

# PITERBARG'S MAX-DISCRETISATION THEOREM FOR STATIONARY VECTOR GAUSSIAN PROCESSES OBSERVED ON DIFFERENT GRIDS

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**Abstract:** In this paper we derive Piterbarg's max-discretisation theorem for two different grids considering centered stationary vector Gaussian processes. So far in the literature results in this direction have been derived for the joint distribution of the maximum of Gaussian processes over  $[0, T]$  and over a grid  $\mathfrak{R}(\delta_1(T)) = \{k\delta_1(T) : k = 0, 1, \dots\}$ . In this paper we extend the recent findings by considering additionally the maximum over another grid  $\mathfrak{R}(\delta_2(T))$ . We derive the joint limiting distribution of maximum of stationary Gaussian vector processes for different choices of such grids by letting  $T \rightarrow \infty$ . As a by-product we find that the joint limiting distribution of the maximum over different grids, which we refer to as the Piterbarg distribution, is in the case of weakly dependent Gaussian processes a max-stable distribution.

**Key Words:** Piterbarg's max-discretisation theorem; Limiting distribution; Piterbarg distribution; Pickands constant; Extremes of Gaussian processes; Gumbel limit law; Berman condition.

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## 1. INTRODUCTION

Let  $\{X(t), t \geq 0\}$  be a centered stationary Gaussian process with continuous sample paths, unit variance and correlation function  $r(\cdot)$  which satisfies for some  $\alpha \in (0, 2]$

$$(1) \quad r(t) = 1 - C|t|^\alpha + o(|t|^\alpha) \text{ as } t \rightarrow 0 \text{ and } r(t) < 1 \text{ for } t \neq 0,$$

where  $C$  is some positive constant. In various applications only realisations of  $X$  on a discrete time grid are possible. For simplicity, in this paper we shall consider uniform grids of points  $\mathfrak{R}(\delta) = \{k\delta : k = 0, 1, \dots\}$  where  $\delta := \delta(T) > 0$  depends on the parameter  $T > 0$ . In view of the findings of Berman (see [5, 7]) the maximum of  $X$  taken over such a discrete grid has a limiting Gumbel distribution if

$$(2) \quad \lim_{T \rightarrow \infty} (2 \ln T)^{1/\alpha} \delta(T) = D,$$

with  $D = \infty$  and the Berman condition

$$(3) \quad \lim_{T \rightarrow \infty} r(T) \ln T = r$$

holds for  $r = 0$ . Specifically, for the maximum  $M(\delta, T) = \max_{i: 0 \leq i\delta \leq T} X(i\delta)$  over  $\mathfrak{R}(\delta) \cap [0, T]$  we have

$$\lim_{T \rightarrow \infty} \sup_{x \in \mathbb{R}} \left| \mathbb{P} \{a_T(M(\delta, T) - b_{\delta, T}) \leq x\} - e^{-e^{-x}} \right| = 0,$$

provided that both (2) and (3) hold, where

$$(4) \quad a_T = \sqrt{2 \ln T}, \quad b_{\delta, T} = a_T - \frac{\ln(a_T \delta \sqrt{2\pi})}{a_T}, \quad T > 0.$$

For the maximum over  $[0, T]$  defined thus as  $M(T) = \max_{t \in [0, T]} X(t)$  it is well-known (see e.g., [21, 1, 2, 7, 26]) that (1) and (3) imply

$$(5) \quad \lim_{T \rightarrow \infty} \sup_{x \in \mathbb{R}} \left| \mathbb{P} \{a_T(M(T) - b_T) \leq x\} - e^{-e^{-x}} \right| = 0,$$

where

$$(6) \quad b_T = a_T + a_T^{-1} \ln((2\pi)^{-1/2} C^{1/\alpha} H_\alpha a_T^{-1+2/\alpha})$$

and  $H_\alpha \in (0, \infty)$  denotes Pickands constant, see [24, 25, 6, 21, 2, 26, 11, 3, 14, 12, 9, 17] for more details and generalisations of  $H_\alpha$ .

The seminal contribution [27] derives the joint convergence as  $T \rightarrow \infty$  of  $M(T)$  and  $M(\delta, T)$  showing their asymptotic independence, i.e.,

$$\lim_{T \rightarrow \infty} \sup_{x, y \in \mathbb{R}} \left| \mathbb{P} \{a_T(M(T) - b_T) \leq x, a_T(M(\delta, T) - b_{\delta, T}) \leq y\} - e^{-e^{-x} - e^{-y}} \right| = 0.$$

Hereafter we set  $B_{\alpha/2}^*(t) := \sqrt{2}B_{\alpha/2}(t) - |t|^\alpha, t \geq 0$  with  $B_\alpha$  a standard fractional Brownian motion with Hurst index  $\alpha/2 \in (0, 1)$ ; recall that  $\delta = \delta(T)$  is given by (2). Define further for any  $D > 0$

$$H_{D, \alpha} = \lim_{\lambda \rightarrow \infty} \lambda^{-1} \mathbb{E} \left\{ e^{\max_{k \in \mathbb{N}: kD \in [0, \lambda]} B_{\alpha/2}^*(kD)} \right\} \in (0, \infty)$$

and set (the constant  $C > 0$  below relates to (1))

$$(7) \quad b_T(D) = a_T + a_T^{-1} \ln((2\pi)^{-1/2} C^{1/\alpha} H_{D, \alpha} a_T^{-1+2/\alpha}).$$

For  $\mathfrak{R}(Da_T^{-2/\alpha}), D > 0$  (in this case the grid is called Pickands grid and  $\delta = \delta(T) = Da_T^{-2/\alpha}$ ), then in view of [27], Theorem 2 the stated asymptotic independence does not hold since

$$\lim_{T \rightarrow \infty} \sup_{x, y \in \mathbb{R}} \left| \mathbb{P} \{a_T(M(T) - b_T) \leq x, a_T(M(\delta, T) - b_T(D)) \leq y\} - e^{-e^{-x} - e^{-y} + H_{D, \alpha}^{\ln H_\alpha + x, \ln H_{D, \alpha} + y}} \right| = 0,$$

where the function  $H_{D, \alpha}^{x, y}$  is defined for any  $x, y \in \mathbb{R}$  as

$$(8) \quad H_{D, \alpha}^{x, y} = \lim_{\lambda \rightarrow \infty} \lambda^{-1} H_{D, \alpha}^{x, y}(\lambda) \in (0, \infty),$$

with

$$H_{D, \alpha}^{x, y}(\lambda) = \int_{s \in \mathbb{R}} e^s \mathbb{P} \left\{ \max_{t \in [0, \lambda]} B_{\alpha/2}^*(t) > s + x, \max_{k \in \mathbb{N}: kD \in [0, \lambda]} B_{\alpha/2}^*(kD) > s + y \right\} ds.$$

Since it follows that for any  $w \in \mathbb{R}$

$$(9) \quad \lim_{x \rightarrow -\infty} H_{D, \alpha}^{x, w} = e^{-w} H_{D, \alpha}, \quad \lim_{y \rightarrow -\infty} H_{D, \alpha}^{w, y} = e^{-w} H_\alpha \in (0, \infty),$$

then

$$Q(x, y) = e^{-e^{-x} - e^{-y} + H_{D, \alpha}^{\ln H_{\alpha} + x, \ln H_{D, \alpha} + y}}, \quad x, y \in \mathbb{R}$$

is a bivariate distribution function which has Gumbel marginals  $Q(z, \infty) = Q(\infty, z) = e^{-e^{-z}}, z \in \mathbb{R}$ . Moreover  $Q$  is a bivariate max-stable distribution, which we shall refer to as Piterbarg distribution. This multivariate distribution is of some independent interest for statistical modelling of dependent multivariate risks.

In the extreme case of a dense grid, which in the terminology of [27] means that (2) holds for  $D = 0$ , then by Theorem 3 in [27]

$$\lim_{T \rightarrow \infty} \sup_{x, y \in \mathbb{R}} \left| \mathbb{P} \{a_T(M(T) - b_T) \leq x, a_T(M(\delta, T) - b_T) \leq y\} - e^{-e^{-\min(x, y)}} \right| = 0$$

thus the continuous time and the discrete time maxima are asymptotically completely dependent.

In case of two different uniform grids  $\mathfrak{R}(\delta_1)$  and  $\mathfrak{R}(\delta_2)$  a natural question that arises is:

What is the joint limiting behaviour of  $M(T), M(\delta_1, T), M(\delta_2, T)$  for different types of grids?

Motivated by this question, our findings this contribution include:

a) We show that  $M(\delta_1, T)$  and  $M(\delta_2, T)$  are always asymptotically independent if one grid is sparse and the other grid is Pickands or dense. Further, we obtain the joint limiting distribution if one of the grids is Pickands, and the other grid is Pickands or dense.

b) The Berman condition is relaxed by assuming that (3) holds for some  $r \in [0, \infty)$ . When  $r > 0$  the Gaussian process  $X$  is said to be strongly dependent, see [22, 26, 23, 32, 29, 8] for details on the extremes of such Gaussian processes. The contribution [34] derives Piterbarg's max-discretisation theorem for strongly dependent Gaussian processes. In applications, often modelling of the maximum of functionals of a Gaussian vector process is of interest, see e.g., [38, 4, 10]. Our results in this paper are derived for the more general framework of Gaussian vector processes extending the recent findings of [31] by considering simultaneously two different grids. This paper highlights the role of different grids in the approximation of the maximum over a continuous interval. Our results are therefore of interest for simulation studies, which was the main motivation of [27, 19, 20, 36, 37, 30, 35].

c) As a by-product we show that for weakly dependent stationary Gaussian processes the limiting distributions are max-stable. In Extreme Value Theory max-stable distributions and processes are characterised in different ways, see e.g., [15, 13]. In order for a multivariate max-stable distribution to be also useful for statistical modelling, it is important to find how that distribution approximates the maxima of certain sequences (or triangular arrays). Piterbarg max-stable distributions are therefore important since we show also their usefulness in the approximations of maxima over different grids.

Organisation of the article is as follows. Our main results are presented in the next section. All the proofs are relegated to Section 3 which is followed by an Appendix.

## 2. MAIN RESULTS

We shall investigate in the following the asymptotics of maxima over different grids of a centered stationary multivariate  $p$ -dimensional Gaussian process  $\{\mathbf{X}(t), t \geq 0\}$ . Each component  $X_k, k \leq p$  of  $\mathbf{X}$  is assumed to have a constant variance function equal to 1, continuous sample paths and correlation function  $r_{kk}(t) =$

$Cov(X_k(s), X_k(s+t))$  which satisfies for any index  $k \leq p$

$$(10) \quad r_{kk}(t) = 1 - C|t|^\alpha + o(|t|^\alpha) \text{ as } t \rightarrow 0 \text{ and } r_{kk}(t) < 1 \text{ for } t \neq 0$$

for some positive constants  $C$ . Hereafter we suppose that  $\mathbf{X}$  has jointly stationary components with cross-correlation function  $r_{kl}(t) = Cov(X_k(s), X_l(s+t))$  which does not depend on  $s$  for any  $s, t$  positive. The strong dependence condition for the vector Gaussian process  $\mathbf{X}$  reads

$$(11) \quad \lim_{T \rightarrow \infty} r_{kl}(T) \ln T = r_{kl} \in [0, \infty), \quad 1 \leq k, l \leq p.$$

In order to exclude the possibility that  $|X_k(t)| = |X_l(t+t_0)|$  for some  $k \neq l, t_0 > 0$

$$(12) \quad \max_{k \neq l} \sup_{t \in [0, \infty)} |r_{kl}(t)| < 1$$

will be further assumed. For simplicity we consider only two uniform grids  $\mathfrak{R}(\delta_1)$  and  $\mathfrak{R}(\delta_2)$ . Recall that  $\delta_i, i = 1, 2$  depend on  $T > 0$ ; in the case of Pickands grid we set

$$\mathfrak{R}(\delta_i) = \mathfrak{R}(D_i a_T^{-2/\alpha})$$

for some constant  $D_i > 0, i = 1, 2$ . The vector of maxima on continuous time will be denoted by  $\mathbf{M}(T)$  and that with respect to the discrete uniform grid  $\mathfrak{R}(\delta_i), i = 1, 2$  by  $\mathbf{M}(\delta_i, T)$ . This means that the  $k$ th components of these two random vectors are  $M_k(T)$  and  $M_k(\delta_i, T)$ , respectively which are defined by

$$M_k(T) = \max_{t \in [0, T]} X_k(t), \quad M_k(\delta_i, T) = \max_{t \in \mathfrak{R}(\delta_i) \cap [0, T]} X_k(t), \quad k \leq p.$$

For notational simplicity we shall set below

$$\widetilde{\mathbf{M}}(T) = \left( a_T(M_1(T) - b_T), \dots, a_T(M_p(T) - b_T) \right)$$

and

$$\widetilde{\mathbf{M}}(\delta_i, T) = \left( a_T(M_1(\delta_i, T) - b_{\delta_i, T}), \dots, a_T(M_p(\delta_i, T) - b_{\delta_i, T}) \right),$$

where  $b_{\delta_i, T}$  is defined in (4) if the grid  $\mathfrak{R}(\delta_i)$  is sparse,  $b_{\delta_i, T} = b_T(D_i)$  is given by (7) if we consider a Pickands grid  $\mathfrak{R}(\delta_i) = \mathfrak{R}(D_i a_T^{-2/\alpha})$  and for a dense grid we set  $b_{\delta_i, T} = b_T$  with  $b_T$  defined in (6).

