Tail Asymptotics of Random Sum and Maximum of Log-Normal Risks

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Abstract: In this paper we derive the asymptotic behaviour of the survival function of both random sum and random maximum of log-normal risks. As for the case of finite sum and maximum investigated in Asmussen and Rojas-Nandaypa (2008) also for the more general setup of random sums and random maximum the principle of a single big jump holds. We investigate both the log-normal sequences and some related dependence structures motivated by stationary Gaussian sequences.

Key words: Risk aggregation; log-normal risks; exact asymptotics; Gaussian distribution; product of random variables.

1. Introduction

Let $Y_i, i \geq 1$ be positive random variables (rv's) which model claim sizes of an insurance portfolio for a given observation period. Denote by N the total number of claims reported during the observation period, thus N is a discrete rv, which we assume to be independent of claim sizes $Y_i, i \geq 1$. The classical risk model $S_N = \sum_{i=1}^N Y_i$ for the total loss amount assumes that Y_i's are independent and identically distributed (iid) rv's. If the assumption of independence of claim sizes is dropped, one faces the problem how to choose a meaningful dependence structure. Further this dependence structure should be tractable from a theoretical point of view. For example Constantinescu et al. (2011) consider a model where the survival copula of claim sizes is assumed to be Archimedean. Such a model has the interpretation that for some positive rv V and iid unit exponential rv's $E_i, i \geq 1$ independent of V, then $Y_i = VE_i, i \geq 1$ form a dependent sequence of claim sizes derived by randomly scaling of iid claim sizes $E_i, i \geq 1$.

In this paper we use dependent Gaussian sequences and related dependence structures to model claim sizes. Specifically, if $X_i, i \geq 1$ are dependent Gaussian rv's with $N(0,1)$ distribution, then $Y_i = e^{X_i}, i \geq 1$ is the corresponding sequence of dependent log-normal rv's that can be used for modeling claim sizes. For instance, if $X_i, i \geq 1$ is a centered stationary Gaussian sequence of $N(0,1)$ components and constant correlation $\rho = \mathbb{E}(X_1 X_i) \in (0,1), i > 1$, then $Y_i = e^{X_i}$ is a sequence of dependent log-normal rv's. Since we have (see e.g., Berman (1992))

$$X_i = \rho Z_0 + \sqrt{1-\rho^2} Z_i, \tag{1.1}$$

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with $Z_i, i \ge 0$ iid $N(0,1)$ rv's, then $Y_i = e^{\rho Z_0} e^{\sqrt{1-\rho^2} Z_i}, i \ge 1$. For such Y_i's, by Asmussen and Rojas-Nandaypa (2008)

$$\mathbb{P}\left(S_n > u\right) \sim n \mathbb{P}\left(X_1 > \log u\right), \quad u \to \infty \tag{1.2}$$

holds for any $n \ge 2$, where $\sim$ stands for asymptotic equivalence of two functions when the argument tends to infinity. In view of Asmussen et al. (2011) (see also Hashorva (2013)) S_n is asymptotically tail equivalent with the maximum $Y_{n:n} = \max_{1 \le i \le n} Y_i$, i.e., $\mathbb{P}\left(S_n > u\right) \sim \mathbb{P}\left(Y_{n:n} > u\right)$ as $u \to \infty$.

Our analysis in this paper is concerned with the probability of observing large values for the random sum S_N, thus we shall investigate $\mathbb{P}\left(S_N > u\right)$ when u is large. Additionally, we shall consider also the tail asymptotics of the maximum claim $Y_{N:N}$ among the claim sizes $Y_1, \ldots, Y_N$; we set $Y_{0:0} = 0$ if $N = 0$. For the case that N is non-random see for recent results on max-sum equivalence Jiang et al. (2014) and the references therein.

For our investigations of the tail behaviours of S_N and $Y_{N:N}$ we shall follow two objectives:

A) We shall exploit the tractable dependence structure implied by (1.1) choosing general Z_i's such that e^{Z_i} has survival function similar to that of a log-normal rv;

B) We consider a log-normal dependence structure induced by a general Gaussian sequence $X_i, i \ge 1$ where X_i, X_n can have a correlation ρ_{in} which is allowed to converge to 1 as $n \to \infty$.

For both cases of dependent Y_i's we show that the principle of a single big jump (see Foss et al. (2013) for details in iid setup) holds if for the discrete rv N we require that

$$\mathbb{E}\left((1+\delta)^N\right) < \infty \tag{1.3}$$

is valid for some $\delta > 0$; a large class of discrete rv's satisfies condition (1.3).

Brief organisation of the rest of the paper: We present our main results in Section 2 followed by the proofs in Section 3.

