ASYMPTOTIC ANALYSIS OF A MEASURE OF VARIATION

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ABSTRACT. Let X_i, $i = 1, \ldots, n$ be a sequence of positive independent identically distributed random variables and define

$$T_n := \frac{X_1^2 + X_2^2 + \ldots + X_n^2}{(X_1 + X_2 + \ldots + X_n)^2}.$$

Utilizing Karamata theory of functions of regular variation, we determine the asymptotic behavior of arbitrary moments $\mathbb{E}(T_n^k)$ $(k \in \mathbb{N})$ for large n, given that X_1 satisfies a tail condition, akin to the domain of attraction condition from extreme value theory. As a side product, the paper offers a new method for estimating the extreme value index of Pareto-type tails.

1. INTRODUCTION

Let X_i, $i = 1, \ldots, n$ be a sequence of positive independent identically distributed (i.i.d.) random variables with distribution function F and define

$$T_n := \frac{X_1^2 + X_2^2 + \ldots + X_n^2}{(X_1 + X_2 + \ldots + X_n)^2}. \tag{1}$$

The asymptotic behavior of $\mathbb{E}(T_n)$ was investigated in [5], simplifying and generalizing earlier results in [4] and [6].
In this paper we extend several results of [5] and derive the limiting behavior of arbitrary moments $\mathbb{E}(T_n^k)$ $(k \in \mathbb{N})$. This is achieved by using an integral representation of $\mathbb{E}(T_n^k)$ in terms of the Laplace transform of X_1, which is derived in Section 2.
Most of our results will be derived under the condition that X_1 satisfies

$$1 - F(x) \sim x^{-\alpha}\ell(x), \quad x \uparrow \infty \tag{2}$$

where $\alpha > 0$ and $\ell(x)$ is slowly varying, i.e. $\lim_{x\to\infty} \ell(tx)/\ell(x) = 1 \, \forall \, t > 0$, see e.g. [3]. It is well known that condition (2) appears as the essential condition in the domain of attraction problem of extreme value theory. For a recent treatment, see [2]. A distribution satisfying (2) is called of *Pareto-type* with *index* α. When $\alpha < 2$, then the condition coincides with the domain of attraction condition for weak convergence to a non-normal stable law. It is then obvious that for $\beta > 0$,

$$E(X_1^\beta) := \mu_\beta = \beta \int_0^\infty x^{\beta-1}(1 - F(x))\, dx \leq \infty \tag{3}$$

will be finite if $\beta < \alpha$ but infinite whenever $\beta > \alpha$. For convenience, we define $\mu_0 := 1$ and $\mu := \mu_1$.

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The results of this paper are based on the theory of functions of regular variation (see e.g. [3]). Clearly, if $\mathbb{E}(X_1) = \infty$, both the numerator and the denominator in (1) will exhibit an erratic behavior, whereas for $\mathbb{E}(X_1) < \infty$ and $\mathbb{E}(X_1^2) = \infty$ this is the case only for the numerator. The results in Section 3 quantify this effect.

As a by-product, the results of this paper suggest a new method for estimating the extreme value index of Pareto-type distributions from a data set of observations, which is discussed in Section 4.

The quantity T_n is a basic ingredient in the study of the sample coefficient of variation of a given set of independent observations $X_1, \ldots, X_n$ from a random variable X, which is a frequently used risk measure in practical applications. In [1], this connection will be used to derive asymptotic properties of the sample coefficient of variation, including a distributional approach.

2. Preliminaries

Let $\varphi(s) := \mathbb{E}(e^{-sX_1}) = \int_0^\infty e^{-sx} dF(x)$, $s \geq 0$ denote the Laplace transform of X_1. Then, following an idea of [5], one can use the identity

$$\frac{1}{x^\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty e^{-s\,x} s^{\beta-1}\, ds, \quad \beta > 0$$

and Fubini's theorem to deduce that

$$\mathbb{E}\frac{1}{X_1^\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty s^{\beta-1} \varphi(s)\, ds.$$

More generally, for i.i.d. random variables $X_1, \ldots, X_n$, one obtains the representation formula

$$\mathbb{E}\frac{\prod_{i=1}^n X_i^{k_i}}{(X_1 + X_2 + \ldots + X_n)^\beta} = \frac{(-1)^{k_1+\ldots+k_n}}{\Gamma(\beta)} \int_0^\infty s^{\beta-1} \prod_{i=1}^n \frac{\partial^{k_i}\varphi(s)}{\partial s^{k_i}}\, ds, \tag{4}$$

for nonnegative integers k_i $(i = 1, \ldots, n)$.
In particular, by symmetry

$$\mathbb{E}(T_n) = \mathbb{E}\frac{X_1^2 + X_2^2 + \ldots + X_n^2}{(X_1 + X_2 + \ldots + X_n)^2} = n \int_0^\infty s\varphi''(s)\varphi^{n-1}(s)\, ds, \tag{5}$$

which formed the basis for the analysis in [5]. The representation (5) can be generalized in the following way:

