_{t-1}, G_{t-2}, \ldots) = P(G_t|G_{t-1})$.

At this point, we should mention that Skyrms and coworkers [109, 111] have proposed a representation of agents' interaction based on evolvable interaction probabilities which is conceptually similar to ours, but the context is very different. Populations are small (ten players) and the network structure is never made explicit as the authors prefer to think in terms of probabilistic "encounters" instead of using the link concept. This approach allows Skyrms and coworkers to establish quantitative models based on stochastic processes for the simplest cases but it does not lead to an explicit description of the actual evolving networks. In addition, when players update their strategies, they have global knowledge of the strategy distribution of the whole population, while in our model this knowledge is strictly local.

## G.4 Simulation Results for Pure Coordination Games

### G.4.1 Simulation Settings

The constant size of the network during the simulations is $N = 1000$. The initial graph $G'_0$ is generated randomly with a mean degree $\bar{k} = 6$. The companion oriented graph $G_0$ is trivially built from $G'_0$ and forces between any pair of neighboring players are initialized at 0.5.

The non-zero diagonal payoff $a$ has been varied in the range $[0.5, 1]$ in steps of 0.05 with $b = 1 - a$; the range $[0, 0.5]$ is symmetrically equivalent. Each value in the phase space reported in the following figures is the average of 50 independent runs and each run has been performed on a fresh realization of the corresponding initial random graph.

To detect steady states of the dynamics [2], i.e. those states with little or no fluctuation over extended periods of time, we first let the system evolve for a transient period of $5000 \times N$ times steps ($= 5 \times 10^6$ time steps when $N = 1000$). After a quasi-equilibrium state is reached past the transient, averages are calculated during $500 \times N$ additional time steps. A steady state has always been reached in all simulations performed within the prescribed amount of time, for most of them well before the limit.
We have experimented with different proportions of uniformly randomly distributed initial strategies $\alpha$ belonging to the set $\{0.05, 0.25, 0.5, 0.75\}$.

### G.4.2 Discussion of Results

Figure G.2 reports the amount of $\alpha$-strategists in the population when a quasi-equilibrium state has been reached as a function of the rewiring frequency $q$. The upper light part of the plots indicate the region of the parameters space where the $\alpha$-strategists are able to completely take over the population. This can happen because $\alpha$ strategy offers the best payoff since $a - b$ is positive, therefore $\beta$-strategists are prone to adapt in order to improve their wealth. Figure G.2(a) shows the case where both $\alpha$ and $\beta$ strategies are present in the same ratio at the beginning of the simulation. The darker region indicates the situations where diversity is able to resist. This clearly happens when the payoff difference $a - b$ is zero. In this case both $\alpha$ and $\beta$ are winning strategies and the players tends to organize in two big clusters to minimize the links with the opposing faction. More surprisingly, even when one of the two strategies has a payoff advantage, the evolution of the topology of the interaction allows the less favorable strategy to resist. The faster the network evolution is (larger $q$), the greater the payoff difference that can be tolerated by the agents.
In figures G.2(b) the case when $\alpha$ represent only 25% of the initial population is presented. When no noise is present the stronger strategy needs an increased payoff advantage to take over the population. When $a - b < 0.3$ the payoff-inferior strategy $\beta$ is able to maintain the majority.

To confirm the stochastic stability of the evolution process we did a series of simulations

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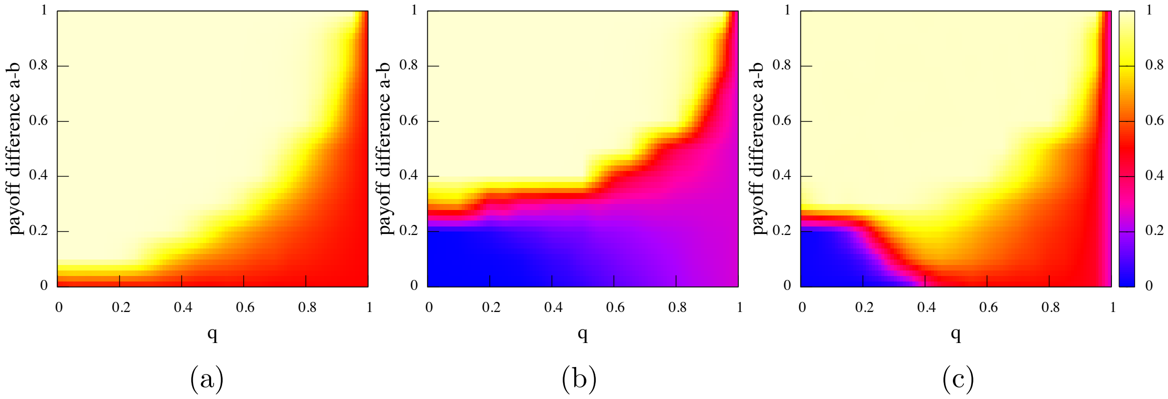
[2] With the above simulated process one cannot properly speak of true equilibrium states in the strict mathematical sense.



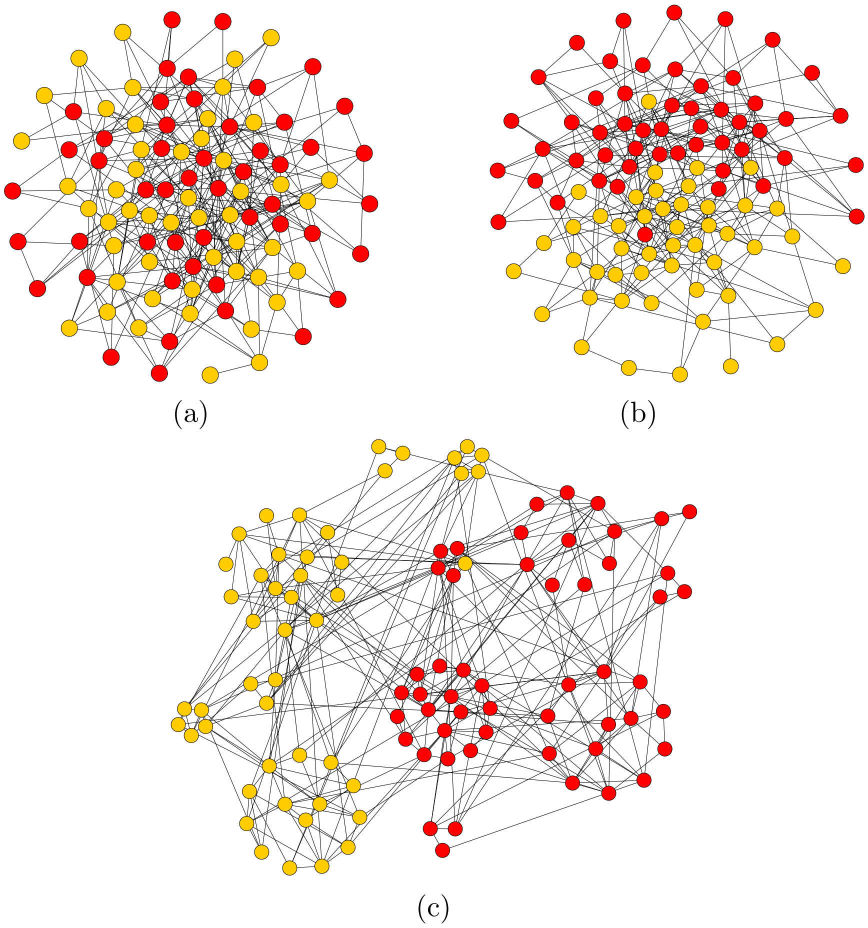
(a) (b) (c)

