On the dual risk model with tax payments

Hansjörg Albrecher[*], Andrei Badescu[†] and David Landriault[‡]

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Abstract

In this paper, we study the dual risk process in ruin theory (see e.g. Cramér (1955), Takacs (1967) and Avanzi et al. (2007)) in the presence of tax payments according to a loss-carry forward system. For arbitrary inter-innovation time distributions and exponentially distributed innovation sizes, an expression for the ruin probability with tax is obtained in terms of the ruin probability without taxation. Furthermore, expressions for the Laplace transform of the time to ruin and arbitrary moments of discounted tax payments in terms of passage times of the risk process are determined. Under the assumption that the inter-innovation times are (mixtures of) exponentials, explicit expressions are obtained. Finally, we determine the critical surplus level at which it is optimal for the tax authority to start collecting tax payments.

1 Introduction

The classical risk model describes the surplus process $\{U(t), t \geq 0\}$ of an insurance company as

$$U(t) = u + ct - S(t), \tag{1}$$

where u is the initial surplus in the portfolio, c is the premium rate and $\{S(t), t \geq 0\}$ represents the aggregate claim amount process that is assumed to be a compound Poisson process (see e.g. Bühlmann (1970), Grandell (1981) or Rolski et al. (1999)).

As pointed out by e.g. Avanzi et al. (2007), its dual process may also be relevant for companies whose inherent business involves a constant flow of expenses while revenues arrive occassionally due to some contingent events (e.g. discoveries, sales). For instance, pharmaceutical or petroleum companies are prime examples of companies for which it is reasonable to model their surplus process $\{R(t), t \geq 0\}$ by

$$R(t) = u - ct + S(t), \tag{2}$$

where u is the company's initial surplus, c is now the constant rate at which expenses are paid out and $\{S(t), t \geq 0\}$ is the aggregate revenue process.

In this paper, we assume that $\{S(t), t \geq 0\}$ is a pure jump process defined as $S(t) = \sum_{i=1}^{N(t)} Y_i$ where

- the innovation number process $\{N(t), t \geq 0\}$ is a renewal process with independent and identically distributed (i.i.d.) inter-innovation times $T_1, T_2, \ldots$ with density function (d.f.) k.

[*]Radon Institute, Austrian Acadamy of Sciences, and Department of Financial Mathematics, University of Linz, Altenbergerstrasse 69, 4040 Linz, Austria. Supported by the Austrian Science Fund Project P18392.

[†]Department of Statistics, University of Toronto, 100 St. George St., Toronto, Canada

[‡]Department of Statistics and Actuarial Science, University of Waterloo, 200 University Avenue West, Waterloo, Canada

- the random variable (r.v.) Y_j corresponds to the revenue associated to the j-th innovation ($j = 1, 2, ...$). The r.v.'s $\{Y_j\}_{j=1}^{\infty}$ form of a sequence of i.i.d. exponentially distributed r.v.'s with d.f. $p(y) = \beta \exp\{-\beta y\}$ ($y \geq 0$).

We also assume that the innovation sizes $\{Y_j\}_{j=1}^{\infty}$ and the inter-innovation times $\{T_j\}_{j=1}^{\infty}$ are mutually independent. For convenience, we define the sequence of innovation times $\{W_j\}_{j=1}^{\infty}$ by $W_0 = 0$ and $W_j = T_1 + ... + T_j$ for $j \in \mathbb{N}^+$.

In Albrecher and Hipp (2007), the effect of tax payments under a loss-carry forward system in the Cramér-Lundberg model was studied and a remarkably simple relationship between the ruin probability of the surplus process with and without tax has been established. In addition, the authors found a simple criterion to determine the optimal surplus level at which taxation starts subject to the maximization of the expected discounted tax payments before ruin. It is natural to question whether similar relations hold in the dual model (2) which is of comparable complexity to the Cramér-Lundberg model and is in several ways closely related.

In this paper, we address the above questions by introducing a tax component of loss-carry-forward type in the dual surplus process (2) with general inter-innovation times and exponential innovation sizes. Hence the company pays tax at rate γ ($0 < \gamma < 1$) on the excess of each new record high of the surplus over the previous one. Due to the structure of the process, a new record high can only be achieved by an innovation which implies that tax payments only occur at the innovation times W_j ($j = 1, 2, ...$). In this paper, we show that whereas the relationship for the ruin probability with and without tax is slightly more complicated in the dual model, the criterion for identifying the optimal starting taxation level is identical to the one known from the Cramér-Lundberg risk model.

Let $\sigma_0 = 0$ and define

$$\sigma_n = \inf_{k \in \mathbb{N}} \left\{ k > \sigma_{n-1} : \sum_{j=\sigma_{n-1}+1}^{k} (Y_j - cT_j) > 0 \right\},$$

to be the number of innovations up to the time of the n-th record high. Let $J_0 = u$ and

$$J_n = J_{n-1} + (1 - \gamma) \sum_{j=\sigma_{n-1}+1}^{\sigma_n} (Y_j - cT_j), \tag{3}$$

be the value of the n-th record high. The resulting surplus process in the dual model with tax is given by

$$\begin{aligned} R_\gamma(t) &= J_{\Xi(t)} - \sum_{j=\sigma_{\Xi(t)}+1}^{N(t)} (cT_j - Y_j) - c\left(t - W_{N(t)}\right) \\ &= J_{\Xi(t)} - c\left(t - W_{\sigma_{\Xi(t)}}\right) + \sum_{j=\sigma_{\Xi(t)}+1}^{N(t)} Y_j \end{aligned} \tag{4}$$

where $\Xi(t) = \sup\{n \in \mathbb{N} : W_{\sigma_n} \leq t\}$. For practical considerations, we assume that the net profit condition

$$c\beta E[T_1] < 1 \tag{5}$$

is satisfied, i.e. the drift of the (before-tax) surplus process (2) is positive.

The time to ruin τ_γ of the surplus process (4) is defined as $\tau_\gamma = \inf\{t \geq 0, R_\gamma(t) = 0\}$ (with the convention $\tau_\gamma = \infty$ if $R_\gamma(t) > 0$ for all $t \geq 0$) and its Laplace transform is denoted by

$$\rho_{\gamma,\delta}(u) = E\left[e^{-\delta\tau_\gamma} 1_{\{\tau_\gamma < \infty\}} | R_\gamma(0) = u\right], \tag{6}$$

where $\delta \geq 0$ can also be interpreted as a discount rate and 1_A denotes the indicator of the event A. An important special case of (6) is the ruin probability $\psi_\gamma(u) = \rho_{\gamma,0}(u)$.