Lemma 2.1. *For an arbitrary positive integer k,*

$$\mathbb{E}\,(T_n^k) = \sum_{r=1}^{k} \sum_{\substack{k_1,\ldots,k_r \geq 1 \\ k_1+\ldots+k_r=k}} \frac{k!}{k_1! \cdots k_r!}\, B(n, k_1, \ldots, k_r) \tag{6}$$

with

$$B(n, k_1, \ldots, k_r) = \frac{\binom{n}{r}}{\Gamma(2k)} \int_0^\infty s^{2k-1} \varphi^{(2k_1)}(s) \cdots \varphi^{(2k_r)}(s)\, \varphi^{n-r}(s)\, ds.$$

Proof. For an arbitrary positive integer k we have

$$\mathbb{E}(T_n^k) = \mathbb{E}\,\frac{(X_1^2 + X_2^2 + \ldots + X_n^2)^k}{(X_1 + X_2 + \ldots + X_n)^{2k}} = \sum_{\substack{k_1,\ldots,k_n \geq 0\\ k_1+\ldots+k_n=k}} \frac{k!}{k_1! \cdots k_n!}\ \mathbb{E}\,\frac{X_1^{2k_1} X_2^{2k_2} \cdots X_n^{2k_n}}{(X_1 + X_2 + \ldots + X_n)^{2k}},$$

where $k_i \leq k$ are nonnegative integers. Choose an n-tuple $(k_1, \ldots, k_n)$ in the above sum and let r denote the number of its non-zero elements $(k_{i_1}, \ldots, k_{i_r})$ (clearly $1 \leq r \leq k$). There are exactly $\binom{n}{r}$ possibilities of extending $(k_{i_1}, \ldots, k_{i_r})$ to an n-tuple by filling in $n - r$ zeroes; each of the resulting n-tuples leads to the same summand in (6). Thus we can write

$$\mathbb{E}(T_n^k) = \sum_{r=1}^{k} \sum_{\substack{k_1,\ldots,k_r \geq 1\\ k_1+\ldots+k_r=k}} \frac{k!}{k_1! \cdots k_r!}\ \underbrace{\binom{n}{r}\,\mathbb{E}\,\frac{X_1^{2k_1} X_2^{2k_2} \cdots X_n^{2k_r}}{(X_1 + X_2 + \ldots + X_n)^{2k}}}_{:=B(n,k_1,\ldots,k_r)}, \tag{7}$$

so that (6) holds in view of (4). □

3. Main Results

As promised, we will assume in the sequel that X_1 satisfies condition (2). Recall that when $\alpha > 1$ then $\mu < \infty$ while $\mu_2 < \infty$ as soon as $\alpha > 2$. The finiteness of μ and/or μ_2 has its influence on the asymptotic behavior of the summands that make up the statistic T_n. It is therefore not surprising that our results will be heavily depending on the range of α. We state a first and general result.

Lemma 3.1. *If X_1 has a regularly varying tail with index $\alpha > 0$ (i.e. $1-F(x) \sim x^{-\alpha}\ell(x)$), then the asymptotic behavior of the m-th derivative of the Laplace transform $\varphi(s)$ as $s \downarrow 0$ is given by*

$$\varphi^{(m)}(s) \sim (-1)^m \alpha\, \Gamma(m - \alpha)\, s^{\alpha - m} \ell(1/s), \quad m > \alpha. \tag{8}$$

Proof. Let $\chi(s) := \int_0^\infty e^{-sx}(1 - F(x))\,dx$. Since $1 - F(x) \sim x^{-\alpha}\ell(x)$, it follows that for $k > \alpha - 1$

$$(-1)^k \chi^{(k)}(s) = \int_0^\infty x^k e^{-sx}(1 - F(x))\,dx \sim \Gamma(k + 1 - \alpha) s^{-k-1}(1 - F(\frac{1}{s})) \quad \text{as } s \to 0.$$

