LUNDBERG'S RISK PROCESS WITH TAX

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Abstract

In this paper we extend the classical Cramér-Lundberg risk model by including tax payments. The considered tax rule is to pay a certain proportion of the premium income, whenever the portfolio is in a profitable situation. It is shown that the resulting survival probability is a power of the survival probability without tax. Furthermore, an explicit expression for the expected discounted total sum of tax payments until ruin according to this taxation rule is derived and the optimal starting level for taxation is determined. Finally, numerical illustrations of the results are given for the case of exponential claim amounts.

1 Introduction and Summary

We consider an insurance company with insurance risk modeled by a classical Lundberg process $R_0(t) = s+ct-S(t)$, in which the process $S(t)$ is compound Poisson with intensity λ and claim size distribution F, and the premium intensity $c > 0$ has a positive safety loading:

$$c > \lambda\mu,$$

where μ is the mean claim size. For initial surplus $s \geq 0$ let $\psi_0(s)$ be the infinite time ruin probability

$$\psi_0(s) = \mathbb{P}\{s + ct < S(t) \text{ for some } t \geq 0\}.$$

We investigate how tax influences the qualitative and quantitative behavior of the infinite time ruin probability. We assume that tax is paid at a fixed rate γ of the insurers income (premia) whenever he is in a profitable situation: he is in a profitable situation at time t if for his risk process R_γ with tax we have

$$R_\gamma(t) = \max\{R_\gamma(u) : \ u \leq t\}.$$

So the insurer will pay a tax rate $c\gamma$ at profitable times, and zero at times without profit (loss carried forward system). Our premium income is reduced by tax payment from c – which is collected at times without profit – to $c(1-\gamma)$ at times with profit.

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Our model is far from being realistic: carried forward losses are used in real world, but in addition one can use claims reserves (IBNR and RBNS) as well as equalization reserves to reduce tax liability. We have chosen not to include these additional specifications here to keep the model simple and tractable. The same is true for other specifications which are not included: investment returns, dividend payment, expenses for acquisition, and others.

If $\gamma \geq 1$ in our model then ruin will be certain for the insurer (note that for $\gamma = 1$ the resulting risk process is identical to one with horizontal dividend barrier strategy where the initial capital is at the barrier, see for instance Bühlmann [2, Ch.6.4]). A priori it is not clear whether ruin is certain for $\gamma < 1$ also. We will show in this paper that for $\gamma < 1$ the ruin probability is less than one and, in particular, is changed from $\psi_0(s)$ to

$$\psi_\gamma(s) = 1 - (1 - \psi_0(s))^{1/(1-\gamma)}. \tag{1}$$

Asymptotically, the ruin probability hence grows by the factor $1/(1-\gamma)$,

$$\psi_\gamma(s) \sim \frac{1}{1-\gamma}\psi_0(s). \tag{2}$$

In a situation with tax, a higher initial surplus is needed to have the same ruin probability, compared to the case without tax. So, for light-tailed claims admitting the adjustment coefficient R, the additional capital needed is asymptotically equal to

$$-\frac{1}{R}\log(1-\gamma).$$

As another example, for Pareto claim sizes with tail index α we need an additional capital which is asymptotically equal to

$$\left((1-\gamma)^{-1/(\alpha-1)} - 1\right)s.$$

With cost of capital ν we obtain a higher premium rate which is the result of tax payments. If c is the original net premium rate and s is the initial capital, then the gross premium rate equals $c + \nu s$. For a tax rate $\gamma > 0$ we obtain an asymptotic premium rate of

$$c + \nu s - \nu\frac{1}{R}\log(1-\gamma)$$

and

$$c + \nu(1-\gamma)^{-1/(\alpha-1)}s,$$

respectively.
The expected accumulated discounted tax for given initial surplus s is given by

$$v(s) = E\left[\int_0^\tau e^{-\delta t}\gamma(t)dt\right], \tag{3}$$

where τ is the time of ruin, $\delta > 0$ is the discount rate, and $\gamma(t)$ is the tax rate paid at time t which equals $c\gamma$ at profitable times and zero elsewhere. The quantity $v(s)$ allows to evaluate the collected tax in the given tax system. In Section 2.2 we shall give an explicit formula for the function $v(s)$. Moreover, in Section 2.3 we discuss how the optimal surplus level can be determined from which on the tax authority should collect taxes in order to maximize $v(s)$. Finally, in Section 3 numerical illustrations of the results are given for exponential claim amounts.

