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Corrigendum to "Resource-monotonicity for house allocation problems"

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Abstract Ehlers and Klaus (Int J Game Theory 32:545–560, 2003) study so-called allocation problems and claim to characterize all rules satisfying *efficiency*, *independence of irrelevant objects*, and *resource-monotonicity* on two preference domains (Ehlers and Klaus 2003, Theorem 1). They explicitly prove Theorem 1 for preference domain $\mathcal{R}_0$ which requires that the null object is always the worst object and mention that the corresponding proofs for the larger domain $\mathcal{R}$ of unrestricted preferences "are completely analogous." In Example 1 and Lemma 1, this corrigendum provides a counterexample to Ehlers and Klaus (2003, Theorem 1) on the general domain $\mathcal{R}$. We also propose a way of correcting the result on the general domain $\mathcal{R}$ by strengthening *independence of irrelevant objects*: in addition to requiring that the chosen allocation should depend only on preferences over the set of available objects (which always includes the null object), we add a situation in which the allocation should also be invariant when preferences over the null object change. Finally, we offer a short proof of the corrected result that uses the established result of Theorem 1 for the restricted domain $\mathcal{R}_0$.

Keywords Corrigendum · Indivisible objects · Resource-monotonicity

JEL Classification D63 · D70

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1 House allocation with variable resources

For completeness and the convenience of the reader, we briefly state the model and the main result of Ehlers and Klaus (2003).

Let N denote a finite set of agents, $|N| \geq 2$. Let K denote a set of potential real objects. Not receiving any real object is called "receiving the null object." Let 0 represent the *null object*. Each agent $i \in N$ is equipped with a preference relation R_i over all objects $K \cup \{0\}$. Given $x, y \in K \cup \{0\}$, $x\ R_i\ y$ means that agent i weakly prefers x to y, and $x\ P_i\ y$ means that agent i strictly prefers x to y. We assume that R_i is strict, i.e., R_i is a linear order over $K \cup \{0\}$. Let $\mathcal{R}$ denote the class of all linear orders over $K \cup \{0\}$, and $\mathcal{R}^N$ the set of *(preference) profiles* $R = (R_i)_{i \in N}$ such that for all $i \in N$, $R_i \in \mathcal{R}$. Given $K' \subseteq K \cup \{0\}$, let $R_i|_{K'}$ denote the restriction of R_i to K' and $R|_{K'} = (R_i|_{K'})_{i \in N}$. Let $\mathcal{R}_0 \subsetneq \mathcal{R}$ denote the class of preference relations where the null object is the worst object. That is, if $R_i \in \mathcal{R}_0$, then all real objects are "goods": for all $x \in K$, $x\ P_i\ 0$.

An *allocation* is a list $a = (a_i)_{i \in N}$ such that for all $i \in N$, $a_i \in K \cup \{0\}$, and none of the real objects in K is assigned to more than one agent. Note that 0, the null object, can be assigned to any number of agents and that not all real objects have to be assigned. Let $\mathcal{A}$ denote the set of all allocations. Let $\mathcal{H}$ denote the set of all non-empty subsets H of K. A *(house allocation) problem* consists of a preference profile $R \in \mathcal{R}^N$ and a set of real objects $H \in \mathcal{H}$. Note that the associated set of *available objects* $H \cup \{0\}$ includes the null object which is available in any economy. An *(allocation) rule* is a function $\varphi : \mathcal{R}^N \times \mathcal{H} \to \mathcal{A}$ such that for all problems $(R, H) \in \mathcal{R}^N \times \mathcal{H}$, $\varphi(R, H) \in \mathcal{A}$ is *feasible*, i.e., for all $i \in N$, $\varphi_i(R, H) \in H \cup \{0\}$. By feasibility, each agent receives an available object. Given $i \in N$, we call $\varphi_i(R, H)$ the *allotment* of agent i at $\varphi(R, H)$.

A natural requirement for a rule is that the chosen allocation depends only on preferences over the set of available objects.

Independence of irrelevant objects: For all $(R, H) \in \mathcal{R}^N \times \mathcal{H}$ and all $R' \in \mathcal{R}^N$ such that $R|_{H \cup \{0\}} = R'|_{H \cup \{0\}}$, $\varphi(R, H) = \varphi(R', H)$.