Figure G.2: Fraction of $\alpha$-strategists in the population as a function of the relinking probability $q$ when the quasi-equilibrium has been reached. (a) shows the case where the initial fraction of $\alpha$ is 0.5 and noise is not present. In (b) and (c) the initial fraction of $\alpha$ is 0.25. (b) shows the noiseless case and (c) the case where noise is 0.01. Results are averages over 50 independent runs.

using a noisy version of the strategy evolution rule [51]. The amount of noise used is 0.01, which means that an agent will pick the wrong strategy once every 100 updates on average. This quantity is rather small and does not change the results obtained when the two populations are equally represented in the initial network, the graphic representation is almost the same of the one in fig. G.2(a) with respect to stochastic fluctuations. However, when the initial share is not the same, the presence of noise allows a considerable increase in the performance of the Pareto-superior strategy when this strategy is less represented in the beginning. Figure G.2(c) shows the case when the initial ratio of $\alpha$-strategists is 25% of the population. We can clearly see that the strategy that offers the higher payoff ($\alpha$ in this case but the results for $\beta$ would obviously be symmetrical) can recover a considerable amount of the parameters space even when it starts from an unfavorable situation. The coexistence of stochastic errors and network plasticity allows the more advantageous strategy to improve its share. In this case, when $q > 0.4$ the situation is almost the same as when the initial shares are the same. The same phenomenons happen when the initial ratio of $\alpha$ is smaller. The case of an initial ratio of 5% has been verified but is not shown here.

For visualization purposes, figures G.3 and G.4 show one typical instance of the evolution of the network $G'$ and of the strategy distribution from the initial state in which strategies are distributed uniformly at random to a final quasi-equilibrium steady state for a smaller $N = 100$ network. in spite of the relatively small size, the phenomena are qualitatively the same for $N = 100$ and $N = 1000$, the major difference is just the time to convergence which is much shorter for $N = 100$.

These results have been obtained for a symmetric payoff of the strategies $a = b$ and for an equal initial fraction of $\alpha$-strategists and $\beta$-strategists. It is visually clear that the system



(a) (b) (c)

Figure G.3: (a) The simulation starts from a random network with $N = 100$ and 50 players for each type. (b) In the first short part of the simulation ($\sim 500$ time steps) the strategies reach an equilibrium, the network however is still unorganized. (c) The community structure starts then to emerge, many small clusters with nearly uniform strategy appears.

goes from a random state of both the network and the strategy distribution to a final one in which the network is no longer completely random and, even more important, the strategies are distributed in a completely polarized way. In other words, the system evolves toward an equilibrium where individuals following the same convention are clustered together. Since both norms are equivalent in the sense that their respective payoffs are the same, agents tend to pair-up with other agents playing the same strategy since playing the opposite one is a dominated strategy. The process of polarization and, in some cases, even the splitting of the graph into two distinct connected components of different colors, is facilitated by the possibility of breaking and forming links when an interaction is judged unsatisfactory by an agent. Even with a relatively small rewiring frequency of $q = 0.15$ as for the case represented in the figures, polarization is reached relatively quickly. In fact, since our graphs $G$ and $G'$ are purely relational entities devoid of any metric structure, breaking a link and forming another one may also be interpreted as "moving away", which is what would physically happen in certain social contexts. If, on the other hand, the environment is say, belonging to one of two forums on the Internet, then link rewiring would not represent any physical reconfiguration of the agents, just a different web connection. Although our model is an abstract one and does not claim any social realism, still one could imagine how conceptually similar phenomena may take place in society. For example, the two norms might represent two different dress codes. People dressing in a certain way, if they go to a public place, say a bar or a concert in which individuals dress in the other way in the majority, will tend to change place in order to feel more adapted to their surroundings. Of course, one can find many other examples that would fit this description. An early model capable of qualitatively represent this kind of phenomena was Schelling's segregation cellular automaton [106] which was based on a simple majority rule. However, Schelling's model, being based on a two-dimensional grid, is not realistic as a social network. Furthermore, the game theory approach allows to adjust the payoffs for a given strategy and is analytically solvable for homogeneous or regular graphs.

The above qualitative observations can be rendered more statistically rigorous by using the concept of *communities*. Communities or clusters in networks can be loosely defined as being groups of nodes that are strongly connected between them and poorly connected with the rest of the graph. These structures are extremely important in social networks and may determine to a large extent the properties of dynamical processes such as diffusion, search, and rumor spreading among others. Several methods have been proposed to uncover the clusters present in a network (for a recent review see, for instance, [28]). To detect communities, here we have used the divisive method of Girvan and Newman [79] which is based on iteratively removing edges with a high value of edge betweennes. A commonly used statistical indicator of the presence of a recognizable community structure is the *modularity* $Q$. According to Newman [77] modularity is proportional to the number of edges falling within clusters minus the expected number in an equivalent network with edges placed at random. In general, networks with strong community structure tend to have values of $Q$ in the range $0.4-0.7$. In the case of our simulations $Q = 0.19$ for the initial random networks with $N = 100$ like the one shown in fig. G.3(a). $Q$ progressively increases and reaches $Q = 0.29$ for fig. G.3(c) and $Q = 0.45$ for the final polarized network of fig. G.4. In the case of the larger networks with $N = 1000$ the modularity is slightly higher during the evolution, $Q \sim 0.3$ at the beginning of the simulation and $Q \sim 0.5$ when the network has reached a polarized state. This is due to the more sparse structure of these networks.

To confirm the stability of this topological evolution we performed several simulation using the noisy strategy update rule. Even in this situation the network will attain a polarized state but due to the stochastic strategy fluctuations the two main clusters almost never reach a completely disconnected state and the modularity remains slightly lower ($\sim 0.4$) compared to the noiseless case.

