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# Almost Sure Central Limit Theorems of Extremes and the Partial Sum of Gaussian Sequences①

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**Abstract**: For a standardized stationary Gaussian sequence, the joint version of the almost sure central limit theorem related to maximum, minimum and the partial sum is considered when the covariance function satisfies some weak dependence conditions.

**Key words**: almost sure central limit theorem; extreme; partial sum; stationary Gaussian sequence

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Let $(X_n)$ be a standardized stationary Gaussian sequence with marginal distribution function $\Phi(x)$. Suppose that the covariance function $r(t) = \mathrm{Cov}(X_1, X_{1+t})$ satisfies

$$r(t) = \frac{L(t)}{t^\alpha} \qquad t = 1, 2, \cdots \tag{1}$$

where $\alpha > 0$ and $L(\cdot)$ is a positive, slowly varying function at infinity. Moreover, suppose that there exist numerical sequences $(u_n)$, $(v_n)$ and $0 < \tau_i < \infty$ for $i = 1, 2$, such that

$$n(1 - \Phi(u_n)) \longrightarrow \tau_1 \qquad n\Phi(v_n) \longrightarrow \tau_2 \tag{2}$$

as $n \to \infty$. Define $M_n = \max\{X_1, \cdots, X_n\}$, $S_n = X_1 + \cdots + X_n$ and $\sigma_n = (\mathrm{Var}(S_n))^{1/2}$. [1] obtained the almost sure central limit theorem (ASCLT) for the maximum of weakly dependent sequence. The joint version of ASCLT on the random vector $(M_n, S_n)$ is proved in [2], i. e.

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I\{M_n \leqslant u_n, S_n/\sigma_n \leqslant y\} = \mathrm{e}^{-\tau}\Phi(y) \text{ a. s.}$$

if (1) holds. In this paper, we are interested in the ASCLT of random vector $(M_n, m_n, S_n)$ if (1) holds, where $m_n = \min\{X_1, \cdots, X_n\}$. The main result is:

**Theorem 1** Let $(X_n)$ be a standardized stationary Gaussian sequence and its covariance function satisfies (1). Suppose that (2) holds for numerical sequences $(u_n)$ and $(v_n)$. Then for any $z \in R$, we have

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I\left\{v_n < m_n \leqslant M_n \leqslant u_n, \frac{S_n}{\sigma_n} \leqslant z\right\} = \mathrm{e}^{-(\tau_1+\tau_2)}\Phi(z) \text{ a. s.}$$

Furthermore, for all $x, y, z \in R$,

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$$\lim_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}I\left\{-a_ny-b_n<m_n\leqslant M_n\leqslant a_nx+b_n,\ \frac{S_n}{\sigma_n}\leqslant z\right\}$$
$$=\exp(-e^{-(x+y)})\Phi(z)\ \text{a. s.}$$

where

$$a_n=(2\log n)^{-1/2}\qquad b_n=(2\log n)^{1/2}-(2\log n)^{-1/2}(\log\log n+\log 4\pi)/2$$

Before proving the main result, we need some lemmas. For convenience, denote $\mathscr{A}_{k,n}=\{v_n<m_{k,n}\leqslant M_{k,n}\leqslant u_n\}$ for $k<n$, $\mathscr{A}_n=\mathscr{A}_{0,n}=\{v_n<m_n\leqslant M_n\leqslant u_n\}$ and $\mathscr{C}_n=\{S_n/\sigma_n\leqslant z\}$ and let the positive absolute constant $\mathbb{C}$ change from line to line.

**Lemma 1** Under the conditions of Theorem 1, for $1\leqslant k<n$ and $z\in R$, there exists some $\gamma>0$, such that

$$E\mid I\{\mathscr{A}_n\mathscr{C}_n\}-I\{\mathscr{A}_{k,n}\mathscr{C}_n\}\mid\leqslant\mathbb{C}\left(\frac{1}{n^\gamma}+\frac{k}{n}\right)$$

**Proof** Notice that

$$\begin{aligned}&E\mid\{\mathscr{A}_n\mathscr{C}_n\}-I\{\mathscr{A}_{k,n}\mathscr{C}_n\}\mid\\ \leqslant&\mid P\{\mathscr{A}_{k,n}\}-[\Phi(u_n)-\Phi(v_n)]^{n-k}\mid+\mid P\{\mathscr{A}_n\}-[\Phi(u_n)-\Phi(v_n)]^n\mid+\\ &\mid[\Phi(u_n)-\Phi(v_n)]^{n-k}-[\Phi(u_n)-\Phi(v_n)]^n\mid\\ =&A_1+A_2+A_3\end{aligned}$$

Note that $n(1-\Phi(u_n))\longrightarrow\tau_1$ and $n\Phi(v_n)\longrightarrow\tau_2$ as $n\to\infty$ imply (see [3]),

$$\exp\left(-\frac{u_n^{\ 2}}{2}\right)\sim\frac{\mathbb{C}u_n}{n}\qquad\exp\left(-\frac{v_n^{\ 2}}{2}\right)\sim\frac{\mathbb{C}v_n}{n}\qquad u_n^2\sim 2\log n\qquad v_n^2\sim 2\log n$$