In the following  $\mathbf{x}, \mathbf{y}_1, \mathbf{y}_2 \in \mathbb{R}^p$  are fixed vectors and  $\mathbf{Z}$  is a  $p$ -dimensional centered Gaussian random vector with covariances

$$(13) \quad Cov(Z_k, Z_l) = \frac{r_{kl}}{\sqrt{r_{kk}r_{ll}}}, \quad 1 \leq l \leq k \leq p.$$

When  $r_{kk}r_{ll} = 0$  we assume that  $Z_k$  and  $Z_l$  are independent, i.e., we shall set

$$Cov(Z_k, Z_l) = 0.$$

The operations with vectors are meant componentwise, for instance  $\mathbf{x} \leq \mathbf{y}$  means  $x_k \leq y_k$  for any index  $k \leq p$ , with  $x_k$  and  $y_k$  the  $k$ th component of  $\mathbf{x}$  and  $\mathbf{y}$ , respectively. Hereafter we define

$$p_{T,\mathbf{x},\mathbf{y},\delta} := \mathbb{P} \left\{ \widetilde{\mathbf{M}}(T) \leq \mathbf{x}, \widetilde{\mathbf{M}}(\delta_i, T) \leq \mathbf{y}_i, i = 1, 2 \right\}.$$

In the first theorem below we discuss the case when one of the grids is sparse. Our results shall establish that

$$(14) \quad \lim_{T \rightarrow \infty} \sup_{\mathbf{x}, \mathbf{y}_1, \mathbf{y}_2 \in \mathbb{R}^p} \left| p_{T,\mathbf{x},\mathbf{y},\delta} - \mathbb{E} \left\{ \exp \left( - \sum_{k=1}^p f(x_k, y_{k1}, y_{k2}) e^{-r_{kk} + \sqrt{2r_{kk}} Z_k} \right) \right\} \right| = 0,$$

where the function  $f$  is given below explicitly for each particular case.

**Theorem 2.1.** *Let  $\{\mathbf{X}(t), t \geq 0\}$  be a centered stationary Gaussian vector process as defined above and let  $\mathfrak{R}(\delta_1)$  be a sparse grid. Assume that (10), (11) and (12) hold and the Gaussian random vector  $\mathbf{Z}$  has a positive-definite covariance matrix with elements defined in (13).*

*i) If  $\mathfrak{R}(\delta_2)$  is another sparse grid such that  $\mathfrak{R}(\delta_1) \cap \mathfrak{R}(\delta_2) = \emptyset$  or  $\lim_{T \rightarrow \infty} \delta_1(T)/\delta_2(T) = \infty$ , then (14) holds with*

$$f(x_k, y_{k1}, y_{k2}) = e^{-x_k} + e^{-y_{k1}} + e^{-y_{k2}}.$$

*ii) Let  $\mathfrak{R}(\delta_2)$  be a sparse grid such that  $\mathfrak{R}(\delta_1) \cap \mathfrak{R}(\delta_2) = \mathfrak{R}(\delta_3)$ . If  $\mathfrak{R}(\delta_3)$  is a non-empty grid such that*

$$\lim_{T \rightarrow \infty} \ln \left( \frac{\delta_3(T)}{\delta_1(T)} \right) = \theta_1 \in [0, \infty), \quad \lim_{T \rightarrow \infty} \ln \left( \frac{\delta_3(T)}{\delta_2(T)} \right) = \theta_2 \in [0, \infty),$$

*then (14) holds with (write  $\theta = \theta_2 - \theta_1$ )*

$$f(x_k, y_{k1}, y_{k2}) = e^{-x_k} + e^{-y_{k1}} + e^{-y_{k2}} - e^{-y_{k1} - \theta} I(y_{k1} > y_{k2} + \theta) - e^{-y_{k2} - \theta} I(y_{k1} \leq y_{k2} + \theta),$$

*where  $I(\cdot)$  is the indicator function.*

*iii) If  $\mathfrak{R}(\delta_2) = \mathfrak{R}(D_2 a_T^{-2/\alpha})$  is a Pickands grid, then (14) holds with*

$$f(x_k, y_{k1}, y_{k2}) = e^{-x_k} + e^{-y_{k1}} + e^{-y_{k2}} - H_{D_2, \alpha}^{\ln H_\alpha + x_k, \ln H_{D_2, \alpha} + y_{k2}}.$$

*iv) If  $\mathfrak{R}(\delta_2)$  is a dense grid, then again (14) holds with*

$$f(x_k, y_{k1}, y_{k2}) = e^{-y_{k1}} + e^{-\min(x_k, y_{k2})}.$$

We consider next the cases that one grid is a Pickands grid and the second one is either a Pickands or a dense grid. For positive constants  $D_1, D_2, \lambda$  and  $x, z_1, z_2 \in \mathbb{R}$  define (recall  $B_{\alpha/2}^*(t) := \sqrt{2}B_{\alpha/2}(t) - |t|^\alpha$ )

$$H_{D_1, D_2, \alpha}^{z_1, z_2}(\lambda) = \int_{s \in \mathbb{R}} e^{s\mathbb{P}} \left\{ \max_{k \in \mathbb{N}: kD_i \in [0, \lambda]} B_{\alpha/2}^*(kD_i) > s + z_i, i = 1, 2 \right\} ds$$

and

$$H_{D_1, D_2, \alpha}^{x, z_1, z_2}(\lambda) = \int_{s \in \mathbb{R}} e^{s\mathbb{P}} \left\{ \max_{t \in [0, \lambda]} B_{\alpha/2}^*(t) > s + x, \max_{k \in \mathbb{N}: kD_i \in [0, \lambda]} B_{\alpha/2}^*(kD_i) > s + z_i, i = 1, 2 \right\} ds.$$

**Theorem 2.2.** *Under the assumptions of Theorem 2.1 suppose further that  $\mathfrak{R}(\delta_1) = \mathfrak{R}(D_1 a_T^{-2/\alpha})$ ,  $D_1 > 0$  is a Pickands grid.*

i) *If  $\mathfrak{R}(\delta_2) = \mathfrak{R}(D_2 a_T^{-2/\alpha})$ ,  $D_2 \in (0, \infty) \setminus \{D_1\}$  is also a Pickands grid, then for any  $x, z_1, z_2 \in \mathbb{R}$*

$$H_{D_1, D_2, \alpha}^{z_1, z_2} = \lim_{\lambda \rightarrow \infty} \frac{H_{D_1, D_2, \alpha}^{z_1, z_2}(\lambda)}{\lambda} \in (0, \infty) \quad \text{and} \quad H_{D_1, D_2, \alpha}^{x, z_1, z_2} = \lim_{\lambda \rightarrow \infty} \frac{H_{D_1, D_2, \alpha}^{x, z_1, z_2}(\lambda)}{\lambda} \in (0, \infty)$$

and further (14) holds with  $f$  given by

$$\begin{aligned} f(x_k, y_{k1}, y_{k2}) &= e^{-x_k} + e^{-y_{k1}} + e^{-y_{k2}} - H_{D_1, \alpha}^{\ln H_\alpha + x_k, \ln H_{D_1, \alpha} + y_{k1}} - H_{D_2, \alpha}^{\ln H_\alpha + x_k, \ln H_{D_2, \alpha} + y_{k2}} \\ &\quad - H_{D_1, D_2, \alpha}^{\ln H_{D_1, \alpha} + y_{k1}, \ln H_{D_2, \alpha} + y_{k2}} + H_{D_1, D_2, \alpha}^{\ln H_\alpha + x_k, \ln H_{D_1, \alpha} + y_{k1}, \ln H_{D_2, \alpha} + y_{k2}}. \end{aligned}$$

ii) *If  $\mathfrak{R}(\delta_2)$  is a dense grid, then (14) holds with*

$$f(x_k, y_{k1}, y_{k2}) = e^{-\min(x_k, y_{k2})} + e^{-y_{k1}} - H_{D_1, \alpha}^{\ln H_\alpha + \min(x_k, y_{k2}), \ln H_{D_1, \alpha} + y_{k1}}.$$

iii) *If both  $\mathfrak{R}(\delta_1)$  and  $\mathfrak{R}(\delta_2)$  are dense grids, then again (14) holds with*

$$f(x_k, y_{k1}, y_{k2}) = e^{-\min(x_k, y_{k1}, y_{k2})}.$$

**Remarks:** a) From the above results it follows that the joint convergence stated therein is determined by the choice of the grids. The dependence parameters  $r_{lk}, l, k \leq p$  determine the covariance of the Gaussian random vector  $\mathbf{Z}$  and appears explicitly in the definition of the limiting distribution.

Clearly, if each  $r_{kk}$  equals 0, i.e., the Berman condition holds for each component of the vector process, then  $\mathbf{Z}$  does not appear in any of the limiting results above. For such cases the maxima over a sparse grid is independent of that taken over a Pickands or a dense grid.

b) Condition (9) can be stated in a slightly more general form putting therein  $C_k$  instead of  $C$ . Our results can be restated then with some obvious modifications on the constants involved.

c) In [33] a particular case of Piterbarg's max-discretisation theorem was investigated, which in our notation corresponds to  $r_{kk} = \infty$ . Considering for simplicity  $p = 1$ , so we assume that  $r_{11} = \infty$ , then if (1) holds with  $\alpha \in (0, 1]$  and  $r(t) = o(1), t \rightarrow \infty$  a convex function, and  $(r(t) \ln t)^{-1}$  is monotone for large  $t$  and  $o(1)$ , then for any two different sparse, Pickands or dense grids  $\mathfrak{R}(\delta_1)$  and  $\mathfrak{R}(\delta_2)$  we have

$$(15) \quad \lim_{T \rightarrow \infty} \mathbb{P} \{ a_T^*(M(T) - b_T^*) \leq x, a_T^*(M(\delta_1, T) - b_{\delta_1, T}^*) \leq y, a_T^*(M(\delta_2, T) - b_{\delta_2, T}^*) \leq z \} = \Phi(\min(x, y, z))$$

for any  $x, y, z \in \mathbb{R}$  as  $T \rightarrow \infty$ , where

$$a_T^* = 1/\sqrt{r(T)}, \quad b_{\delta_i, T}^* = \sqrt{(1 - r(T))/r(T)} b_{\delta_i, T}$$

and  $\Phi$  denotes the distribution function of an  $N(0, 1)$  random variable. The proof of the above claim follows by Theorem 2.1 in [33] and Lemma 4.5.

Consequently, for this case different grids do not play a role in the limiting distribution. Note however that the normalisation constant  $b_{\delta_i, T}^*$  depends on the type of the grid.

iv) Set for  $\mathbf{x}, \mathbf{y}_1, \mathbf{y}_2 \in \mathbb{R}^p$

$$G(\mathbf{x}, \mathbf{y}_1, \mathbf{y}_2) = \mathbb{E} \left\{ \exp \left( - \sum_{k=1}^p f(x_k, y_{k1}, y_{k2}) e^{-r_{kk} + \sqrt{2r_{kk}} Z_k} \right) \right\},$$

where  $f$  and  $\mathbf{Z}$  are as in Theorem 2.1 and Theorem 2.2. It follows that

$$\begin{aligned} \lim_{x \rightarrow -\infty} H_{D_1, D_2, \alpha}^{x, y_1, y_2} &= H_{D_1, D_2, \alpha}^{y_1, y_2}, & \lim_{y_1 \rightarrow -\infty, y_2 \rightarrow -\infty} H_{D_1, D_2, \alpha}^{x, y_1, y_2} &= e^{-x} H_{D_1, D_2, \alpha}, \\ \lim_{x \rightarrow -\infty, y_1 \rightarrow -\infty} H_{D_1, D_2, \alpha}^{x, y_1, y_2} &= \lim_{y_1 \rightarrow -\infty} H_{D_1, D_2, \alpha}^{y_1, y_2} = H_{D_2, \alpha}^{y_2} &= e^{-y_2} H_{D_2, \alpha}. \end{aligned}$$

Hence, using further (9) we conclude that  $G$  is a non-degenerate multivariate distribution in  $\mathbb{R}^{3p}$ , which we refer to as the Piterbarg distribution. One important property of  $G$  is that when  $r_{kk} = 0$  for all indices  $k \leq p$ , then it has unit Gumbel marginals  $\Lambda(x) = e^{-e^{-x}}$ ,  $x \in \mathbb{R}$ . Moreover,  $G$  is a max-stable distribution since

$$(G(\mathbf{x} + \ln n, \mathbf{y}_1 + \ln n, \mathbf{y}_2 + \ln n))^n = G(\mathbf{x}, \mathbf{y}_1, \mathbf{y}_2), \quad \mathbf{x}, \mathbf{y}_1, \mathbf{y}_2 \in \mathbb{R}^p, n \in \mathbb{N}.$$

In Extreme Value Theory max-stable distributions are important for modelling of extremes and rare events, see e.g., [28, 15] for details.

### 3. PROOFS

In this section we present several lemmas needed for the proof of the main results. In order to establish Piterbarg's max-discretisation theorem for multivariate stationary Gaussian processes we need to closely follow [27], and of course to strongly rely on the deep ideas and the techniques presented in [26]. First, for  $1 \leq k, l \leq p$  define

$$\rho_{kl}(T) = r_{kl} / \ln T.$$

Following the former reference, we divide the interval  $[0, T]$  onto intervals of length  $S$  alternating with shorter intervals of length  $R$ . Let  $0 < b < a < 1$  be two positive constants, where  $b$  will be chosen below (see (29)). We shall denote in the sequel

$$S = T^a, \quad R = T^b, \quad T > 0.$$

Denote the long intervals by  $\mathcal{S}_l$ ,  $l = 1, \dots, n_T$ , and the short intervals by  $\mathcal{R}_l$ ,  $l = 1, \dots, n_T$  where

$$(16) \quad n_T := \lceil T / (S + R) \rceil.$$

It will be seen from the proofs, that a possible remaining interval with length different than  $S$  or  $R$  plays no role in our asymptotic considerations; we call also this interval a short interval. Define further  $\mathbf{S} = \cup_{l=1}^{n_T} \mathcal{S}_l$ ,  $\mathbf{R} = \cup_{l=1}^{n_T} \mathcal{R}_l$  and thus  $[0, T] = \mathbf{S} \cup \mathbf{R}$ .