As a second kind of numerical experiment, we asked how the population would react when, in a polarized social situation, a few connected players of one of the clusters suddenly switch to the opposite strategy. The results of a particular but typical simulation are shown in Figs. G.5. Starting from the clusters obtained as a result of the co-evolution of strategies and network leading to Fig. G.4, a number of "red" individuals replace some "yellow" ones in the corre-

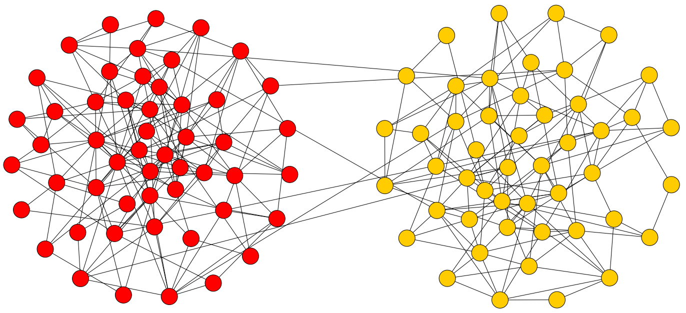
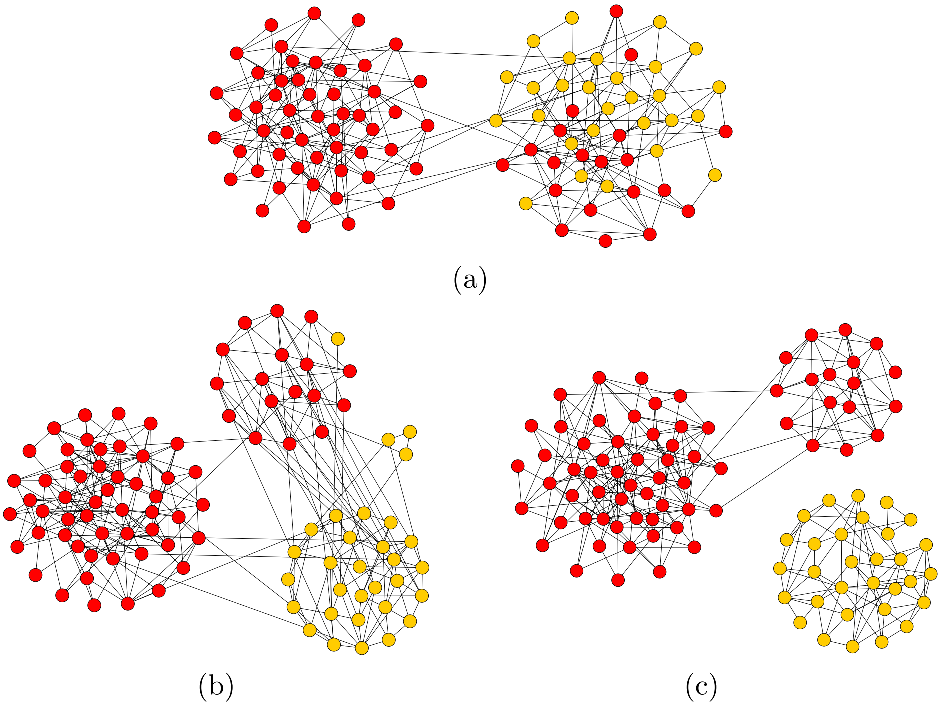


Figure G.4: In the last phase the network is entirely polarized in two homogeneous clusters. If the simulation is long enough all the links between the two poles will disappear.



(a)

(b)

(c)

Figure G.5: (a) A consistent amount of mutant is inserted in one of the two clusters. (b) This invasion perturbs the structure of the population that starts to reorganize. (c) With enough evolution time the topology reaches a new polarized quasi-equilibrium.

sponding cluster. The evolution is very interesting: after some time the two-cluster structure disappears and is replaced by a different network in which several clusters with a majority of one or the other strategies coexist. However, these intermediate structures are unstable and, at steady state one recovers essentially a situation close to the initial one, in which the two poles form again but with small differences with respect to the original one. Clearly the size of the clusters is different from that of before the invasion. Even in this case, if the evolution time is long enough, the two components can become disconnected at the end. This means that, once formed, polar structures are rather stable, except for noise and stochastic errors. Moreover, we observed that at the beginning the invasion process the modularity drops slightly due to the strong reorganization of the network but then it increases again and often reaches a higher value with respect to the previous state. In the case shown here, the final modularity is 0.56. The same happens in the larger networks where, after the invasion process $Q$ reaches values of $Q \sim 0.55$.

## G.5 Results for General Coordination Games

In this section we show the numerical results for the Stag Hunt class of coordination games. We recall that, unlike pure coordination games, in Stag Hunt games there is risk in coordinating on the Pareto-efficient strategy and thus agents may wish to reduce their aspirations by playing the socially inferior strategy for fear of being "betrayed" (see Sect. G.2.1).

The simulation parameters are the same as for coordination games, see Sect. G.4.1, except that now the game parameter space is more complex. For the Stag Hunt the ordering of payoffs is $a > d > b > c$, and we have studied the portion of the parameters' space defined by $c \in [-1, 0]$ and $d \in [0, 1]$, $a = 1$, and $b = 0$, as is customarily done [96]. The $c - d$ plane has been sampled with a grid step of 0.05.

In order to find out whether a change in the strategy update dynamics would make a difference in the results, we have used, besides the already described best response dynamics, another update rule which is related to replicator dynamics [122, 30]. Instead of considering a mixing population, the version of replicator dynamics used here is modified to take into account the local nature of interaction networks as proposed by [41]. It assumes that the probability of switching strategy is a monotonic increasing function $\phi$ of the payoff difference; here $\phi$ is a linear function. First, a player $i$ is randomly chosen from the population to be updated with uniform probability and with replacement. To update its strategy another player $j$ is next drawn uniformly at random from $i$'s neighborhood $\mathcal{N}_i$. Then, strategy $s_i$ is replaced by $s_j$ with probability

$$p_i = \phi(\Pi_j - \Pi_i), \tag{G.1}$$

in which

$$\Pi_i(t) = \sum_{j \in \mathcal{N}_i} \pi_{ij}(t)$$

is the accumulated payoff collected by player $i$ at time step $t$ after having played with all his neighbors $\mathcal{N}_i$. The major difference with standard replicator dynamics is that two-person encounters between players are only possible among neighbors, instead of being drawn from the whole population.

### G.5.1 Strategy Distribution at Steady State

The following Figures G.6 and G.7 show the average fraction of strategy $\alpha$ (light color) and $\beta$ (darker color) respectively at steady state for best response dynamics and replicator dynamics, and for three values of the rewiring frequency $q$ increasing from left to right. Initially there is an equal amount of $\alpha$ and $\beta$ players randomly distributed in the network nodes. The first thing to notice is that the trend is the same, i.e. higher frequencies of link rewiring favor the Pareto-efficient result for both dynamics, although this happens to a lesser extent for best response. The reason is that best response confirms the players in what they are doing: the best response to $\alpha$ is $\alpha$ and to $\beta$ it is $\beta$ but, on the whole, the possibility of link rewiring allows unsatisfied $\alpha$-strategists to cut a link with a $\beta$-strategist and to search for another $\alpha$ in the next to first neighborhood. Since the data plotted in the figures are average values, it is also important to point out that, actually, in all runs the final steady state is constituted by a monomorphic population, i.e. only one strategy is present. This is coherent with standard results on Stag Hunt games which show that polymorphic populations are unstable and that the dynamics should converge to one of the pure states [30, 51, 26, 95].