Since $\lim\limits_{t\to\infty}r(t)=\lim\limits_{t\to\infty}L(t)/t^\alpha=0$, there exists number $\delta$ such that $0<\sup\limits_{t\geqslant 1}r(t)=\delta<1$. For $0<\alpha<1$, there exist numbers $\delta_1$ and $n_1$, such that $0<\sup\limits_{t\geqslant n_1}r(t)=\delta_1<\alpha/(2-\alpha)<1$. By Normal Comparison Lemma (cf. Lemma 11.1.2 of [3]), we get

$$\begin{aligned}A_1+A_2&\leqslant\mathbb{C}n\sum_{t=1}^{n-1}r(t)\exp\left(-\frac{\omega^2}{1+r(t)}\right)\\ &\leqslant\mathbb{C}n\left[\sum_{t=1}^{n_1}\exp\left(-\frac{\omega^2}{1+\delta}\right)+\sum_{t=n_1+1}^{n-1}\frac{L(t)}{t^\alpha}\exp\left(-\frac{\omega^2}{1+\delta_1}\right)\right]\\ &\leqslant\mathbb{C}\left[\frac{(\log n)^{\frac{1}{1+\delta}}}{n^{\frac{1-\delta}{1+\delta}}}+\frac{L(n)(\log n)^{\frac{1}{1+\delta_1}}}{n^{\frac{2}{1+\delta_1}+\alpha-2}}\right]\end{aligned}$$

where $\omega=\min\{\mid u_n\mid,\mid v_n\mid\}$. For $\alpha\geqslant 1$, we get

$$A_1+A_2\leqslant\mathbb{C}n\sum_{t=1}^{n-1}\frac{L(t)}{t}\exp\left(-\frac{\omega^2}{1+\delta}\right)\leqslant\mathbb{C}\frac{L(n)(\log n)^{\frac{1}{1+\delta}+1}}{n^{\frac{2}{1+\delta}-1}}$$

Note $2/(1+\delta_1)+\alpha-2>0$ and $L(n)<\mathbb{C}n^\varepsilon$ for arbitrary $\varepsilon>0$. For some $\alpha>0$ and $\gamma>0$, we have $A_1+A_2\leqslant\mathbb{C}/n^\gamma$. Further, $A_3\leqslant k/n$ follows from $x^{n-k}-x^n\leqslant k/n$ for $0\leqslant x\leqslant 1$. Hence we get the desired result.

**Lemma 2** Under the conditions of Theorem 1, for any $z\in R$ and some $\gamma>0$, as $k<n$ we have

$$\mid\mathrm{Cov}(I\{\mathscr{A}_k\mathscr{C}_k\},I\{\mathscr{A}_{k,n}\mathscr{C}_n\})\mid$$
$$\leqslant B(k,n)=\begin{cases}\mathbb{C}\left[\dfrac{1}{n^\gamma}+\dfrac{L(k)^{1/2}k^{1-\alpha/2}}{L(n)^{1/2}n^{1-\alpha/2}}+\dfrac{k(n+2k)^{1-\alpha}L(n+2k)}{L(k)^{1/2}k^{1-\alpha/2}L(n)^{1/2}n^{1-\alpha/2}}\right] & 0<\alpha<1\\ \mathbb{C}\left[\dfrac{1}{n^\gamma}+\left(\dfrac{k}{n}\right)^{1/2}\right] & \alpha>1\\ \mathbb{C}\left[\dfrac{1}{n^\gamma}+\dfrac{\widetilde{L}(k)^{1/2}k^{1/2}}{\widetilde{L}(n)^{1/2}n^{1/2}}+\dfrac{k^{1/2}(\widetilde{L}(n+2k)-\widetilde{L}(k+1))}{n^{1/2}\widetilde{L}(k)^{1/2}\widetilde{L}(n)^{1/2}}\right] & \alpha=1\end{cases}$$

Where $\widetilde{L}(n) = 1 + 2\sum_{t=1}^{n-1} r(t)$.

**Proof**　By Normal Comparison Lemma (see [3]), we have

$$\begin{aligned}
&|\operatorname{Cov}(I\{\mathscr{A}_k\mathscr{C}_k\}, I\{\mathscr{A}_{k,n}\mathscr{C}_n\})| = |P\{\mathscr{A}_k\mathscr{C}_k\mathscr{A}_{k,n}\mathscr{C}_n\} - P\{\mathscr{A}_k\mathscr{C}_k\}P\{\mathscr{A}_{k,n}\mathscr{C}_n\}| \\
&\leqslant \mathbb{C}\Bigg[k\sum_{t=1}^{n} r(t)\exp\Big(-\frac{\omega^2}{1+r(t)}\Big) + \sum_{i=1}^{k}\operatorname{Cov}\Big(X_i, \frac{S_n}{\sigma_n}\Big)\exp\Bigg[-\frac{\omega^2+z^2}{2\Big(1+\operatorname{Cov}\Big(X_i, \frac{S_n}{\sigma_n}\Big)\Big)}\Bigg] + \\
&\sum_{j=k+1}^{n}\operatorname{Cov}\Big(X_j, \frac{S_k}{\sigma_k}\Big)\exp\Bigg[-\frac{\omega^2+z^2}{2\Big(1+\operatorname{Cov}\Big(X_j, \frac{S_k}{\sigma_k}\Big)\Big)}\Bigg] + \\
&\operatorname{Cov}\Big(\frac{S_k}{\sigma_k}, \frac{S_n}{\sigma_n}\Big)\exp\Bigg[-\frac{z^2}{1+\operatorname{Cov}\Big(\frac{S_k}{\sigma_k}, \frac{S_n}{\sigma_n}\Big)}\Bigg]\Bigg] \\
&\leqslant B(k, n)
\end{aligned}$$