2 Results and Proofs

The risk process $R_\gamma(t)$ with tax, under a loss written forward system, evolves as follows: If s is the initial surplus, then we have a period with profit in which the premium rate is reduced to $c(1-\gamma)$ until the first claim of size X_1 at time W_1. The gains level is set to $M_1 = s + c(1-\gamma)W_1$. Then there is a period without profit in which the premium rate is c until the risk process reaches M_1 again, at time σ_1, say. Then we have a period with profit until the first claim after time σ_1, which happens at $\sigma_1 + W_2$ and has size X_2, say. The new gains level is set to $M_2 = s + c(1-\gamma)(W_1 + W_2)$, and so on (cf. Figure 1).

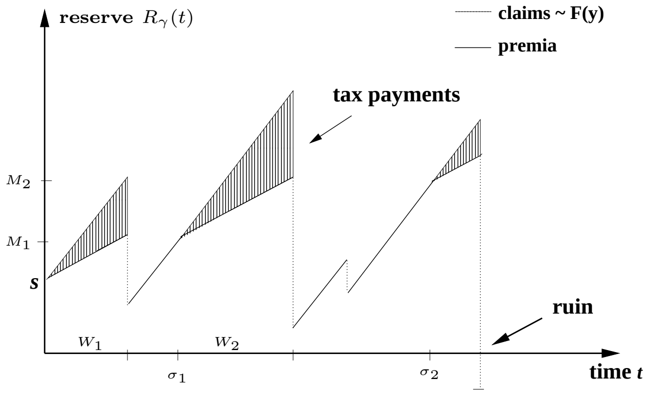


Figure 1: The risk process $R_\gamma(t)$

Let $N(t)$ be the underlying claims arrival process, and assume for the moment that the process $R_\gamma(t)$ is not stopped at its time of ruin. We thus obtain a sequence of gains levels M_n, waiting times W_n, claim sizes X_n and starting times of periods with profit σ_n defined formally as

$$\begin{aligned}
\sigma_0 &= 0, M_0 = s, \\
W_n &= \inf\{t > 0 : N(\sigma_{n-1} + t) > N(\sigma_{n-1})\}, \\
X_n &= R_\gamma(\sigma_{n-1} + W_n -) - R_\gamma(\sigma_{n-1} + W_n), \\
M_n &= M_{n-1} + c(1-\gamma)W_n, \\
\sigma_n &= \inf\{t > \sigma_{n-1} + W_n : R_\gamma(t) = M_n\}, n \geq 1.
\end{aligned}$$

The time intervals with profit are $(\sigma_{n-1}, \sigma_{n-1} + W_n), n \geq 1$. The intervals without profit are

$$C_n = (\sigma_{n-1} + W_n, \sigma_n), n \geq 1.$$

Ruin happens for the process $R_\gamma(t)$ only if $R_\gamma(t) < 0$ for some $t \in C_n, n \geq 1$. The waiting times $W_1, W_2, \ldots$ are independent with distribution $Exp(\lambda)$ – due to the memoryless property of the exponential distribution. So $\sigma_n \to \infty$ and $M_n \to \infty$ almost surely.

2.1 The ruin probability

Theorem 1. *In case of a positive safety loading $c > \lambda\mu$ and $\gamma < 1$ we have $\psi_\gamma(s) < 1$ for all $s \geq 0$. In particular, in this case*

$$1 - \psi_\gamma(s) = (1 - \psi_0(s))^{1/(1-\gamma)}.$$

Proof. Write $\phi_0(s) = 1 - \psi_0(s)$. For $0 \le s < M$ let

$$g(s, M) = \mathbb{P}\{R_0(t) \text{ reaches } M \text{ before ruin}|R_0(0) = s\}.$$