Since $\varphi(s) = 1 - s\,\chi(s)$, we have for $m \geq 1$

$$\varphi^{(m)}(s) = -m\chi^{(m-1)}(s) - s\,\chi^{(m)}(s),$$

so that for $m > \alpha$

$$\begin{aligned}\frac{s^m \varphi^{(m)}(s)}{1 - F(\frac{1}{s})} &= -m\,\frac{s^m \chi^{(m-1)}(s)}{1 - F(\frac{1}{s})} - \frac{s^{m+1}\,\chi^{(m)}(s)}{1 - F(\frac{1}{s})}\\ &\sim (-1)^m (m\,\Gamma(m - \alpha) - \Gamma(m + 1 - \alpha)) = (-1)^m\,\alpha\,\Gamma(m - \alpha),\end{aligned}$$

from which the assertion follows. □

Theorem 3.1. *If X_1 belongs to the domain of attraction of a stable law with index α, $0 < \alpha < 1$, then for all $k \geq 1$*

$$\lim_{n\to\infty} \mathbb{E}(T_n^k) = \frac{k!}{\Gamma(2k)} \sum_{r=1}^{k} \frac{\alpha^{r-1}}{r\,\Gamma(1-\alpha)^r}\, G(r,k), \tag{9}$$

where $G(r,k)$ is the coefficient of x^k in the polynomial

$$\left(\sum_{j=1}^{k-r+1} \frac{\Gamma(2j-\alpha)}{j!}\, x^j \right)^r .$$

Proof. From $1 - F(x) \sim x^{-\alpha}\ell(x)$ it follows that $1 - \varphi(s) \sim \Gamma(1-\alpha)\, s^\alpha \ell(\frac{1}{s})$ (see e.g. Corollary 8.1.7 in [3]). Moreover, for any sequence $(a_n)_{n\geq 1}$ with $a_n \to \infty$ we have

$$\varphi^n(\frac{s}{a_n}) = e^{n \log \varphi(s/a_n)} \sim e^{-n\,(1-\varphi(s/a_n))} \sim \exp[-n \left(\frac{s}{a_n}\right)^\alpha \ell\left(\frac{a_n}{s}\right) \Gamma(1-\alpha)].$$

Choose $(a_n)_{n\geq 1}$ such that

$$n\, a_n^{-\alpha}\, \ell(a_n)\Gamma(1-\alpha) \to 1 \quad \text{for } n \to \infty. \tag{10}$$

Then for all $s \geq 0$

$$\lim_{n\to\infty} \varphi^n(\frac{s}{a_n}) = e^{-s^\alpha}.$$

We will now make use of the representation (6) for $\mathbb{E}(T_n^k)$. We have to investigate the asymptotic behavior of $B(n, k_1, \ldots, k_r)$. The change of variables $s = t/a_n$ together with an application of Potter's Theorem [3, Th.1.5.6], Lebesgue's dominated convergence theorem and Lemma 3.1 leads to

$$\begin{aligned}
B(n,k_1,\ldots,k_r) &= \frac{\binom{n}{r}}{a_n\Gamma(2k)} \int_0^\infty \left(\frac{t}{a_n}\right)^{2k-1} \varphi^{(2k_1)}\left(\frac{t}{a_n}\right) \cdots \varphi^{(2k_r)}\left(\frac{t}{a_n}\right) \underbrace{\varphi^{n-r}\left(\frac{t}{a_n}\right)}_{\to e^{-t^\alpha}} dt \\
&\sim \frac{\alpha^r \binom{n}{r}}{a_n\Gamma(2k)} \int_0^\infty \left(\frac{t}{a_n}\right)^{2k-1} \left(\frac{t}{a_n}\right)^{r\,\alpha-2k} \ell^r\left(\frac{a_n}{t}\right) \left(\prod_{j=1}^r \Gamma(2k_j-\alpha)\right) e^{-t^\alpha}\, dt \\
&\sim \frac{\alpha^r \prod_{j=1}^r \Gamma(2k_j-\alpha)}{\Gamma(2k)} \underbrace{\frac{\binom{n}{r}\, \ell^r(a_n)}{{a_n}^{r\,\alpha}}}_{\to\Gamma(1-\alpha)^{-r}/r!} \underbrace{\int_0^\infty t^{r\,\alpha-1}\, e^{-t^\alpha}\, dt}_{=(r-1)!/\alpha} \\
&\sim \frac{\alpha^{r-1} \prod_{j=1}^r \Gamma(2k_j-\alpha)}{r\,\Gamma(1-\alpha)^r\,\Gamma(2k)}.
\end{aligned}$$

