# Limit Laws for Maxima of Contracted Stationary Gaussian Sequences

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**Abstract**: In this paper we derive the weak and strong limits of maxima of contracted stationary Gaussian random sequences. Due to the random contraction we introduce a modified Berman condition which is sufficient for the weak convergence of the maxima of the scaled sample. Under a stronger assumption the weak convergence is strengthened to almost convergence.

**Key words**: random contraction; stationary Gaussian sequence; Gumbel max-domain of attraction; Davis-Resnick tail property.

## 1 Introduction & Main Result

If $X, X_n, n \geq 1$ are independent $N(0,1)$ random variables, then it is well-known (see e.g., Berman (1992), Piterbarg (1996) or Falk et al. (2010)) that the distribution of sample maxima $M_n = \max_{1 \leq i \leq n} X_i$ converges (after normalisation) to the Gumbel distribution $\Lambda(x) = \exp(-\exp(-x)), x \in \mathbb{R}$, i.e.,

$$\lim_{n \to \infty} \sup_{x \in \mathbb{R}} \left| \mathbb{P}\left(M_n \leq a_n x + b_n\right) - \Lambda(x) \right| = 0, \tag{1.1}$$

where

$$a_n = (2 \ln n)^{-\frac{1}{2}} \text{ and } b_n = (2 \ln n)^{\frac{1}{2}} - \frac{1}{2}(2 \ln n)^{-\frac{1}{2}}(\ln \ln n + \ln 4\pi).$$

Due to some underlying random scaling phenomena, often in applications $Y_i = S_i X_i, i \leq n$ are available and not the original observations $X_i, i \leq n$, where $S_i$ is some random factor. Consider in the following $S, S_n, n \geq 1$ independent non-negative random variables with common distribution function $F$ being independent of $X, X_n, n \geq 1$. We are interested in this paper in contraction-type random scaling, i.e., $F$ has a finite upper endpoint, which for simplicity is assumed to be equal to 1.

If $S$ is regularly varying at 1 with index $\gamma \geq 0$, i.e.,

$$\lim_{u \to \infty} \frac{\mathbb{P}\left(S > 1 - t/u\right)}{\mathbb{P}\left(S > 1 - 1/u\right)} = t^{\gamma}, \quad \forall t > 0, \tag{1.2}$$

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then in view of Theorem 3.1 in Hashorva et al. (2010) (see also Theorem 4.1 in Hashorva (2013)), the limit relation (1.1) still holds for $M_n^* = \max_{1\le i\le n} S_i X_i$ with constants

$$b_n = G^{-1}(1-1/n), \quad a_n = 1/b_n \sim (2\ln n)^{-1/2}, \tag{1.3}$$

where $G^{-1}$ is the inverse of the distribution function $G$ of $SX$ and $\sim$ means asymptotical equivalence when $n\to\infty$. Our first motivating result states that for any $S$ not equal to 0 the approximation (1.1) holds.

**Theorem 1.1.** *If $SX$ has distribution function $G$ with generalised inverse $G^{-1}$, then* (1.1) *holds for $M_n^*$ with constants $a_n, b_n$ as in* (1.3).

The seminal result of Berman (1964) shows that if $X_n, n\ge 1$ is a stationary Gaussian sequence with $\rho(n) = \mathbb{E}(X_1 X_n)$, and $X_1$ is a $N(0,1)$ random variable, then the sample maxima $M_n$ still satisfies (1.1), provided that the Berman condition

$$\lim_{n\to\infty} \rho(n)\ln n = 0 \tag{1.4}$$

is satisfied. In the sequel we refer to $X_n, n\ge 1$ as a standard stationary Gaussian sequence (ssGs).

The main result of this contribution stated below shows that Theorem 1.1 can be stated for any ssGs, provided that the Berman condition is accordingly modified, and further some additional restrictions on the random scaling sequence are imposed via the following constrain:

**Assumption A**. *Let $S$ be a non-negative random variable with distribution function $F$ which has upper endpoint 1. For any $u\in(\nu,1)$ with $\nu\in(0,1)$*

$$\mathbb{P}(S_\tau > u) \ge \mathbb{P}(S>u) \ge \mathbb{P}(S_\gamma > u) \tag{1.5}$$

*holds with $S_\gamma, S_\tau$ two non-negative random variables which have a regularly varying survival function at 1 with non-negative index $\gamma$ and $\tau$, respectively.*

We state next the main result of this paper.