2. Main Results

We consider first X_i's which are in general not Gaussian. So for a given fixed $\rho \in [0,1)$ let $Z_i, i \ge 0$ be independent rv's which define X_i's via the dependence structure (1.1). We shall assume that

$$\mathbb{P}\left(e^{Z_0} > u\right) \sim \mathcal{L}(u) \Psi(\log(u)), \quad u \to \infty, \tag{2.1}$$

with Ψ the survival function of an $N(0,1)$ rv and $\mathcal{L}(\cdot)$ a regularly varying function at ∞ with index $\beta \in \mathbb{R}$, see Bingham et al. (1987) or Mikosch (2009) for details on regularly varying functions. Clearly, (2.1) is satisfied if Z_0 is an $N(0,1)$ rv. Considering Z_0 as a base risk, we shall further assume that with $c_i \in [0, \infty)$ uniformly in i

$$\mathbb{P}\left(Z_i > u\right) \sim c_i \mathbb{P}\left(Z_0 > u\right), \quad u \to \infty. \tag{2.2}$$

For such models the claim sizes $Y_i = e^{X_i}, i \ge 1$ have marginal distributions which are in general neither log-normal nor with tails which are proportional to those of log-normal rv's.

We state next our first result for $Y_{N:N}$ the maximal claim size among $Y_1 = e^{X_1}, \dots, Y_N = e^{X_N}$ and the random sum $S_N = \sum_{i=1}^N Y_i$; we set $Y_{0:0} = 0$ and $S_0 := 0$.

Theorem 2.1. *Let N be an integer-valued rv satisfying $\mathbb{E}\left((1+\delta)^N\right) < \infty$ for some $\delta > 0$. Let $X_i, i \ge 1$ be a sequence of rv's given by* (1.1) *with $Z_i, i \ge 0$ iid rv's and $\rho \in [0,1)$ some given constant. Suppose that* (2.1) *and* (2.2) *hold with $\max_{i\ge 1} c_i < \infty$. If further N is independent of $X_i, i \ge 1$, then*

$$\mathbb{P}(S_N > u) \sim \mathbb{P}(Y_{N:N} > u) \sim \mathbb{E}\left(\sum_{i=1}^N c_i\right) \frac{\mathcal{L}(u^{\rho^2})\mathcal{L}(u^{1-\rho^2})}{\sqrt{2\pi}\log u} \exp\Big(-\frac{(\log u)^2}{2}\Big), \quad u \to \infty. \tag{2.3}$$

Remarks: a) Clearly, if $Y = e^Z$ with Z an $N(0,1)$ rv (thus Y is a log-normal rv with $LN(0,1)$ distribution), then (2.1) holds with $\mathcal{L}(u) = 1, u > 0$.

b) If $\mathcal{L}(\cdot)$ in Theorem 2.1 is constant, then the tail asymptotic behaviour of S_N and $Y_{N:N}$ is not influenced by the value of the dependence parameter ρ, and hence as expected the principle of a single big jump holds. However, for non-constant $\mathcal{L}(\cdot)$ the dependence parameter ρ plays a crucial role in the tail asymptotics derived in (2.3). The reason for this is that by Lemma 3.1

$$\mathbb{P}(Y_i > u) \sim c_i \frac{\mathcal{L}(u^{\rho^2})\mathcal{L}(u^{1-\rho^2})}{\sqrt{2\pi}\log u} \exp\Big(-\frac{(\log u)^2}{2}\Big), \quad u \to \infty. \tag{2.4}$$

Hence also in this case the principle of a single big jump applies.

c) In the proof of Theorem 2.1 we can show $S_N \stackrel{d}{=} e^{\rho Z_0} e^{\sqrt{1-\rho^2} Z^\star}$ for some Z^* independent of Z_0 and then we apply Lemma 3.1. Here we want to mention that after proving (2.4) we can also apply Proposition 2.2 of Foss and Richards (2010) to determine the asymptotic of $\mathbb{P}(S_n > u)$ as $u \to \infty$. If we condition on Z_0 and set $\overline{F}(x) = \mathbb{P}(Y_1 > x)$, $B_i(x) = \{x : e^{\rho Z_0} \le x^\gamma\}$ for some $\gamma \in (\rho, 1)$ and define $h(x) = x^\xi$ with

$$1 - \frac{1}{2}\left(\frac{1-\gamma}{\sqrt{1-\rho^2}}\right) < \xi^2 < 1,$$

then it is straightforward to show that the conditions of Proposition 2.2 of Foss and Richards (2010) are met.