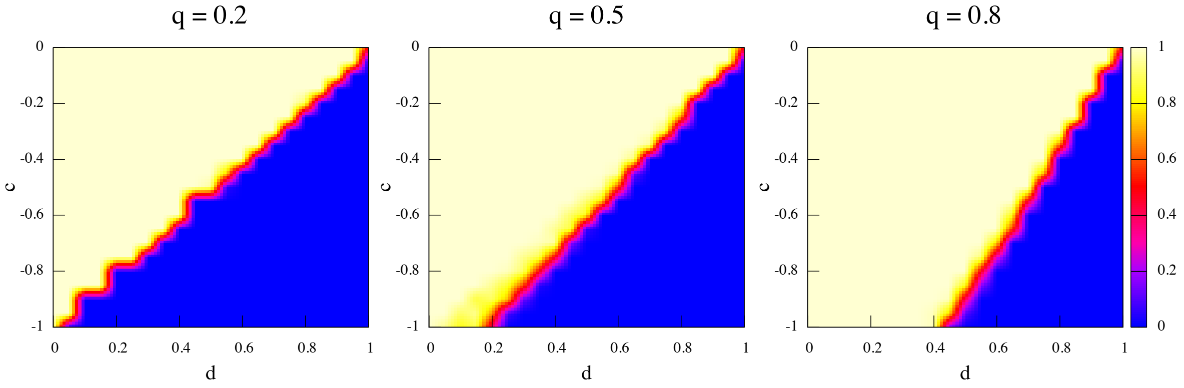


Figure G.6: Average strategy proportions over 50 independent runs in the game's phase space at steady state. Initially $\alpha$ and $\beta$ are equally represented. Update rule is best response and rewiring frequency $q$ increases from left to right.

The update rules used are both noiseless, in the sense that, apart from the implicit probabilities used in the dynamics, no exogenous noise simulating strategic errors or trembles has been added. When a small amount of error probability of 0.01 is added in the best response case the results change very little and are not shown.

One might also ask what happens when initially the strategies are not present in equal amounts and, in particular, whether network and rewiring effects may help the socially efficient strategy $\alpha$ to proliferate when it starts in the minority. Figures G.8 and G.9 are the same as Figs. G.6 and G.7 but this time strategy $\alpha$ is only 1/4 of the total initially. It is apparent that, even when $\alpha$-strategists are the minority, they can occupy a sizeable region of the phase space thanks to rewiring effects. Indeed, by increasing the rewiring frequency the size of the $\alpha$

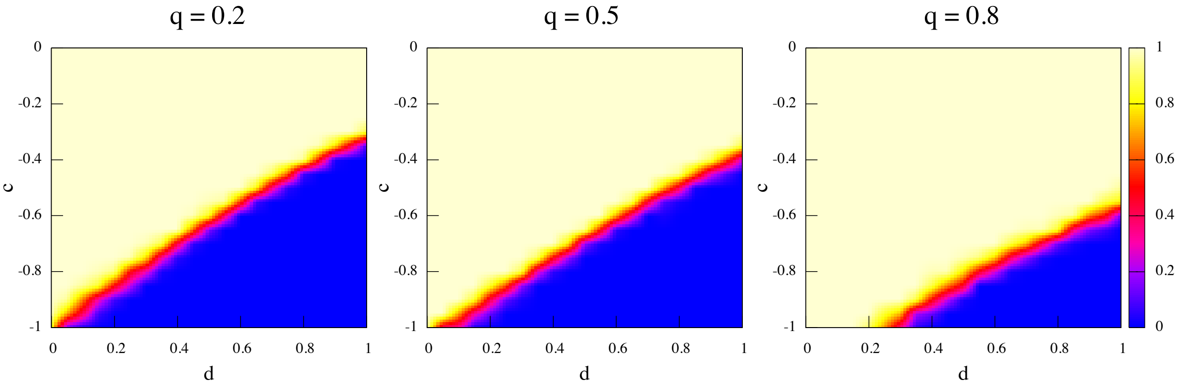


Figure G.7: Average strategy proportions over 50 independent runs in the game's phase space at steady state. Initially $\alpha$ and $\beta$ are equally represented. Update rule is replicator dynamics and rewiring frequency $q$ increases from left to right.

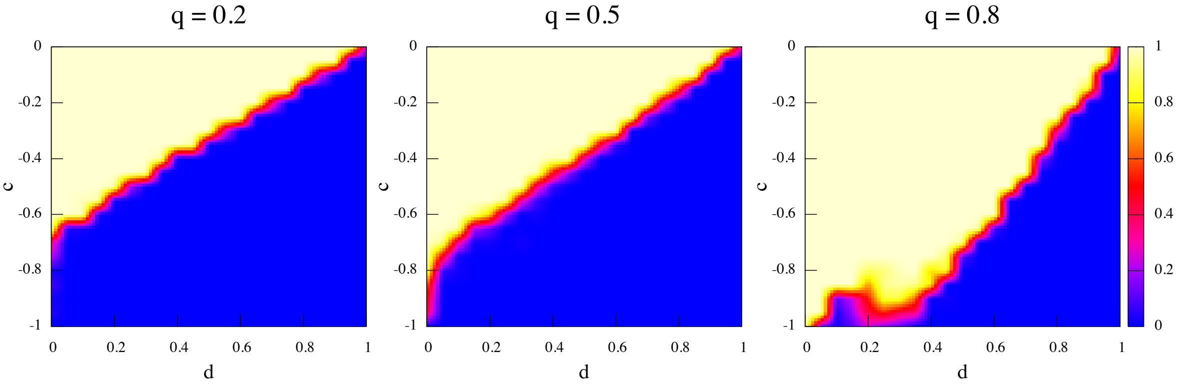


Figure G.8: Average strategy proportions over 50 independent runs in the game's phase space at steady state. Initially $\alpha$ players are 25% of the total. Update rule is best response and rewiring frequency $q$ increases from left to right.

region increases as well. The effects are present for both dynamics but they are more marked in the best response case (Fig. G.8). As a further probe, we have also simulated the evolution of populations with only 5% $\alpha$ initially and the results, not shown here, are that the $\alpha$ strategy proliferates in a non-negligible region of the parameter space and the more so the higher the relinking frequency.

### G.5.2 Network Features

In this section we present a statistical analysis of some of the global and local properties of the networks that emerge when the pseudo-equilibrium states of the dynamics are attained. We do not strive for a complete analysis: this would take too much space; just a few topological features should be already useful to get a clearer picture. The graphs used for the analysis are the undirected, unweighted versions $G'$. An important global network statistics is the *average*

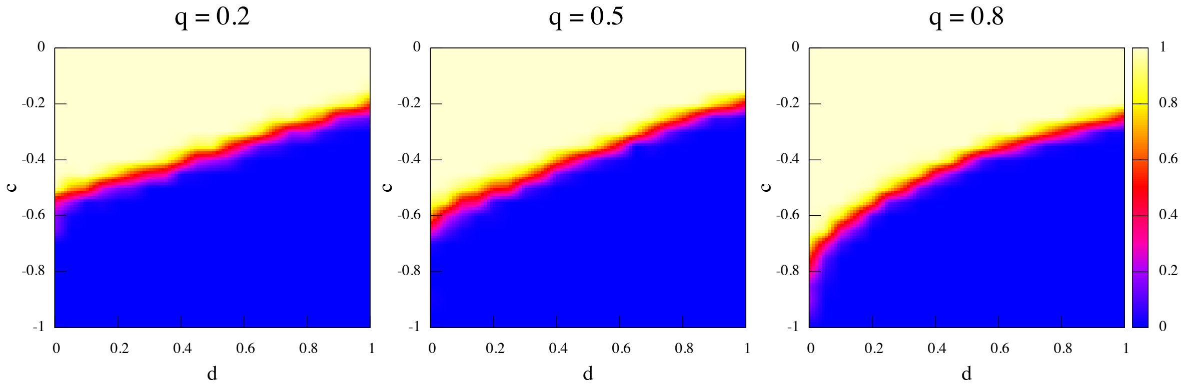


Figure G.9: Average strategy proportions over 50 independent runs in the game's phase space at steady state. Initially $\alpha$ players are 25% of the total. Update rule is replicator dynamics and rewiring frequency $q$ increases from left to right.