Our proofs also rely on the ideas of [22]; we shall construct new Gaussian processes to approximate the original ones. For each index  $k \leq p$  we define a Gaussian process  $\eta_k$  as

$$(17) \quad \eta_k(t) = Y_k^{(j)}(t), \quad t \in \mathcal{R}_j \cup \mathcal{S}_j = [(j-1)(S+R), j(S+R)),$$

where  $\{Y_k^{(j)}(t), t \geq 0\}$ ,  $j = 1, \dots, n_T$  are independent copies of  $\{X_k(t), t \geq 0\}$ . We construct the processes so that  $\eta_k, k = 1, \dots, p$  are independent by taking  $Y_k^{(j)}$  to be independent for any  $j$  and  $k$  two possible indices. The independence of  $\eta_k$  and  $\eta_l$  implies

$$\gamma_{kl}(s, t) := \mathbb{E}\{\eta_k(s)\eta_l(t)\} = 0, \quad k \neq l,$$

whereas for any fixed  $k$

$$\begin{aligned} \gamma_{kk}(s, t) &:= \mathbb{E}\{\eta_k(s)\eta_k(t)\} \\ &= \begin{cases} \mathbb{E}\{Y_k^{(i)}(t), Y_k^{(i)}(s)\} = r_{kk}(s, t), & \text{if } t, s \in \mathcal{R}_i \cup \mathcal{S}_i, \text{ for some } i \leq n_T; \\ \mathbb{E}\{Y_k^{(i)}(t), Y_k^{(j)}(s)\} = 0, & \text{if } t \in \mathcal{R}_i \cup \mathcal{S}_i, s \in \mathcal{R}_j \cup \mathcal{S}_j, \text{ for some } i \neq j \leq n_T. \end{cases} \end{aligned}$$

For  $k = 1, \dots, p$  define

$$\xi_k^T(t) = (1 - \rho_{kk}(T))^{1/2} \eta_k(t) + \rho_{kk}^{1/2}(T) Z_k, \quad 0 \leq t \leq T,$$

where  $\mathbf{Z} = (Z_1, \dots, Z_p)$  is a  $p$ -dimensional centered Gaussian random vector introduced in Section 2, which is independent of  $\{\eta_k(t), t \geq 0\}$ ,  $k = 1, \dots, p$ . Denote by  $\{\varrho_{kl}(s, t), 1 \leq k, l \leq p\}$  the covariance functions of  $\{\xi_k^T(t), 0 \leq t \leq T, k = 1, \dots, p\}$ . We have

$$\varrho_{kl}(s, t) = \mathbb{E}\{\xi_k^T(s)\xi_l^T(t)\} = \rho_{kl}(T), \quad k \neq l$$

and

$$\varrho_{kk}(s, t) = \begin{cases} r_{kk}(s, t) + (1 - r_{kk}(s, t))\rho_{kk}(T), & t \in \mathcal{R}_i \cup \mathcal{S}_i, s \in \mathcal{R}_j \cup \mathcal{S}_j, i = j; \\ \rho_{kk}(T), & t \in \mathcal{R}_i \cup \mathcal{S}_i, s \in \mathcal{R}_j \cup \mathcal{S}_j, i \neq j. \end{cases}$$

For any  $\varepsilon > 0$  set

$$(18) \quad q_\varepsilon = \frac{\varepsilon}{(\ln T)^{1/\alpha}}.$$

For notational simplicity we write

$$\widetilde{\mathbf{M}}_\xi(q_\varepsilon, \mathbf{S}) = \left( a_T(M_{\xi_1}(q_\varepsilon, \mathbf{S}) - b_T), \dots, a_T(M_{\xi_p}(q_\varepsilon, \mathbf{S}) - b_T) \right)$$

and

$$\widetilde{\mathbf{M}}_\xi(\delta_i, \mathbf{S}) = \left( a_T(M_{\xi_1}(\delta_i, \mathbf{S}) - b_{\delta_i, T}), \dots, a_T(M_{\xi_p}(\delta_i, \mathbf{S}) - b_{\delta_i, T}) \right),$$

where

$$M_{\xi_k}(q_\varepsilon, \mathbf{S}) = \max_{t \in \mathfrak{R}(q_\varepsilon) \cap \mathbf{S}} \xi_k^T(t)$$

and  $b_{\delta_i, T}$  is defined in (4) if the grid  $\mathfrak{R}(\delta_i)$  is sparse,  $b_{\delta_i, T} = b_T(D_i)$  is given by (7) if we consider a Pickands grid  $\mathfrak{R}(\delta_i) = \mathfrak{R}(D_i a_T^{-2/\alpha})$  and for a dense grid  $b_{\delta_i, T} = b_T$  with  $b_T$  defined in (6).

We present first four lemmas. Since their proofs are similar to those of Lemmas 3.1-3.4 in [31] we shall not give them here.

**Lemma 3.1.** *If  $\mathfrak{R}(\delta_1)$  and  $\mathfrak{R}(\delta_2)$  are sparse or Pickands grids, then for any  $B > 0$  there exists some  $K > 0$  such that for all  $x_k, y_{ki} \in [-B, B], i = 1, 2, k \leq p$*

$$\left| \mathbb{P} \left\{ \widetilde{\mathbf{M}}(T) \leq \mathbf{x}, \widetilde{\mathbf{M}}(\delta_i, T) \leq \mathbf{y}_i, i = 1, 2 \right\} - \mathbb{P} \left\{ \widetilde{\mathbf{M}}(\mathbf{S}) \leq \mathbf{x}, \widetilde{\mathbf{M}}(\delta_i, \mathbf{S}) \leq \mathbf{y}_i, i = 1, 2 \right\} \right| \leq K(\ln T)^{1/\alpha-1/2} T^{b-a}$$

holds for some  $0 < b < a < 1$  and all  $T$  large.

In the following  $\mathfrak{R}(q_\varepsilon) = \mathfrak{R}(\varepsilon/(\ln T)^{1/\alpha})$  denotes a Pickands grid where  $\varepsilon > 0$  and  $q_\varepsilon$  is defined in (18).

**Lemma 3.2.** *If  $\mathfrak{R}(\delta_1)$  and  $\mathfrak{R}(\delta_2)$  are sparse or Pickands grids, then for any  $B > 0$  and for all  $x_k, y_{ki} \in [-B, B], i = 1, 2, k \leq p$*

$$\left| \mathbb{P} \left\{ \widetilde{\mathbf{M}}(\mathbf{S}) \leq \mathbf{x}, \widetilde{\mathbf{M}}(\delta_i, \mathbf{S}) \leq \mathbf{y}_i, i = 1, 2 \right\} - \mathbb{P} \left\{ \widetilde{\mathbf{M}}(q_\varepsilon, \mathbf{S}) \leq \mathbf{x}, \widetilde{\mathbf{M}}(\delta_i, \mathbf{S}) \leq \mathbf{y}_i, i = 1, 2 \right\} \right| \rightarrow 0$$

as  $\varepsilon \downarrow 0$ .

**Lemma 3.3.** *If  $\mathfrak{R}(\delta_1)$  and  $\mathfrak{R}(\delta_2)$  are sparse or Pickands grids, then for any  $B > 0$  and for all  $x_k, y_{ki} \in [-B, B], i = 1, 2, k \leq p$*

$$\lim_{T \rightarrow \infty} \left| \mathbb{P} \left\{ \widetilde{\mathbf{M}}(q_\varepsilon, \mathbf{S}) \leq \mathbf{x}, \widetilde{\mathbf{M}}(\delta_i, \mathbf{S}) \leq \mathbf{y}_i, i = 1, 2 \right\} - \mathbb{P} \left\{ \widetilde{\mathbf{M}}_\xi(q_\varepsilon, \mathbf{S}) \leq \mathbf{x}, \widetilde{\mathbf{M}}_\xi(\delta_i, \mathbf{S}) \leq \mathbf{y}_i, i = 1, 2 \right\} \right| = 0$$

uniformly for  $\varepsilon > 0$ .

Let in the following  $\Phi_p$  denote the distribution function of the  $p$ -dimensional Gaussian random vector  $\mathbf{Z}$  and set for  $\eta_k$  defined in (17)

$$\widehat{\mathbf{M}}_\eta(\delta_i, \mathcal{S}_j) = \left( \max_{t \in \mathfrak{R}(\delta_i) \cap \mathcal{S}_j} \eta_1(t), \dots, \max_{t \in \mathfrak{R}(\delta_i) \cap \mathcal{S}_j} \eta_p(t) \right), \quad \widehat{\mathbf{M}}_\eta(\mathcal{S}_j) = \left( \max_{t \in \mathcal{S}_j} \eta_1(t), \dots, \max_{t \in \mathcal{S}_j} \eta_p(t) \right).$$

**Lemma 3.4.** *If  $\mathfrak{R}(\delta_1)$  and  $\mathfrak{R}(\delta_2)$  are sparse or Pickands grids, then for any  $B > 0$  for all  $x_k, y_{ki} \in [-B, B], i = 1, 2, k \leq p$*

$$\left| \mathbb{P} \left\{ \widetilde{\mathbf{M}}_\xi(q_\varepsilon, \mathbf{S}) \leq \mathbf{x}, \widetilde{\mathbf{M}}_\xi(\delta_i, \mathbf{S}) \leq \mathbf{y}_i, i = 1, 2 \right\} - \int_{\mathbf{z} \in \mathbb{R}^p} \prod_{j=1}^{n_T} \mathbb{P} \left\{ \widehat{\mathbf{M}}_\eta(\mathcal{S}_j) \leq \mathbf{u}(\mathbf{x}, \mathbf{z}), \widehat{\mathbf{M}}_\eta(\delta_i, \mathcal{S}_j) \leq \mathbf{u}(\mathbf{y}_i, \mathbf{z}), i = 1, 2 \right\} d\Phi_p(\mathbf{z}) \right| \rightarrow 0$$

as  $\varepsilon \downarrow 0$ , where  $\mathbf{u}(\mathbf{x}, \mathbf{z}), \mathbf{u}(\mathbf{y}_i, \mathbf{z}), i = 1, 2$  have components

$$(19) \quad u(x_k, z_k) = \frac{b_T + x_k/a_T - \rho_{kk}^{1/2}(T)z_k}{(1 - \rho_{kk}(T))^{1/2}} = \frac{x_k + r_{kk} - \sqrt{2r_{kk}}z_k}{a_T} + b_T + o(a_T^{-1}),$$

$$(20) \quad u(y_{ki}, z_k) = \frac{b_{\delta_i, T} + y_{ki}/a_T - \rho_{kk}^{1/2}(T)z_k}{(1 - \rho_{kk}(T))^{1/2}} = \frac{y_{ki} + r_{kk} - \sqrt{2r_{kk}}z_k}{a_T} + b_{\delta_i, T} + o(a_T^{-1}),$$

for all  $x_k, y_{ki} \in [-B, B], i = 1, 2, k \leq p$ .

**Proof of Theorem 2.1:** Since all the limits of the probabilities in Lemmas 3.1-3.4 are positive for all  $x_k, y_{ki} \in [-B, B], i = 1, 2, k \leq p$ , by letting  $\varepsilon \downarrow 0$ , we have

$$\begin{aligned} & \mathbb{P} \left\{ \widetilde{\mathbf{M}}(T) \leq \mathbf{x}, \widetilde{\mathbf{M}}(\delta_i, T) \leq \mathbf{y}_i, i = 1, 2 \right\} \\ & \sim \int_{\mathbf{z} \in \mathbb{R}^p} \prod_{j=1}^{n_T} \mathbb{P} \left\{ \widehat{\mathbf{M}}_\eta(\mathcal{S}_j) \leq \mathbf{u}(\mathbf{x}, \mathbf{z}), \widehat{\mathbf{M}}_\eta(\delta_i, \mathcal{S}_j) \leq \mathbf{u}(\mathbf{y}_i, \mathbf{z}), i = 1, 2 \right\} d\Phi_p(\mathbf{z}) \end{aligned}$$

as  $T \rightarrow \infty$ . Thus, if we can prove

$$(21) \quad \lim_{T \rightarrow \infty} \left| \prod_{j=1}^{n_T} \mathbb{P} \left\{ \widehat{\mathbf{M}}_\eta(\mathcal{S}_j) \leq \mathbf{u}(\mathbf{x}, \mathbf{z}), \widehat{\mathbf{M}}_\eta(\delta_i, \mathcal{S}_j) \leq \mathbf{u}(\mathbf{y}_i, \mathbf{z}), i = 1, 2 \right\} - \exp \left( - \sum_{k=1}^p f(x_k, y_{k1}, y_{k2}) e^{-r_{kk} + \sqrt{2r_{kk} z_k}} \right) \right| = 0,$$

where  $f(x_k, y_{k1}, y_{k2})$  is defined in Theorem 2.1, then applying the dominated convergence theorem we complete the proof of Theorem 2.1 for the case  $i) - iii)$ . Define next the events

$$\mathcal{A}_k = \left\{ \max_{t \in [0, S]} \eta_k(t) > u(x_k, z_k) \right\}, \quad \mathcal{A}_{p+k} = \left\{ \max_{t \in \mathfrak{A}(\delta_1) \cap [0, S]} \eta_k(t) > u(y_{k1}, z_k) \right\}$$

and

$$\mathcal{A}_{2p+k} = \left\{ \max_{t \in \mathfrak{A}(\delta_2) \cap [0, S]} \eta_k(t) > u(y_{k2}, z_k) \right\}, \quad k = 1, \dots, p.$$