Our second result is for log-normal rv's where we remove the assumptions of equi-correlations. Specifically, we consider for each n claim sizes $Y_{1,n} = e^{X_{1,n}}, \dots, Y_{n,n} = e^{X_{n,n}}$, where $(X_{1,n}, \dots, X_{n,n})$ is a normal random vector with mean zero and covariance matrix $\Sigma^{(n)}$ which is a correlation matrix with entries $\sigma_{i,j}^{(n)}$. We shall assume that $\rho_{i,j}^n := \sigma_{i,j}^{(n)}$ is bounded by some sequence ρ_n and some $\rho \in (0,1)$, i.e.,

$$\rho_{i,j}^n \le \max(\rho_n, \rho), \quad n \ge 1 \tag{2.5}$$

for all $i \neq j$. Further, we suppose that the sequence $\rho_n, n \geq 1$ satisfies for some $c^* > 8$ and some $\eta > 0$

$$\rho_{n(u)} \leq 1 - \frac{c^* \log(\log(u))}{\log(u)}, \quad \text{with } n(u) = \left\lfloor (1+\eta) \frac{(\log(u))^2}{2\log(1+\delta)} \right\rfloor. \tag{2.6}$$

If for instance all $\rho_{i,j}^n$ are bounded, then clearly condition (2.6) is valid; it holds also if for some c large enough $\rho_n \leq 1 - c\log(n)/\sqrt{n}$.

We present next our final result.

Theorem 2.2. *Let $Y_{1,n}, \ldots, Y_{n,n}, n \geq 1$ be claim sizes as above being further independent of some integer-valued rv N which satisfies* (1.3) *for some $\delta > 0$. If further* (2.5) *holds with ρ_n satisfying* (2.6)*, then*

$$\mathbb{P}\left(\max_{1 \leq i \leq N} Y_{i,N} > u\right) \sim \mathbb{P}(S_N > u) \sim \frac{\mathbb{E}(N)}{\sqrt{2\pi}\log u} \exp\Big(-\frac{(\log u)^2}{2}\Big), \quad u \to \infty. \tag{2.7}$$

Remarks: a) Our second result in Theorem 2.2 shows that the principle of a single big jump still holds even if we allow for a more general dependence structure.

b) Kortschak (2012) derives second order asymptotic results for subexponential risks. Similar ideas as therein are utilised to derive second order asymptotic results for the aggregation of log-normal random vectors in Kortschak and Hashorva (2013,2014). In the setup of randomly weighted sums it is also possible to derive such results.

3. Proofs

We give next two lemmas needed in the proofs below. The first lemma is of some interest on its own, in particular it implies Lemma 2.3 in Farkas and Hashorva (2013) (see also Lemma 8.6 in Piterbarg (1996)).

Lemma 3.1. *Let $\mathcal{L}_i(\cdot), i = 1, 2$ be some regularly varying functions at infinity with index β_i. If Z_1, Z_2 are two independent rv such that $\mathbb{P}\left(e^{Z_i} > u\right) \sim \mathcal{L}_i(u)\Psi(\log(u)), i = 1, 2,$ then for any σ_1, σ_2 two positive constants*

$$\mathbb{P}\left(e^{\sigma_1 Z_1 + \sigma_2 Z_2} > u\right) \sim \sigma^2 e^{\frac{\sigma_1^2\sigma_2^2}{2\sigma^2}(\beta_1 - \beta_2)^2} \mathcal{L}_1(u^\gamma)\mathcal{L}_2(u^{1-\gamma})\Psi((\log u)/\sigma) \tag{3.1}$$

holds as $u \to \infty$, where $\gamma = \sigma_1^2/(\sigma_1^2 + \sigma_2^2)$ and $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$.

Proof of Lemma 3.1 Choose an $\alpha > 0$ such that

$$\frac{\sigma_1^2}{\sigma_2^2} < \frac{1+\alpha}{1-\alpha}.$$

Then for any $a > 0$ we have

$$\begin{aligned}
\frac{\mathbb{P}\left(e^{\sigma_1 Z_1 + \sigma_2 Z_2} > u, e^{\sigma_2 Z_2} \leq a\right)}{\mathbb{P}\left(e^{\sigma_1 Z_1 + \sigma_2 Z_2} > u, e^{\sigma_2 Z_2} > a\right)} \quad &\leq \quad \frac{\mathbb{P}\left(e^{\sigma_1 Z_1} > u/a\right)}{\mathbb{P}\left(e^{\sigma_1 Z_1} > u^\alpha\right)\mathbb{P}\left(e^{\sigma_2 Z_2} > u^{1-\alpha}\right)} \\
&\sim \quad \frac{\mathcal{L}_1((u/a)^{1/\sigma_1})\Psi(\frac{1}{\sigma_1}\log(u/a))}{\mathcal{L}_1(u^{\alpha/\sigma_1})\mathcal{L}_2(u^{(1-\alpha)/\sigma_1})\Psi(\frac{\alpha}{\sigma_1}\log(u))\Psi(\frac{1-\alpha}{\sigma_2}\log(u))}
\end{aligned}$$

$$\to \quad 0, \quad u \to \infty, \tag{3.2}$$

with Ψ the survival function of an $N(0,1)$ rv. With the same argument we get that for any $a > 0$ we have

$$\frac{\mathbb{P}\left(e^{\sigma_1 Z_1+\sigma_2 Z_2} > u, e^{\sigma_1 Z_1} \le a\right)}{\mathbb{P}\left(e^{\sigma_1 Z_1+\sigma_2 Z_2} > u, e^{\sigma_1 Z_1} > a\right)} \to 0, \quad u \to \infty,$$