**Theorem 1.2.** *If $S$ is such that Assumption A is satisfied, then Theorem 1.1 holds for any ssGs $X_n, n\ge 1$ such that for some $\Delta > 2(\gamma-\tau)$*

$$\lim_{n\to\infty} \rho(n)(\ln n)^{1+\Delta} = 0. \tag{1.6}$$

This paper is organized as follows: we continue below with a new Section discussing our main findings and then presenting an extension which strengthens the distributional convergence of maxima $(M_n^* - b_n)/a_n$ to almost sure convergence. Proofs and auxiliary results are displayed in Section 3.

## 2 Discussion & Extensions

In the light of extreme value theory (see e.g., Resnick (1987), Embrechts et al. (1997), Falk et al. (2010)) the result (1.1) means that the distribution function $\Phi$ is in the Gumbel max-domain of attraction (MDA). A general univariate distribution function $H$ with upper endpoint $\infty$ is in the Gumbel MDA (abbreviated $H \in GMDA(w)$) if (set $\overline{H} = 1 - H$)

$$\frac{\overline{H}(u + x/w(u))}{\overline{H}(u)} \sim \exp(-x), \quad \forall x \geq 0, \tag{2.1}$$

with $w(\cdot)$ some positive scaling function. Again we write $\sim$ to mean asymptotic equivalence of two functions when the argument (typically $u$) approaches infinity. For the standard Gaussian distribution function $\Phi$ on $I\!R$ we have $\Phi \in GMDA(w)$ where $w(x) = x$. Consequently, Theorem 1.1 means that $SX$ has distribution function $G \in GMDA(w)$ with scaling function $w(x) = x$ whenever the random variable $S \geq 0$ is bounded and independent of $X$ which has distribution function $\Phi$.

Regarding Assumption A we mention that it is satisfied by a large class of random contraction $S$, for instance if $S$ is a Beta random variable, or $\mathbb{P}(S = 1) = c \in (0, 1)$ and for some $s < 1$ we have $\mathbb{P}(S < s) = 1 - c$. Another example is when

$$\mathbb{P}\left(S > 1 - \frac{1}{u}\right) = (1 + o(1))cu^{-\gamma}, \quad u \to \infty \tag{2.2}$$

for some $c > 0$. In this particular case, the constants $a_n, b_n$ in (1.3) can be calculated explicitly as

$$a_n = (2\ln n)^{-\frac{1}{2}}, \qquad b_n = b_{n,\gamma} = (2\ln n)^{\frac{1}{2}} + (2\ln n)^{-\frac{1}{2}}\left(\ln \varpi - \frac{2\gamma + 1}{2}(\ln\ln n + \ln 2)\right), \tag{2.3}$$

with $\varpi = c(2\pi)^{-\frac{1}{2}}\Gamma(1 + \gamma)$.

In numerous contributions (see e.g., Cheng et al. (1998), Fahrner and Stadtmüller (1998), Csáki and Gonchigdanzan (2002), Peng et al. (2010), Tan and Wang (2012), Weng et al. (2012), Hashorva and Weng (2013)) the convergence in distribution for the maxima is strengthen to almost sure convergence. We present such an extension of our main result in the next theorem:

**Theorem 2.1.** *Under the assumptions and notation of Theorem 1.2, if further*

$$\rho(n)(\ln n)^{1+\Delta}(\ln\ln n)^{1+\epsilon} = O(1), \quad n \to \infty, \tag{2.4}$$

*for some $\Delta > 2(\gamma - \tau)$ and $\epsilon$ positive, then for $x \in I\!R$*

$$\frac{1}{\ln n}\sum_{k=1}^{n}\frac{1}{k}\mathbb{I}\left(M_k^* \leq a_k x + b_k\right) \to \Lambda(x), \quad n \to \infty \tag{2.5}$$

*holds almost surely, with $\mathbb{I}(\cdot)$ the indicator function.*

**Remarks 2.2.** *i) If $S$ satisfies (2.2), then Theorem 1.2 holds under the Berman condition, i.e., we just need to assume therein that (1.6) is true when $\Delta = 0$. Crucial for the proof is that (3.7) holds with $\epsilon = 0$ if (2.2) holds.*