$i)$  By the definition of  $\{\eta_k(t), k = 1, \dots, p\}$  (we write  $\mathcal{A}_k^c$  for the complimentary event of  $\mathcal{A}_k$ )

$$\begin{aligned} \prod_{j=1}^{n_T} \mathbb{P} \left\{ \widehat{\mathbf{M}}_\eta(\mathcal{S}_j) \leq \mathbf{u}(\mathbf{x}, \mathbf{z}), \widehat{\mathbf{M}}_\eta(\delta_i, \mathcal{S}_j) \leq \mathbf{u}(\mathbf{y}_i, \mathbf{z}), i = 1, 2 \right\} &= (\mathbb{P}\{\cap_{k=1}^{3p} \mathcal{A}_k^c\})^{n_T} \\ &= \exp(n_T \ln(\mathbb{P}\{\cap_{k=1}^{3p} \mathcal{A}_k^c\})) \\ &= \exp(-n_T \mathbb{P}\{\cup_{k=1}^{3p} \mathcal{A}_k\} + W_{n_T}), \end{aligned}$$

where  $n_T$  is defined in (16). Since  $\lim_{T \rightarrow \infty} \mathbb{P}\{\cap_{k=1}^{3p} \mathcal{A}_k^c\} = 1$  we get that the remainder  $W_{n_T}$  satisfies

$$W_{n_T} = o(n_T \mathbb{P}\{\cup_{k=1}^{3p} \mathcal{A}_k\}), \quad T \rightarrow \infty.$$

Next, by Bonferroni inequality

$$\begin{aligned} \sum_{k=1}^{3p} \mathbb{P}\{\mathcal{A}_k\} &\geq \mathbb{P}\{\cup_{k=1}^{3p} \mathcal{A}_k\} \geq \sum_{k=1}^{3p} \mathbb{P}\{\mathcal{A}_k\} - \sum_{1 \leq k < l \leq 3p} \mathbb{P}\{\mathcal{A}_k, \mathcal{A}_l\} \\ &= \sum_{k=1}^{3p} \mathbb{P}\{\mathcal{A}_k\} - \sum_{1 \leq k < l \leq p} \mathbb{P}\{\mathcal{A}_k, \mathcal{A}_l\} - \sum_{1 \leq k < l \leq p} \mathbb{P}\{\mathcal{A}_{p+k}, \mathcal{A}_{p+l}\} - \sum_{1 \leq k < l \leq p} \mathbb{P}\{\mathcal{A}_{2p+k}, \mathcal{A}_{2p+l}\} \\ &\quad - \sum_{1 \leq k < l \leq p} \mathbb{P}\{\mathcal{A}_k, \mathcal{A}_{p+l}\} - \sum_{1 \leq k < l \leq p} \mathbb{P}\{\mathcal{A}_k, \mathcal{A}_{2p+l}\} - \sum_{1 \leq k < l \leq p} \mathbb{P}\{\mathcal{A}_{p+k}, \mathcal{A}_{2p+l}\} \\ &\quad - \sum_{k=1}^p \mathbb{P}\{\mathcal{A}_k, \mathcal{A}_{p+k}\} - \sum_{k=1}^p \mathbb{P}\{\mathcal{A}_k, \mathcal{A}_{2p+k}\} - \sum_{k=1}^p \mathbb{P}\{\mathcal{A}_{p+k}, \mathcal{A}_{2p+k}\} \\ (22) \quad &=: A_1 - A_2 - A_3 - A_4 - A_5 - A_6 - A_7 - A_8 - A_9 - A_{10}. \end{aligned}$$

Further, Lemma 2 in [27] and (19), (20) imply (recall  $S = T^a$ )

$$A_1 \sim \sum_{k=1}^p ST^{-1}(e^{-x_k} + e^{-y_{k1}} + e^{-y_{k2}})e^{-r_{kk} + \sqrt{2r_{kk}}z_k}, \quad T \rightarrow \infty.$$

For  $A_2$ , by the independence of  $\eta_k(t)$  and  $\eta_l(t)$ ,  $k \neq l$ , Lemma 2 of [27] and (19), (20), we have

$$\begin{aligned} A_2 &= \sum_{1 \leq k < l \leq p} \mathbb{P} \left\{ \max_{t \in [0, S]} \eta_k(t) > u(x_k, z_k), \max_{t \in [0, S]} \eta_l(t) > u(x_l, z_l) \right\} \\ &= \sum_{1 \leq k < l \leq p} \mathbb{P} \left\{ \max_{t \in [0, S]} \eta_k(t) > u(x_k, z_k) \right\} \mathbb{P} \left\{ \max_{t \in [0, S]} \eta_l(t) > u(x_l, z_l) \right\} \\ &\sim \sum_{1 \leq k < l \leq p} ST^{-1}e^{-x_k - r_{kk} + \sqrt{2r_{kk}}z_k} ST^{-1}e^{-y_l - r_{ll} + \sqrt{2r_{ll}}z_l} = o(A_1). \end{aligned}$$

Since  $\mathfrak{R}(\delta_i)$ ,  $i = 1, 2$  is a sparse grid, similar arguments as for  $A_2$  lead to

$$A_k = o(A_1), \quad k = 3, 4, 5, 6, 7.$$

Further, Lemma 2 of [27] implies  $A_i = o(A_1)$ ,  $i = 8, 9$ . By the first assertion of Lemma 4.1 we have

$$A_{10} = o(T^{a-1}) = o(A_1).$$

Consequently, as  $T \rightarrow \infty$

$$n_T \mathbb{P}\{\cup_{k=1}^{3p} \mathcal{A}_k\} \sim \sum_{k=1}^p (e^{-x_k} + e^{-y_{k1}} + e^{-y_{k2}})e^{-r_{kk} + \sqrt{2r_{kk}}z_k},$$

which completes the proof of (21).

*ii)* We proceed as for the proof of case *i*) using the lower bound (22); we have thus

$$\begin{aligned} \mathbb{P}\{\cup_{k=1}^{3p} \mathcal{A}_k\} &= \sum_{k=1}^{3p} \mathbb{P}\{\mathcal{A}_k\} - \sum_{1 \leq k, l \leq 3p} \mathbb{P}\{\mathcal{A}_k, \mathcal{A}_l\} + \sum_{1 \leq k, \dots, l \leq 3p} \mathbb{P}\{\cdot\} \\ &= \sum_{k=1}^{3p} \mathbb{P}\{\mathcal{A}_k\} - \sum_{1 \leq k < l \leq p} \mathbb{P}\{\mathcal{A}_k, \mathcal{A}_l\} - \sum_{1 \leq k < l \leq p} \mathbb{P}\{\mathcal{A}_{p+k}, \mathcal{A}_{p+l}\} - \sum_{1 \leq k < l \leq p} \mathbb{P}\{\mathcal{A}_{2p+k}, \mathcal{A}_{2p+l}\} \\ &\quad - \sum_{1 \leq k < l \leq p} \mathbb{P}\{\mathcal{A}_k, \mathcal{A}_{p+l}\} - \sum_{1 \leq k < l \leq p} \mathbb{P}\{\mathcal{A}_k, \mathcal{A}_{2p+l}\} - \sum_{1 \leq k < l \leq p} \mathbb{P}\{\mathcal{A}_{p+k}, \mathcal{A}_{2p+l}\} \\ &\quad - \sum_{k=1}^p \mathbb{P}\{\mathcal{A}_k, \mathcal{A}_{p+k}\} - \sum_{k=1}^p \mathbb{P}\{\mathcal{A}_k, \mathcal{A}_{2p+k}\} - \sum_{k=1}^p \mathbb{P}\{\mathcal{A}_{p+k}, \mathcal{A}_{2p+k}\} + \sum_{1 \leq k < \dots, l \leq 3p} \mathbb{P}\{\cdot\} \\ (23) \quad &=: A_1 - A_2 - A_3 - A_4 - A_5 - A_6 - A_7 - A_8 - A_9 - A_{10} + A_{11}. \end{aligned}$$

The estimates for  $A_i$ ,  $i = 1, \dots, 9$  are the same as for case *i*), therefore we only need to deal with the terms  $A_{10}$  and  $A_{11}$ . It follows that each term of  $A_{11}$  can be bounded by  $A_5$ ,  $A_6$  or  $A_7$  implying

$$A_{11} = o(A_1), \quad T \rightarrow \infty.$$

Next, the definition of  $u(y_{ki}, z_k)$ ,  $i = 1, 2$  implies

$$(24) \quad u(y_{ki}, z_k) = \sqrt{2 \ln T} - \frac{1}{2} \frac{\ln \ln T}{\sqrt{2 \ln T}} + \frac{\ln \delta_i^{-1}(T)}{\sqrt{2 \ln T}} + \frac{\ln(2^{-1} \pi^{-1/2})}{\sqrt{2 \ln T}} + \frac{y_{ki} + r_{kk} - \sqrt{2r_{kk}z_k}}{\sqrt{2 \ln T}} + \frac{o(1)}{\sqrt{2 \ln T}}$$

for sparse grids. From the assumptions we know that  $\lim_{T \rightarrow \infty} \ln(\frac{\delta_1(T)}{\delta_2(T)}) = \theta = \theta_2 - \theta_1$ . Consequently, we have

$$\begin{aligned} u(y_{k1}, z_k) - u(y_{k2}, z_k) &= \left[ \ln\left(\frac{\delta_2(T)}{\delta_1(T)}\right) + y_{k1} - y_{k2} \right] (2 \ln T)^{-1/2} + o(1)(2 \ln T)^{-1/2} \\ &\sim [-\theta + y_{k1} - y_{k2}] (2 \ln T)^{-1/2} + o(1)(2 \ln T)^{-1/2} \end{aligned}$$

as  $T \rightarrow \infty$ . Letting first  $y_{k1} > y_{k2} + \theta$ , we thus have  $u(y_{k1}, z_k) > u(y_{k2}, z_k)$  for sufficiently large  $T$ . Further,

$$\begin{aligned} A_{10} &= \sum_{k=1}^p \mathbb{P} \left\{ \max_{t \in \mathcal{R}(\delta_1) \cap [0, S]} \eta_k(t) > u(y_{k1}, z_k), \max_{t \in \mathcal{R}(\delta_2) \cap [0, S]} \eta_k(t) > u(y_{k2}, z_k) \right\} \\ &= \sum_{k=1}^p \left[ \mathbb{P} \left\{ \max_{t \in \mathcal{R}(\delta_1) \cap [0, S]} \eta_k(t) > u(y_{k1}, z_k) \right\} + \mathbb{P} \left\{ \max_{t \in \mathcal{R}(\delta_2) \cap [0, S]} \eta_k(t) > u(y_{k2}, z_k) \right\} \right. \\ &\quad \left. - \left( 1 - \mathbb{P} \left\{ \max_{t \in \mathcal{R}(\delta_1) \cap [0, S]} \eta_k(t) \leq u(y_{k1}, z_k), \max_{t \in \mathcal{R}(\delta_2) \cap [0, S]} \eta_k(t) \leq u(y_{k2}, z_k) \right\} \right) \right] \\ &= \sum_{k=1}^p \left[ \mathbb{P} \left\{ \max_{t \in \mathcal{R}(\delta_1) \cap [0, S]} \eta_k(t) > u(y_{k1}, z_k) \right\} + \mathbb{P} \left\{ \max_{t \in \mathcal{R}(\delta_2) \cap [0, S]} \eta_k(t) > u(y_{k2}, z_k) \right\} \right. \\ &\quad \left. - \left( 1 - \mathbb{P} \left\{ \max_{t \in \mathcal{R}(\delta_1) \cap [0, S] \setminus \mathcal{R}(\delta_2) \cap [0, S]} \eta_k(t) \leq u(y_{k1}, z_k), \max_{t \in \mathcal{R}(\delta_2) \cap [0, S]} \eta_k(t) \leq u(y_{k2}, z_k) \right\} \right) \right] \\ &= \sum_{k=1}^p \left[ \mathbb{P} \left\{ \max_{t \in \mathcal{R}(\delta_1) \cap [0, S]} \eta_k(t) > u(y_{k1}, z_k) \right\} - \mathbb{P} \left\{ \max_{t \in \mathcal{R}(\delta_1) \cap [0, S] \setminus \mathcal{R}(\delta_3) \cap [0, S]} \eta_k(t) > u(y_{k1}, z_k) \right\} \right. \\ &\quad \left. + \mathbb{P} \left\{ \max_{t \in \mathcal{R}(\delta_1) \cap [0, S] \setminus \mathcal{R}(\delta_3) \cap [0, S]} \eta_k(t) > u(y_{k1}, z_k), \max_{t \in \mathcal{R}(\delta_2) \cap [0, S]} \eta_k(t) > u(y_{k2}, z_k) \right\} \right]. \end{aligned}$$

By Lemma 4.2 and (24) we have for  $i = 1, 2$  as  $T \rightarrow \infty$

$$\begin{aligned} \mathbb{P} \left\{ \max_{t \in \mathcal{R}(\delta_i) \cap [0, S] \setminus \mathcal{R}(\delta_3) \cap [0, S]} \eta_k(t) > u(y_{ki}, z_k) \right\} &\sim \delta_i \frac{S}{T} \left( \frac{1}{\delta_i} - \frac{1}{\delta_3} \right) e^{-y_{ki} - r_{kk} + \sqrt{2r_{kk}z_k}} \\ &\sim ST^{-1} (1 - e^{-\theta_i}) e^{-y_{ki} - r_{kk} + \sqrt{2r_{kk}z_k}}, \quad T \rightarrow \infty. \end{aligned}$$

Further, applying Lemma 2 in [27] (recall (24)) we obtain as  $T \rightarrow \infty$

$$\mathbb{P} \left\{ \max_{t \in \mathcal{R}(\delta_i) \cap [0, S]} \eta_k(t) > u(y_{ki}, z_k) \right\} \sim ST^{-1} e^{-y_{ki} - r_{kk} + \sqrt{2r_{kk}z_k}}, \quad i = 1, 2.$$

By the second assertion of Lemma 4.1, the third term is  $o(T^{a-1})$ .