and hence

$$\begin{aligned}\mathbb{P}\left(e^{\sigma_1 Z_1+\sigma_2 Z_2} > u\right) =& \mathbb{P}\left(e^{\sigma_1 Z_1+\sigma_2 Z_2} > u, e^{\sigma_1 Z_1} > a, e^{\sigma_2 Z_2} > a\right)\\ &+ \mathbb{P}\left(e^{\sigma_1 Z_1+\sigma_2 Z_2} > u, e^{\sigma_1 Z_1} \le a\right) + \mathbb{P}\left(e^{\sigma_1 Z_1+\sigma_2 Z_2} > u, e^{\sigma_2 Z_2} \le a\right)\\ \sim& \mathbb{P}\left(e^{\sigma_1 Z_1+\sigma_2 Z_2} > u, e^{\sigma_1 Z_1} > a, e^{\sigma_2 Z_2} > a\right), \quad u \to \infty. \end{aligned} \tag{3.3}$$

In view of (3.3) we have

$$\mathbb{P}\left(e^{\sigma_1 Z_1+\sigma_2 Z_2} > u\right) \quad \sim \quad \mathbb{P}\left(e^{\sigma_1 Z_1+\sigma_2 Z_2} > u, e^{\sigma_1 Z_1} > \xi, e^{\sigma_2 Z_2} > \xi\right), \quad u \to \infty.$$

Assume next without loss of generalty that $\sigma_1 \ge \sigma_2$. If H denotes the distribution of $e^{\sigma_1 Z_1}$, then for any $\xi > 0$ with $u > 2\xi$

$$\begin{aligned}\mathbb{P}\left(e^{\sigma_1 Z_1+\sigma_2 Z_2} > u, e^{\sigma_1 Z_1} > \xi, e^{\sigma_2 Z_2} > \xi\right) &= \mathbb{P}\left(e^{\sigma_1 Z_1+\sigma_2 Z_2} > u, u/\xi \ge e^{\sigma_1 Z_1} > \xi, e^{\sigma_2 Z_2} > \xi\right)\\ &\quad + \mathbb{P}\left(e^{\sigma_1 Z_1+\sigma_2 Z_2} > u, e^{\sigma_1 Z_1} > u/\xi, e^{\sigma_2 Z_2} > \xi\right)\\ &= \mathbb{P}\left(e^{\sigma_1 Z_1+\sigma_2 Z_2} > u, u/\xi \ge e^{\sigma_1 Z_1} > \xi\right) + \mathbb{P}\left(e^{\sigma_1 Z_1} > u/\xi, e^{\sigma_2 Z_2} > \xi\right)\\ &= \int_\xi^{u/\xi} \mathbb{P}\left(e^{\sigma_2 Z_2} > u/s\right)\, dH(s) + \mathbb{P}\left(e^{\sigma_1 Z_1} > u/\xi, e^{\sigma_2 Z_2} > \xi\right).\end{aligned}$$

For all u and ξ large enough

$$\int_\xi^{u/\xi} \mathbb{P}\left(e^{\sigma_2 Z_2} > u/s\right)\, dH(s) \ge \frac{1}{2}\mathbb{P}\left(e^{\sigma_2 Z_2} > u/\xi\right) \ge \mathbb{P}\left(e^{\sigma_1 Z_1} > u/\xi, e^{\sigma_2 Z_2} > \xi\right)$$

implying as $u \to \infty$

$$\int_\xi^{u/\xi} \mathbb{P}\left(e^{\sigma_2 Z_2} > u/s\right)\, dH(s) + \mathbb{P}\left(e^{\sigma_1 Z_1} > u/\xi, e^{\sigma_2 Z_2} > \xi\right) \quad \sim \quad \int_\xi^{u/\xi} \mathbb{P}\left(e^{\sigma_2 Z_2} > u/s\right)\, dH(s).$$

Further, since again the constant ξ can be chosen arbitrary large we get for $\gamma = \sigma_1^2/(\sigma_1^2+\sigma_2^2)$

$$\begin{aligned}&\int_\xi^{u/\xi} \mathbb{P}\left(e^{\sigma_2 Z_2} > u/s\right)\, dH(s)\\ &\sim \int_\xi^{u/\xi} \frac{\sigma_2^2 \mathcal{L}_2(u/s)}{\sqrt{2\pi\sigma_2^2}\log(u/s)} \exp\left(-\frac{(\log(u/s))^2}{2\sigma_2^2}\right) dH(s)\\ &= \frac{\sigma_2^2 \mathcal{L}_2\left(u^{1-\gamma}\right)}{\sqrt{2\pi\sigma_2^2}\log(u^{1-\gamma})} \int_{\xi u^{-\gamma}}^{\frac{1}{\xi}u^{1-\gamma}} \frac{\mathcal{L}_2\left(\frac{u^{1-\gamma}}{s}\right)}{\mathcal{L}_2\left(u^{1-\gamma}\right)} \frac{\log\left(u^{1-\gamma}\right)}{\log\left(\frac{u^{1-\gamma}}{s}\right)} \exp\left(-\frac{\left(\log\left(\frac{u^{1-\gamma}}{s}\right)\right)^2}{2\sigma_2^2}\right) dH(u^\gamma s)\end{aligned}$$