We point out that the surplus process $R_0(t)$ (i.e. $\gamma = 0$) is the dual equivalent of the Sparre Andersen risk model in ruin theory and can also be viewed as a $GI/M/1$ queueing system (see e.g. Cohen (1982) and Prabhu (1998)). Thus, the time to ruin in the surplus process (4) with $\gamma = 0$ can be interpreted as the length of the first busy period in the $GI/M/1$ queue. Here, the positive security loading condition (5) translates into a traffic intensity $\rho > 1$ in the queueing system $GI/M/1$ (congested queue). However, it seems that most of the explicit results in queueing theory that are relevant for the present purpose are based on the assumption $\rho < 1$, so that we will derive some results for the congested queue in the Appendix.

For the limit $\gamma = 1$, it is immediate that the surplus process $\{R_1(t), t \geq 0\}$ corresponds to a dual model with a horizontal barrier strategy, where the initial surplus level is at the barrier (see Avanzi et al. (2007) for a detailed study of that case). This gives rise to an alternative interpretation of tax payments, as they can also be viewed as dividend payments to shareholders who ask for a proportion γ of each new profit.

The paper is organized as follows: in Section 2, we derive a relation between the ruin probability with and without tax payments in this dual model and give some more explicit expressions for (mixtures of) exponential inter-innovation times. In Section 3, the Laplace transform of the time to ruin for the surplus process $R_\gamma(t)$ in (4) is studied and an explicit expression is obtained under exponential inter-innovation times. Section 4 is then devoted to the analysis of the moments of the discounted tax payments before ruin. Finally, in Section 5, we determine the critical surplus level at which it is optimal for a tax authority to start collecting taxes.

2 Ruin probability

First, we consider the impact of the defined tax system on the ruin probability of the surplus process of the dual model (4). The analysis will be carried out by a study of its complement, namely the non-ruin probability $\phi_\gamma(u) = 1 - \psi_\gamma(u)$. Indeed, starting with an initial surplus u, the surplus process $\{R_\gamma(t), t \geq 0\}$ shall (up-) cross level u at least once in order to avoid ruin. Let

$$\xi_u = \inf\{t > 0 : R_\gamma(t) \geq u\}, \tag{7}$$

be the time of the first up-crossing of $R_\gamma(t)$ above a level exceeding its initial value u. Clearly, ξ_u is the first time at which the surplus process (4) reaches a new record high. It is well known that (5) is sufficient to guarantee the existence of such a passage time. Let $L = \inf\{R_\gamma(t) : 0 \leq t < \xi_u\}$ be the minimum surplus level up to this passage time and define the Laplace transform of ξ_u, conditioned on avoiding ruin up to time ξ_u, by

$$g_\delta(u) = E\left[e^{-\delta\xi_u} 1_{\{L>0\}} \,|R_\gamma(0) = u\right]. \tag{8}$$

Note that none of the quantities ξ_u, L and $g_\delta(u)$ depends on γ, as the first taxation only starts at ξ_u. Clearly, $g_0(u)$ is equivalent to the probability that the classical risk process (1) starting at $u = 0$ (with negative safety loading) drops below level 0 prior to reach level u. We also point out that $g_0(u)$ is a crucial quantity in the fluid flow literature (where often the terminology ${}^u\Psi(0)$ is used, see Ahn et al. (2007)).

Lemma 2.1 *Under the net profit condition (5), we have* $\lim_{u\to\infty} \psi_\gamma(u) = 0$ *for* $\gamma < 1$.

Proof: Under (5) and with $\gamma < 1$, the probability that $R_\gamma(t)$ reaches a new record high (ignoring a possible ruin event in the interim) is 1. Consequently $R_\gamma(t)$ reaches a new record high infinitely often which implies that $\sup_{n\in\mathbb{N}}\left(W_{\sigma_{n+1}} - W_{\sigma_n}\right)$ is finite a.s. From (4), an upper bound for the ruin probability $\psi_\gamma(u)$ is given by

$$\psi_\gamma(u) \leq \Pr\left(\inf_{n\in\mathbb{N}}\left\{J_n - c\left(W_{\sigma_{n+1}} - W_{\sigma_n}\right)\right\} < 0\right).$$

Also, from (3), it is clear that $\inf_{n\in\mathbb{N}}\{J_n\} = u$ from which

$$\psi_\gamma(u) \leq \Pr\left(\sup_{n\in\mathbb{N}} c\left(W_{\sigma_{n+1}} - W_{\sigma_n}\right) > u\right).$$

Finally,

$$\lim_{u\to\infty}\psi_\gamma(u) \leq \lim_{u\to\infty}\Pr\left(\sup_{n\in\mathbb{N}} c\left(W_{\sigma_{n+1}} - W_{\sigma_n}\right) > u\right) = 0.$$

$\square$

Proposition 2.2 *For the surplus process (4) and $\gamma < 1$, the infinite-time ruin probability $\psi_\gamma(u)$ $(u \geq 0)$ is given by*

$$\psi_\gamma(u) = 1 - (1 - \psi_0(u))^{\frac{1}{1-\gamma}} (g_0(u))^{-\frac{\gamma}{1-\gamma}}. \tag{9}$$

Proof: Given that the (after-tax) excess of the surplus level over u at time ξ_u is exponentially distributed with mean $(1-\gamma)/\beta$, the survival probability $\phi_\gamma(u)$ can be written as

$$\begin{aligned}\phi_\gamma(u) &= g_0(u) \int_0^\infty \phi_\gamma(u+x) \frac{\beta}{1-\gamma} e^{-\frac{\beta}{1-\gamma}x} dx \\ &= g_0(u) \int_u^\infty \phi_\gamma(x) \frac{\beta}{1-\gamma} e^{-\frac{\beta}{1-\gamma}(x-u)} dx.\end{aligned} \tag{10}$$

Differentiating (10) with respect to (w.r.t.) u yields

$$\begin{aligned}\phi_\gamma'(u) &= g_0'(u) \int_u^\infty \phi_\gamma(x) \frac{\beta}{1-\gamma} e^{-\frac{\beta}{1-\gamma}(x-u)} dx + \frac{\beta}{1-\gamma} \phi_\gamma(u) (1 - g_0(u)) \\ &= \left(\frac{d}{du} \ln g_0(u) + \frac{\beta}{1-\gamma} (1 - g_0(u)) \right) \phi_\gamma(u).\end{aligned} \tag{11}$$

The solution of the differential equation (11) can be expressed as

$$\phi_\gamma(u) = \frac{\phi_\gamma(\infty)}{g_0(\infty)} g_0(u) e^{-\frac{1}{1-\gamma} \int_u^\infty \beta(1-g_0(u))du},$$

where $\phi_\gamma(\infty) = \lim_{u\to\infty} \phi_\gamma(u)$ and $g_0(\infty) = \lim_{u\to\infty} g_0(u)$. Using Lemma 2.1, $\phi_\gamma(\infty) = 1$ and $g_0(\infty) = 1$ which leads to

$$\phi_\gamma(u) = g_0(u) e^{-\frac{1}{1-\gamma} \int_u^\infty \beta(1-g_0(u))du}. \tag{12}$$

Since (12) also holds for $\gamma = 0$ (the model without tax), we arrive at

$$\phi_\gamma(u) = (g_0(u))^{1-\frac{1}{1-\gamma}} (\phi_0(u))^{\frac{1}{1-\gamma}}.$$

Finally, note that $\phi_\gamma(u) = 0$ for all $u \geq 0$ also solves the differential equation (11), but can be ruled out due to Lemma 2.1. $\square$

In what follows, we consider two particular cases of the dual Sparre Andersen risk model with taxation (4). For that purpose, we first discuss a link between $g_\delta(u)$ and the expected discounted dividends $V_\delta(u;b)$ paid before ruin in the surplus process (2), if a dividend barrier strategy at level b is applied (see Avanzi et al. (2007)). Using a simple sample path argument, it is immediate that for exponentially distributed innovation sizes with mean $1/\beta$,

$$V_\delta(u;u) = g_\delta(u) \left(\frac{1}{\beta} + V_\delta(u,u) \right),$$

or equivalently

$$g_\delta(u) = \frac{V_\delta(u;u)}{\frac{1}{\beta} + V_\delta(u,u)}, \tag{13}$$

for $u \geq 0$.