Summing up over all $r = 1, \ldots, k$ in (6), we arrive at

$$\lim_{n\to\infty} \mathbb{E}(T_n^k) = \frac{k!}{(2k-1)!} \sum_{r=1}^{k} \frac{\alpha^{r-1}}{r\,\Gamma(1-\alpha)^r} \sum_{\substack{k_1,\ldots,k_r\geq 1 \\ k_1+\ldots+k_r=k}} \prod_{j=1}^{r} \frac{\Gamma(2k_j-\alpha)}{k_j!}. \tag{11}$$

Now observe that

$$G(r,k) := \sum_{\substack{k_1,\ldots,k_r \geq 1 \\ k_1+\ldots+k_r=k}} \prod_{j=1}^{r} \frac{\Gamma(2k_j-\alpha)}{k_j!}$$

can be determined by generating functions. Concretely, if we look at the r-fold product

$$\left(\Gamma(2-\alpha)\,x + \frac{\Gamma(4-\alpha)}{2!}\,x^2 + \ldots + \frac{\Gamma(2m-\alpha)}{m!}\,x^m\right)^r$$

for m sufficiently large, then $G(r,k)$ can be read off as its coefficient of x^k, since the kth power exactly comprises all contributions of combinations $k_1,\ldots,k_r \geq 1$ with $k_1+\ldots+k_r = k$ in the above sum. It suffices to choose $m = k-r+1$, since larger powers do not contribute to the coefficient of x^k any more. Hence Theorem 3.1 follows from (11). □

Remark 3.1. For $k=1$, we obtain $\lim_{n\to\infty} \mathbb{E}(T_n) = 1-\alpha$, which is Theorem 5.3 of [5]. The limit of moments of higher order can now be calculated from (9):

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}(T_n^2) &= \tfrac{1}{3}(1-\alpha)(3-2\,\alpha),\\
\lim_{n\to\infty} \mathbb{E}(T_n^3) &= \tfrac{1}{15}(1-\alpha)(15-17\,\alpha+5\,\alpha^2),\\
\lim_{n\to\infty} \mathbb{E}(T_n^4) &= \tfrac{1}{105}(1-\alpha)(105-155\,\alpha+79\,\alpha^2-14\,\alpha^3),\\
\lim_{n\to\infty} \mathbb{E}(T_n^5) &= \tfrac{1}{945}(1-\alpha)(945-1644\,\alpha+1106\,\alpha^2-344\,\alpha^3+42\,\alpha^4).
\end{aligned}$$

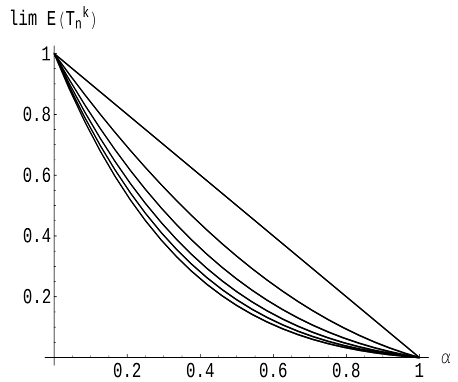

FIGURE 1. $\lim_{n\to\infty} \mathbb{E}(T_n^k)$ as a function of α ($k=1,\ldots,5$ from top to bottom)

The following result generalizes Theorem 5.5 of [5], where the case $k=1$ was covered:

Theorem 3.2. *If X_1 belongs to the domain of attraction of a stable law with index $\alpha = 1$ and $\mathbb{E}(X_1) = \infty$, then for all $k \geq 1$*

$$\mathbb{E}(T_n^k) \sim \frac{1}{2k-1}\,\frac{\ell(a_n)}{\tilde{\ell}(a_n)}, \tag{12}$$

where $\tilde{\ell}(x) = \int^x (\ell(t)/t)\,dt$ and $(a_n)_{n\geq 1}$ is a sequence satisfying $a_n \sim n\,\tilde{\ell}(a_n)$.