Next, for  $y_{k1} \leq y_{k2} + \theta$ , we have  $u(y_{k1}, z_k) \leq u(y_{k2}, z_k)$  for sufficient large  $T$ . Similarly, we have

$$\begin{aligned} A_{10} &= \sum_{k=1}^p \mathbb{P} \left\{ \max_{t \in \mathcal{R}(\delta_1) \cap [0, S]} \eta_k(t) > u(y_{k1}, z_k), \max_{t \in \mathcal{R}(\delta_2) \cap [0, S]} \eta_k(t) > u(y_{k2}, z_k) \right\} \\ &= \sum_{k=1}^p \left[ \mathbb{P} \left\{ \max_{t \in \mathcal{R}(\delta_2) \cap [0, S]} \eta_k(t) > u(y_{k2}, z_k) \right\} - \mathbb{P} \left\{ \max_{t \in \mathcal{R}(\delta_2) \cap [0, S] \setminus \mathcal{R}(\delta_3) \cap [0, S]} \eta_k(t) > u(y_{k2}, z_k) \right\} \right. \\ &\quad \left. + \mathbb{P} \left\{ \max_{t \in \mathcal{R}(\delta_1) \cap [0, S]} \eta_k(t) > u(y_{k1}, z_k), \max_{t \in \mathcal{R}(\delta_2) \cap [0, S] \setminus \mathcal{R}(\delta_3) \cap [0, S]} \eta_k(t) > u(y_{k2}, z_k) \right\} \right]. \end{aligned}$$

Again, in view of the second assertion of Lemma 4.1 the third term is also  $o(T^{a-1})$ . Consequently,

$$A_{10} = \sum_{k=1}^p T^{a-1} [e^{-y_{k1}-\theta_1} I(y_{k1} > y_{k2} + \theta) + e^{-y_{k2}-\theta_2} I(y_{k1} \leq y_{k2} + \theta)] e^{-r_{kk} + \sqrt{2r_{kk}} z_k} + o(T^{a-1}), \quad T \rightarrow \infty$$

implying that as  $T \rightarrow \infty$

$$n_T \mathbb{P}\{\cup_{k=1}^{3p} \mathcal{A}_k\} \sim \sum_{k=1}^p (e^{-x_k} + e^{-y_{k1}} + e^{-y_{k2}} - e^{-y_{k1}-\theta_1} I(y_{k1} > y_{k2} + \theta) - e^{-y_{k2}-\theta_2} I(y_{k1} \leq y_{k2} + \theta)) e^{-r_{kk} + \sqrt{2r_{kk}} z_k},$$

which completes the proof of (21).

*iii)* We proceed as for the proof of cases *i)* and *ii)* using the bound (23). By Lemmas 2 and 3 in [27] and (19), (20) we obtain

$$A_1 \sim T^{a-1} \sum_{k=1}^p (e^{-x_k} + e^{-y_{k1}} + e^{-y_{k2}}) e^{-r_{kk} + \sqrt{2r_{kk}} z_k}, \quad T \rightarrow \infty.$$

With similar argument as for  $A_2$  in the proof of case *i)*, we conclude that

$$A_k = o(A_1), \quad k = 2, 3, 4, 5, 6, 7.$$

Further, Lemma 2 in [27] implies  $A_8 = o(A_1)$  and Lemma 4.3 yields

$$A_{10} = o(T^{a-1}) = o(A_1), \quad T \rightarrow \infty.$$

Similar arguments as for  $A_{11}$  in the proof of case *ii)* imply

$$A_{11} = o(A_1), \quad T \rightarrow \infty.$$

Borrowing the arguments of [26], p. 176 and using Lemma 3 in [27] it follows that

$$\begin{aligned} A_9 &= \sum_{k=1}^p \mathbb{P} \left\{ \max_{t \in [0, S]} \eta_k(t) > u(x_k, z_k), \max_{t \in \mathfrak{R}(\delta_2) \cap [0, S]} \eta_k(t) > u(y_{k2}, z_k) \right\} \\ &\sim T^{a-1} \sum_{k=1}^p H_{D_2, \alpha}^{\ln H_\alpha + x_k, \ln H_{D_2, \alpha} + y_{k2}} e^{-r_{kk} + \sqrt{2r_{kk}} z_k}, \quad T \rightarrow \infty. \end{aligned}$$

Consequently, as  $T \rightarrow \infty$

$$n_T P\{\cup_{k=1}^{3p} \mathcal{A}_k\} \sim \sum_{k=1}^p (e^{-x_k} + e^{-y_{k1}} + e^{-y_{k2}} - H_{D_2, \alpha}^{\ln H_\alpha + x_k, \ln H_{D_2, \alpha} + y_{k2}}) e^{-r_{kk} + \sqrt{2r_{kk}} z_k},$$

which completes the proof of the claim in (21).

*iv)* By Lemma 5 in [27], we have

$$\begin{aligned} &\left| \mathbb{P} \left\{ \widetilde{\mathbf{M}}(T) \leq \mathbf{x}, \widetilde{\mathbf{M}}(\delta_1, T) \leq \mathbf{y}_1, \widetilde{\mathbf{M}}(\delta_2, T) \leq \mathbf{y}_2 \right\} - \mathbb{P} \left\{ \widetilde{\mathbf{M}}(T) \leq \mathbf{x}, \widetilde{\mathbf{M}}(\delta_1, T) \leq \mathbf{y}_1, \widetilde{\mathbf{M}}(T) \leq \mathbf{y}_2 \right\} \right| \\ &\leq \left| \mathbb{P} \left\{ \widetilde{\mathbf{M}}(\delta_2, T) \leq \mathbf{y}_2 \right\} - \mathbb{P} \left\{ \widetilde{\mathbf{M}}(T) \leq \mathbf{y}_2 \right\} \right| \rightarrow 0, \quad T \rightarrow \infty. \end{aligned}$$

Now, by Theorem 2.1 of [31], we have

$$\begin{aligned} \mathbb{P}\left\{\widetilde{\mathcal{M}}(T) \leq \mathbf{x}, \widetilde{\mathcal{M}}(\delta_1, T) \leq \mathbf{y}_1, \widetilde{\mathcal{M}}(T) \leq \mathbf{y}_2\right\} &= \mathbb{P}\left\{\widetilde{\mathcal{M}}(T) \leq \min(\mathbf{x}, \mathbf{y}_2), \widetilde{\mathcal{M}}(\delta_1, T) \leq \mathbf{y}_1\right\} \\ &\rightarrow \mathbb{E}\left\{\exp\left(-\sum_{k=1}^p f(x_k, y_{k1}, y_{k2})e^{-r_{kk}+\sqrt{2r_{kk}}Z_k}\right)\right\}, \end{aligned}$$

as  $T \rightarrow \infty$  with

$$f(x_k, y_{k1}, y_{k2}) = e^{-\min(x_k, y_{k2})} + e^{-y_{k1}}$$

establishing the proof.  $\square$

**Proof of Theorem 2.2:** *i*) The limiting properties of the two constants can be found in Lemma 4.4. We give the proof of the relation of (14). As for the proof of Theorem 2.1, in view of Lemmas 3.1-3.4 and the dominated convergence theorem in order to establish the proof we need to show that (21) holds with

$$\begin{aligned} f(x_k, y_{k1}, y_{k2}) &= e^{-x_k} + e^{-y_{k1}} + e^{-y_{k2}} - H_{D_1, \alpha}^{\ln H_{\alpha} + x_k, \ln H_{D_1, \alpha} + y_{k1}} - H_{D_2, \alpha}^{\ln H_{\alpha} + x_k, \ln H_{D_2, \alpha} + y_{k2}} \\ &\quad - H_{D_1, D_2, \alpha}^{\ln H_{D_1, \alpha} + y_{k1}, \ln H_{D_2, \alpha} + y_{k2}} + H_{D_1, D_2, \alpha}^{\ln H_{\alpha} + x_k, \ln H_{D_1, \alpha} + y_{k1}, \ln H_{D_2, \alpha} + y_{k2}}. \end{aligned}$$

We proceed as in the proof of case *ii*) of Theorem 2.1 using the bound (23); we have thus

$$\begin{aligned} \mathbb{P}\left\{\bigcup_{k=1}^{3p} \mathcal{A}_k\right\} &= \sum_{k=1}^{3p} \mathbb{P}\{\mathcal{A}_k\} - \sum_{1 \leq k, l \leq 3p} \mathbb{P}\{\mathcal{A}_k, \mathcal{A}_l\} + \sum_{1 \leq k, l, j \leq 3p} \mathbb{P}\{\mathcal{A}_k, \mathcal{A}_l, \mathcal{A}_j\} + \sum_{1 \leq k, \dots, l \leq 3p} \mathbb{P}\{\cdot\} \\ (25) \quad &=: \Sigma_1 - \Sigma_2 + \Sigma_3 + \Sigma_4. \end{aligned}$$

By Lemmas 2 and 3 in [27] and (19), (20) we obtain that

$$\Sigma_1 \sim T^{a-1} \sum_{k=1}^p (e^{-x_k} + e^{-y_{k1}} + e^{-y_{k2}}) e^{-r_{kk} + \sqrt{2r_{kk}}z_k}, \quad T \rightarrow \infty.$$

Further, write

$$(26) \quad \Sigma_2 = A_2 + A_3 + A_4 + A_5 + A_6 + A_7 + A_8 + A_9 + A_{10},$$

where  $A_i, i = 2, \dots, 10$  are defined in the proof of *ii*) of Theorem 2.1. Hence, with similar arguments as above  $A_i = o(A_1), i = 1, \dots, 7$  and

$$\begin{aligned} A_8 &\sim T^{a-1} \sum_{k=1}^p H_{D_1, \alpha}^{\ln H_{\alpha} + x_k, \ln H_{D_1, \alpha} + y_{k1}} e^{-r_{kk} + \sqrt{2r_{kk}}z_k}, \\ A_9 &\sim T^{a-1} \sum_{k=1}^p H_{D_2, \alpha}^{\ln H_{\alpha} + x_k, \ln H_{D_2, \alpha} + y_{k2}} e^{-r_{kk} + \sqrt{2r_{kk}}z_k}, \\ A_{10} &\sim T^{a-1} \sum_{k=1}^p H_{D_1, D_2}^{\ln H_{D_1, \alpha} + y_{k1}, \ln H_{D_2, \alpha} + y_{k2}} e^{-r_{kk} + \sqrt{2r_{kk}}z_k} \end{aligned}$$

as  $T \rightarrow \infty$ , where for the estimates of  $A_8$  and  $A_9$  we applied Lemma 3 in [27] and for the estimate of  $A_{10}$  we have used Lemma 4.4. Further

$$\begin{aligned} \Sigma_3 &= \sum_{\substack{1 \leq k < l < j \leq 3p \\ l \neq k+p, j \neq l+p, j \neq k+2p}} \mathbb{P}\{\mathcal{A}_k, \mathcal{A}_l, \mathcal{A}_j\} + \sum_{\substack{1 \leq k < l < j \leq 3p \\ l=k+p, j \neq l+p, j \neq k+2p}} \mathbb{P}\{\mathcal{A}_k, \mathcal{A}_l, \mathcal{A}_j\} + \sum_{\substack{1 \leq k < l < j \leq 3p \\ l \neq k+p, j=l+p, j \neq k+2p}} \mathbb{P}\{\mathcal{A}_k, \mathcal{A}_l, \mathcal{A}_j\} \\ &+ \sum_{\substack{1 \leq k < l < j \leq 3p \\ l \neq k+p, j \neq l+p, j=k+2p}} \mathbb{P}\{\mathcal{A}_k, \mathcal{A}_l, \mathcal{A}_j\} + \sum_{\substack{1 \leq k < l < j \leq 3p \\ l=k+p, j=l+p}} \mathbb{P}\{\mathcal{A}_k, \mathcal{A}_l, \mathcal{A}_j\} \\ &=: B_1 + B_2 + B_3 + B_4 + B_5. \end{aligned}$$

For  $B_1$ , by the independence of  $\eta_k(t)$  and  $\eta_l(t)$ ,  $k \neq l$ , Lemma 3 of [27] and (19), (20), we have for some constant  $K > 0$

$$B_1 = \sum_{\substack{1 \leq k < l < j \leq 3p \\ l \neq k+p, j \neq l+p, j \neq k+2p}} \mathbb{P}\{\mathcal{A}_k\} P\{\mathcal{A}_l\} P\{\mathcal{A}_j\} \sim KT^{3(a-1)} = o(A_1).$$

Similarly, we can show that

$$B_i \sim KT^{2(a-1)} = o(A_1), \quad i = 2, 3, 4.$$

For  $B_5$ , using Lemma 4.4, we have

$$B_5 \sim T^{a-1} \sum_{k=1}^p H_{D_1, D_2, \alpha}^{\ln H_{\alpha+x_k, \ln H_{D_1, \alpha+y_{k1}, \ln H_{D_2, \alpha+y_{k2}}}} e^{-r_{kk} + \sqrt{2r_{kk} z_k}}.$$