$$= \frac{(\sigma_1^2+\sigma_2^2)\mathcal{L}_2\left(u^{1-\gamma}\right)}{\sqrt{2\pi\sigma_2^2}\log(u)} \int_{\xi u^{-\gamma}}^{\frac{1}{\xi}u^{1-\gamma}} q(u,\gamma,s)\exp\left(-\frac{\left(\log\left(\frac{u^{1-\gamma}}{s}\right)\right)^2}{2\sigma_2^2}\right) dH(u^\gamma s),$$

with $q(u,\gamma,s) = \frac{\mathcal{L}_2\left(\frac{u^{1-\gamma}}{s}\right)}{\mathcal{L}_2(u^{1-\gamma})}\frac{\log\left(u^{1-\gamma}\right)}{\log\left(\frac{u^{1-\gamma}}{s}\right)}$. For some $c>0$, by the uniform convergence theorem for regularly varying functions (see Theorem A3.2 in Embrechts et al. (1997)) we get uniformly in $1/c < s < c$

$$\lim_{u\to\infty} q(u,\gamma,s) = s^{-\beta_2}.$$

Further note that in the light of Potter's bound (see Bingham et al. (1987)) for every $\epsilon>0$ and $A>1$ we can find a positive constant ξ such that for all $\xi u^{-\gamma} < s < \frac{1}{\xi}u^{1-\gamma}$

$$\frac{1}{A}s^{-\beta_2}\min(s^\epsilon, s^{-\epsilon}) \le q(u,\gamma,s) \le A s^{-\beta_2}\max(s^\epsilon, s^{-\epsilon}).$$

Consequently, for different values of $0<a<b$ (that might depend on u) and β we want to find the asymptotics of

$$\begin{aligned}
&\int_a^b s^\beta \exp\left(-\frac{\left(\log\left(\frac{u^{1-\gamma}}{s}\right)\right)^2}{2\sigma_2^2}\right) dH(u^\gamma s)\\
&= -s^\beta \exp\left(-\frac{\left(\log\left(\frac{u^{1-\gamma}}{s}\right)\right)^2}{2\sigma_2^2}\right)\mathbb{P}\left(e^{\sigma_1 Z_1} > u^\gamma s\right)\Bigg|_{s=a}^{b}\\
&\quad + \int_a^b s^{\beta-1}\left(\beta + \frac{\log\left(\frac{u^{1-\gamma}}{s}\right)}{\sigma_2^2}\right)\exp\left(-\frac{\left(\log\left(\frac{u^{1-\gamma}}{s}\right)\right)^2}{2\sigma_2^2}\right)\mathbb{P}\left(e^{\sigma_1 Z_1} > u^\gamma s\right)\, ds.
\end{aligned}$$

Since we can choose ξ arbitrary large we can replace $\mathbb{P}\left(e^{\sigma_1 Z_1} > u^\gamma s\right)$ by its asymptotic form and hence we can use the approximation (set $\sigma_* := \sigma_1\sigma_2/\sqrt{\sigma_1^2+\sigma_2^2}$)

$$\begin{aligned}
&\exp\left(-\frac{\left(\log\left(\frac{u^{1-\gamma}}{s}\right)\right)^2}{2\sigma_2^2}\right)\mathbb{P}\left(e^{\sigma_1 Z_1} > u^\gamma s\right)\\
&\approx \sigma_1^2\frac{\mathcal{L}_1(u^\gamma s)}{\sqrt{2\pi\sigma_1^2}\log(u^\gamma s)}\exp\left(-\frac{\left(\log\left(\frac{u^{1-\gamma}}{s}\right)\right)^2}{2\sigma_2^2} - \frac{(\log(u^\gamma s))^2}{2\sigma_1^2}\right)\\
&= \sigma_1^2\frac{\mathcal{L}_1(u^\gamma s)}{\sqrt{2\pi\sigma_1^2}\log(u^\gamma s)}\exp\left(-\frac{\left(\sigma_1^2(1-\gamma)^2+\sigma_2^2\gamma^2\right)(\log(u))^2 + 2\left(\sigma_1^2(\gamma-1)+\sigma_2^2\gamma\right)\log(u)\log(s) + (\sigma_1^2+\sigma_2^2)(\log(s))^2}{2\sigma_1^2\sigma_2^2}\right)\\
&= \sigma_1^2\frac{\mathcal{L}_1(u^\gamma s)}{\sqrt{2\pi\sigma_1^2}\log(u^\gamma s)}\exp\left(-\frac{(\log(u))^2}{2(\sigma_1^2+\sigma_2^2)}\right)\exp\left(-\frac{(\log(s))^2}{2\sigma_*^2}\right).
\end{aligned}$$