Proof. Since X_1 belongs to the domain of attraction of a stable law with index $\alpha = 1$, we have $1-F(x) \sim x^{-1}\ell(x)$ for some slowly varying function $\ell(x)$. Moreover $1-\varphi(s) \sim s\,\tilde{\ell}(\frac{1}{s})$

with $\tilde{\ell}(x) = \int^x (\ell(t)/t)\, dt$ (see e.g. [3]). Note that $\tilde{\ell}(x)$ is again a slowly varying function. For any sequence $(a_n)_{n\geq 1}$ with $a_n \to \infty$ we have

$$\varphi^n(\frac{s}{a_n}) = e^{n \log \varphi(s/a_n)} \sim e^{-n\,(1-\varphi(s/a_n))} \sim \exp[-n \left(\frac{s}{a_n}\right) \tilde{\ell}\left(\frac{a_n}{s}\right)].$$

If we choose a_n such that

$$n\, a_n^{-1}\, \tilde{\ell}(a_n) \to 1 \quad \text{for } n \to \infty, \tag{13}$$

then

$$\lim_{n\to\infty} \varphi^n(\frac{s}{a_n}) = e^{-s}. \tag{14}$$

Take a_n as in (13) and replace s by t/a_n in the representation (6). An application of Potter's Theorem, Lebesgue's dominated convergence theorem and Lemma 3.1 yields

$$\begin{aligned}
B(n, k_1, \ldots, k_r) &= \frac{\binom{n}{r}}{a_n \Gamma(2k)} \int_0^\infty \left(\frac{t}{a_n}\right)^{2k-1} \varphi^{(2k_1)}\left(\frac{t}{a_n}\right) \cdots \varphi^{(2k_r)}\left(\frac{t}{a_n}\right) \underbrace{\varphi^{n-r}\left(\frac{t}{a_n}\right)}_{\to\, e^{-t}} dt \\
&\sim \frac{\binom{n}{r}}{a_n \Gamma(2k)} \int_0^\infty \left(\frac{t}{a_n}\right)^{2k-1} \left(\frac{t}{a_n}\right)^{r-2k} \ell^r\left(\frac{a_n}{t}\right) \left(\prod_{j=1}^r \Gamma(2k_j - 1)\right) e^{-t}\, dt \\
&\sim \frac{\prod_{j=1}^r \Gamma(2k_j-1)}{r!\, \Gamma(2k)}\, \frac{n^r\, \ell^r(a_n)}{a_n{}^r} \underbrace{\int_0^\infty t^{r-1}\, e^{-t}\, dt}_{=(r-1)!} \\
&\sim \frac{\prod_{j=1}^r \Gamma(2k_j-1)}{r\, \Gamma(2k)} \left(\frac{\ell(a_n)}{\tilde{\ell}(a_n)}\right)^r
\end{aligned}$$

Note that $\ell(a_n)/\tilde{\ell}(a_n) \to 0$ for $n \to \infty$ and thus, opposed to the case $\alpha < 1$, only the summand with $r = 1$ contributes to the dominating asymptotic term of (6). Therefore we obtain

$$\mathbb{E}\,(T_n^k) \sim \frac{1}{2k-1} \frac{\ell(a_n)}{\tilde{\ell}(a_n)}.$$

□

Theorem 3.3. *Let X_1 belong to the domain of attraction of a stable law with index α, $1 \leq \alpha < 2$ and $\mu := \mathbb{E}(X_1) < \infty$. Then for all $k \geq 1$*

$$\mathbb{E}(T_n^k) \sim \frac{\Gamma(2k-\alpha)\Gamma(1+\alpha)}{\Gamma(2k)\, \mu^\alpha}\, n^{1-\alpha} \ell(n). \tag{15}$$

Proof. Since μ is finite, it follows that

$$\lim_{n\to\infty} \varphi^n(t/n) = e^{-\mu t} \quad \text{for all } t \geq 0. \tag{16}$$

However, in view of (16), we will use the change of variables $s = t/n$ in the representation (6). By virtue of Potter's Theorem, Lebesgue's dominated convergence theorem and Lemma 3.1 we then obtain

$$
\begin{aligned}
B(n,k_1,\ldots,k_r) &= \frac{\binom{n}{r}}{n\Gamma(2k)} \int_0^\infty \left(\frac{t}{n}\right)^{2k-1} \varphi^{(2k_1)}\left(\frac{t}{n}\right)\cdots\varphi^{(2k_r)}\left(\frac{t}{n}\right) \underbrace{\varphi^{n-r}\left(\frac{t}{n}\right)}_{\to e^{-\mu t}} dt \\
&\sim \frac{\alpha^r \binom{n}{r}}{n\Gamma(2k)} \int_0^\infty \left(\frac{t}{n}\right)^{2k-1} \left(\frac{t}{n}\right)^{r\,\alpha-2k} \ell^r\left(\frac{n}{t}\right) \left(\prod_{j=1}^r \Gamma(2k_j-\alpha)\right) e^{-\mu t}\, dt \\
&\sim \frac{\alpha^r \prod_{j=1}^r \Gamma(2k_j-\alpha)}{\Gamma(2k)} \underbrace{\frac{\binom{n}{r}\ell^r(n)}{n^{r\,\alpha}}}_{\sim n^{r(1-\alpha)}\ell^r(n)/r!} \underbrace{\int_0^\infty t^{r\,\alpha-1} e^{-\mu t}\, dt}_{=\Gamma(r\,\alpha)/\mu^{r\,\alpha}} \\
&\sim \frac{\alpha^r\,\Gamma(r\,\alpha)\prod_{j=1}^r \Gamma(2k_j-\alpha)}{r!\,\mu^{r\alpha}\,\Gamma(2k)}\, n^{r(1-\alpha)}\ell^r(n).
\end{aligned}
$$