Finally, it is easy to see that  $\Sigma_4 = o(A_1)$  as  $T \rightarrow \infty$ . Thus, we have as  $T \rightarrow \infty$

$$n_T \mathbb{P}\left\{\bigcup_{k=1}^{3p} \mathcal{A}_k\right\} \sim \sum_{k=1}^p f(x_k, y_{k1}, y_{k2}) e^{-r_{kk} + \sqrt{2r_{kk} z_k}},$$

with

$$\begin{aligned} f(x_k, y_{k1}, y_{k2}) &= e^{-x_k} + e^{-y_{k1}} + e^{-y_{k2}} - H_{D_1, \alpha}^{\ln H_{\alpha+x_k, \ln H_{D_1, \alpha+y_{k1}}}} - H_{D_2, \alpha}^{\ln H_{\alpha+x_k, \ln H_{D_2, \alpha+y_{k2}}}} \\ &\quad - H_{D_1, D_2, \alpha}^{\ln H_{D_1, \alpha+y_{k1}, \ln H_{D_2, \alpha+y_{k2}}}} + H_{D_1, D_2, \alpha}^{\ln H_{\alpha+x_k, \ln H_{D_1, \alpha+y_{k1}, \ln H_{D_2, \alpha+y_{k2}}}}}, \end{aligned}$$

which completes the proof of (21).

ii) Applying Lemma 5 in [27] we obtain

$$\begin{aligned} &\left| \mathbb{P}\left\{\widetilde{\mathbf{M}}(T) \leq \mathbf{x}, \widetilde{\mathbf{M}}(\delta_1, T) \leq \mathbf{y}_1, \widetilde{\mathbf{M}}(\delta_2, T) \leq \mathbf{y}_2\right\} - \mathbb{P}\left\{\widetilde{\mathbf{M}}(T) \leq \mathbf{x}, \widetilde{\mathbf{M}}(\delta_1, T) \leq \mathbf{y}_1, \widetilde{\mathbf{M}}(T) \leq \mathbf{y}_2\right\} \right| \\ &\leq \left| \mathbb{P}\left\{\widetilde{\mathbf{M}}(\delta_2, T) \leq \mathbf{y}_2\right\} - \mathbb{P}\left\{\widetilde{\mathbf{M}}(T) \leq \mathbf{y}_2\right\} \right| \rightarrow 0, \quad T \rightarrow \infty. \end{aligned}$$

Further, Theorem 2.2 in [31] yields

$$\begin{aligned} \mathbb{P}\left\{\widetilde{\mathbf{M}}(T) \leq \mathbf{x}, \widetilde{\mathbf{M}}(\delta_1, T) \leq \mathbf{y}_1, \widetilde{\mathbf{M}}(T) \leq \mathbf{y}_2\right\} &= \mathbb{P}\left\{\widetilde{\mathbf{M}}(T) \leq \min(\mathbf{x}, \mathbf{y}_2), \widetilde{\mathbf{M}}(\delta_1, T) \leq \mathbf{y}_1\right\} \\ &\rightarrow \mathbb{E}\left\{\exp\left(-\sum_{k=1}^p f(x_k, y_{k1}, y_{k2}) e^{-r_{kk} + \sqrt{2r_{kk} z_k}}\right)\right\} \end{aligned}$$

with

$$f(x_k, y_{k1}, y_{k2}) = e^{-\min(x_k, y_{k2})} + e^{-y_{k1}} - H_{D_1, \alpha}^{\ln H_\alpha + \min(x_k, y_{k2}), \ln H_{D_1, \alpha} + y_{k1}},$$

which completes the proof.

iii) By Theorem 2.3 in [31] for the dense grid  $\mathfrak{R}(\delta_i)$ ,  $i = 1, 2$  and any  $\mathbf{y}_i \in \mathbb{R}^p$

$$\lim_{T \rightarrow \infty} \mathbb{P} \left\{ \widetilde{\mathcal{M}}(\delta_i, T) \leq \mathbf{y}_i \right\} = \mathbb{E} \left\{ \exp \left( - \sum_{k=1}^p e^{-y_{ki} - r_{kk} + \sqrt{2r_{kk}} Z_k} \right) \right\}$$

and further

$$\lim_{T \rightarrow \infty} \mathbb{P} \left\{ \widetilde{\mathcal{M}}(T) \leq \mathbf{x} \right\} = \mathbb{E} \left\{ \exp \left( - \sum_{k=1}^p e^{-x_k - r_{kk} + \sqrt{2r_{kk}} Z_k} \right) \right\}, \quad \forall \mathbf{x} \in \mathbb{R}^p,$$

hence the claim follows immediately from Lemma 4.5.  $\square$

#### 4. APPENDIX

For the proof of the main results, we need the following technical lemmas. Let in the sequel  $\mathcal{C}$  be a positive constant whose value will change from place to place and  $\overline{\Phi}$ ,  $\varphi$  be the survival function and the density function of an  $N(0, 1)$  random variable, respectively.

**Lemma 4.1.** *Suppose that  $\mathfrak{R}(\delta_1)$  and  $\mathfrak{R}(\delta_2)$  are sparse grids and  $a \in (0, 1)$ .*

i) *If  $\lim_{T \rightarrow \infty} \delta_1(T)/\delta_2(T) = \infty$  or  $\mathfrak{R}(\delta_1) \cap \mathfrak{R}(\delta_2) = \emptyset$ , then we have for  $k \leq p$  as  $T \rightarrow \infty$*

$$\mathbb{P} \left\{ \max_{t \in \mathcal{R}(\delta_1) \cap [0, S]} \eta_k(t) > u(y_{k1}, z_k), \max_{t \in \mathcal{R}(\delta_2) \cap [0, S]} \eta_k(t) > u(y_{k2}, z_k) \right\} = o(T^{a-1}).$$

ii) *Let  $\mathfrak{R}(\delta_1) \cap \mathfrak{R}(\delta_2) = \mathfrak{R}(\delta_3)$  and  $\lim_{T \rightarrow \infty} \ln(\frac{\delta_3(T)}{\delta_1(T)}) = \theta_1 \in [0, \infty)$ ,  $\lim_{T \rightarrow \infty} \ln(\frac{\delta_3(T)}{\delta_2(T)}) = \theta_2 \in [0, \infty)$  hold. If  $y_{k1} > y_{k2} + \theta_2 - \theta_1$ , then we have for  $k \leq p$  as  $T \rightarrow \infty$*

$$\mathbb{P} \left\{ \max_{t \in \mathcal{R}(\delta_1) \cap [0, S] \setminus \mathcal{R}(\delta_3) \cap [0, S]} \eta_k(t) > u(y_{k1}, z_k), \max_{t \in \mathcal{R}(\delta_2) \cap [0, S]} \eta_k(t) > u(y_{k2}, z_k) \right\} = o(T^{a-1}),$$

whereas if  $y_{k1} \leq y_{k2} + \theta_2 - \theta_1$

$$\mathbb{P} \left\{ \max_{t \in \mathcal{R}(\delta_1) \cap [0, S]} \eta_k(t) > u(y_{k1}, z_k), \max_{t \in \mathcal{R}(\delta_2) \cap [0, S] \setminus \mathcal{R}(\delta_3) \cap [0, S]} \eta_k(t) > u(y_{k2}, z_k) \right\} = o(T^{a-1})$$

holds.

**Proof of Lemma 4.1:** The following fact will be extensively used in the proof. From assumption (10), we can choose an  $\epsilon > 0$  such that for all  $|s - t| \leq \epsilon < 2^{-1/\alpha}$

$$(27) \quad \frac{1}{2}|s - t|^\alpha \leq 1 - r_{kk}(s, t) \leq 2|s - t|^\alpha.$$

i) We first deal with the case  $\lim_{T \rightarrow \infty} \delta_1(T)/\delta_2(T) = \infty$ . It is easy to check that

$$\sum_{k=1}^p \mathbb{P} \left\{ \max_{t \in \mathcal{R}(\delta_1) \cap [0, S]} \eta_k(t) > u(y_{k1}, z_k), \max_{t \in \mathcal{R}(\delta_2) \cap [0, S]} \eta_k(t) > u(y_{k2}, z_k) \right\}$$

$$\begin{aligned} &\leq \sum_{k=1}^p \left[ \mathbb{P} \left\{ \max_{t \in \mathcal{R}(\delta_1) \cap [0, S]} \eta_k(t) > u(y_{k2}, z_k) \right\} \right. \\ &\quad \left. + \mathbb{P} \left\{ \max_{t \in \mathcal{R}(\delta_1) \cap [0, S]} \eta_k(t) > u(y_{k1}, z_k), \max_{t \in \mathcal{R}(\delta_2) \cap [0, S] \setminus \mathcal{R}(\delta_1) \cap [0, S]} \eta_k(t) > u(y_{k2}, z_k) \right\} \right]. \end{aligned}$$

By Lemma 2 of [27] and the definition of  $u(y_{k2}, z_k)$ , we have as  $T \rightarrow \infty$

$$\begin{aligned} \mathbb{P} \left\{ \max_{t \in \mathcal{R}(\delta_1) \cap [0, S]} \eta_k(t) > u(y_{k2}, z_k) \right\} &\sim S\delta_1^{-1}(T)\bar{\Phi}(u(y_{k2}, z_k)) \\ &= \mathcal{C}S\delta_1^{-1}(T)T^{-1}\delta_2(T) \\ &= \mathcal{C}T^{a-1}\frac{\delta_2(T)}{\delta_1(T)} = o(T^{a-1}). \end{aligned}$$

Now, for  $m, n \in \mathbb{N}$  and the  $\epsilon$  chosen in (27), we have

$$\begin{aligned} &\mathbb{P} \left\{ \max_{t \in \mathcal{R}(\delta_1) \cap [0, S]} \eta_k(t) > u(y_{k1}, z_k), \max_{t \in \mathcal{R}(\delta_2) \cap [0, S]} \eta_k(t) > u(y_{k2}, z_k) \right\} \\ &= \mathbb{P} \left\{ \max_{t \in \mathcal{R}(\delta_1) \cap [0, S]} \eta_k(t) > u(y_{k1}, z_k), \max_{t \in \mathcal{R}(\delta_2) \cap [0, S] \setminus \mathcal{R}(\delta_1) \cap [0, S]} \eta_k(t) > u(y_{k2}, z_k) \right\} + o(T^{a-1}) \\ &\leq \sum_{n=0}^{\lfloor S/\delta_1 \rfloor + 1} \mathbb{P} \left\{ \eta_k(n\delta_1) > u(y_{k1}, z_k), \max_{t \in \mathcal{R}(\delta_2) \cap [0, S] \setminus \mathcal{R}(\delta_1) \cap [0, S]} \eta_k(t) > u(y_{k2}, z_k) \right\} + o(T^{a-1}) \\ &\leq \sum_{n=0}^{\lfloor S/\delta_1 \rfloor + 1} \mathbb{P} \left\{ \eta_k(n\delta_1) > u(y_{k1}, z_k), \max_{\substack{0 \leq t \leq S \\ |t - n\delta_1| \leq \epsilon}} \eta_k(t) > u(y_{k2}, z_k) \right\} \\ &\quad + \sum_{n=0}^{\lfloor S/\delta_1 \rfloor + 1} \mathbb{P} \left\{ \eta_k(n\delta_1) > u(y_{k1}, z_k), \max_{\substack{0 \leq m\delta_2 \leq S \\ |n\delta_1 - m\delta_2| > \epsilon}} \eta_k(m\delta_2) > u(y_{k2}, z_k) \right\} + o(T^{a-1}) \\ &=: S_{T,1} + S_{T,2} + o(T^{a-1}), \end{aligned}$$

where  $[x]$  denotes the integer part of  $x$ . By stationarity we have setting  $\eta_{nk}^*(t) = \eta_k(n\delta_1) + \eta_k(t)$

$$\begin{aligned} S_{T,1} &\leq \sum_{n=0}^{\lfloor S/\delta_1 \rfloor + 1} \mathbb{P} \left\{ \max_{n\delta_1 - \epsilon \leq t \leq n\delta_1 + \epsilon} \eta_{nk}^*(t) > u(y_{k1}, z_k) + u(y_{k2}, z_k) \right\} \\ &= \mathcal{C}\frac{S}{\delta_1} \mathbb{P} \left\{ \max_{0 \leq t < \epsilon} \eta_{0k}^*(t) > u(y_{k1}, z_k) + u(y_{k2}, z_k) \right\}. \end{aligned}$$

For the correlation function of  $\eta_{0k}^*(t) = \eta_k(0) + \eta_k(t)$ ,  $t \in [0, \epsilon]$  we have

$$\begin{aligned} 1 - \frac{\mathbb{E}(\eta_{0k}^*(s))\mathbb{E}(\eta_{0k}^*(t))}{\sqrt{\mathbb{E}((\eta_{0k}^*(s))^2)\mathbb{E}((\eta_{0k}^*(t))^2)}} &\leq \frac{1 - r_{kk}(t-s)}{2\sqrt{1+r_{kk}(t)}\sqrt{1+r_{kk}(s)}} \\ &\leq \frac{2|t-s|^\alpha}{2-2\epsilon^\alpha} \leq 1 - \exp(-|t-s|^\alpha). \end{aligned}$$