*ii) If (2.2) is satisfied and (2.4) holds with $\Delta = 0$, then we have (2.5) also holds with $a_n$ and $b_n$ satisfying (2.3).*

*iii) Extension of our results to the case that $X_n, n \ge 1$ is a non-stationary Gaussian sequence is possible. Various results for extremes of non-stationary Gaussian processes are derived by Hüsler and his co-authors, see for more details Falk et al. (2010).*

# 3 Proofs of the Main Results

Proof of Theorem 1.1 The independence of $S$ and $X$ implies for any $\nu \in (1, \infty)$ and $u > 0$

$$\overline{\Phi}(u\nu)\mathbb{P}(S > 1/\nu) = \mathbb{P}(X > u\nu)\mathbb{P}(S > 1/\nu) \le \mathbb{P}(SX > u, S > 1/\nu) \le \mathbb{P}(SX > u) \le \mathbb{P}(X > u), \quad (3.1)$$

where $\Phi$ is the standard Gaussian distribution on $I\!R$ and $\overline{\Phi} = 1 - \Phi$. Since for any $1 < \nu^* < \nu$ we have $\lim_{u\to\infty} \frac{\overline{\Phi}(u\nu)}{\overline{\Phi}(u\nu^*)} = 0$, then

$$\lim_{u\to\infty} \frac{\overline{\Phi}(u\nu)}{\mathbb{P}(SX > u)} = \lim_{u\to\infty} \frac{\overline{\Phi}(u\nu)}{\overline{\Phi}(u\nu^*)} \frac{\overline{\Phi}(u\nu^*)}{\mathbb{P}(SX > u)} \le \frac{1}{\mathbb{P}(S > 1/\nu^*)} \lim_{u\to\infty} \frac{\overline{\Phi}(u\nu)}{\overline{\Phi}(u\nu^*)} = 0$$

implying thus for any $s \in (0, 1)$

$$\begin{aligned} \mathbb{P}(SX > u) &= \int_0^s \overline{\Phi}(u/x)\, dF(x) + \int_s^1 \overline{\Phi}(u/x)\, dF(x) \\ &= O\Big(\overline{\Phi}(u/s)\Big) + \int_s^1 \overline{\Phi}(u/x)\, dF(x) \sim \int_s^1 \overline{\Phi}(u/x)\, dF(x), \quad u \to \infty. \end{aligned} \quad (3.2)$$

Now, uniformly for $x \in [1/2, 1]$ and some fixed $t \in I\!R$

$$\frac{\overline{\Phi}(u/x + (t/x^2)(x/u))}{\overline{\Phi}(u/x)} \to \exp(-t/x^2), \quad u \to \infty.$$

Consequently, for $u$ large and any $\varepsilon^* \in (0, 1)$ and $x \in (s, 1)$

$$(1 - \varepsilon^*)\overline{\Phi}(u/x)\exp(-t/s^2) \le \overline{\Phi}(u/x + t/(xu)) \le (1 + \varepsilon^*)\overline{\Phi}(u/x)\exp(-t)$$

implying thus for all $u$ large and any $s \in (1/2, 1)$

$$(1 - \varepsilon^*)\exp(-t/s^2) \le \frac{\int_s^1 \overline{\Phi}((u + t/u)/x)\, dF(x)}{\int_s^1 \overline{\Phi}(u/x)\, dF(x)} \le (1 + \varepsilon^*)\exp(-t).$$

Hence for any $\varepsilon \in (0,1)$, since $s$ can be close enough to 1 and by (3.2), we obtain

$$(1-\varepsilon)\exp(-t) \le \frac{\mathbb{P}(SX > u + t/u)}{\mathbb{P}(SX > u)} \le (1+\varepsilon)\exp(-t)$$

and thus $SX$ has distribution function in the Gumbel MDA with scaling function $w(u) = u$.