Next, a rule chooses only (Pareto) efficient allocations.

Efficiency: For all $(R, H) \in \mathcal{R}^N \times \mathcal{H}$, there is no feasible allocation $a \in \mathcal{A}$ such that for all $i \in N$, $a_i\ R_i\ \varphi_i(R, H)$, with strict preference holding for some $j \in N$.

A rule satisfies resource-monotonicity, if more resources become available, all agents (weakly) gain.

Resource-monotonicity: For all $R \in \mathcal{R}^N$ and all $H, H' \in \mathcal{H}$, if $H \subseteq H'$, then for all $i \in N$, $\varphi_i(R, H')\ R_i\ \varphi_i(R, H)$.

2 Mixed dictator-pairwise-exchange rules

Mixed dictator-pairwise-exchange rules are defined via the use of "(endowment) inheritance tables" (Pápai 2000). For each real object $x \in K$, a one-to-one function π_x :

$\{1, \ldots, |N|\} \to N$ specifies the inheritance of object x. An *inheritance table* is a profile $\pi = (\pi_x)_{x \in K}$ specifying the inheritance of each real object. We call an inheritance table π a *mixed dictator-pairwise-exchange inheritance table* if it induces a partition of agents into singletons and pairs $S = (S_1, \ldots, S_m)$ and the corresponding *mixed dictator-pairwise-exchange inheritance rule* $\varphi^{(\pi,S)}$ works as follows (see Ehlers and Klaus 2003, Sect. 3 for details).

First, if S_1 is a singleton, then the agent who initially owns all objects picks his best available object. If S_1 specifies a pair of agents, the two agents who initially own all objects obtain their best objects if these are different; otherwise whoever owns the (common) best available object is assigned that object, and the other agent picks his best available object from the remaining objects. This process of either dictatorship or pairwise-exchange steps is repeated with S_2, S_3, etc. for the set of remaining available objects after each step.

For a formal definition of mixed dictator-pairwise-exchange inheritance table we refer to Ehlers and Klaus (2003).

Theorem 1 Ehlers and Klaus (2003, Theorem 1)
Let $|K| > |N|$. On the domain $\mathcal{R}^N$ ($\mathcal{R}_0^N$), mixed dictator-pairwise-exchange rules are the only rules satisfying efficiency, independence of irrelevant objects, and resource-monotonicity.

3 Problems concerning Theorem 1 on domain $\mathcal{R}$

3.1 Counterexample to Theorem 1

Example 1 Let $N = \{1, 2, 3\}$, $x \in K$, and $|K| \geq 4$. Rule φ is defined as follows: for all $(R, H) \in \mathcal{R}^N \times \mathcal{H}$,

(a) if $x \in H$, $x \; P_2 \; 0$, and x is what agents 1 and 3 prefer most in H, the serial dictatorship corresponding to the order $(3, 1, 2)$ is used; and

(b) otherwise the serial dictatorship corresponding to the order $(1, 3, 2)$ is used.

Note that φ is not a mixed dictator-pairwise-exchange rule because for any problem (R, H) such that $x \in H$ is what agents 1 and 3 prefer most in H, we have $\varphi_3(R, H) = x$ if $x \; P_2 \; 0$ and $\varphi_1(R, H) = x$ if $0 \; P_2 \; x$.

Lemma 1 A counterexample to Theorem 1 on Domain $\mathcal{R}$
Rule φ as defined in Example 1 satisfies efficiency, independence of irrelevant objects, and resource-monotonicity.

Proof It is obvious that rule φ as defined in Example 1 satisfies *efficiency* and *independence of irrelevant objects*. In order to check *resource-monotonicity*, let $R \in \mathcal{R}^N$ and $H, H' \in \mathcal{H}$ be such that $H \subseteq H'$. If (R, H) and (R, H') are both of type (a) (or both of type (b)), then *resource-monotonicity* is satisfied by φ because serial dictatorship rules are *resource-monotonic*.

We are left with the case where (R, H) and (R, H') are not of the same "type" (a) or (b).