Further

$$\text{Var}(\eta_{0k}^*(t)) = 2 + 2r_{kk}(t) = 4 - 2|t|^\alpha(1 + o(1))$$

as  $t \rightarrow 0$ . Hence by Slepian's inequality (see e.g. Theorem 7.4.2 of [21]) we have

$$\mathbb{P} \left\{ \max_{0 \leq t < \epsilon} \eta_{0k}^*(t) > u(y_{k1}, z_k) + u(y_{k2}, z_k) \right\}$$

$$\begin{aligned}
&= \mathbb{P} \left\{ \max_{0 \leq t < \epsilon} \frac{\eta_{0k}^*(t)}{\sqrt{\mathbb{E}((\eta_{0k}^*(t))^2)}} \sqrt{\mathbb{E}((\eta_{0k}^*(t))^2)} > u(y_{k1}, z_k) + u(y_{k2}, z_k) \right\} \\
&\leq \mathbb{P} \left\{ \max_{0 \leq t < \epsilon} W(t) \sqrt{\mathbb{E}((\eta_{0k}^*(t))^2)} > u(y_{k1}, z_k) + u(y_{k2}, z_k) \right\},
\end{aligned}$$

where  $W$  is a Gaussian zero mean stationary process with covariance function  $\exp(-|t|^\alpha)$ , thus the condition of Theorem D.3 in [26] for the case  $\alpha = \beta$  hold. By that theorem

$$S_{T,1} \leq C \frac{S}{\delta_1} \bar{\Phi} \left( \frac{u(y_{k1}, z_k) + u(y_{k2}, z_k)}{2} \right).$$

The definition of  $u(y_{ki}, z_k), i = 1, 2$  implies thus for sparse grids

$$(28) \quad [u(y_{ki}, z_k)]^2 = 2 \ln T - \ln \ln T + 2 \ln \delta_i^{-1}(T) + O(1).$$

Consequently, from the fact that  $\lim_{T \rightarrow \infty} \delta_1(T)/\delta_2(T) = \infty$

$$S_{T,1} \leq C \frac{S}{\delta_1(T)} \frac{1}{\sqrt{\ln T}} T^{-1} \sqrt{\ln T} \delta_1^{1/2}(T) \delta_2^{1/2}(T) = CT^{a-1} \left( \frac{\delta_2(T)}{\delta_1(T)} \right)^{1/2} = o(T^{a-1}), \quad T \rightarrow \infty.$$

Now, let  $\vartheta_{kk}(t) = \sup_{t \leq s \leq S} r_{kk}(s)$ . Assumption (10) implies that  $\vartheta_{kk}(\epsilon) < 1$  for all  $T$  and any  $\epsilon \in (0, 2^{-1/\alpha})$ .

Consequently, we may choose some positive constant  $\beta_{kk}$  such that

$$\beta_{kk} < \frac{1 - \vartheta_{kk}(\epsilon)}{1 + \vartheta_{kk}(\epsilon)} < 1$$

for all sufficiently large  $T$ . In the following we choose

$$(29) \quad 0 < a < b < \min_{1 \leq k \leq p} \beta_{kk}.$$

For the second term, by stationarity and Berman's inequality (see eg. Theorem 4.2.1 of [21], Theorem C.2 of [26]), we have

$$\begin{aligned}
S_{T,2} &\leq \sum_{n=0}^{\lfloor S/\delta_1 \rfloor + 1} \sum_{\substack{0 \leq m\delta_2 \leq S \\ |n\delta_1 - m\delta_2| > \epsilon}} \mathbb{P} \{ \eta_k(n\delta_1) > u(y_{k1}, z_k), \eta_k(m\delta_2) > u(y_{k2}, z_k) \} \\
&\leq \sum_{n=0}^{\lfloor S/\delta_1 \rfloor + 1} \sum_{\substack{0 \leq m\delta_2 \leq S \\ |n\delta_1 - m\delta_2| > \epsilon}} \left[ \bar{\Phi}(u(y_{k1}, z_k)) \bar{\Phi}(u(y_{k2}, z_k)) + C \exp \left( -\frac{u^2(y_{k1}, z_k) + u^2(y_{k2}, z_k)}{2(1 + r_{kk}(|n\delta_1 - m\delta_2|))} \right) \right] \\
&\leq \frac{S}{\delta_1} \frac{S}{\delta_2} \left[ \bar{\Phi}(u(y_{k1}, z_k)) \bar{\Phi}(u(y_{k2}, z_k)) + C \exp \left( -\frac{u^2(y_{k1}, z_k) + u^2(y_{k2}, z_k)}{2(1 + \vartheta_{kk}(\epsilon))} \right) \right] \\
&=: S_{T,21} + S_{T,22}.
\end{aligned}$$

Utilising again (28)

$$\begin{aligned}
S_{T,21} &\leq C \frac{S}{\delta_1} \frac{S}{\delta_2} \frac{\varphi(u(y_{k1}, z_k))}{u(y_{k1}, z_k)} \frac{\varphi(u(y_{k2}, z_k))}{u(y_{k2}, z_k)} \\
&\leq C \frac{S}{\delta_1} \frac{S}{\delta_2} \frac{1}{\ln T} \exp \left( -\frac{1}{2} u^2(y_{k1}, z_k) \right) \exp \left( -\frac{1}{2} u^2(y_{k2}, z_k) \right) \\
&\leq C \frac{S}{\delta_1} \frac{S}{\delta_2} \frac{1}{\ln T} T^{-1} (\ln T)^{1/2} \delta_1 T^{-1} (\ln T)^{1/2} \delta_2
\end{aligned}$$

$$= CT^{2(a-1)}$$

as  $T \rightarrow \infty$ . Since  $u(y_{ki}, z_k) \sim (2 \ln T)^{1/2}, i = 1, 2$

$$\begin{aligned} S_{T,22} &\leq C \frac{S}{\delta_1} \frac{S}{\delta_2} \exp\left(-\frac{u^2(y_{k1}, z_k) + u^2(y_{k2}, z_k)}{2(1 + \vartheta_{kk}(\epsilon))}\right) \\ &\leq C \frac{T^a}{\delta_1} \frac{T^a}{\delta_2} T^{-\frac{2}{1 + \vartheta_{kk}(\epsilon)}} \\ &\leq CT^{a-1} T^{a - \frac{1 - \vartheta_{kk}(\epsilon)}{1 + \vartheta_{kk}(\epsilon)}} (\delta_1 \delta_2)^{-1}. \end{aligned}$$

Both (29) and  $\lim_{T \rightarrow \infty} (\ln T)^{1/\alpha} \delta_i(T) = \infty$  imply  $S_{T,22} = o(T^{a-1})$  as  $T \rightarrow \infty$ .

Let us consider now the case that  $\mathfrak{R}(\delta_1) \cap \mathfrak{R}(\delta_2) = \emptyset$ . Without loss of generality, we suppose that  $u(y_{k1}, z_k) < u(y_{k2}, z_k)$  holds for sufficient large  $T$ . By stationarity, for  $m, n \in \mathbb{N}$  and  $\epsilon > 0$  we have

$$\begin{aligned} &\mathbb{P} \left\{ \max_{t \in \mathcal{R}(\delta_1) \cap [0, S]} \eta_k(t) > u(y_{k1}, z_k), \max_{t \in \mathcal{R}(\delta_2) \cap [0, S]} \eta_k(t) > u(y_{k2}, z_k) \right\} \\ &\leq \sum_{n=0}^{[S/\delta_1]+1} \mathbb{P} \left\{ \eta_k(n\delta_1) > u(y_{k1}, z_k), \max_{0 < m\delta_2 \leq S} \eta_k(m\delta_2) > u(y_{k2}, z_k) \right\} \\ &\leq \sum_{n=0}^{[S/\delta_1]+1} \mathbb{P} \left\{ \eta_k(n\delta_1) > u(y_{k1}, z_k), \max_{\substack{0 < m\delta_2 \leq S \\ |n\delta_1 - m\delta_2| \leq \epsilon}} \eta_k(m\delta_2) > u(y_{k1}, z_k) \right\} \\ &\quad + \sum_{n=0}^{[S/\delta_1]+1} \mathbb{P} \left\{ \eta_k(n\delta_1) > u(y_{k1}, z_k), \max_{\substack{0 < m\delta_2 \leq S \\ |n\delta_1 - m\delta_2| > \epsilon}} \eta_k(m\delta_2) > u(y_{k2}, z_k) \right\} \\ &= C \frac{S}{\delta_1} \mathbb{P} \left\{ \eta_k(0) > u(y_{k1}, z_k), \max_{0 < m\delta_2 \leq \epsilon} \eta_k(m\delta_2) > u(y_{k1}, z_k) \right\} \\ &\quad + \sum_{n=0}^{[S/\delta_1]+1} \mathbb{P} \left\{ \eta_k(n\delta_1) > u(y_{k1}, z_k), \max_{\substack{0 < m\delta_2 \leq S \\ |n\delta_1 - m\delta_2| > \epsilon}} \eta_k(m\delta_2) > u(y_{k2}, z_k) \right\} \\ &=: R_{T,1} + R_{T,2}. \end{aligned}$$

Using the well-known results for bivariate Gaussian tail probability (see e.g., [16]) setting  $r = r_{kk}(m\delta_2)$  we have

$$\begin{aligned} R_{T,1} &\leq C \frac{S}{\delta_1} \sum_{0 < m\delta_2 \leq \epsilon} \mathbb{P} \{ \eta_k(0) > u(y_{k1}, z_k), \eta_k(m\delta_2) > u(y_{k1}, z_k) \} \\ &= C \frac{S}{\delta_1} \sum_{0 < m\delta_2 \leq \epsilon} \left[ \bar{\Phi}(u(y_{k1}, z_k)) \bar{\Phi} \left( u(y_{k1}, z_k) \frac{\sqrt{1-r}}{\sqrt{1+r}} \right) \right]. \end{aligned}$$

Since by (27)

$$\frac{1-r}{1+r} = \frac{1 - r_{kk}(m\delta_2)}{1 + r_{kk}(m\delta_2)} \geq \frac{1}{4} (m\delta_2)^\alpha$$

and using (28) we obtain

$$\begin{aligned} R_{T,1} &\leq C \frac{S}{\delta_1} \sum_{0 < m\delta_2 \leq \epsilon} \left[ \bar{\Phi}(u(y_{k1}, z_k)) \bar{\Phi} \left( \frac{1}{2} (m\delta_2)^{\alpha/2} u(y_{k1}, z_k) \right) \right] \\ &= CT^{a-1} \sum_{0 < m\delta_2 \leq \epsilon} \bar{\Phi} \left( \frac{1}{2} (m\delta_2)^{\alpha/2} u(y_{k1}, z_k) \right) \end{aligned}$$

$$\begin{aligned}
&= \mathcal{C}T^{a-1} \sum_{0 < m\delta_2 \leq \epsilon} \frac{1}{(m\delta_2)^{\alpha/2} u(y_{k_1}, z_k)} \exp\left(-\frac{1}{8}(m\delta_2)^\alpha u^2(y_{k_1}, z_k)\right) \\
&= \mathcal{C}T^{a-1} \sum_{0 < m\delta_2 \leq \epsilon} \frac{1}{[m\delta_2(\ln T)^{1/\alpha}]^{\alpha/2}} \exp\left(-\frac{1}{4}[m\delta_2(\ln T)^{1/\alpha}]^\alpha\right) \\
&\leq \mathcal{C}T^{a-1} \frac{1}{[\delta_2(\ln T)^{1/\alpha}]^{\alpha/2}} \sum_{0 < m \leq [\epsilon/\delta_2] + 1} \exp\left(-\frac{1}{4}[m\delta_2(\ln T)^{1/\alpha}]^\alpha\right) \\
&\leq \mathcal{C}T^{a-1} \frac{1}{[\delta_2(\ln T)^{1/\alpha}]^{\alpha/2}} \\
&= T^{a-1} o(1),
\end{aligned}$$

where we used additionally the fact that  $\lim_{T \rightarrow \infty} (\ln T)^{1/\alpha} \delta_i(T) = \infty$ ,  $i = 1, 2$ . By repeating the calculations for  $S_{T,2}$  we obtain further  $R_{T,2} = o(T^{a-1})$  as  $T \rightarrow \infty$ , which completes the proof.