*clustering coefficient* $C$. The clustering coefficient $C_i$ of a node $i$ is defined as $C_i = 2E_i/k_i(k_i-1)$, where $E_i$ is the number of edges in the neighborhood of $i$ and $k_i$ is $i$'s degree. Thus $C_i$ measures the amount of "cliquishness" of the neighborhood of node $i$ and it characterizes the extent to which nodes adjacent to node $i$ are connected to each other. The clustering coefficient of the graph is simply the average over all nodes: $C = \frac{1}{N}\sum_{i=1}^{N} C_i$ [75]. In Figure G.10 we report for each grid point the average value of $C$ over 50 realizations of $G'$. The figure shows that in general $C$ increases with increasing $q$ which is reasonable, as higher $q$ values mean more rewiring and thus more transitive closure of triangles in the neighborhood. In particular, we have remarked that $C$ is especially high in the proximity of the transition zone between $\beta$ and $\alpha$ regions. We also note that the clustering of $\alpha$-strategists networks are higher than those of $\beta$-strategists. Indeed, in the relinking process, $\alpha-\alpha$ links tends to be stable as both players get the maximum payoff. On the other hand, $\alpha-\beta$ links will be unstable since the $\alpha$ end will try to dismiss the link, while the $\beta$ end tries to keep it. As for the $\beta-\beta$ links, they also tend to be unstable for both agents will in the average try to chase for an $\alpha$. When $\alpha$s become rearer in the population the links are more difficult to stabilize and local structure does not emerge.

To give a qualitative idea of the network self-organization, we compare $C$ values in the $\alpha$ and $\beta$ regions with the values expected for the initial random graph. Random graphs are locally homogeneous and for them $C$ is simply equal to the probability of having an edge between any pair of nodes independently. In contrast, real networks have local structures and thus higher values of $C$. For example, with $q = 0.5$, $C$ can be as high as 0.3 in the $\alpha$ region close to the transition, and as low as $C = 0.009$ in the $\beta$ region. It is thus apparent that the networks self-organize and acquire local structure for $\alpha$ networks as $C$ is much higher than that of the random graph with the same number of nodes and edges, which is $\bar{k}/N = 6/1000 = 0.006$. In the $\beta$ region there is barely more clustering than for a random graph. Given that the model favors rewiring a link toward a neighbor's neighbor, it is obvious that the clustering coefficient will tend to increase and thus the effect was expected. Nevertheless, this transitive closure is a

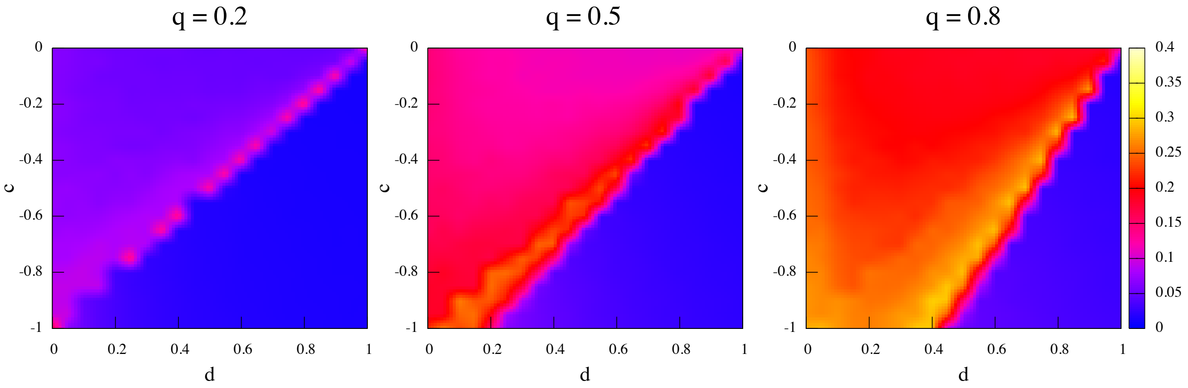


Figure G.10: Average clustering coefficients of steady-state networks; relinking frequency $q$ increases from left to right. Equal initial proportions of $\alpha$ and $\beta$ strategies and strategy update is by best response.

well known social phenomenon and the model successfully simulates it.

Another important quantity is the *degree distribution function*. The degree distribution function $p(k)$ of a graph represents the probability that a randomly chosen node has degree $k$ [75]. Random graphs are characterized by distributions of Poisson form $p(k) = \bar{k}^k e^{-\bar{k}}/k!$, while social and technological real networks often show long tails to the right, i.e. there are nodes that have an unusually large number of neighbors [75]. In some extreme cases the distribution has a power-law form $p(k) \propto k^{-\gamma}$; the tail is particularly extended and there is no characteristic degree.

In our simulations the population graph $G^{'}$ always starts random, i.e. $G^{'}_0$ has a Poisson degree distribution. It would be interesting to see whether the graphs remain random after the co-evolutionary process stabilizes in a steady state, or whether they acquire some more structure. Fig. G.11 shows the degree distribution functions sampled at two points in the $c - d$ plane. One point is in the $\alpha$-stable region and the other is in the $\beta$-stable one. The third dotted curve is shown for comparison and corresponds to the initial random graph which has a Poisson $p(k)$ with $\bar{k} = 6$. Both curves at steady state deviate from the random graph distribution but, while the degree distribution of the network of $\beta$ players is still rather close to Poisson, the $\alpha$ network (thick curve) shows a distribution that has a longer tail to the right, i.e. there is a non-negligible quantity of nodes that have more connections. Together with the increase of the clustering coefficient seen above, this shows that $\alpha$ networks have acquired more structure than $\beta$ networks during the co-evolutionary process. It appears that $\alpha$ strategists use the relinking possibility in such a way that more $\alpha$ clusters are created, thus protecting them from $\beta$ "exploiters". The curves shown are for $q = 0.8$; for lower values of $q$ the effect is the same but less marked as $q \to 0$.