Let $b(t) = G^{-1}(1-1/t)$ with $G^{-1}$ the generalised inverse of the distribution function of $SX$. In view of (3.1) for all $t$ large (write $\Phi^{-1}$ for the inverse of $\Phi$)

$$\Phi^{-1}(1-1/t) \ge b(t) \ge \frac{1}{\nu}\Phi^{-1}\Big(1 - \frac{1}{t\mathbb{P}(S > 1/\nu)}\Big)$$

and since $\nu > 1$ can be close enough to 1

$$b(n) \sim \Phi^{-1}(1-1/n) \sim (2\ln n)^{\frac{1}{2}}, \quad a_n \sim (2\ln n)^{-\frac{1}{2}}, \quad n \to \infty,$$

hence the claim follows. □

**Lemma 3.1.** *Suppose that Assumption A holds for $S, S_\gamma, S_\tau$ which are independent of the random variable $X$ with distribution function $H$. If $H$ has an infinite upper endpoint and further $H \in GMDA(w)$, then*

$$\mathbb{P}(S_\tau X > u) \ge \mathbb{P}(SX > u) \ge \mathbb{P}(S_\gamma X > u) \tag{3.3}$$

*holds for all $u$ large.*

Proof of Lemma 3.1 By the independence of $S$ and $X$ and the fact that $S$ has distribution function with upper endpoint equal 1 for any $\nu > 1, u > 0$ we have

$$\mathbb{P}(SX > u) = \int_u^{u\nu} \mathbb{P}(S > u/x)\, dH(x) + O(\overline{H}(u\nu)).$$

Hence by (1.5), for all $u$ large

$$\int_u^{u\nu} \mathbb{P}(S_\tau > u/x)\, dH(x) + O(\overline{H}(u\nu)) \ge \mathbb{P}(SX > u) \ge \int_u^{u\nu} \mathbb{P}(S_\gamma > u/x)\, dH(x) + O(\overline{H}(u\nu)).$$

A key property of $H \in GMDA(w)$ is the so-called *Davis-Resnick tail property*, see e.g., Hashorva (2012). Specifically, by Proposition 1.1 of Davis and Resnick (1988)

$$\lim_{u\to\infty} (uw(u))^{\mu} \frac{\overline{H}(xu)}{\overline{H}(u)} \quad = \quad 0 \tag{3.4}$$

holds for any $\mu \ge 0$ and $x > 1$, hence the claim follows now by Theorem 3.1 in Hashorva et al. (2010). □

**Lemma 3.2.** *Let the positive random variables $Z_n, n \ge 1$ have df $H_n$ such that for all large $z$*

$$1 - H_n(z) = \exp\left(-\vartheta_n z^q\right) \tag{3.5}$$

*holds with $q > 0, \vartheta_n$ positive constants satisfying $\vartheta_n \in [a, b], \forall n \ge 1$ with $a < b$ two finite positive constants. If further $Z_n$ is independent of $S$ which has a regularly varying survival function at 1 with index $\gamma \ge 0$ and $u_n, n \ge 1$ are positive constants such that $\lim_{n\to\infty} u_n = \infty$, then we have*

$$\mathbb{P}\left(SZ_n > u_n\right) \sim \Gamma(\gamma + 1) \exp\left(-\vartheta_n u_n^q\right) \mathbb{P}\left(S > 1 - \frac{1}{q\vartheta_n u_n^q}\right). \tag{3.6}$$

Proof of Lemma 3.2 Let $H(x) = 1 - \exp(-x^q), x > 0$ and let $Z$ with distribution function $H$ be independent of $S$. By Davis-Resnick tail property of $H$ given in (3.4) for all large $u_n$, all $\varepsilon > 0$

$$\begin{aligned}
\mathbb{P}\left(SZ_n > u_n\right) &= \int_{u_n}^{\infty} \mathbb{P}\left(S > \frac{u_n}{z}\right) dH_n(z) \\
&\sim \int_{u_n}^{u_n(1+\varepsilon)} \mathbb{P}\left(S > \frac{u_n}{z}\right) dH_n(z) \\
&\sim \int_{\vartheta_n^{1/q} u_n}^{\vartheta_n^{1/q} u_n(1+\varepsilon)} \mathbb{P}\left(S > \frac{\vartheta_n^{1/q} u_n}{z}\right) dH(z) \\
&\sim \mathbb{P}\left(SZ > \vartheta_n^{1/q} u_n\right) \\
&\sim \Gamma(\gamma + 1) \exp\left(-\vartheta_n u_n^q\right) \mathbb{P}\left(S > 1 - \frac{1}{q\vartheta_n u_n^q}\right),
\end{aligned}$$

where the last step follows from Theorem 3.1 in Hashorva at al. (2010). □

**Remarks 3.1.** *If $S$ has a regularly varying survival function at 1 with index $\gamma \ge 0$, by the Karamata representation (see e.g., Resnick (1987), p.17), we have*

$$\mathbb{P}\left(S > 1 - \frac{1}{q\vartheta_n u_n^q}\right) \le c\left(\frac{1}{q\vartheta_n u_n^q}\right)^{\gamma - \epsilon}$$