Example 2.1 *(Exponential inter-innovation times) Let $T_j \sim Exp(\lambda)$ $(j = 1, 2, \ldots)$. From Eq. (3.5) in Avanzi et al. (2007), (13) becomes*

$$g_\delta(u) = \frac{e^{-Ru} - e^{\rho u}}{\frac{\beta}{\beta+R} e^{-Ru} - \frac{\beta}{\beta-\rho} e^{\rho u}}, \quad u \geq 0, \tag{14}$$

where $\rho \geq 0$ and $-R < 0$ are the two solutions to the characteristic equation $s^2 - \left(\beta - \frac{\lambda+\delta}{c}\right) s - \frac{\beta}{c}\delta = 0$. Consequently,

$$g_0(u) = \frac{1 - e^{-\left(\frac{\lambda}{c} - \beta\right)u}}{1 - \frac{c\beta}{\lambda} e^{-\left(\frac{\lambda}{c} - \beta\right)u}}. \tag{15}$$

Since in the dual model with exponential inter-innovation times we have

$$\psi_0(u) = e^{-\left(\frac{\lambda}{c}-\beta\right)u} \tag{16}$$

(see e.g. Cramér (1955) or Gerber (1979)), this together with (9) leads to

$$\psi_\gamma(u) = 1 - \left(1 - e^{-\left(\frac{\lambda}{c}-\beta\right)u}\right)\left(1 - \frac{c\beta}{\lambda} e^{-\left(\frac{\lambda}{c}-\beta\right)u}\right)^{\frac{\gamma}{1-\gamma}}.$$

Remark 2.1 *By reflection, it is clear that $g_\delta(u)$ in the dual risk model (2) with exponential inter-innovation times with mean $1/\lambda$ and general innovation sizes with d.f. q is equivalent to the Laplace transform of the time to ruin in the Cramér-Lundberg risk model (with initial surplus 0, Poisson parameter λ, general claim sizes with d.f. q and negative safety loading) avoiding level u in the interim. From Gerber (1979), it follows that*

$$g_0(u) = 1 - \frac{f_0(0)}{f_0(u)}, \qquad u \geq 0, \tag{17}$$

where $\{f_0(u), u \geq 0\}$ is the solution of the integro-differential equation

$$\left(\mathcal{D} - \frac{\lambda}{c}\right) f_0(u) + \frac{\lambda}{c}\int_0^u f_0(u-y)\, q(y)\, dy = 0, \tag{18}$$

which is unique up to a constant. Here $\mathcal{D}$ denotes the differential operator. For instance, for $q(y) = \beta e^{-\beta y}$, one readily obtains (15) as the solution of the integro-differential equation (18) with (17).

Example 2.2 *(Mixture of exponentials inter-innovation times) Let $k(t) = \sum_{i=1}^n p_i \lambda_i \exp(-\lambda_i t)$. In the Appendix, we prove that*

$$\psi_0(u) = \sum_{j=0}^m d_j e^{s_j u} \tag{19}$$

and

$$V_\delta(u;b) = \sum_{j=0}^m C_j(b)\, e^{s_j u}, \quad (0 < u < b),$$

where $s_0, s_1, ..., s_n$ denote the solutions to the fundamental equation (63) and $C_j(b), d_j$ $(j = 0, \dots, m)$ are constants (which complements results of Avanzi et al. (2007)). Thus,

$$g_\delta(u) = 1 - \left(1 + \beta \sum_{j=0}^m C_j(u)\, e^{s_j u}\right)^{-1} \tag{20}$$

and finally

$$\psi_\gamma(u) = 1 - \left(1 - \sum_{j=0}^m d_j e^{s_j u}\right)^{\frac{1}{1-\gamma}} \left(1 - \left(1 + \beta \sum_{j=0}^m C_j(u)\, e^{s_j u}\right)^{-1}\right)^{-\frac{\gamma}{1-\gamma}}.$$

Remark 2.2 *Exploiting the connection between fluid flows and risk processes (see e.g. Ramaswami (2006) and Badescu et al. (2007)), a probabilistic approach can alternatively be used to identify the components $g_0(u)$ and $\psi_0(u)$ in (9). Indeed, for phase-type distributed $PH(\alpha, A)$ inter-innovation times, a reflection of the dual risk model (2) readily leads to*

$$g_0(u) = \alpha\ {}^{u}\mathbf{\Psi}(0)\ \mathbf{1},$$

and

$$\psi_0(u) = \alpha\left[I - {}^{u}\mathbf{\Psi}(0)\,\mathbf{\Psi}^r(0)\right]^{-1} f_{12}(0,u,0)\,\mathbf{1},$$

where ${}^{u}\mathbf{\Psi}(\delta)$ is the Laplace transform of the busy period in a finite buffer (at level u) fluid flow, $\mathbf{\Psi}^{r}(\delta)$ is the Laplace transform of a busy period in an infinite buffer reflected fluid flow and $f_{12}(0,u,\delta)$ is the Laplace transform of the first passage from level 0 to level u (avoiding level 0 en route) of a fluid flow with generator

$$Q = \begin{bmatrix} A & -A\mathbf{1} \\ \beta\alpha & -\beta \end{bmatrix}$$

We refer the reader to Badescu et al. (2007) for the calculation of ${}^{u}\mathbf{\Psi}(\delta)$, $\mathbf{\Psi}^{r}(\delta)$ and $f_{12}(\delta,u,0)$ in more general Markovian arrival risk processes.

3 Laplace transform of the time to ruin τ_γ

Let us now consider the Laplace transform $\rho_{\gamma,\delta}(u)$ of the time to ruin τ_γ. Starting with an initial surplus u, the surplus process $\{R_\gamma(t), t \geq 0\}$ can either reach a new record high at time ξ_u avoiding ruin en route or reach level 0 before any visit to levels greater than u. For the latter, the Laplace transform $h_\delta(u)$ of the corresponding passage time ν_u is defined as

$$h_\delta(u) = E\left[e^{-\delta\nu_u} 1_{\{Z\leq u\}} \,|R_\gamma(0) = u\right], \tag{21}$$

where $Z = \sup\{R_\gamma(t) : 0 \leq t < \nu_u < \infty\}$ corresponds to the maximum surplus level before ruin.