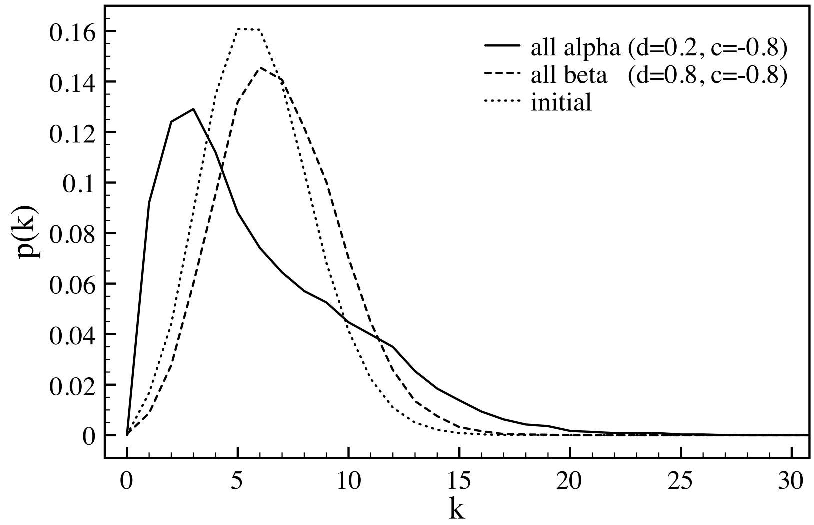


Figure G.11: Average empirical degree distribution functions for the initial random graphs (dotted line), steady-state graphs in the $\alpha$-dominant region of the parameters' space (thick curve), and in the $\beta$-dominant region (dashed line) All distributions are discrete, continuous curves are just a guide for the eye.

## G.6 Conclusions

In this paper we propose and simulate numerically a model in which a population of agents interacting according to a network of contacts play games of coordination with each other. The agents can update their game strategy according to their payoff and the payoff of their neighbors by using simple rules such as best response and replicator dynamics. In addition, the links between agents have a strength that changes dynamically and independently as a function of the relative satisfaction of the two end points when playing with their immediate neighbors in the network. A player may wish to break and redirect a tie to a neighbor if it is unsatisfied. As a result, there is co-evolution of strategies in the population and also of the graph that represents the network of contacts.

We have applied the above model to the class of coordination games, which are important paradigms for collaboration and social efficiency. For pure coordination games, the networks co-evolve towards the polarization and, in some cases, even the splitting of the graph into two distinct connected components of different strategies. Even with a relatively small rewiring frequency polarization is reached relatively quickly. This metaphorically represents the segregation of norm-following subpopulations in larger populations. In the case of general coordination games the issue is whether the socially efficient strategy, i.e the Pareto-dominant one, may establish in the population. While results in well-mixed and static networked populations tend to favor the risk-dominant, and thus socially inefficient outcome, our simulation results show that the possibility of refusing neighbors and choosing different partners increases the success rate of the Pareto-dominant equilibrium. Although the model is extremely simplified, the possibility of link redirection is a real one in society and thus these results mean that some plasticity in the network contacts may have positive global social effects.

## Acknowledgements

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# Part V

# Discussion and Conclusions

# Chapter 5

# Conclusions

> Only Robinson Crusoe had everything done by Friday.
>
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This work has been conducted in the last five years. It's a quite long time span and although some of the ideas developed here where already planned, some developments have been decided on the way.

In the following, we shall try to point out which are the main results and conclusions reached in this work in the light of the motivations and objectives that we set for ourselves in Chapter 1.

### On update rules

From a statistical point if view, update rules and timing of events have only limited effect on the amount of cooperation in a structured population. When we look at the mean values, the transition from cooperation to defection may happen for slightly different values of the payoff matrix but in general the global picture seen at the equilibrium is only slightly affected by timing and update rule (see figures A.1 and A.2).

The numbers of time steps needed to reach an equilibrium are however strongly dependent on the update rule. For example, when using a deterministic rule like "Imitate the best" the evolution is much quicker.

On the other hand, when we look more closely at the behavior of the agents on a single population instead of analyzing the average behavior of many populations, the update rule can have a greater influence on the stable states and on the behavior of agents. With little or no stochastic noise, i.e with deterministic update rules like the "Imitate the best", the outcome can be drastically different even for the same parameters. For the same payoff values, the population can converge to either full cooperation or full defection, i.e. in these cases we observe bistability in the dynamic as a population starting from the same situation can reach two different stable states (for an example see figure A.3(a)). In this case the specific network topology and a particular initial distribution of the different strategies on the network are the reason for this bistable behavior: even small differences in connections or initial strategies can thus have an influence on the final state.

Conversely, when the decisions of the agents are based on a stochastic rule like "Replicator dynamics" , the evolution can reach a steady state even when the population still include agents playing different strategies. These stable situation are not however completely frozen as small fluctuations like those that can be seen in figure A.3(b) may continue to be observed in these populations. This happens when agents use stochastic rules because their decisions, even if statistically good, are dictated by probabilities and in some cases leads to errors. These behaviors are not apparent when we look at the mere average cooperation levels.

### On utility calculation

The method used to calculate the individual's payoff can have a great impact on the cooperation level in a population, in particular when coupled with an heterogeneous network. When the payoff of the encounters is accumulated, a highly connected agent has a strong influence on his neighborhood and on the dynamics of the population. On the other hand, when the payoff is divided by the number of links, i.e. when the average payoff is considered, the influence of the heterogeneity is almost nullified. This is an important consideration in our opinion, as many works use accumulated payoff without taking into account link maintaining costs. In real life this is a significant issue as, even considering that different agents can have different skills, the involvement needed to maintain a relationship is such that a great difference in the quantity of links should imply a slight degradation in the effort to maintain these relationships healthy.

Moreover, if accumulated payoff is coupled with replicator dynamics, an affine transformation of the payoff matrix can change completely the fixed points of the dynamics. To obviate this situation we proposed a modified version of the replicator dynamics to maintain the advantage of heterogeneous degree and restore the invariance of replicator dynamics [61].

### On network structure

As expected, the structure of the population plays an important role on the quasi-stable states of the simulations. In general, degree heterogeneous networks such as scale-free and social networks allow cooperative behavior to emerge in a wider area of the parameter space than more homogeneous networks such as rings, grids, and random networks with Poisson degree distribution. The conclusions explained here are based on the assumption that the payoff is calculated in an accumulated way, as average payoff does not take advantage from the degree heterogeneity of the structured population.

In the case of Barabási-Albert scale-free networks the evolution of cooperation essentially depends on the colonization of highly connected nodes by cooperators [101]. Clustering and community structure plays only a little role for these kinds of structure as these features are almost absent. When we look at the initial progression of cooperators in a scale-free network, shown in figure A.3(c), we can see how the initial number of cooperators drops quickly and then eventually starts a slower recovery. This drop is proportional to the value of the temptation to defect compared to the reward for cooperate, the black lines in the left images of figure 5.1 shows this initial drop for two different values of $T$. In this first phase of the evolution, isolated cooperators are absorbed by the defectors as their payoff is considerably lower, moreover, isolated defectors are able to exploit the cooperators and diffuse their strategy. This initial fluctuation is due to the initial random distribution of the agents, because cooperators are only able to survive by sticking together and only when they manage to achieve a tight cluster they are able to resist to defectors.

When this first initial drop does not lead to cooperator's extinction, the clustered cooperators are able to slowly expand and eventually regain a majority position. This happens for two main reasons: cooperators are able to obtain high payoff when clustered together and they became even stronger when they successfully transmit their strategy; on the other hand, defectors are stronger when isolated between cooperators, but by diffusing their strategy they weaken their position as they cannot get the same benefit from a defector than from a cooperator. Of course these phenomenon can have a greater of smaller amplitude depending on the payoff values.