Hence the first-order asymptotic behavior of (6) is solely determined by the term with $r = 1$ and we obtain

$$\mathbb{E}(T_n^k) \sim \frac{\Gamma(2k-\alpha)\Gamma(1+\alpha)}{\Gamma(2k)\,\mu^\alpha}\, n^{1-\alpha}\ell(n).$$

□

Remark 3.2. For the special case $k = 1$, (15) yields $\mathbb{E}(T_n) \sim \frac{\Gamma(2-\alpha)\Gamma(1+\alpha)}{\mu^\alpha}\, n^{1-\alpha}\ell(n)$, which is Theorem 5.1. of [5].

We pass to the case $\alpha > 2$.

Theorem 3.4. *Let $1-F(x) \sim x^{-\alpha}\,\ell(x)$ for some slowly varying function $\ell(x)$ and $\alpha > 2$. Then for all integers $k < \alpha - 1$*

$$\mathbb{E}(T_n^k) \sim \left(\frac{\mu_2}{\mu^2}\right)^k n^{-k} \tag{17}$$

and for $k > \alpha - 1$

$$\mathbb{E}(T_n^k) \sim \frac{\Gamma(2k-\alpha)\Gamma(1+\alpha)}{\Gamma(2k)\,\mu^\alpha}\, n^{1-\alpha}\ell(n). \tag{18}$$

If $k = \alpha - 1$, then

(i) (17) *holds if $\ell(x) = o(1)$ (and in particular if $\mathbb{E}(X_1^{k+1}) < \infty$),*

(ii) $\mathbb{E}(T_n^k) \sim \left(\left(\frac{\mu_2}{\mu^2}\right)^k + C\,\frac{\Gamma(k-1)\Gamma(k+2)}{\Gamma(2k)\,\mu^{1+k}}\right) n^{-k}$ *holds if $\ell(x) \sim C$ for a constant C,*

(iii) *and else* (18) *holds.*

Proof. Let us look at the quantity $B(n,k_1,\ldots,k_r)$. By Lemma 3.1 and the Bingham-Doney Lemma (see e.g. [3, Th.8.1.6]) the asymptotic behavior of $\varphi^{(m)}(s)$ at the origin is given by

$$(-1)^m\,\varphi^{(m)}(s) \sim \begin{cases} \alpha\,\Gamma(m-\alpha)\,s^{\alpha-m}\ell(1/s) & \text{if } m > \alpha \\ \alpha\,\tilde{\ell}(1/s) & \text{if } m = \alpha \text{ and } \mathbb{E}(X_1^m) = \infty \\ \mu_m & \text{if } m \le \alpha \text{ and } \mathbb{E}(X_1^m) < \infty \end{cases},$$

where $\tilde{\ell}(x) = \int_0^x (\ell(u)/u)du$ is itself a slowly varying function. For simplicity, let us first assume that $\alpha \notin \mathbb{N}$. Then one can conclude in an analogous way as in the proof of Theorem 3.3 that the asymptotic behavior of $B(n, k_1, \ldots, k_r)$ is given by

$$B(n, k_1, \ldots, k_r) \sim C_1\, n^{r-\alpha\, r_1 - 2(k-u_1)} \ell^{r_1}(n),$$

where r_1 is the number of integers among $k_1, \ldots, k_r$ that are greater than $\alpha/2$, u_1 is the sum of these and C_1 is some constant. It remains to determine the dominating asymptotic term among all possible $B(n, k_1, \ldots, k_r)$: If $r_1 > 0$, then $r_1 = 1$, $u_1 = k$ and thus $r = 1$ yields the largest exponent, so that the asymptotic order is $n^{1-\alpha}\ell(n)$. Note that $r_1 > 0$ is possible for $2k > \alpha$ only. For $r_1 = 0$, on the other hand, $r = k$ and thus $k_1 = \ldots = k_r = 1$ dominates leading to asymptotic order n^{-k}. Hence the asymptotically dominating power among $B(n, k_1, \ldots, k_r)$ is given by $\max(1-\alpha, -k)$. From this we see that for $k < \alpha - 1$, $r = k$ dominates and we obtain from (6)