Thus, by conditioning on these two scenarios, we obtain

$$\rho_{\gamma,\delta}(u) = g_\delta(u) \int_u^\infty \frac{\beta}{1-\gamma} e^{-\frac{\beta}{1-\gamma}(x-u)} \rho_{\gamma,\delta}(x)\, dx + h_\delta(u), \tag{22}$$

for $u \geq 0$. Differentiating (22) w.r.t. u yields

$$\rho'_{\gamma,\delta}(u) = g'_\delta(u) \int_u^\infty \frac{\beta}{1-\gamma} e^{-\frac{\beta}{1-\gamma}(x-u)} \rho_{\gamma,\delta}(x)\, dx + \frac{\beta}{1-\gamma}(1-g_\delta(u))\rho_{\gamma,\delta}(u) + h'_\delta(u) - \frac{\beta}{1-\gamma} h_\delta(u). \tag{23}$$

With (22) this can be rewritten as

$$\rho'_{\gamma,\delta}(u) = \left(\frac{d}{du}\ln g_\delta(u)\right)(\rho_{\gamma,\delta}(u) - h_\delta(u)) + \frac{\beta}{1-\gamma}(1-g_\delta(u))\rho_{\gamma,\delta}(u) + h'_\delta(u) - \frac{\beta}{1-\gamma} h_\delta(u),$$

and further

$$\begin{aligned} &\rho'_{\gamma,\delta}(u) - \left(\frac{d}{du}\ln g_\delta(u) + \frac{\beta}{1-\gamma}(1-g_\delta(u))\right)\rho_{\gamma,\delta}(u) \\ &= h'_\delta(u) - \left(\frac{d}{du}\ln g_\delta(u) + \frac{\beta}{1-\gamma}(1-g_\delta(u))\right) h_\delta(u) - \frac{\beta}{1-\gamma} g_\delta(u) h_\delta(u). \end{aligned} \tag{24}$$

The non-homogeneous differential equation (24) can be solved by applying the multiplicative factor $\exp\left\{-\frac{\beta}{1-\gamma}\int_0^u (1-g_\delta(z))\, dz\right\} \Big/ g_\delta(u)$ on both sides, leading to

$$\frac{d}{du}\left(e^{-\frac{\beta}{1-\gamma}\int_0^u (1-g_\delta(z))dz} \frac{\rho_{\gamma,\delta}(u)}{g_\delta(u)}\right) = \frac{d}{du}\left(e^{-\frac{\beta}{1-\gamma}\int_0^u (1-g_\delta(z))dz} \frac{h_\delta(u)}{g_\delta(u)}\right) - \frac{\beta}{1-\gamma} h_\delta(u)\, e^{-\frac{\beta}{1-\gamma}\int_0^u (1-g_\delta(z))dz}. \tag{25}$$

Integrating from y to ∞ together with $\lim_{u\to\infty}\rho_{\gamma,\delta}(u) = \lim_{u\to\infty} h_\delta(u) = 0$ and $\lim_{u\to\infty} g_\delta(u) > 0$ (due to (5)), one concludes

$$\rho_{\gamma,\delta}(y) = h_\delta(y) + \frac{\beta}{1-\gamma} g_\delta(y) \int_y^\infty h_\delta(u)\, e^{-\frac{\beta}{1-\gamma}\int_y^u (1-g_\delta(z))dz} du. \tag{26}$$

Exponential inter-innovation times

For the case $T_j \sim \text{Exp}(\lambda), j = 1, 2, ..,$ an expression for $g_\delta(u)$ has been obtained in (14), which can be rewritten as

$$\begin{aligned} 1 - g_\delta(u) &= \frac{1}{\beta}\frac{d}{du}\ln\left(\frac{1}{\beta+R}e^{-Ru} - \frac{1}{\beta-\rho}e^{\rho u}\right) \\ &= \frac{1}{\beta}\left(\rho + \frac{d}{du}\ln(1-\eta_\delta(u))\right) \end{aligned} \tag{27}$$

with $\eta_\delta(u) = (\beta - \rho)\, e^{-(R+\rho)u}/(\beta + R)$ and further

$$g_\delta(u) = \frac{\beta-\rho}{\beta}\left(1 - e^{-(R+\rho)u}\right)(1-\eta_\delta(u))^{-1}. \tag{28}$$

Substituting the latter expressions in (26) yields

$$\rho_{\gamma,\delta}(y) = h_\delta(y) + \frac{\beta-\rho}{1-\gamma}\left(e^{\rho y} - e^{-Ry}\right)(1-\eta_\delta(y))^{\frac{\gamma}{1-\gamma}}\int_y^\infty h_\delta(u)\, e^{-\frac{\rho}{1-\gamma}u}\left(\frac{1}{1-\eta_\delta(u)}\right)^{\frac{1}{1-\gamma}} du. \tag{29}$$

We now need to identify an explicit expression for $h_\delta(u)$. We again reflect the surplus process (4) to relate it with the surplus process (1) and find from Gerber (1979, p.147) that

$$h_\delta(u) = \frac{\rho+R}{\beta+R}e^{-Ru}(1-\eta_\delta(u))^{-1}. \tag{30}$$

Using (30), (29) turns into

$$\rho_{\gamma,\delta}(y) = h_\delta(y) + \frac{\beta-\rho}{1-\gamma}\frac{\rho+R}{\beta+R}\left(e^{\rho y} - e^{-Ry}\right)(1-\eta_\delta(y))^{\frac{\gamma}{1-\gamma}}\int_y^\infty e^{-\left(R+\frac{\rho}{1-\gamma}\right)u}\left(\frac{1}{1-\eta_\delta(u)}\right)^{1+\frac{1}{1-\gamma}} du. \tag{31}$$