In the diffusion of the cooperative strategy, the highly connected hubs of the network play a crucial role: when cooperators are able to cluster and maintain the control of at least one hub, the cooperator hub will achieve an higher benefit. On the other hand, a defector hub, is only able to gather a relatively small benefit. In this situation, the defector hub will have a high probability to imitate the strategy of a cooperator hub or even of another well connected cooperator. When this imitation happens the payoff of this new cooperator hub is increased by the neighboring cooperators and the neighboring defectors will now slowly imitate the strategy of the hub. The more a cooperator hub is able to diffuse his strategy and the less he will tend to imitate a defector agent.

This phenomenon can be observed in the upper panel of figure C.10, here we can see the number of cooperators in the neighborhood of a defector hub. As mentioned before, the initial ratio of cooperators decreases, this lowers the payoff of the hub. When the hub changes to cooperation, his payoff increases, and thus he is able to diffuse his strategy to some of his defectors neighbors. This effect is amplified by the correlation between the hubs. Especially in structures like the Barabási-Albert scale-free networks, the construction of the network introduces correlation between high connected nodes that may amplify the diffusion of cooperation by allowing cooperator hubs to spread their strategy directly to defector hubs. The inverse phenomenon is in general not possible because of the higher benefit obtained by cooperator hubs. Indeed when the correlationless configuration model is used [101], the global amount of cooperation is lower.

In social networks individuals with very high degree are almost inexistent. However these structures are still able to boost cooperation, even if not to a level as high as in scale-free networks. The phenomenon is however very important as real social structures are different from scale-free network but they are still capable of sustaining cooperation.

In social network the diffusion of cooperation has to rely on other features of the topology, and we identified these properties in the higher presence of clustering, but also on the organization of the population in communities. As can be seen in Article B and in Article F, the presence of a community structure in the network strongly influences the diffusion of the strategies. These two properties allow cooperators to reorganize and stick together during the initial phase, when defectors are able to expand. When the initial regression is contained, the cooperators may exploit their cohesion to expand their strategy. In these networks highly connected hubs are absent and cooperators cannot exploit this preferential channel to diffuse their strategy. However, the lack of defector hubs is in this case even profitable for cooperators, combined with the higher clustering of cooperators, well connected cooperators have an high benefit and are easily imitated by low and high connected defectors.

When we look closely at mixed stable populations, i.e. populations where both strategies are still present when the stable state has been reached, we see that the clusters of the networks are often monomorphic, or at least with one of the strategies more strongly represented as the low connectivity between different clusters allows an high degree of diversity to exist in what can be interpreted as different social groups. This phenomenon is particularly evident in the case of coordination games.

To sum up, the presence of communities and high clustering coefficient is a key factor for sustaining an high level of cooperation, and an heterogeneous degree is the channel that allows cooperators to expand their influence in the population, especially when combined with some assortativity, i.e. highly connected nodes have more connection to other highly connection nodes than with weakly connected nodes.

## On conformity

Conformity, i.e. complying to the action of the majority, also has consequences that depend on the network structure of the population. However, in general, the effect of conformity coupled with replicator dynamics produce an acceleration of the evolution towards the steady state, and a reduction of the occurrence of the situations with stable mixed populations.

For the Prisoner's Dilemma there is also a sizable reduction of cooperation especially on scale-free networks which are otherwise the more favorable structure for cooperators.

This phenomenon happens because of the dynamical evolution of the strategies in a population. If we look at figure C.9 and at figure 5.1 we can clearly see that at the very beginning of the evolution, defectors are able to increase their share of diffusion in the populations; this is caused by the fact that the initial random distribution of the agents does not favor the cooperators, that need to be together to succeed. After the initial phase, when cooperators tend to cluster together, they may be able to expand their influence because by clustering together they are able to greatly increase their payoff.

When we add conformity to the picture, isolated defectors, that are stronger than cooperators when the only rule is payoff-based, they became vulnerable to adopt the cooperative strategy because of their isolated position between cooperators. This phenomenon leads, especially in regular networks like rings but also on other networks, to a situation where the two strategies

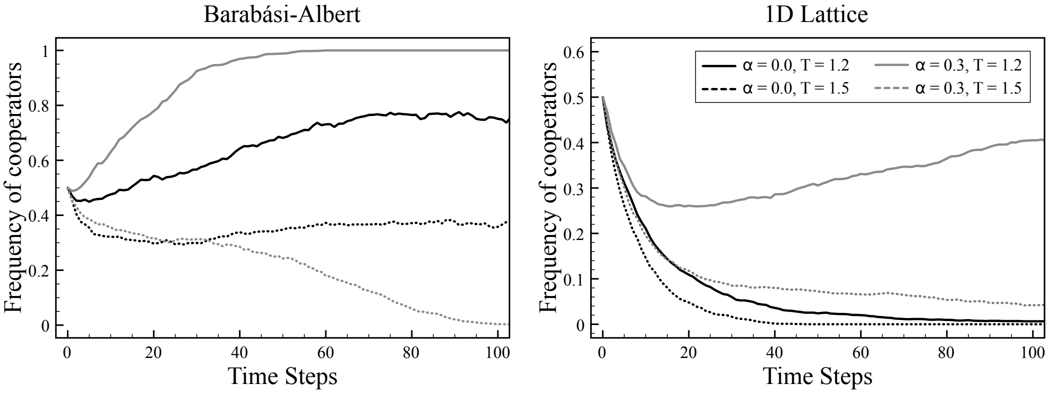

Figure 5.1: First time steps of the evolution of a single population of agents interacting according to the rules of the Prisoner's Dilemma on Barabási-Albert scale-free networks (left image) and 1D lattices(right image). The four lines in each image represent the cases without conformity (in black) and with 30% of conformity (in gray), for two different values of the temptation value T (the solid lines are for $T = 1.2$ and the dotted lines for $T = 1.5$). The used networks have $N = 2000$ agents and an average degree $\bar{k} = 8$.

are clustered together. Conformity is good for cooperators in this initial situation as it allows the suppression of isolated defectors; because of this the initial drop of cooperation is smaller than in cases without conformity. In figure 5.1 the gray lines drop slower at the beginning than the black lines.

Especially in the case of scale-free networks, the initial increase in the number of defectors can trigger the "majority rule" implied by conformity. Even if the initial drop in cooperation is slower than when only replicator dynamics is used, the frequency of cooperators drops to a value so low than the effect of the hubs is not enough to recover. It's also important to take into account that a cooperator hub, with few cooperator neighbors and many defector neighbors can spread his strategy because of his higher accumulate payoff, but he can be easily mutated into a defector under the effect of conformity. That's why by increasing the amount of conformity the success of cooperators on scale-free network is reduced.

On the other hand, on regular networks like lattices, the initial phenomenon that reduces the devastating effects of initial isolated defectors is able to produce a situation where cooperators can succeed. Once the initial situation stabilizes and the population reaches a state where only big homogeneous clusters exist, due to the regular degree of the network imitation by conformity becomes almost impossible. In this case the clusters of cooperators are bigger when conformity is higher because isolated defectors disappear quickly. When these clusters are formed, if the temptation to defect is low enough, there will be more defectors that may imitate an efficient cooperator than the contrary, as can be seen in figure 5.2. The cause of the growth in cooperation when the ratio $\alpha$ of conformity is higher is exactly this faster absorption of defectors invaders. When invaders are eliminated and cooperative population is not fragmented anymore, the task for defectors is harder, and they will slowly adopt the cooperative strategy.