$$\mathbb{E}(T_n^k) \sim k! \frac{n^k\, \mu_2^k\, \Gamma(2k)}{k!\, \Gamma(2k)\, n^{2k}\, \mu^{2k}} \sim \left(\frac{\mu_2}{\mu^2}\right)^k n^{-k}.$$

Alternatively, if $k > \alpha - 1$, the term with $r = 1$ dominates and we obtain (18) in just the same way as in Theorem 3.3.
Finally, the above conclusions also hold for $\alpha \in \mathbb{N}$ except when $k = \alpha - 1$. In the latter case the slowly varying function $\ell(x)$ determines which of the two terms $n^{1-\alpha}\ell(n)$ (corresponding to $r = 1$) and n^{-k} (corresponding to $r = k$) dominates the asymptotic behavior: if $\ell(x) = o(1)$ (which due to $\mathbb{E}(X_1^{k+1}) \sim (k+1) \int_0^n x^{-1}\ell(x)\, dx$ is in particular fulfilled for $\mathbb{E}(X_1^{k+1}) < \infty$), the second one dominates. If $\ell(x) \sim$ *const.*, then both terms matter and the assertion of the theorem follows. □

Corollary 3.1. *If* $1 - F(x) \sim x^{-2}\,\ell(x)$, *then for* $k \geq 2$

$$\mathbb{E}(T_n^k) \sim \frac{1}{(k-1)(2k-1)\mu^2} \frac{\ell(n)}{n}$$

and

$$\mathbb{E}(T_n) \sim \begin{cases} \frac{\mu_2}{\mu^2\, n} & \text{if } \mathbb{E}(X_1^2) < \infty \\ \frac{2}{\mu^2} \frac{\tilde{\ell}(n)}{n} & \text{if } \mathbb{E}(X_1^2) = \infty. \end{cases}$$

Proof. One can easily verify that Theorem 3.4 remains true for $\alpha = 2$ except for $k = 1$ in the case $\mathbb{E}(X_1^2) = \infty$. In the latter case obviously $r = 1$ and one obtains (using $\varphi''(s) \sim 2\,\tilde{\ell}(1/s)$)

$$\mathbb{E}(T_n) \sim B(n, 1) \sim \frac{2\, n\, \tilde{\ell}(n)}{n^2} \int_0^\infty t e^{-\mu\, t}\, dt \sim \frac{2}{\mu^2} \frac{\tilde{\ell}(n)}{n},$$

which is already contained in [5, Theorem 5.2]. □

Remark 3.3. One might wonder whether a general limit result for $\mathbb{E}(T_n^k)$ for X_1 in the domain of attraction of a normal law (in the spirit of Theorem 5.2 of [5] for $k = 1$) can be obtained with the integral representation approach used in this paper. This is however not the case: From $\int_0^x y^2\, dF(y) \sim \ell_2(x)$ (where $\ell_2(x)$ is a slowly varying function) it follows by partial integration that $\varphi^{(2k)}(s)/\ell_2(1/s) = o(s^{2-2k})$ for $k > 1$ as $s \to 0$, but the latter