From Newton's generalized binomial theorem (see Graham et al. (1994)), (31) becomes

$$\rho_{\gamma,\delta}(y) = h_\delta(y) + \frac{\beta-\rho}{1-\gamma}\frac{\rho+R}{\beta+R}\left(e^{\rho y} - e^{-Ry}\right)(1-\eta_\delta(y))^{\frac{\gamma}{1-\gamma}}\int_y^\infty e^{-\left(R+\frac{\rho}{1-\gamma}\right)u}\sum_{j=0}^\infty C_j^{-\left(1+\frac{1}{1-\gamma}\right)}\left(-\frac{\beta-\rho}{\beta+R}\right)^j e^{-j(R+\rho)u} du. \tag{32}$$

where $C_j^r = \frac{1}{j!}\prod_{k=0}^{j-1}(r-k)$ for $k \in \mathbb{N}$. Simple modifications of (32) give

$$\begin{aligned} \rho_{\gamma,\delta}(y) &= h_\delta(y) + \frac{\beta-\rho}{1-\gamma}\frac{\rho+R}{\beta+R}\left(e^{\rho y} - e^{-Ry}\right)(1-\eta_\delta(y))^{\frac{\gamma}{1-\gamma}}\sum_{j=0}^\infty C_j^{\frac{1}{1-\gamma}+j}\left(\frac{\beta-\rho}{\beta+R}\right)^j \frac{e^{-\left((j+1)R+\left(j+\frac{1}{1-\gamma}\right)\rho\right)y}}{(j+1)R+\left(j+\frac{1}{1-\gamma}\right)\rho} \\ &= h_\delta(y) + \frac{\beta-\rho}{1-\gamma}\frac{\rho+R}{\beta+R}\left(e^{\rho y} - e^{-Ry}\right)e^{-\left(R+\frac{1}{1-\gamma}\rho\right)y}(1-\eta_\delta(y))^{\frac{\gamma}{1-\gamma}}\sum_{j=0}^\infty C_j^{\frac{1}{1-\gamma}+j}\frac{(\eta_\delta(y))^j}{j+\frac{R+\frac{\rho}{1-\gamma}}{\rho+R}}. \end{aligned} \tag{33}$$

Using the Gauss Hypergeometric function defined as

$$F(a,b;\xi;x) = \frac{\Gamma(\xi)}{\Gamma(a)\Gamma(b)}\sum_{j=0}^\infty \frac{\Gamma(a+j)\Gamma(b+j)}{\Gamma(\xi+j)\Gamma(j+1)}x^j, \tag{34}$$

(see e.g. Abramowitz and Stegun (1972)), (33) can be written as

$$\begin{aligned} \rho_{\gamma,\delta}(y) &= \frac{\rho+R}{\beta+R}e^{-Ry}(1-\eta_\delta(y))^{-1} \\ &+ \frac{\beta-\rho}{1-\gamma}\frac{R+\frac{\rho}{1-\gamma}}{\beta+R}\left(e^{\rho y} - e^{-Ry}\right)e^{-\left(R+\frac{1}{1-\gamma}\rho\right)y}(1-\eta_\delta(y))^{\frac{\gamma}{1-\gamma}} F\left(1+\frac{1}{1-\gamma}, \frac{R+\frac{\rho}{1-\gamma}}{\rho+R}; \frac{R+\frac{\rho}{1-\gamma}}{\rho+R}+1; \eta_\delta(y)\right), \end{aligned}$$

for $y \geq 0$.

4 Discounted tax payments

Let $D_{\gamma,\delta}(u)$ denote the discounted tax payments before ruin in the surplus process (4) defined as

$$D_{\gamma,\delta}(u) := \gamma \sum_{n=1}^{\infty} e^{-\delta W_{\sigma_n}} \left(\sum_{j=\sigma_{n-1}+1}^{\sigma_n} (Y_j - cT_j) \right) 1_{\{\tau_\gamma > W_{\sigma_n}\}}.$$

In this section, we will analyze the nth moment of $D_{\gamma,\delta}(u)$, namely

$$M_n(u) = E\left[(D_{\gamma,\delta}(u))^n \right],$$

for $n = 1, 2, ...$ By conditioning on the first upper exit time ξ_u of the surplus process (4), one finds

$$\begin{aligned} M_n(u) &= g_{n\delta}(u) \int_0^\infty \frac{\beta}{1-\gamma} e^{-\frac{\beta}{1-\gamma}x} E\left[\left(D_{\gamma,\delta}(u+x) + \frac{\gamma}{1-\gamma}x \right)^n \right] dx \\ &= g_{n\delta}(u) \int_u^\infty \frac{\beta}{1-\gamma} e^{-\frac{\beta}{1-\gamma}(x-u)} E\left[\left(D_{\gamma,\delta}(x) + \frac{\gamma}{1-\gamma}(x-u) \right)^n \right] dx. \end{aligned} \tag{35}$$

Differentiating (35) w.r.t. to u gives

$$\begin{aligned} M_n'(u) &= \left(\frac{d}{du} \ln g_{n\delta}(u) + \frac{\beta}{1-\gamma}(1 - g_{n\delta}(u)) \right) M_n(u) \\ &\quad - \frac{n\gamma}{1-\gamma} g_{n\delta}(u) \int_u^\infty \frac{\beta}{1-\gamma} e^{-\frac{\beta}{1-\gamma}(x-u)} E\left[\left(D_{\gamma,\delta}(x) + \frac{\gamma}{1-\gamma}(x-u) \right)^{n-1} \right] dx. \end{aligned} \tag{36}$$

From (35) with n replaced by $n-1$, (36) can be simplified to

$$M_n'(u) = \left(\frac{d}{du} \ln g_{n\delta}(u) + \frac{\beta}{1-\gamma}(1 - g_{n\delta}(u)) \right) M_n(u) - \frac{n\gamma}{1-\gamma} \frac{g_{n\delta}(u)}{g_{(n-1)\delta}(u)} M_{n-1}(u). \tag{37}$$

Multiplying both sides of (37) by the multiplicative factor $\exp\left\{ -\frac{\beta}{1-\gamma} \int_0^u (1 - g_{n\delta}(z))\, dz \right\} / g_{n\delta}(u)$, it follows that

$$\frac{d}{du} \left(\frac{M_n(u)}{g_{n\delta}(u)} e^{-\frac{\beta}{1-\gamma} \int_0^u (1-g_{n\delta}(z))dz} \right) = -\frac{n\gamma}{1-\gamma} \frac{M_{n-1}(u)}{g_{(n-1)\delta}(u)} e^{-\frac{\beta}{1-\gamma} \int_0^u (1-g_{n\delta}(z))dz}. \tag{38}$$

Integrating (38) from y to ∞ then yields

$$M_n(y) = g_{n\delta}(y) \left(\frac{M_n(\infty)}{g_{n\delta}(\infty)} e^{-\frac{\beta}{1-\gamma} \int_y^\infty (1-g_{n\delta}(z))dz} + \int_y^\infty \frac{n\gamma}{1-\gamma} \frac{M_{n-1}(u)}{g_{(n-1)\delta}(u)} e^{-\frac{\beta}{1-\gamma} \int_y^u (1-g_{n\delta}(z))dz} du \right), \tag{39}$$

where $M_n(\infty) := \lim_{\varsigma\to\infty} M_n(\varsigma)$ and $g_{n\delta}(\infty) := \lim_{\varsigma\to\infty} g_{n\delta}(\varsigma)$. Under (5) and $\delta > 0$, we have $\lim_{\varsigma\to\infty} g_{n\delta}(\varsigma) > 0$, $\lim_{\varsigma\to\infty} M_n(\varsigma)$ is finite a.s. and

$$\lim_{\varsigma\to\infty} \exp\left\{ -\frac{\beta}{1-\gamma} \int_y^\varsigma (1 - g_{n\delta}(z))\, dz \right\} \le \lim_{\varsigma\to\infty} \exp\left\{ -\frac{\beta}{1-\gamma} (1 - g_{n\delta}(\varsigma))(\varsigma - y) \right\} = 0,$$

which implies that (39) becomes

$$M_n(y) = \frac{n\gamma}{1-\gamma} g_{n\delta}(y) \int_y^\infty \frac{M_{n-1}(u)}{g_{(n-1)\delta}(u)} e^{-\frac{\beta}{1-\gamma} \int_y^u (1-g_{n\delta}(z))dz} du. \tag{40}$$

Of special interest is the case $n = 1$ which leads to an expression for the expected discounted tax payments until ruin

$$M_1(y) = \frac{\gamma}{1-\gamma} g_\delta(y) \int_0^\infty e^{-\frac{\beta}{1-\gamma} \int_0^u (1-g_\delta(z+y))dz} du, \tag{41}$$

using $M_0(u) = g_0(u)$ (cf. (35)).