*with $c > 1$, $\epsilon \in (0, \gamma)$. Consequently, by (3.6)*

$$\mathbb{P}\left(SZ_n > u_n\right) \le c\Gamma(\gamma + 1) \exp\left(-\vartheta_n u_n^q\right) \left(\frac{1}{q\vartheta_n u_n^q}\right)^{\gamma - \epsilon} = O\Big(\left(u_n\right)^{-q(\gamma - \epsilon)} \exp\left(-\vartheta_n u_n^q\right)\Big) \tag{3.7}$$

*holds for any positive sequence $u_n, n \ge 1$ such that $\lim_{n\to\infty} u_n = \infty$.*

**Lemma 3.3.** *Under the conditions of Theorem 1.2, we have*

$$n \sum_{k=1}^{n-1} |\rho(k)| \int_0^1 \int_0^1 \exp\left(-\frac{\left(u_n(x)/s\right)^2 + \left(u_n(x)/t\right)^2}{2(1 + |\rho(k)|)}\right) dF(s)\, dF(t) \to 0, \quad n \to \infty, \tag{3.8}$$

*where $u_n(x) = a_n x + b_n$ with $a_n$ and $b_n$ are defined in (1.3) and $x \in \mathbb{R}$.*

Proof of Lemma 3.3 Denote $u_{n,c}(z) = a_n z + b_{n,c}$ with $a_n$ and $b_{n,c}$ defined in (2.3). By Lemma 3.1, we have for all large $n$ $b_{n,\gamma} \le b_n \le b_{n,\tau}$, hence

$$u_{n,\gamma}(z) \le u_n(z) \le u_{n,\tau}(z) \qquad \text{and} \qquad \tau \le \gamma.$$

Consequently, using the assumption of $Z_n$ with $q = 2$ and $\vartheta_n = 1/(2 + 2|\rho(n)|), n \ge 1$ in (3.5) and (1.5), along the lines of the proof of Lemma 3.2 we obtain for all large $n$

$$\mathbb{P}\left(SZ_n > u_n(z)\right) \le \mathbb{P}\left(SZ_n > u_{n,\gamma}(z)\right) \le \mathbb{P}\left(S_\tau Z_n > u_{n,\gamma}(z)\right). \tag{3.9}$$

Define next

$$\sigma = \max_{k\ge 1} |\rho(k)|, \quad \kappa_n = [n^r],$$

where $r$ is any positive constant such that $r < (1-\sigma)/(1+\sigma)$. This choice of $r$ is possible since by Berman condition and stationarity of the sequence $\sigma < 1$ follows easily.

Hereafter $C_1, C_2, C_3$ are positive constants and $\varepsilon \in (0, \tau)$ is taken to be sufficiently small. By the inequality (3.9) and (3.7) (denote $F_\tau$ the distribution function of $S_\tau$) for all large $n$

$$\begin{aligned}
& n \sum_{k=1}^{n-1} |\rho(k)| \int_0^1 \int_0^1 \exp\left(-\frac{(u_n(x)/s)^2 + (u_n(x)/t)^2}{2(1+|\rho(k)|)}\right) dF(s)\, dF(t) \\
\le\ & n \sum_{k=1}^{n-1} |\rho(k)| \int_0^1 \int_0^1 \exp\left(-\frac{(u_{n,\gamma}(x)/s)^2 + (u_{n,\gamma}(x)/t)^2}{2(1+|\rho(k)|)}\right) dF_\tau(s)\, dF_\tau(t) \\
\le\ & C_1 n \sum_{k=1}^{n-1} |\rho(k)| (u_{n,\gamma}(x))^{-4(\tau-\epsilon)} \exp\left(-\frac{u_{n,\gamma}^2(x)}{1+|\rho(k)|}\right) \\
=\ & C_1 n \left(\sum_{k=1}^{\kappa_n} + \sum_{k=\kappa_n+1}^{n-1}\right) |\rho(k)| (u_{n,\gamma}(x))^{-4(\tau-\epsilon)} \exp\left(-\frac{u_{n,\gamma}^2(x)}{1+|\rho(k)|}\right) =: S_{n1} + S_{n2}.
\end{aligned}$$