First, suppose that (R, H) is of type (a) and (R, H') is of type (b). Then, by definition of φ, $\varphi_3(R, H) = x$. Now, if $\varphi_3(R, H') = x$, then *resource-monotonicity* is satisfied because the order in the serial dictatorship between agents 1 and 2 is unchanged in (a) and (b). Otherwise $\varphi_3(R, H') \neq x$ and because (R, H') is of type (b), at $\varphi(R, H')$ agent 1 or agent 3 receives an object in $H' \setminus H$ and the other agent either receives x or also an object in $H' \setminus H$. Now, *resource-monotonicity* is satisfied because agents 1 and 3 are weakly better off and agent 2 can choose from a larger set of objects under (R, H') than under (R, H).

Second, suppose that (R, H) is of type (b) and (R, H') is of type (a). Then, by definition of φ, $\varphi_3(R, H') = x$. Since (R, H) is of type (b) and (R, H) not of type (a), we must have $x \in H' \setminus H$. Then, $\varphi_3(R, H')\ R_3\ \varphi_3(R, H)$ and both agents 1 and 2 can choose from a larger set of objects at (R, H') than at (R, H). Since the order in the serial dictatorship between agents 1 and 2 is unchanged in (a) and (b), *resource-monotonicity* is satisfied. □

3.2 Correction of Theorem 1

In order to correct Theorem 1 on domain $\mathcal{R}^N$, we strengthen *independence of irrelevant objects* by additionally requiring that if only the ranking of the null object changes below agents' allotments, then the allocation does not change.

Strong independence of irrelevant objects: For all $(R, H) \in \mathcal{R}^N \times \mathcal{H}$ and all $R' \in \mathcal{R}^N$ such that

(i) $R|_{H\cup\{0\}} = R'|_{H\cup\{0\}}$ or
(ii) $R|_H = R'|_H$ and for all $i \in N$, $\varphi_i(R, H)\ P_i\ 0$ implies $\varphi_i(R, H)\ P'_i\ 0$,

we have $\varphi(R, H) = \varphi(R', H)$.

We are now ready to present a correction of Theorem 1 on Domain $\mathcal{R}$.

Theorem 2 *Let $|K| > |N|$. On the domain $\mathcal{R}^N$, mixed dictator-pairwise-exchange rules are the only rules satisfying efficiency, strong independence of irrelevant objects, and resource-monotonicity.*

We next offer a short proof of Theorem 2 that uses the established result of Ehlers and Klaus (2003, Theorem 1) for the restricted domain $\mathcal{R}_0^N$.

Before proving Theorem 2, we establish two useful implications of *efficiency* and *resource-monotonicity*.

The following lemma states that *efficiency* and *resource-monotonicity* imply that taking out an unassigned object does not change the assigned allocation.

Lemma 2 *Let φ be an efficient and resource-monotonic rule and $(R, H) \in \mathcal{R}^N \times \mathcal{H}$. If for all $i \in N$, $\varphi_i(R, H) \neq x \in H$, then $\varphi(R, H) = \varphi(R, H \setminus \{x\})$.*

Proof Suppose by contradiction that $(R, H) \in \mathcal{R}^N \times \mathcal{H}$ is such that for all $i \in N$, $\varphi_i(R, H) \neq x \in H$ and $\varphi(R, H) \neq \varphi(R, H \setminus \{x\})$. Note that object x is neither assigned at (R, H) nor at $(R, H \setminus \{x\})$, i.e., $\varphi(R, H)$ is feasible for $(R, H \setminus \{x\})$.

Hence, by *efficiency*, there exists an agent $j \in N$ such that $\varphi_j(R, H \backslash \{x\}) P_j \varphi_j(R, H)$, contradicting *resource-monotonicity*. □

The following lemma states that *efficiency* and *resource-monotonicity* imply that adding an object that none of the agents who are assigned objects would prefer will not change the assigned allocation for these agents. For $(R, H) \in \mathcal{R}^N \times \mathcal{H}$, we denote the set of agents who are assigned objects in H at $\varphi(R, H)$ by $N^+(\varphi, R, H) \equiv \{i \in N : \varphi(R, H) \in H\}$.