Exponential inter-innovation times

Assuming exponential inter-innovation times with mean $1/\lambda$, the substitution of (27) and (28) in (41) leads to

$$M_1(y) = \frac{\gamma}{1-\gamma}\frac{\beta-\rho}{\beta}\left(1-e^{-(R+\rho)y}\right)e^{\frac{\rho}{1-\gamma}y}\left(1-\eta_\delta(y)\right)^{\frac{\gamma}{1-\gamma}}\int_y^\infty e^{-\frac{\rho}{1-\gamma}u}\left(1-\eta_\delta(u)\right)^{-\frac{1}{1-\gamma}}du. \tag{42}$$

From Newton's generalized binomial theorem, (42) becomes

$$\begin{aligned}
M_1(y) &= \frac{\gamma}{1-\gamma}\frac{\beta-\rho}{\beta}\left(1-e^{-(R+\rho)y}\right)e^{\frac{\rho}{1-\gamma}y}\left(1-\eta_\delta(y)\right)^{\frac{\gamma}{1-\gamma}}\int_y^\infty e^{-\frac{\rho}{1-\gamma}u}\sum_{k=0}^\infty C_k^{-\frac{1}{1-\gamma}}\left(-\frac{\beta-R}{\beta+\rho}\right)^k e^{-k(\rho+R)u}du \\
&= \frac{\gamma}{1-\gamma}\frac{\beta-\rho}{\beta}\left(1-e^{-(R+\rho)y}\right)e^{\frac{\rho}{1-\gamma}y}\left(1-\eta_\delta(y)\right)^{\frac{\gamma}{1-\gamma}}\int_y^\infty \sum_{k=0}^\infty C_k^{\frac{1}{1-\gamma}+k-1}\left(\frac{\beta-R}{\beta+\rho}\right)^k e^{-\left(\left(k+\frac{1}{\gamma}\right)\rho+kR\right)u}du. \\
&= \frac{\gamma}{1-\gamma}\frac{\beta-\rho}{\beta}\left(1-e^{-(R+\rho)y}\right)\left(1-\eta_\delta(y)\right)^{\frac{\gamma}{1-\gamma}}\sum_{k=0}^\infty C_k^{\frac{1}{1-\gamma}+k-1}\frac{\left(\eta_\delta(y)\right)^k}{\left(k+\frac{1}{1-\gamma}\right)\rho+kR} \\
&= \frac{\gamma}{\rho}\frac{\beta-\rho}{\beta}\left(1-e^{-(R+\rho)y}\right)\left(1-\eta_\delta(y)\right)^{\frac{\gamma}{1-\gamma}}F\left(\frac{1}{1-\gamma},\frac{\rho}{(\rho+R)(1-\gamma)};\frac{\rho}{(\rho+R)(1-\gamma)}+1;\eta_\delta(y)\right),
\end{aligned}$$

for $y \geq 0$.

5 Delayed start of tax payments

In this section we consider a variant of the tax system where tax payments start only after the surplus is greater than a threshold level b ($b > u$). Let $v_b(u)$ denote the resulting expected discounted tax payments. By a probabilistic argument, one easily shows

$$v_b(u) = B_\delta(u,b)\left[\frac{\gamma}{\beta}+\int_0^\infty \frac{\beta}{1-\gamma}e^{-\frac{\beta}{1-\gamma}x}M_1(b+x)\,dx\right], \tag{43}$$

where $B_\delta(u,b)$ is the Laplace transform of the first passage from level u to any level above b avoiding ruin en route. Furthermore, an expression for $B_\delta(u,b)$ can be obtained in terms of $g_\delta(x)$ for $u \leq x \leq b$. Indeed, by conditioning on the ascending ladder height, one obtains

$$\begin{aligned}
B_\delta(u,b) &= g_\delta(u)\left[\int_0^{b-u}\frac{\beta}{1-\gamma}e^{-\frac{\beta}{1-\gamma}x}B_\delta(u+x,b)\,dx + e^{-\frac{\beta}{1-\gamma}(b-u)}\right] \\
&= g_\delta(u)\left[\int_u^b\frac{\beta}{1-\gamma}e^{-\frac{\beta}{1-\gamma}(x-u)}B_\delta(x,b)\,dx + e^{-\frac{\beta}{1-\gamma}(b-u)}\right].
\end{aligned} \tag{44}$$

Differentiating (44) w.r.t. u, we obtain

$$B'_\delta(u,b) = \left(\frac{g'_\delta(u)}{g_\delta(u)}+\frac{\beta}{1-\gamma}-\frac{\beta}{1-\gamma}g_\delta(u)\right)B_\delta(u,b),$$

from which

$$B_\delta(u,b) = g_\delta(u)\,e^{-\frac{\beta}{1-\gamma}\int_u^b(1-g_\delta(x))dx}. \tag{45}$$

Combining (35) at $n=1$, (43) and (45), one deduces

$$v_b(u) = \frac{B_\delta(u,b)}{g_\delta(b)}M_1(b) = \frac{g_\delta(u)}{g_\delta(b)}e^{-\beta\int_u^b(1-g_\delta(x))dx}M_1(b). \tag{46}$$

Theorem 5.1 *If there is an optimal level $b^* > 0$ to start taxation at rate $0 < \gamma < 1$ in the dual risk model, it has to fulfill the condition*

$$\frac{d}{db}\left(\frac{M_1(b)}{g_\delta(b)}\right)\bigg|_{b=b^*} = 1 \tag{47}$$

together with

$$g_\delta'(b^*) > \beta\left(1 - g_\delta(b^*)\right)^2, \tag{48}$$

and the optimal expected discounted tax payment is then given by

$$v_{b^*}(u) = \begin{cases} \frac{B_\delta(u,b^*)}{\beta(1-g_\delta(b^*))}, & u < b^*, \\ v(u), & u \geq b^*. \end{cases}$$

A sufficient condition for the existence of such an optimal positive level $b^* > 0$ is $\lim_{u\to 0}\frac{M_1(u)}{g_\delta(u)} > \frac{1}{\beta}$.
On the other hand, if such a $b^* > 0$ does not exist, then the optimal level to start taxation is $b^* = 0$ (i.e. start immediately), so that in this case $v_{b^*}(u) = v(u)$.