Since $\sigma_1^2(\gamma-1)+\sigma_2^2\gamma = 0$, using again Potter's bounds (see Bingham et al. (1987)) and the fact that $\mathcal{L}_1(\cdot)$ is regularly varying at infinity, the above derivations imply

$$\begin{aligned}
&\mathbb{P}(e^{\sigma_1 Z_1+\sigma_2 Z_2} > u)\\
&\sim \frac{\sigma_1^2(\sigma_1^2+\sigma_2^2)\mathcal{L}_1(u^\gamma)\mathcal{L}_2(u^{1-\gamma})}{\sigma_2^2\sqrt{2\pi\sigma_2^2\sigma_1^2}\log(u)}\frac{1-\gamma}{\gamma\sqrt{2\pi}}\exp\left(-\frac{(\log(u))^2}{2(\sigma_1^2+\sigma_2^2)}\right)\int_0^\infty s^{\beta_1-\beta_2-1}\exp\left(-\frac{(\log(s))^2}{2\sigma_*^2}\right) ds
\end{aligned}$$

$$= \frac{\sqrt{\sigma_1^2+\sigma_2^2}\mathcal{L}_1(u^\gamma)\mathcal{L}_2(u^{1-\gamma})}{\sqrt{2\pi}\log(u)} \exp\left(-\frac{(\log(u))^2}{2(\sigma_1^2+\sigma_2^2)}\right) \int_0^\infty \frac{1}{\sqrt{2\pi\sigma_*^2}} s^{\beta_1-\beta_2-1} \exp\left(-\frac{(\log(s))^2}{2\sigma_*^2}\right) ds$$

$$= \sqrt{\sigma_1^2+\sigma_2^2} e^{\frac{\sigma_*^2}{2}(\beta_1-\beta_2)^2} \frac{\mathcal{L}_1(u^\gamma)\mathcal{L}_2(u^{1-\gamma})}{\sqrt{2\pi}\log(u)} \exp\left(-\frac{(\log(u))^2}{2(\sigma_1^2+\sigma_2^2)}\right),$$

hence the proof is complete. □

Lemma 3.2. *Assume that $n \le n(u)$ with $n(u)$ defined in* (2.6) *and set $\epsilon(u) = 4\log(\log(u))/\log(u)$. If Y_1 is an $LN(0,1)$ rv and $X_{i,n}, i \le n$ are as in Theorem 2.2, then as $u \to \infty$*

$$\mathbb{P}(Y_1 > u - nu^{1-\epsilon(u)}) \sim \mathbb{P}(Y_1 > u)$$

and for $i \neq j$

$$\mathbb{P}(Y_{i,n} > u^{1-\epsilon(u)}, Y_{j,n} > u^{1-\epsilon(u)}) = o(\mathbb{P}(Y_1 > u)).$$

Proof of Lemma 3.2 By the assumptions on n and $n(u)$ as $u \to \infty$ we have

$$\begin{aligned}
\mathbb{P}(Y_1 > u) \le \mathbb{P}(Y_1 > u - nu^{1-\epsilon(u)}) &\le \mathbb{P}(Y_1 > u - n(u)u^{1-\epsilon(u)}) \\
&= \mathbb{P}\left(Y_1 > u - \frac{u}{(\log(u))^4}(1+\eta)\frac{(\log(u))^2}{2\log(1+\delta)}\right) \\
&= \mathbb{P}\left(Y_1 > u - \frac{(1+\eta)}{2\log(1+\delta)}\frac{u}{(\log(u))^2}\right) \\
&\sim \mathbb{P}(Y_1 > u).
\end{aligned}$$

Next, denote by f the probability density function of Y_1. Let further W_1 and W_2 be two independent $N(0,1)$ rv's, and write ρ_* for the correlation between $\log Y_{i,n}$ and $\log Y_{j,n}$. We may write for $u > 0$