The last inequality follows from Lemma 2 of [2]. Thus, the desired bound is obtained.

**Lemma 3**　Under the conditions of Theorem 1, for all $z \in R$, we have

$$\lim_{n\to\infty} P\Big\{v_n < m_n \leqslant M_n \leqslant u_n, \frac{S_n}{\sigma_n} \leqslant z\Big\} = \mathrm{e}^{-(\tau_1+\tau_2)}\Phi(z)$$

**Proof**　Let $(X_n^*)$ be the associated random sequence of $(X_n)$ and $Y_n$ denote a random variable, which has the same distribution as $S_n/\sigma_n$, but is independent of $(X_n^*)$. Then

$$\begin{aligned}
&\Big|P\Big\{v_n < m_n \leqslant M_n \leqslant u_n, \frac{S_n}{\sigma_n} \leqslant z\Big\} - P\{v_n < m_n^* \leqslant M_n^* \leqslant u_n\}P\{Y_n \leqslant z\}\Big| \\
&\leqslant \mathbb{C}\Bigg[n\sum_{t=1}^{n} r(t)\exp\Big(-\frac{\omega^2}{1+r(t)}\Big) + \sum_{i=1}^{n}\operatorname{Cov}\Big(X_i, \frac{S_n}{\sigma_n}\Big)\exp\Bigg[-\frac{\omega^2+z^2}{2\Big(1+\operatorname{Cov}\Big(X_i, \frac{S_n}{\sigma_n}\Big)\Big)}\Bigg]\Bigg]
\end{aligned}$$

By using the similar arguments provided in Lemma 3 of [2], we have

$$\lim_{n\to\infty} P\Big\{v_n < m_n \leqslant M_n \leqslant u_n, \frac{S_n}{\sigma_n} \leqslant z\Big\} = \lim_{n\to\infty} P\{v_n < m_n^* \leqslant M_n^* \leqslant u_n\}\Phi(z)$$

According to Theorem 1.8.2 in [3], if (2) holds, we have

$$\lim_{n\to\infty} P\{v_n < m_n^* \leqslant M_n^* \leqslant u_n\} = \mathrm{e}^{-(\tau_1+\tau_2)}$$

The proof is complete.

**Proof of Theorem 1**　By Lemma 3, we will show that Lemma 3.1 in [4] holds for $(\xi_n)$, where $\xi_n = I\{v_n < m_n \leqslant M_n \leqslant u_n, S_n/\sigma_n \leqslant z\}$. Notice that

$$\operatorname{Var}\Big(\sum_{n=1}^{N}\frac{1}{n}\xi_n\Big) = \sum_{n=1}^{N}\frac{1}{n^2}\operatorname{Var}(\xi_n) + 2\sum_{1\leqslant k<n\leqslant N}\frac{1}{kn}\operatorname{Cov}(\xi_k, \xi_n) = \Sigma_1 + \Sigma_2 \tag{3}$$

Since $|\xi_n| \leqslant 1$, we obtain that $\Sigma_1 \leqslant \sum_{n=1}^{\infty}\frac{1}{n^2} < \infty$. Thus, we only need to estimate $\Sigma_2$ in (3). For $k < n$, we have

$$\begin{aligned}
\operatorname{Cov}(\xi_k, \xi_n) &\leqslant 2E|I\{\mathscr{A}_n\mathscr{C}_n\} - I\{\mathscr{A}_{k,n}\mathscr{C}_n\}| + |\operatorname{Cov}(I\{\mathscr{A}_k\mathscr{C}_k\}, I\{\mathscr{A}_{k,n}\mathscr{C}_n\})| \\
&\leqslant \mathbb{C}\Big(B(k, n) + \frac{k}{n}\Big)
\end{aligned}$$

where $B(k, n)$ is defined in Lemma 2. Hence by using the same arguments of that in Theorem 1 of [2], we have

$$\sigma_2 \leqslant \mathbb{C} \sum_{1\leqslant k<n\leqslant N} \frac{1}{kn}\left(B(k,n)+\frac{k}{n}\right) \leqslant \mathbb{C}\log N$$

The proof is complete.

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# 高斯序列极值与部分和的几乎处处中心极限定理


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**摘要**：对于标准化平稳高斯序列，当协方差函数满足弱相依条件时，证明了最大值、最小值及部分和联合的几乎处处中心极限定理.



**关键词**：几乎处处中心极限定理；极值；部分和；平稳高斯序列



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