Proof: To identify the optimal surplus level b^* for the authority to start tax collection, we shall look for the solution of

$$\frac{\partial}{\partial b}v_b(u) = 0. \tag{49}$$

Using (46), (49) becomes

$$\left(\frac{M_1'(b^*)}{M_1(b^*)} - \left(\frac{g_\delta'(b^*)}{g_\delta(b^*)} + \beta\left(1 - g_\delta(b^*)\right)\right)\right)v_{b^*}(u) = 0. \tag{50}$$

Since $v_b(u) > 0$ for $u, b > 0$, we get

$$\frac{M_1'(b^*)}{M_1(b^*)} - \left(\frac{g_\delta'(b^*)}{g_\delta(b^*)} + \beta\left(1 - g_\delta(b^*)\right)\right) = 0. \tag{51}$$

On the other hand, we know from (37) at $n = 1$ that

$$\frac{M_1'(u)}{M_1(u)} = \left(\frac{g_\delta'(u)}{g_\delta(u)} + \frac{\beta}{1-\gamma}\left(1 - g_\delta(u)\right)\right) - \frac{\gamma}{1-\gamma}\frac{g_\delta(u)}{M_1(u)}, \tag{52}$$

for any $u \geq 0$. Hence (51) can be written as

$$\frac{M_1(b^*)}{g_\delta(b^*)} = \frac{1}{\beta\left(1 - g_\delta(b^*)\right)}. \tag{53}$$

Replacing (53) in (51) eventually leads to (47).

In order to ensure that b^* is indeed a maximum, we have to prove that

$$\frac{\partial^2}{\partial b^2}v_b(u)\bigg|_{b=b^*} < 0.$$

From (46) we have

$$v_b''(b)|_{b=b^*} = \left(\left(\frac{M_1'(b)}{M_1(b)}\right)' - \left(\frac{g_\delta'(b)}{g_\delta(b)}\right)' + \beta g_\delta'(b)\right)v_b(u)\bigg|_{b=b^*}.$$

Differentiating (52) w.r.t. b we also get

$$\left(\frac{M_1'(b)}{M_1(b)}\right)' = \left(\frac{g_\delta'(b)}{g_\delta(b)}\right)' - \frac{\beta}{1-\gamma}g_\delta'(b) - \frac{\gamma}{1-\gamma}\left(\frac{g_\delta(b)}{M_1(b)}\right)'.$$

and combining the last two equations, one arrives at

$$\begin{aligned} v_b''(b^*) &= -\frac{\gamma}{1-\gamma}\left(\beta g_\delta'(b^*) + \left(\frac{g_\delta(b^*)}{M_1(b^*)}\right)'\right) v_b(b^*) \\ &= -\frac{\gamma}{1-\gamma}\left(\beta g_\delta'(b^*) - \left(\frac{g_\delta(b^*)}{M_1(b^*)}\right)^2\right) v_b(b^*), \end{aligned} \tag{54}$$

or, by virtue of (53),

$$v_b''(b^*) = -\frac{\gamma}{1-\gamma}\left(\beta g_\delta'(b^*) - \beta^2 (1-g_\delta(b^*))^2\right) v_b(b^*).$$

Hence $g_\delta'(b^*) - \beta(1-g_\delta(b^*))^2 > 0$ guarantees $v_b''(b^*) < 0$, identifying b^* as a (local) maximum. Note that (48) also translates into

$$\frac{d}{db}\left(\frac{1}{\beta}\frac{1}{1-g_\delta(b)}\right)\bigg|_{b=b^*} > 1, \tag{55}$$

which means that the derivative of the right-hand side exceeds the one of the left-hand side of (53) in the intersection point b^*. From (41),

$$\lim_{u\to\infty}\frac{M_1(u)}{g_\delta(u)} = \frac{\gamma}{1-\gamma}\int_0^\infty e^{-\frac{\beta}{1-\gamma}\int_0^z (1-g_\delta(\infty))dy} dz = \frac{\gamma}{\beta(1-g_\delta(\infty))}. \tag{56}$$

Note that (56) can also be obtained directly by probabilistic reasoning (in the absence of ruin):

$$M_1(\infty) = g_\delta(\infty)\left(\frac{\gamma}{\delta} + M_1(\infty)\right).$$

Altogether it is then clear that

$$\lim_{u\to\infty}\frac{M_1(u)}{g_\delta(u)} < \lim_{u\to\infty}\frac{1}{\beta(1-g_\delta(u))}, \tag{57}$$

where $g_\delta(\infty) := \lim_{u\to\infty} g_\delta(u)$. Hence, for $\lim_{u\to 0}\frac{M_1(u)}{g_\delta(u)} > \frac{1}{\beta}$, the continuity of the functions $M_1(u)/g_\delta(u)$ and $1/\beta(1-g_\delta(u))$ guarantees the existence of an optimal $b^* > 0$ (in case there should be several positive solutions of (47) with (48), one would have to pick the one leading to the largest value of $v_b(u)$).

Finally, in the absence of a positive local maximum, the fact that $v_\infty(u) = 0$ then establishes $b^* = 0$ as the optimal taxation level. □

Remark 5.1 *Using* (13), *criterion* (47) *can easily be translated into*

$$M_1(b^*) = V(b^*; b^*). \tag{58}$$

In other words, the optimal taxation starting level can only be the initial value for which the expected discounted tax payments M_1 equals the expected discounted dividend payments under a horizontal barrier strategy with barrier at the initial surplus level. Note that criterion (58) *is identical to the one obtained for the optimal taxation level in the Cramér-Lundberg model (cf. Albrecher and Hipp (2007)).*

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Appendix

Assume that the inter-innovation times are a mixture of exponentials with density function $k(t)$ as given in Example 2.2 and the innovation sizes are exponentially distributed with mean $1/\beta$. By conditioning on the time and size of the first innovation, the expected discounted dividends for the surplus process (2) are given by

$$V_\delta(u;b) = \sum_{i=1}^m p_i \int_0^u \frac{\lambda_i}{c} e^{-\frac{\lambda_i+\delta}{c}t} \left(\int_0^{t+b-u} V_\delta(u+y-t;b)\,\beta e^{-\beta y} dy + \left(\frac{1}{\beta} + V_\delta(b;b)\right) e^{-\beta(t+b-u)} \right) dt, \tag{59}$$

for $0 < u < b$. Applying the operator $(\mathcal{D}-\beta)$ on both sides of (59) leads to