Lemma 3 *Let φ be an efficient and resource-monotonic rule and $(R, H) \in \mathcal{R}^N \times \mathcal{H}$. Let $x \in K \setminus H$ and for all $i \in N^+(\varphi, R, H)$, $\varphi_i(R, H)\ P_i\ x$. Then, for all $i \in N^+(\varphi, R, H)$, $\varphi_i(R, H) = \varphi_i(R, H \cup \{x\})$.*

Proof Let $(R, H) \in \mathcal{R}^N \times \mathcal{H}$, $x \in K \setminus H$ and for all $i \in N^+(\varphi, R, H)$, $\varphi_i(R, H) P_i\, x$. By *resource-monotonicity*, for all $i \in N^+(\varphi, R, H)$, $\varphi_i(R, H \cup \{x\})\ R_i\ \varphi_i(R, H)$. Hence, object x at $(R, H \cup \{x\})$ is not assigned to any agent in $N^+(\varphi, R, H)$ (it might be assigned to any of the agents who did not receive an object). Since $\varphi(R, H)$ is *efficient* and only agents in $N^+(\varphi, R, H)$ received objects in H, the only way to satisfy *resource-monotonicity* is to not change the assigned allocation for agents in $N^+(\varphi, R, H)$, i.e., for all $i \in N^+(\varphi, R, H)$, $\varphi_i(R, H) = \varphi_i(R, H \cup \{x\})$. □

3.3 Proof of Theorem 2

Throughout the proof, when referring to Ehlers and Klaus (2003, Theorem 1), we refer to Ehlers and Klaus (2003, Theorem 1) on the domain $\mathcal{R}_0^N$.

It is easy to verify that mixed dictator-pairwise-exchange rules satisfy *efficiency*, *strong independence of irrelevant objects*, and *resource-monotonicity* since no more than two agents "trade" at any step. In proving the converse, let $|K| > |N|$ and let φ be a rule satisfying *efficiency*, *strong independence of irrelevant objects*, and *resource-monotonicity*.

By Ehlers and Klaus (2003, Theorem 1), φ equals a mixed dictator-pairwise-exchange rules on the domain $\mathcal{R}_0^N$. Hence, on the domain $\mathcal{R}_0^N, \varphi = \varphi^{(\pi,S)}$. Suppose, by contradiction, that on the general domain $\mathcal{R}^N, \varphi \neq \varphi^{(\pi,S)}$. Then, there exists $(R, H) \in \mathcal{R}^N \times \mathcal{H}$ such that $\varphi(R, H) \neq \varphi^{(\pi,S)}(R, H)$. In particular, by *efficiency*, there exists $j \in N$ such that $\varphi_j(R, H)\ P_j\ \varphi_j^{(\pi,S)}(R, H)$.

Let $\left(\bigcup_{i \in N}\{\varphi_i(R, H)\}\right) \setminus \{0\} = H'$. By Lemma 2, for all $i \in N, \varphi_i(R, H') = \varphi_i(R, H)$. By *resource-monotonicity* of $\varphi^{(\pi,S)}$ and $H' \subseteq H, \varphi_j^{(\pi,S)}(R, H)\ R_j\ \varphi_j^{(\pi,S)}(R, H')$. Thus, $\varphi_j(R, H')\ P_j\ \varphi_j^{(\pi,S)}(R, H')$ and at $\varphi(R, H')$ all real objects are assigned. To simplify notation, and without loss of generality, we assume that $H' = H$ in the sequel.

Case 1 ($|H| = |N|$): If $|H| = |N|$, then we move the null object to the bottom of each agent's preference relation and obtain a profile $R' \in \mathcal{R}_0^N$. Formally, let R' be such that $R'|_H = R|_H$ and for all $i \in N$ and all $x \in H, x\ P'_i\ 0$. Note that for all $i \in N, \varphi_i(R, H) \neq 0$ and by *efficiency*, for all $i \in N, \varphi_i(R, H)\ P_i\ 0$. Using (ii) in *strong independence of irrelevant objects* we then have $\varphi(R, H) = \varphi(R', H)$. By

$R' \in \mathcal{R}_0^N$ and Ehlers and Klaus (2003, Theorem 1), $\varphi(R', H) = \varphi^{(\pi,S)}(R', H)$. Then, $\varphi(R, H) = \varphi(R', H) = \varphi^{(\pi,S)}(R', H)$ implies that for all $i \in N$, $\varphi_i^{(\pi,S)}(R, H)\ P_i\ 0$ and $\varphi_i^{(\pi,S)}(R, H)\ P_i'\ 0$. Using (ii) in *strong independence of irrelevant objects* we then have $\varphi^{(\pi,S)}(R', H) = \varphi^{(\pi,S)}(R, H)$. Hence, $\varphi(R, H) = \varphi^{(\pi,S)}(R, H)$; a contradiction to $\varphi_j(R, H)\ P_j\ \varphi_j^{(\pi,S)}(R, H)$.