$$\begin{aligned}
\mathbb{P}(Y_{i,n} > u^{1-\epsilon(u)}, Y_{j,n} > u^{1-\epsilon(u)}) &= \mathbb{P}(e^{W_1} > u^{1-\epsilon(u)}, e^{\rho_* W_1 + \sqrt{1-\rho_*^2}W_2} > u^{1-\epsilon(u)}) \\
&= \mathbb{P}\left(e^{W_1} > \frac{u}{(\log(u))^4}, e^{\rho_* W_1} e^{\sqrt{1-\rho_*^2}W_2} > \frac{u}{(\log(u))^4}\right) \\
&\le \mathbb{P}\left(\frac{u}{(\log(u))^4} < e^{W_1} < 2u, e^{\rho_* W_1} e^{\sqrt{1-\rho_*^2}W_2} > \frac{u}{(\log(u))^4}\right) + \mathbb{P}(e^{W_1} > 2u) \\
&= \int_{\frac{u}{(\log(u))^4}}^{2u} \mathbb{P}\left(e^{W_2} > \left(\frac{u}{(\log(u))^4 x^{\rho_*}}\right)^{1/\sqrt{1-\rho_*^2}}\right) f(x)dx + \mathbb{P}(e^{W_1} > 2u) \\
&\le \int_{\frac{u}{(\log(u))^4}}^{2u} \mathbb{P}\left(e^{W_2} > \left(\frac{u^{1-\rho_*}}{(\log(u))^4 2^{\rho_*}}\right)^{1/\sqrt{1-\rho_*^2}}\right) f(x)dx + \mathbb{P}(e^{W_1} > 2u) \\
&\le \mathbb{P}\left(Y_1 > \frac{u^{\sqrt{\frac{1-\rho_*}{1+\rho_*}}}}{2^{\rho_*}(\log(u))^4}\right) \mathbb{P}\left(Y_1 > \frac{u}{(\log(u))^4}\right) + \mathbb{P}(e^{W_1} > 2u) \\
&= o(\mathbb{P}(Y_1 > u)), \quad u \to \infty
\end{aligned}$$

since

$$\begin{aligned}\left(1+\frac{1-\rho_*}{1+\rho_*}\right)\log(u) &= \frac{2}{1+\rho_*}\log(u)\\ &\geq \frac{2}{1+\rho_{n(u)}}\log(u)\\ &\geq \frac{2}{2-\frac{c^*\log(\log(u))}{\log(u)}}\log(u)\\ &= \log(u)+\frac{2}{2-\frac{c^*\log(\log(u))}{\log(u)}}c^*\log(\log(u))\\ &\sim \log(u)+c^*\log(\log(u)).\end{aligned}$$

Consequently, the assumption $c^* > 8$ entails

$$\begin{aligned}&2\log\left(\mathbb{P}\left(Y_1>\frac{u^{\sqrt{\frac{1-\rho_*}{1+\rho_*}}}}{2^{\rho_*}(\log(u))^4}\right)\mathbb{P}\left(Y_1>\frac{u}{(\log(u))^4}\right)\right)\\ &\sim \log\left(\frac{u^{\sqrt{\frac{1-\rho_*}{1+\rho_*}}}}{2^{\rho_*}(\log(u))^4}\right)^2+\log\left(\frac{u}{(\log(u))^4}\right)^2\\ &\sim \left(1+\frac{1-\rho_*}{1+\rho_*}\right)\log(u)^2-8\log(u)\log(\log(u))-8\sqrt{\frac{1-\rho_*}{1+\rho_*}}\log(u)\log(\log(u))\\ &\lesssim \log(u)^2+(c^*-8)\log(\log(u))\end{aligned}$$

establishing the proof. □

Proof of Theorem 2.1 For any $u > 0$ we have

$$\begin{aligned}\mathbb{P}(S_N>u) &= \mathbb{P}\left(e^{\rho Z_0}\sum_{i=1}^{N}e^{\sqrt{1-\rho^2}Z_i}>u\right)\\ &=: \mathbb{P}\left(e^{\rho Z_0}W_N>u\right).\end{aligned}$$

Since $e^{\sqrt{1-\rho^2}Z_i}, i \geq 1$ are subexponential risks, then along the lines of the proof of Theorem 3.37 in Foss et al. (2013) (see also for similar result Theorem 1.3.9 in Embrechts et al. (1997))

$$\mathbb{P}(W_N>u)\sim\Theta\mathbb{P}\left(e^{\sqrt{1-\rho^2}Z^*}>u\right),\quad \Theta:=\mathbb{E}\left(\sum_{i=1}^{N}c_i\right)$$

as $u \to \infty$, with Z^* an independent copy of Z_0. It can be easily checked that Z_0 and $\log(W_N)/(1-\rho^2)$ fulfill the conditions of Lemma 3.1, hence the asymptotic of $\mathbb{P}(S_N > u)$ follows. Similarly,

$$Y_{N:N} = \max_{1\leq i\leq N} e^{\rho Z_0+\sqrt{1-\rho^2}Z_i} = e^{\rho Z_0}\max_{1\leq i\leq N} e^{\sqrt{1-\rho^2}Z_i} =: e^{\rho Z_0}W_N^*.$$

Since we have

$$\mathbb{P}(W_N^*>u)\sim\mathbb{P}(W_N>u)\sim\Theta\mathbb{P}\left(\exp(\sqrt{1-\rho^2}Z^*)>u\right),\quad u\to\infty$$

the proof follows by applying once again Lemma 3.1. □

Proof of Theorem 2.2 Denote next Y_1 an $LN(0,1)$ rv and let $\mathcal{I}_{\{\cdot\}}$ denote the indicator function. Since for all fixed $n \ge 1$ we get by interchanging limit and finite sum that

$$\begin{aligned}\mathbb{P}(S_N > u) &= \mathbb{P}(S_N > u, N \le n) + \mathbb{P}(S_N > u, N > n)\\ &\sim \mathbb{E}\left(N\mathcal{I}_{\{N \le n\}}\right)\mathbb{P}(Y_1 > u) + \mathbb{P}(S_N > u, N > n)\end{aligned}$$

we can assume w.l.o.g. that $\rho_{i,j}^n \le \rho_n$. From (1.3) it follows that there exist $C_1, C_2 > 0$ such that

$$p_n := \mathbb{P}(N = n) \le C_1(1+\delta)^{-n} \quad \text{and} \quad \mathbb{P}(N > n) \le C_2(1+\delta)^{-n}.$$