According to (2.3) we have

$$\exp\left(-\frac{u_{n,\gamma}^2(x)}{2}\right) \sim C_2 n^{-1} (u_{n,\gamma}(x))^{1+2\gamma}, \quad u_{n,\gamma}(x) \sim \sqrt{2\ln n}, \quad n \to \infty.$$

As in Lemma 4.3.2 in Leadbetter et al. (1983)

$$\begin{aligned}
S_{n1} &\le C_3 n^{1+r} (u_{n,\gamma}(x))^{-4(\tau-\epsilon)} \exp\left(-\frac{u_{n,\gamma}^2(x)}{1+\sigma}\right) \\
&= O\left(n^{1+r} (u_{n,\gamma}(x))^{-4(\tau-\epsilon)} \left(\frac{(u_{n,\gamma}(x))^{1+2\gamma}}{n}\right)^{\frac{2}{1+\sigma}}\right) \\
&= O\left(n^{1+r-\frac{2}{1+\sigma}} (\ln n)^{\frac{1+2\gamma}{1+\sigma} - 2(\tau-\epsilon)}\right) \to 0, \quad n \to \infty
\end{aligned}$$

by our choice $1 + r - \frac{2}{1+\sigma} < 0$. Next, with $\sigma(l) = \max_{k\ge l} |\rho(k)| < 1$, we have

$$S_{n2} \le C_1 n \sigma(\kappa_n) (u_{n,\gamma}(x))^{-4(\tau-\epsilon)} \exp\left(-u_{n,\gamma}^2(x)\right) \sum_{k=\kappa_n+1}^{n-1} \exp\left(\frac{u_{n,\gamma}^2(x)|\rho(k)|}{1+|\rho(k)|}\right)$$

$$
\begin{aligned}
&\leq C_1 n^2 \sigma(\kappa_n)(u_{n,\gamma}(x))^{-4(\tau-\epsilon)} \exp\left(-u_{n,\gamma}^2(x)\right) \exp\left(\sigma(\kappa_n) u_{n,\gamma}^2(x)\right) \\
&= O\Big(\sigma(\kappa_n)(u_{n,\gamma}(x))^{2+4\gamma-4(\tau-\epsilon)} \exp\left(\sigma(\kappa_n) u_{n,\gamma}^2(x)\right)\Big).
\end{aligned}
$$

By (1.6) and the fact that $\lim_{n\to\infty} \kappa_n = \infty$ we have for some $\Delta > 2(\gamma - \tau)$

$$
\sigma(\kappa_n)(u_{n,\gamma}(x))^{2+2\Delta} \sim \sigma(\kappa_n)(2\ln n)^{1+\Delta} \leq \left(\frac{2}{r}\right)^{1+\Delta} \max_{k\geq\kappa_n} |\rho(k)|(\ln k)^{1+\Delta} \to 0
$$

and

$$
\sigma(\kappa_n)(u_{n,\gamma}(x))^2 \sim 2\sigma(\kappa_n)\ln n \leq \sigma(\kappa_n)(2\ln n)^{1+\Delta} \to 0
$$

as $n \to \infty$. Since the exponential term above tends to one and the remaining product tends to zero, the claim follows. □

Proof of Theorem 1.2 Let $\hat{X}_n, n \geq 1$ be independent random variables with the same distribution as $X_1$ and define $\hat{\mathbb{M}}_n^* = \max_{1\leq i\leq n} S_i \hat{X}_i$. If (1.5) holds, by the independence of the scaling factors with the Gaussian random variables and Berman's Normal Comparison Lemma (see e.g., Piterbarg (1996)), and using Lemma 3.3 we obtain