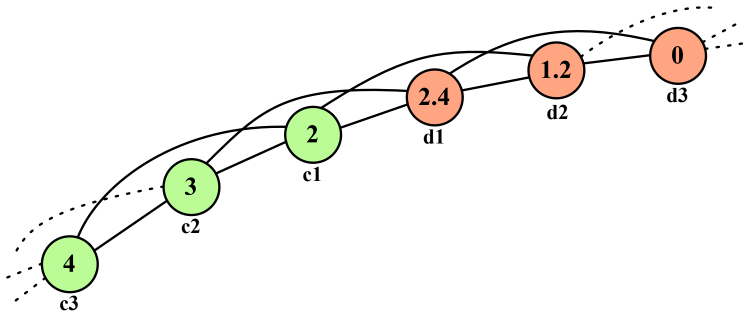

Figure 5.2: The border region between two population in a lattice. No one will change strategy because of conformity, because in this situation each agent has more neighbors like himself then of the other strategy, this because he counts himself too in the distribution share. When we look at the payoff based imitation only cooperators c1 has a possibility to imitate a defector, d1. On the defector side however, both d1 and d2 can imitate a cooperator, the former may imitate c2 and the latter c1. The numerical values inside the nodes represent their accumulated payoff, this is for the case where $T = 1.2$, $R = 1$, and $P = S = 0$; the degree of the nodes is $k = 4$.

### On dynamic networks

To try to further extend the understanding of the relationship between structure and strategy evolution we introduced a supplementary degree of freedom consisting in allowing links to be cut and formed, as well as the update of player strategies. The co-evolution of the network topology, starting from a random network, showed that in a dynamic population, the network of the interactions between the agents evolves towards a more complex structure and allows different strategies to coexist inside different clusters that can became disconnected to protect their diversity in the case of pure coordination games.

Of course our implementation of dynamic networks is biased and represents an important factor that leads to an increased success for the cooperative strategies. By changing these mechanisms the direction of evolution of the dynamical system can be completely different, e.g. by using random rewire instead of our trust–based rewire, the formation of clusters is not promoted in the population, thus cooperators are not able to cultivate one of the key factory that allows cooperation to be sustained in a population.

We believe that we have only scratched the surface in the issue of network/strategy co-evolution and our feelings are that this field will rapidly expand in the close future.

### Future developments

Among the possible future developments, one is the investigation of the influence of weighted links in the dynamic. The reason is very simple: in a society, some links are more important than others, by using weighted edges this aspect would be included in the model. Of course, the weight should be first taken as constants of the system and then, in a second step, allowed to evolve. We already include weighted links in our co-evolutionary models but there is still more to be investigated.

A second development is the introduction of more complex games such as games with more than two strategies or repeated games, where players have a memory of past encounters.

Another likely extension is the inclusion of more sophisticated forms of learning that should allow the agents to adapt over time and to take more efficient decisions.

# List of Publications

## International Journal Publications

*Evolution of Coordination in Social Networks: A Numerical Study*
Marco Tomassini and Enea Pestelacci
**International Journal Of Modern Physics C**, 2010, Volume 21, Number 10, pp. 1277-1296

*Coordination Games on Dynamical Networks*
Marco Tomassini and Enea Pestelacci
**Games**, 2010, Volume 1, Number 3, pp. 242-261

*Mutual trust and cooperation in the evolutionary hawks-doves game*
Marco Tomassini, Enea Pestelacci and Leslie Luthi
**BioSystems**, 2010, Volume 99, Number 1, pp. 50-59

*Evolutionary Games on Networks and Payoff Invariance under Replicator Dynamics*
Leslie Luthi, Marco Tomassini and Enea Pestelacci
**BioSystems**, 2009, Volume 96, Number 3, pp. 213-222

*Conformity hinders the evolution of cooperation on scale-free networks*
Jorge Peña, Henri Volken, Enea Pestelacci and Marco Tomassini
**Physical Review E**, 2009, Volume 80, Number 1, pp. 016110-1 - 016110-11

*Cooperation and community structure in social networks*
Leslie Luthi, Enea Pestelacci and Marco Tomassini
**Physica A - Statistical Mechanics And Its Applications**, 2008, Volume 387, Number 4, pp. 955-966

*Evolution of Cooperation and Coordination in a Dynamically Networked Society*
Enea Pestelacci, Marco Tomassini and Leslie Luthi
**Journal of Biological Theory**, 2008, Volume 3, Number 2, pp. 139-153

*Social dilemmas and cooperation in complex networks*
Marco Tomassini, Enea Pestelacci and Leslie Luthi
**International Journal Of Modern Physics C**, 2007, Volume 18, Number 7, pp. 1173-1185

## Peer Reviewed Conferences & Workshops Publications

*Evolution of Conventions and Social Polarization in Dynamical Complex Networks*
Enea Pestelacci and Marco Tomassini
**PPSN 2010** - Proceedings of Parallel Problem Solving from Nature 2010, Lecture Notes in Computer Science, Volume 6239, Springer Verlag, pp. 512-522

*Conformity and Network Effects in the Prisoner's Dilemma*
Jorge Peña, Enea Pestelacci, Marco Tomassini and Henri Volken
**CEC 2009** - IEEE Congress on Evolutionary Computation, pp. 506-513)

*Hawks and doves in an artificial dynamically structured society*
Enea Pestelacci and Marco Tomassini
**ALife 2008** - Artificial Life XI: Proceedings of the Eleventh International Conference on the Simulation and Synthesis of Living Systems, MIT Press, pp. 466-473

*Cooperation in Coevolving Networks: the Prisoner's Dilemma and Stag-Hunt Games*
Enea Pestelacci and Marco Tomassini
**PPSN 2008** - Proceedings of Parallel Problem Solving from Nature 2008, Lecture Notes in Computer Science, Volume 5199, Springer Verlag, pp. 539-548

*Evolutionary Dilemmas in a Social Network*
Enea Pestelacci, Leslie Luthi and Marco Tomassini
**ECAL 2007** - Proceedings of Advances in Artificial Life, 9th European Conference, Lecture Notes in Computer Science, Volume 4648, Springer Verlag, pp. 545-554

*Dinamica di Reti Booleane con Topologia Scale-Free*
Christian Darabos, Mario Giacobini, Enea Pestelacci and Marco Tomassini
**Wiva 3** - Italian Workshop on Artificial Life, Siena, ISSN 1970-5077 (CD), 2006

*Dynamics of Scale-Free Semi-Synchronous Boolean Networks*
Mario Giacobini, Marco Tomassini, Paolo De Los Rios and Enea Pestelacci
**ALife 2006** - Proceedings of Artificial Life X, Tenth International Conference on the Simulation And Synthesis of Living Systems, MIT Press, pp. 1-7

## Peer Reviewed Abstracts Presented at International Conferences

*Evolution of Coordination in Social Networks: A Numerical Study*

Enea Pestelacci and Marco Tomassini

**NetSci2010** - International Workshop on Network Science, Boston, USA, 2010

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