By the independence of N and the claim sizes

$$\mathbb{P}(S_N > u) = \sum_{n=1}^{\infty} p_n \mathbb{P}(S_n > u)$$

and for $n(u)$ defined in (2.6)

$$\begin{aligned}\sum_{n=n(u)}^{\infty} p_n \mathbb{P}(S_n > u) &\le \mathbb{P}(N > n(u))\\ &\le C_2(1+\delta)^{-n(u)}\\ &\le C_2 \exp\left(-\frac{1+\eta}{2}(\log(u))^2\right)\\ &= o(\mathbb{P}(Y_1 > u)).\end{aligned}$$

Since

$$\mathbb{P}(S_n > u) \ge n\mathbb{P}(Y_1 > u) - \sum_{i \ne j} \mathbb{P}(Y_i > u, Y_j > u)$$

and by Lemma 3.2

$$\mathbb{P}(Y_i > u, Y_j > u) = o(\mathbb{P}(Y_1 > u)), \quad u \to \infty$$

it follows that

$$\begin{aligned}\sum_{n=0}^{n(u)} p_n \mathbb{P}(S_n > u) &\ge \mathbb{P}(Y_1 > u)\left(\sum_{n=0}^{n(u)} np_n - o(1)\sum_{n=0}^{n(u)} n^2 p_n\right)\\ &\sim \mathbb{E}(N)\mathbb{P}(Y_1 > u), \quad u \to \infty.\end{aligned}$$

So we are left with finding an asymptotic upper bound. For $n \le n(u)$ we use the following decomposition (c.f. Asmussen and Rojas-Nandaypa (2008))

$$\mathbb{P}(S_n > u) = \sum_{i=1}^{n} \mathbb{P}\left(S_n > u, Y_{i,n} \ge Y_{j,n}, \max_{j \ne i} Y_{j,n} > u^{1-\epsilon(u)}\right) + \mathbb{P}\left(S_n > u, Y_{i,n} \ge Y_{j,n}, \max_{j \ne i} Y_{j,n} \le u^{1-\epsilon(u)}\right),$$

where $\epsilon(u) = 4\log(\log(u))/\log(u)$. By Lemma 3.2 we have

$$\sum_{i=1}^{n} \mathbb{P}\left(S_n > u, Y_{i,n} \ge Y_{j,n}, \max_{j \ne i} Y_{j,n} > u^{1-\epsilon(u)}\right) \le \sum_{i=1}^{n}\sum_{j \ne i} \mathbb{P}\left(S_n > u, Y_{i,n} \ge Y_{j,n}, Y_{j,n} > u^{1-\epsilon(u)}\right)$$

$$\leq \sum_{i=1}^{n} \sum_{j \neq i} \mathbb{P}\left(Y_{i,n} > u^{1-\epsilon(u)}, Y_{j,n} > u^{1-\epsilon(u)}\right)$$

$$= n(n-1) o(\mathbb{P}(Y_1 > u)).$$

Further

$$\begin{aligned}
\mathbb{P}\left(S_n > u, Y_{i,n} \geq Y_{j,n}, \max_{j \neq i} Y_{j,n} \leq u^{1-\epsilon(u)}\right) &\leq \mathbb{P}\left(Y_{i,n} > u - \sum_{i=1}^{n} Y_{j,n}, \max_{j \neq i} Y_{j,n} \leq u^{1-\epsilon(u)}\right) \\
&\leq \mathbb{P}\left(Y_{i,n} > u - n u^{1-\epsilon(u)}, \max_{j \neq i} Y_{j,n} \leq u^{1-\epsilon(u)}\right) \\
&\leq \mathbb{P}\left(Y_{i,n} > u - n u^{1-\epsilon(u)}\right) \\
&\sim \mathbb{P}(Y_1 > u)
\end{aligned}$$

as $u \to \infty$, hence the proof for the tail asymptotics of S_N follows by applying (3.2). Since for any $u > 0$

$$n\mathbb{P}(Y_1 > u) - \sum_{i \neq j} \mathbb{P}(Y_i > u, Y_j > u) \leq \mathbb{P}\left(\max_{1 \leq i \leq n} Y_{i,n} > u\right) \leq \mathbb{P}(S_n > u)$$

the tail asymptotics of $\max_{1 \leq i \leq N} Y_{i,N}$ can be easily established, and thus the proof is complete. □

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