$$
\begin{aligned}
&\left|\mathbb{P}\left(M_n^* \leq u_n(x)\right) - \mathbb{P}\left(\hat{\mathbb{M}}_n^* \leq u_n(x)\right)\right| \\
&\leq \int_{[0,1]^n} \left|\mathbb{P}\left(\bigcap_{k=1}^n \left\{X_k \leq \frac{u_n(z)}{s_k}\right\}\right) - \mathbb{P}\left(\bigcap_{k=1}^n \left\{\hat{X}_k \leq \frac{u_n(z)}{s_k}\right\}\right)\right| dF(s_1)\cdots dF(s_n) \\
&\leq \frac{1}{4} n \sum_{k=1}^{n-1} |\rho(k)| \int_0^1 \int_0^1 \exp\left(-\frac{(u_n(x)/s)^2 + (u_n(s)/t)^2}{2(1+|\rho(k)|)}\right) dF(s)dF(t) \\
&\to 0
\end{aligned}
$$

as $n \to \infty$, and thus by Theorem 1.1 the claim follows. □

Proof of Theorem 2.1 In order to show the claim, by Theorem 1.2 it suffices to prove that

$$
\frac{1}{\ln n} \sum_{k=1}^n \frac{1}{k} \Big(\mathbb{I}\left(M_k^* \leq u_k(x)\right) - \mathbb{P}\left(M_k^* \leq u_k(x)\right)\Big) \to 0, \quad n \to \infty \tag{3.10}
$$

holds almost surely, which according to Lemma 3.1 in Csáki and Gonchigdanzan (2002) follows if for some $\epsilon > 0$

$$
\sum_{1\leq k<l\leq n}^{n} \frac{1}{kl} \operatorname{Cov}\left(\mathbb{I}\left(M_k^* \leq u_k(x)\right), \mathbb{I}\left(M_l^* \leq u_l(x)\right)\right) = O\Big((\ln n)^2 (\ln\ln n)^{-1-\epsilon}\Big).
$$

Next, for any $k < l$ (write below $M_{l,k}^* = \max_{k<i\leq l} S_i X_i$ and $\hat{\mathbb{M}}_{l,k}^* = \max_{k<i\leq l} S_i \hat{X}_i$)

$$
\operatorname{Cov}\left(\mathbb{I}\left(M_k^* \leq u_k(x)\right), \mathbb{I}\left(M_l^* \leq u_l(x)\right)\right)
$$

$$
\begin{aligned}
&\leq \; 2\,\mathbb{E}\left|\mathbb{I}\left(M_l^* \leq u_l(x)\right) - \mathbb{I}\left(M_{l,k}^* \leq u_l(x)\right)\right| + \left|\mathrm{Cov}\left(\mathbb{I}(M_k^* \leq u_k(x)), \mathbb{I}(M_{l,k}^* \leq u_l(x))\right)\right| \\
&\leq \; 2\left|\mathbb{P}\left(M_{l,k}^* \leq u_l(x)\right) - \mathbb{P}\left(\hat{\mathbb{M}}_{l,k}^* \leq u_l(x)\right)\right| + 2\left|\mathbb{P}\left(M_l^* \leq u_l(x)\right) - \mathbb{P}\left(\hat{\mathbb{M}}_l^* \leq u_l(x)\right)\right| \\
&\quad +2\left|\mathbb{P}\left(\hat{\mathbb{M}}_{l,k}^* \leq u_l(x)\right) - \mathbb{P}\left(\hat{\mathbb{M}}_l^* \leq u_l(x)\right)\right| \\
&\quad + \left|\mathbb{P}\left(M_k^* \leq u_k(x), M_{l,k}^* \leq u_l(x)\right) - \mathbb{P}\left(M_k^* \leq u_k(x)\right)\mathbb{P}\left(M_{l,k}^* \leq u_l(x)\right)\right| \\
&= \; P_1 + P_2 + P_3 + P_4.
\end{aligned}
$$

In view of Berman's Normal Comparison Lemma and (2.4), along the same lines of the proof of Lemma 3.3, we have

$$P_i = O\biggl((\ln\ln n)^{-1-\epsilon}\biggr), \qquad i = 1, 2, 4.$$

Further, since

$$P_3 = \mathbb{G}^{l-k}(u_l(x)) - \mathbb{G}^l(u_l(x)) \leq \frac{k}{l},$$

where $\mathbb{G}$ is the df of $S_1\hat{X}_1$, we establish the claim. □

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