*ii)* If  $y_{k_1} \leq y_{k_2} + \theta_2 - \theta_1$ , then we have  $u(y_{k_1}, z_k) \leq u(y_{k_2}, z_k)$  for sufficient large  $T$ . By stationarity we have for  $m, n \in \mathbb{N}$  and  $\epsilon > 0$

$$\begin{aligned}
&\mathbb{P} \left\{ \max_{t \in \mathcal{R}(\delta_1) \cap [0, S]} \eta_k(t) > u(y_{k_1}, z_k), \max_{t \in \mathcal{R}(\delta_2) \cap [0, S] \setminus \mathcal{R}(\delta_3) \cap [0, S]} \eta_k(t) > u(y_{k_2}, z_k) \right\} \\
&\leq \sum_{n=0}^{[S/\delta_1]+1} \mathbb{P} \left\{ \eta_k(n\delta_1) > u(y_{k_1}, z_k), \max_{t \in \mathcal{R}(\delta_2) \cap [0, S] \setminus \mathcal{R}(\delta_3) \cap [0, S]} \eta_k(t) > u(y_{k_2}, z_k) \right\} \\
&\leq \sum_{n=0}^{[S/\delta_1]+1} \mathbb{P} \left\{ \eta_k(n\delta_1) > u(y_{k_1}, z_k), \max_{\substack{0 < m\delta_2 \leq S \\ |n\delta_1 - m\delta_2| \leq \epsilon}} \eta_k(m\delta_2) > u(y_{k_2}, z_k) \right\} \\
&\quad + \sum_{n=0}^{[S/\delta_1]+1} \mathbb{P} \left\{ \eta_k(n\delta_1) > u(y_{k_1}, z_k), \max_{\substack{0 < m\delta_2 \leq S \\ |n\delta_1 - m\delta_2| > \epsilon}} \eta_k(m\delta_2) > u(y_{k_2}, z_k) \right\} \\
&\leq \mathcal{C} \frac{S}{\delta_1} \mathbb{P} \left\{ \eta_k(0) > u(y_{k_1}, z_k), \max_{0 < m\delta_2 \leq \epsilon} \eta_k(m\delta_2) > u(y_{k_1}, z_k) \right\} \\
&\quad + \sum_{n=0}^{[S/\delta_1]+1} \mathbb{P} \left\{ \eta_k(n\delta_1) > u(y_{k_1}, z_k), \max_{\substack{0 < m\delta_2 \leq S \\ |n\delta_1 - m\delta_2| > \epsilon}} \eta_k(m\delta_2) > u(y_{k_2}, z_k) \right\} \\
&=: M_{T,1} + M_{T,2}.
\end{aligned}$$

Using the same estimates for  $R_{T,1}$  and  $R_{T,2}$ , we get that both  $M_{T,1}$  and  $M_{T,2}$  are  $o(T^{a-1})$ . The proof when  $y_{k_1} > y_{k_2} + (\theta_2 - \theta_1)$  is similar. This completes the proof of the lemma.  $\square$

The next lemma extends Lemma 2 of [27] to the non-uniform sparse grid. Let  $\mathcal{R}(\delta^*) = \{t_1(T) < t_2(T) < \dots\}$  be a non-uniform grid on  $[0, T]$  such that

$$\delta_{\max} := \max_{t_k(T) \in [0, T]} (t_k(T) - t_{k-1}(T)) \leq \delta_0 \quad \text{and} \quad \delta_{\min}(\ln T)^{1/\alpha} := \min_{t_k(T) \in [0, T]} (t_k(T) - t_{k-1}(T))(\ln T)^{1/\alpha} \rightarrow \infty$$

as  $T \rightarrow \infty$ .

**Lemma 4.2.** *For  $S = T^a$ ,  $a \in (0, 1)$  we have for any  $k \leq p$*

$$\mathbb{P} \left\{ \max_{t \in \mathcal{R}(\delta^*) \cap [0, S]} \eta_k(t) > u_T \right\} = \sharp(\mathcal{R}(\delta^*) \cap [0, S]) \bar{\Phi}(u_T) (1 + o(1))$$

as  $T \rightarrow \infty$ , where  $u_T = (2 \ln T)^{1/2} (1 + o(1))$  and  $\sharp(A)$  denotes the number of the elements of the set  $A$ .

**Proof of Lemma 4.2:** By Bonferroni inequality for all  $T$  large (set  $\Theta_T := \#(\mathcal{R}(\delta^*) \cap [0, S])$  and  $u := u_T$ )

$$\begin{aligned} \sum_{l=1}^{\Theta_T} \mathbb{P} \{ \eta_k(t_l(T)) > u \} &\geq \mathbb{P} \left\{ \max_{t \in \mathcal{R}(\delta^*) \cap [0, S]} \eta_k(t) > u \right\} \\ &\geq \sum_{l=1}^{\Theta_T} \mathbb{P} \{ \eta_k(t_l(T)) > u \} - \sum_{1 \leq m < l \leq \Theta_T} \mathbb{P} \{ \eta_k(t_m(T)) > u, \eta_k(t_l(T)) > u \} \\ &=: P_{T,1} - P_{T,2}. \end{aligned}$$

By the stationarity of  $\eta_k$

$$P_{T,1} = \Theta_T \bar{\Phi}(u),$$

whereas for the second term we have for all  $\varepsilon > 0$  sufficiently small

$$\begin{aligned} P_{T,2} &= \sum_{\substack{1 \leq m < l \leq \Theta_T \\ t_l(T) - t_m(T) \leq \varepsilon}} \mathbb{P} \{ \eta_k(t_m(T)) > u, \eta_k(t_l(T)) > u \} + \sum_{\substack{1 \leq m < l \leq \Theta_T \\ t_l(T) - t_m(T) > \varepsilon}} \mathbb{P} \{ \eta_k(t_m(T)) > u, \eta_k(t_l(T)) > u \} \\ &=: P_{T,21} + P_{T,22}. \end{aligned}$$

Similarly as in the calculations of  $R_{T,1}$  setting  $r = r(t_l(T) - t_m(T))$  we have

$$\begin{aligned} P_{T,21} &= \sum_{\substack{1 \leq m < l \leq \Theta_T \\ t_l(T) - t_m(T) \leq \varepsilon}} \mathbb{P} \{ \eta_k(0) > u, \eta_k(t_l(T) - t_m(T)) > u \} \\ &\leq \sum_{\substack{1 \leq m < l \leq \Theta_T \\ t_l(T) - t_m(T) \leq \varepsilon}} \bar{\Phi}(u) \bar{\Phi} \left( u \frac{\sqrt{1-r}}{\sqrt{1+r}} \right). \end{aligned}$$

Since by (27)

$$\frac{1-r}{1+r} = \frac{1 - r_{kk}(t_l(T) - t_m(T))}{1 + r_{kk}(t_l(T) - t_m(T))} \geq \frac{1}{4} (t_l(T) - t_m(T))^\alpha \geq \frac{1}{4} \delta_{\min}^\alpha$$

and the fact that  $u = u_T = (2 \ln T)^{1/2} (1 + o(1))$  and  $\delta_{\min} (\ln T)^{1/\alpha} \rightarrow \infty$  we get

$$\begin{aligned} P_{T,21} &\leq \Theta_T \bar{\Phi}(u) \frac{\varepsilon}{\delta_{\min}} \bar{\Phi} \left( \frac{1}{2} u \delta_{\min}^{\alpha/2} \right) \\ &\leq \mathcal{C} \Theta_T \bar{\Phi}(u) \frac{\varepsilon}{\delta_{\min}} \frac{1}{u \delta_{\min}^{\alpha/2}} \exp \left( -\frac{1}{8} u^2 \delta_{\min}^\alpha \right) \\ &\leq \mathcal{C} \Theta_T \bar{\Phi}(u) \frac{1}{u \delta_{\min}^{\alpha/2}} \\ &= \Theta_T \bar{\Phi}(u) o(1). \end{aligned}$$

Recalling the bound derived for  $S_{T,2}$ , by stationarity and Berman's inequality

$$\begin{aligned} P_{T,22} &\leq \sum_{\substack{1 \leq m < l \leq \Theta_T \\ t_l(T) - t_m(T) > \varepsilon}} \left[ \bar{\Phi}^2(u) + \mathcal{C} \exp \left( -\frac{u^2}{1 + \vartheta_{kk}(t_l(T) - t_m(T))} \right) \right] \\ &\leq \mathcal{C} \Theta_T^2 \left[ \bar{\Phi}^2(u) + \mathcal{C} \exp \left( -\frac{u^2}{1 + \vartheta_{kk}(\varepsilon)} \right) \right], \end{aligned}$$

where  $\vartheta_{kk}(\cdot)$  is defined in the proof of Lemma 4.1. Noting that  $\Theta_T \leq S/\delta_{\min} = T^a/\delta_{\min}$  and by repeating the calculations for  $S_{T,2}$  we obtain further  $P_{T,22} = \Theta_T \bar{\Phi}(u) o(1)$  as  $T \rightarrow \infty$ , which completes the proof.  $\square$

**Lemma 4.3.** *If  $\mathfrak{R}(\delta_1)$  and  $\mathfrak{R}(\delta_2)$  are sparse and Pickands girds, respectively, then for  $k \leq p$  as  $T \rightarrow \infty$*

$$\mathbb{P} \left\{ \max_{t \in \mathcal{R}(\delta_1) \cap [0, S]} \eta_k(t) > u(y_{k1}, z_k), \max_{t \in \mathcal{R}(\delta_2) \cap [0, S]} \eta_k(t) > u(y_{k2}, z_k) \right\} = o(T^{a-1}).$$

**Proof of Lemma 4.3:** Since  $\mathfrak{R}(\delta_1)$  and  $\mathfrak{R}(\delta_2)$  are sparse and Pickands girds, respectively, we have

$$\lim_{T \rightarrow \infty} \delta_1(T)/\delta_2(T) = \infty.$$

Consequently, the proof is similar to that of the case that  $\lim_{T \rightarrow \infty} \delta_1(T)/\delta_2(T) = \infty$  of Lemma 4.1, and therefore we omit further details.  $\square$

Let  $X$  be a centered stationary Gaussian process which satisfies condition (1) (as in the Introduction). For the proof of Theorem 2.2 we shall determine the asymptotic behaviours, as  $u \rightarrow \infty$ , of the following probabilities

$$P_S(u, x) = \mathbb{P} \left\{ \max_{t \in \mathfrak{R}(\delta_1) \cap [0, S]} X(t) > u, \max_{t \in \mathfrak{R}(\delta_2) \cap [0, S]} X(t) > u + \frac{x}{u} \right\}$$

and

$$P_S(u, x, y) = \mathbb{P} \left\{ \max_{t \in \mathfrak{R}(\delta_1) \cap [0, S]} X(t) > u, \max_{t \in \mathfrak{R}(\delta_2) \cap [0, S]} X(t) > u + \frac{x}{u}, \max_{t \in [0, S]} X(t) > u + \frac{y}{u} \right\},$$

where  $\mathfrak{R}(\delta_1) = \mathfrak{R}(cu^{-2/\alpha})$  and  $\mathfrak{R}(\delta_2) = \mathfrak{R}(du^{-2/\alpha})$  with  $c > d > 0$ .

For  $\lambda \in (c, \infty)$  along the lines of the proof of Lemma D.1 in [26] (see also the proof of Lemma 12.2.3 of [21])

$$P_{\lambda u^{-2/\alpha}}(u, x) \sim H_{c,d,\alpha}^{0,x}(\lambda) \overline{\Phi}(u) \quad \text{and} \quad P_{\lambda u^{-2/\alpha}}(u, x, y) \sim H_{c,d,\alpha}^{0,x,y}(\lambda) \overline{\Phi}(u)$$

as  $u \rightarrow \infty$ , where

$$H_{c,d,\alpha}^{0,x}(\lambda) = \int_{s \in \mathbb{R}} e^s \mathbb{P} \left\{ \max_{k \in \mathbb{N}: kc \in [0, \lambda]} B_{\alpha/2}^*(kc) > s, \max_{k \in \mathbb{N}: kd \in [0, \lambda]} B_{\alpha/2}^*(kd) > s + x \right\} ds$$

and

$$H_{c,d,\alpha}^{0,x,y}(\lambda) = \int_{s \in \mathbb{R}} e^s \mathbb{P} \left\{ \max_{k \in \mathbb{N}: kc \in [0, \lambda]} B_{\alpha/2}^*(kc) > s, \max_{k \in \mathbb{N}: kd \in [0, \lambda]} B_{\alpha/2}^*(kd) > s + x, \max_{t \in [0, \lambda]} B_{\alpha/2}^*(t) > s + y \right\} ds.$$

The next result can be shown along the same lines of the proof of Theorem D.2 in [26].

**Lemma 4.4.** *For any  $x, y \in \mathbb{R}$  we have*

$$0 < H_{c,d,\alpha}^{0,x} = \lim_{\lambda \rightarrow \infty} \frac{H_{c,d,\alpha}^{0,x}(\lambda)}{\lambda} < \infty \quad \text{and} \quad 0 < H_{c,d,\alpha}^{0,x,y} = \lim_{\lambda \rightarrow \infty} \frac{H_{c,d,\alpha}^{0,x,y}(\lambda)}{\lambda} < \infty.$$

Furthermore, for any  $S > 0$

$$P_S(u, x) \sim SH_{c,d,\alpha}^{0,x} u^{2/\alpha} \overline{\Phi}(u) \quad \text{and} \quad P_S(u, x, y) \sim SH_{c,d,\alpha}^{0,x,y} u^{2/\alpha} \overline{\Phi}(u)$$

as  $u \rightarrow \infty$ .

**Lemma 4.5.** Let  $\{Z_{T,ij}, 1 \leq i \leq p, 1 \leq j \leq m\}, T > 0$  be a random matrix. Suppose that the following convergence in distribution

$$\mathbf{Z}_{T,j} := (Z_{T,1j}, \dots, Z_{T,pj}) \xrightarrow{d} (W_1, \dots, W_p) =: \mathbf{W}, \quad T \rightarrow \infty$$

is valid for any index  $j \leq m$ . If further  $Z_{T,ij} \leq Z_{i1}$  holds almost surely for any index  $i \leq p, 2 \leq j \leq m$ , then we have the joint convergence in distribution

$$(\mathbf{Z}_{T,1}, \dots, \mathbf{Z}_{T,k}) \xrightarrow{d} (\mathbf{W}, \dots, \mathbf{W}), \quad T \rightarrow \infty.$$

**Proof of Lemma 4.5:** Assume for simplicity that  $m = p = 2$ . By the assumptions, Lemma 2.3 in [18] implies the convergence in distributions

$$(Z_{T,11}, Z_{T,12}) \xrightarrow{d} (W_1, W_1), \quad (Z_{T,21}, Z_{T,22}) \xrightarrow{d} (W_2, W_2), \quad T \rightarrow \infty.$$

Hence we have the convergence in probability

$$Z_{T,12} - Z_{T,11} \xrightarrow{P} 0, \quad Z_{T,22} - Z_{T,21} \xrightarrow{P} 0, \quad T \rightarrow \infty,$$

which then entails that

$$(Z_{T,11}, Z_{T,21}, Z_{T,12}, Z_{T,22}) \xrightarrow{d} (W_1, W_2, W_1, W_2), \quad T \rightarrow \infty$$

establishing thus the proof. □

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