Case 2 ($|H| < |N|$): If $|H| < |N|$, then let $l \in N$ be such that $\varphi_l(R, H) = 0$. Let $x \in K \setminus H$ and change agents' preferences from R to $R' \in \mathcal{R}^N$ such that

(i) $R'|_{H\cup\{0\}} = R|_{H\cup\{0\}}$,

(ii) agent l ranks x just above 0, i.e., $x\ P_l'\ 0$ and for no $y \in H$, $x\ P_l'\ y\ P_l'\ 0$, and

(iii) all other agents, rank x worst, i.e., for all $i \in N\setminus\{l\}$ and all $y \in H\cup\{0\}$, $y\ P_i'\ x$.

By (i) of *strong independence of irrelevant objects*, $\varphi(R', H) = \varphi(R, H)$ and $\varphi^{(\pi,S)}(R', H) = \varphi^{(\pi,S)}(R, H)$. Hence, $\varphi_l(R', H) = 0$. Furthermore, there exists $j \in N$ such that $\varphi_j(R', H)\ P_j'\ \varphi_j^{(\pi,S)}(R', H)$.

Resource-monotonicity implies that for all $i \in N$, $\varphi_i(R', H \cup \{x\})\ R_i'\ \varphi_i(R', H)$. Now by construction and *efficiency*, $\varphi_l(R', H \cup \{x\}) = x$ and for all $i \in N \setminus \{l\}$, $\varphi_i(R', H \cup \{x\}) = \varphi_i(R', H)$.

If $\varphi_l^{(\pi,S)}(R', H) = 0$, then the same arguments as above imply $\varphi_l^{(\pi,S)}(R', H \cup \{x\}) = x$ and for all $i \in N \setminus \{l\}$, $\varphi_i^{(\pi,S)}(R', H \cup \{x\}) = \varphi_i^{(\pi,S)}(R', H)$. Now if $\varphi_l^{(\pi,S)}(R', H) \neq 0$, then Lemma 3 implies that $\varphi^{(\pi,S)}(R', H \cup \{x\}) = \varphi^{(\pi,S)}(R', H)$.

Hence, we have that $\varphi_j(R', H\cup\{x\})\ P_j'\ \varphi_j^{(\pi,S)}(R', H\cup\{x\})$ and $\left(\bigcup_{i\in N}\{\varphi_i(R', H\cup\{x\})\}\right) \setminus \{0\} = H \cup \{x\}$. If for some $l' \in N$, $\varphi_{l'}(R', H \cup \{x\}) = 0$, then we repeat the Case 2 step above by adding an object $x' \in K \setminus (H \cup \{x\})$ etc.

Repeating this step will eventually lead to a problem $(\tilde{R}, \tilde{H})$ where $\varphi_j(\tilde{R}, \tilde{H})\ \tilde{P}_j\ \varphi_j^{(\pi,S)}(\tilde{R}, \tilde{H})$ and for all $i \in N$, $\varphi_i(\tilde{R}, \tilde{H}) \neq 0$. But then, $|\tilde{H}| = |N|$ and we can apply the arguments of Case 1 to obtain a contradiction. □

Remark 1 ($|K| \leq |N|$) The following example demonstrates that the statement of Ehlers and Klaus (2003, p. 552) "For the larger domain $\mathcal{R}$, all our results remain true." concerning the adjustment of results for $|K| \leq |N|$ is not correct. Let $N = \{1, 2, 3\}$ and $K = \{x\}$. Rule φ is defined as follows: for all $(R, H) \in \mathcal{R}^N \times \mathcal{H}$,

(a) if $x\ P_3\ 0$, the serial dictatorship corresponding to the order $(1, 2, 3)$ is used; and

(b) if $0\ P_3\ x$, the serial dictatorship corresponding to the order $(2, 1, 3)$ is used.

Note that φ is not a mixed dictator-pairwise-exchange rule, but it satisfies *efficiency*, *strong independence of irrelevant objects*, and *resource-monotonicity*.

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