$$\begin{aligned}(\mathcal{D}-\beta) V_\delta(u;b) = &\sum_{i=1}^m p_i \frac{\lambda_i}{c} e^{-\frac{\lambda_i+\delta}{c}u} \left(\int_0^b V_\delta(y;b)\,\beta e^{-\beta y} dy + \left(\frac{1}{\beta} + V_\delta(b;b)\right) e^{-\beta b} \right) \\ &- \beta \sum_{i=1}^m p_i \int_0^u \frac{\lambda_i}{c} e^{-\frac{\lambda_i+\delta}{c}(u-t)} V_\delta(t;b)\, dt,\end{aligned} \tag{60}$$

for $0 < u < b$. An application of the operator $A(\mathcal{D}) = \prod_{i=1}^{m} \left(\mathcal{D} + \frac{\lambda_i+\delta}{c}\right)$ on both sides of (60) yields the linear homogeneous differential equation of order $m+1$

$$A(\mathcal{D})(\mathcal{D}-\beta) V_\delta(u;b) + \sum_{i=1}^{m} p_i \frac{\lambda_i}{c} A_{-i}(\mathcal{D}) \beta V_\delta(u;b) = 0, \tag{61}$$

for $0 < u < b$ where $A_{-i}(\mathcal{D})$ corresponds to the operator $A(\mathcal{D})$ with the i-th term removed from the product. Hence $V_\delta(u;b)$ $(0 < u < b)$ can be expressed as

$$V_\delta(u;b) = \sum_{j=0}^{m} C_j(b) e^{s_j u}, \tag{62}$$

where $\{s_j\}_{j=0}^{m}$ are the $m+1$ solutions of the characteristic equation

$$A(s)(s-\beta) + \sum_{i=1}^{m} p_i \frac{\lambda_i}{c} A_{-i}(s) \beta = 0. \tag{63}$$

A substitution of (62) in (25) allows the identification of the constants $\{C_j(b)\}_{j=0}^{m}$ in (62). Indeed, one finds

$$\begin{aligned}\sum_{j=0}^{m} C_j(b)(s_j-\beta) e^{s_j u} &= \sum_{i=1}^{m} p_i \frac{\lambda_i}{c} e^{-\frac{\lambda_i+\delta}{c}u} \beta \sum_{j=0}^{m} C_j(b) \frac{1-e^{-(\beta-s_j)b}}{\beta-s_j} - \sum_{i=1}^{m} p_i \frac{\lambda_i}{c} \beta \sum_{j=0}^{m} C_j(b) \frac{e^{s_j u} - e^{-\frac{\lambda_i+\delta}{c}u}}{s_j + \frac{\lambda_i+\delta}{c}} \\ &+ \sum_{i=1}^{m} p_i \frac{\lambda_i}{c} e^{-\frac{\lambda_i+\delta}{c}u} \left(\frac{1}{\beta} + \sum_{j=0}^{m} C_j(b) e^{s_j b}\right) e^{-\beta b},\end{aligned} \tag{64}$$

for $0 < u < b$. For (64) to be valid for all u in $(0,b)$, the coefficients of $\exp(s_j u)$ on both sides of (64) must be equal which is guaranteed from (63). Analogously, the coefficients of $\exp(-(\lambda+\delta)u/c)$ on both sides of (64) have to coincide, i.e. the constants $\{C_j\}_{j=0}^{m}$ must satisfy

$$0 = \beta \sum_{j=0}^{m} C_j(b) \left(\frac{1-e^{-(\beta-s_j)b}}{\beta-s_j} + \frac{1}{s_j + \frac{\lambda_i+\delta}{c}}\right) + \left(\frac{1}{\beta} + \sum_{j=0}^{m} C_j(b) e^{s_j b}\right) e^{-\beta b}, \tag{65}$$

for $j = 0, 1, ..., m$. Combining (65) with the initial value condition

$$V_\delta(0;b) = \sum_{j=0}^{m} C_j(b) = 0,$$

we obtain a system of $m+1$ linear equations that allows the identification of the constants $\{C_j(b)\}_{j=0}^{m}$ in (62).

Finally, we derive an expression for the Laplace transform of the time to ruin $\rho_{0,\delta}(u)$ under these assumptions in dual risk model. By conditioning on the time and size of the first innovation, it follows that

$$\rho_{0,\delta}(u) = \sum_{i=1}^{m} p_i \int_0^u \frac{\lambda_i}{c} e^{-\frac{\lambda_i+\delta}{c}t} \int_u^\infty \rho_{0,\delta}(y-t) \beta e^{-\beta(y-u)} dy dt + \sum_{i=1}^{m} p_i e^{-\frac{\lambda_i+\delta}{c}u}. \tag{66}$$

Applying the operator $(\mathcal{D}-\beta)$ on both sides of (66) leads to

$$(\mathcal{D}-\beta)\rho_{0,\delta}(u) = \sum_{i=1}^{m} p_i \frac{\lambda_i}{c} \beta e^{-\frac{\lambda_i+\delta}{c}u} \left(\int_0^\infty \rho_{0,\delta}(y) e^{-\beta y} dy - \int_0^u e^{\frac{\lambda_i+\delta}{c}t} \rho_{0,\delta}(t) dt\right) + \sum_{i=1}^{m} p_i e^{-\frac{\lambda_i+\delta}{c}u}. \tag{67}$$

By an application of the operator $A(\mathcal{D})$ on both sides of (67), one readily finds that $\{\rho_{0,\delta}(u), u \geq 0\}$ satisfies the homogeneous differential equation (61) for $u > 0$ which implies that its general solution is given by

$$\rho_{0,\delta}(u) = \sum_{j=0}^{m} d_j e^{s_j u}, \tag{68}$$

for $u > 0$. A substitution of (68) in (67) allows the identification of the constants $\{d_j\}_{j=0}^m$ in (68). Indeed, one finds

$$\sum_{j=0}^m d_j \left(s_j - \beta\right) e^{s_j u} = \sum_{i=1}^m p_i \frac{\lambda_i}{c} \beta \sum_{j=0}^m d_j \left(\frac{e^{-\frac{\lambda_i+\delta}{c}u}}{\beta - s_j} - \frac{e^{s_j u} - e^{-\frac{\lambda_i+\delta}{c}u}}{\frac{\lambda_i+\delta}{c} + s_j} \right) + \sum_{i=1}^m p_i \, e^{-\frac{\lambda_i+\delta}{c}u}, \tag{69}$$

for $u > 0$. By equaling the coefficients of $\exp\left(-\left(\lambda_i + \delta\right) u/c\right)$ on both sides of (69), the constants $\{d_j\}_{j=0}^m$ have to satisfy

$$0 = \frac{\lambda_i}{c} \beta \sum_{j=0}^m d_j \frac{\frac{\lambda_i+\delta}{c} + \beta}{\left(\frac{\lambda_i+\delta}{c} + s_j\right)\left(\beta - s_j\right)} + 1, \tag{70}$$

for $i = 1, ..., m$. Combining (70) with the initial value condition

$$\rho_{0,\delta}\left(0\right) = \sum_{j=0}^m d_j = 1,$$

the constants $\{d_j\}_{j=0}^m$ can be obtained by a system of $m+1$ linear equations akin to (62).