	$\mathbb{E}(T_n)$	$\mathrm{Var}\,(T_n)$	$\mathrm{Var}\,(T_n)/\mathbb{E}(T_n)$
$0 < \alpha < 1$	$1 - \alpha$	$\frac{\alpha(1-\alpha)}{3}$	$\frac{\alpha}{3}$
$\alpha = 1$	$\frac{\ell(a_n)}{\tilde{\ell}(a_n)}\ (\to 0)$	$\frac{1}{3}\,\frac{\ell(a_n)}{\tilde{\ell}(a_n)}\ (\to 0)$	$\frac{1}{3}$
$1 < \alpha < 2$	$\frac{\Gamma(2-\alpha)\Gamma(1+\alpha)}{\mu^2}\, n^{1-\alpha}\ell(n)$	$\frac{\Gamma(4-\alpha)\Gamma(1+\alpha)}{6\mu^\alpha}\, n^{1-\alpha}\ell(n)$	$\frac{(3-\alpha)(2-\alpha)}{6}$
$\alpha = 2$	$\frac{2}{\mu^2}\,\frac{\tilde{\ell}(n)}{n}$	$\frac{\ell(n)}{3\,n\,\mu^2}$	$\frac{1}{6}\,\frac{\ell(n)}{\tilde{\ell}(n)}\ (\to 0)$
$2 < \alpha < 4$	$\frac{\mu_2}{\mu^2\,n}$	$\frac{\Gamma(4-\alpha)\Gamma(1+\alpha)}{6\,\mu^\alpha}\, n^{1-\alpha}\ell(n)$	$\frac{\Gamma(4-\alpha)\Gamma(1+\alpha)}{6\,\mu^{\alpha-2}\,\mu_2}\, n^{2-\alpha}\ell(n)$
$\alpha \geq 4$	$\frac{\mu_2}{\mu^2\,n}$	$\frac{\mu_4\mu^2-\mu_2^2\mu^2+4\,\mu_2^3-4\mu\mu_2\mu_3}{\mu^6}\,\frac{1}{n^3}$	$\frac{\mu_4\mu^2/\mu_2-\mu_2\mu^2+4\,\mu_2^2-4\mu\mu_3}{\mu^4}\,\frac{1}{n^2}$

TABLE 1. First order asymptotic terms of $\mathbb{E}(T_n)$, $\mathrm{Var}\,(T_n)$ and $\mathrm{Var}\,(T_n)/\mathbb{E}(T_n)$ for $1 - F(x) \sim x^{-\alpha}\ell(x)$ as a function of α

is not strong enough to identify the dominating term among the $B(n, k_1, \ldots, k_r)$ without any further assumptions on the distribution of X_1.

As an illustration of the results of this paper, Table 1 gives the first order asymptotic terms of $\mathbb{E}(T_n)$, $\mathrm{Var}\,(T_n)$ and the dispersion $\mathrm{Var}\,(T_n)/\mathbb{E}(T_n)$ as a function of α. Note that the entries for $\alpha > 2$ have been obtained by calculating second-order asymptotic terms. The result for $\alpha > 4$ in the table actually holds whenever $\mu_4 < \infty$, since in this case the derivation of second-order terms does not rely on the assumption of regular variation and one obtains $\mathbb{E}(T_n^2) = \frac{\mu_2^2}{\mu^4}\frac{1}{n^2} + (\frac{10\mu_2^3-3\mu_2^2\mu^2-8\mu\mu_2\mu_3+\mu^2\mu_4}{\mu^6})\frac{1}{n^3} + O(\frac{1}{n^4})$ and $\mathbb{E}^2(T_n) = \frac{\mu_2^2}{\mu^4}\frac{1}{n^2} + (\frac{6\mu_2^3-4\mu\mu_2\mu_3-2\mu^2\mu_2^2}{\mu^6})\frac{1}{n^3} + O(\frac{1}{n^4}))$.
From Table 1 we see that the dispersion of T_n is a continuous function in α with its maximum in $\alpha = 1$ (see Figure 2).

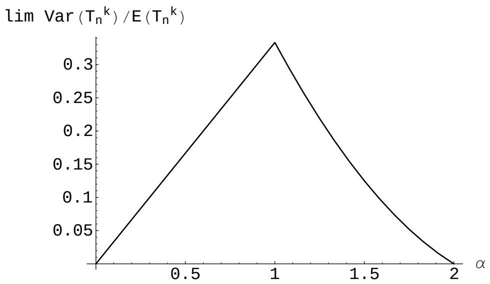

FIGURE 2. Limit of the dispersion of T_n as a function of α

4. ESTIMATION OF THE EXTREME VALUE INDEX FOR PARETO-TYPE TAILS

The results of Section 3 also give rise to an alternative and seemingly new method for estimating the extreme value index $1/\alpha$ for Pareto-type tails $1 - F(x) \sim x^{-\alpha}\ell(x)$ with

$0 < \alpha < 2$ from a given data set of independent observations (see e.g. [2] for other estimators of the extreme value index). In fact, plotting $n\,T_n$ against n will tend to a line with slope $1-\alpha$, if $0 < \alpha < 1$ and plotting $\log(n\,T_n)$ against $\log n$ will tend to a line with slope $2-\alpha$, if $1 < \alpha < 2$. The asymptotic behavior of higher order moments of $n\,T_n$ available from Section 3 can then be used to increase the efficiency of the estimation procedure.
At the same time, this provides a technique to test the finiteness of the mean of a distribution in the domain of attraction of a stable law.

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