Then $\phi_0(s) = g(s, M)\phi_0(M)$ yields

$$g(s, M) = \phi_0(s)/\phi_0(M). \tag{4}$$

For $\phi_\gamma(s) = 1 - \psi_\gamma(s)$ we have

$$\begin{aligned} \phi_\gamma(s) &= E[1_{\{\text{ no ruin in } C_1\}}\phi_\gamma(M_1)] \\ &= E[g(M_1 - X_1, M_1)\phi_\gamma(M_1)] \\ &= E[\phi_0(M_1 - X_1)/\phi_0(M_1)\phi_\gamma(M_1)]; \end{aligned}$$

with $\sigma_n \to \infty$ we obtain

$$\phi_\gamma(s) = E[\prod_{n=1}^{\infty} \phi_0(M_n - X_n)/\phi_0(M_n)].$$

Since $X_1, X_2, ...$ and $W_1, W_2, ...$ are independent, we have

$$\phi_\gamma(s) = E[\prod_{n=1}^{\infty} g_0(M_n)/\phi_0(M_n)],$$

where $g_0(x) = E[\phi_0(x - X)]$. From the integro-differential equation for $\phi_0(s)$

$$0 = \lambda E[\phi_0(s - X) - \phi_0(s)] + c\phi_0'(s), \quad s \ge 0, \tag{5}$$

we obtain for $x \ge 0$

$$g_0(x) = \phi_0(x) - \frac{c}{\lambda}\phi_0'(x),$$

and with $f(x) = (c/\lambda)\phi_0'(s + x)/\phi_0(s + x)$ we arrive at

$$\phi_\gamma(s) = E\left[\prod_{n=1}^{\infty}(1 - f(c(1 - \gamma)(W_1 + ... + W_n)))\right].$$

The variables $c(1-\gamma)W_n, n = 1, 2, ...$ are independent exponential with parameter $\lambda/(c(1-\gamma))$, and the function $0 \le f(x) \le 1$ is non-negative and integrable; with Lemma A.1 below we hence obtain

$$\phi_\gamma(s) = \exp\left(-\frac{\lambda}{c(1-\gamma)}\int_0^\infty f(x)dx\right) = \exp\left(\frac{1}{(1-\gamma)}\log\phi_0(s)\right) = \phi_0(s)^{1/(1-\gamma)}.$$

□

Remark: At first sight, the following simpler proof of Theorem 1 seems appropriate: By conditioning on the occurrence of the first claim, we have

$$\phi_\gamma(s) = \int_0^\infty \lambda e^{-\lambda t}\, dt \int_0^{s+c(1-\gamma)t} g(s+c(1-\gamma)t-y, s+c(1-\gamma)t)\ \phi_\gamma(s+c(1-\gamma)t)\, dF(y). \tag{6}$$

Changing variables $w = s + c(1-\gamma)t$ gives

$$\phi_\gamma(s) = e^{\frac{\lambda\, s}{c(1-\gamma)}} \int_s^\infty \frac{\lambda}{c(1-\gamma)} e^{-\frac{\lambda\, w}{c(1-\gamma)}}\, dw \int_0^w g(w-y,w)\, \phi_\gamma(w)\, dF(y).$$

Differentiating with respect to s leads to

$$\phi_\gamma'(s) = \frac{\lambda}{c(1-\gamma)}\, \phi_\gamma(s) - \frac{\lambda}{c(1-\gamma)} \int_0^s g(s-y,s)\, \phi_\gamma(s)\, dF(y).$$

Now, using (4) and (5) one obtains

$$\frac{c(1-\gamma)}{\lambda}\phi_\gamma'(s) - \phi_\gamma(s) + \phi_\gamma(s)\, \left(1 - \frac{c}{\lambda}\frac{\phi_0'(s)}{\phi_0(s)}\right) = 0$$

and hence

$$(1-\gamma)\frac{\phi_\gamma'(s)}{\phi_\gamma(s)} = \frac{\phi_0'(s)}{\phi_0(s)},$$

i.e.

$$\phi_\gamma(s) = C\, \phi_0(s)^{1/(1-\gamma)}$$

for some constant C. If the limit $s \to \infty$ on both sides equals one, then $C = 1$ and hence the result would follow. However, this approach does not rule out the possibility $\phi_\gamma(s) = 0\ \forall\, s \geq 0$ (which would also represent a solution of (6)), whereas the proof given above does. But in the next section this conditioning procedure will turn out to be the appropriate tool for establishing an explicit formula for $v(s)$.

2.2 The expected discounted total tax payment

Let $B(s,b) := \mathbb{E}[e^{-\delta\tau^+(s,0,b)}]$ denote the Laplace transform of the upper exit time $\tau^+(s,0,b)$, which is the time until the classical risk process $R_0(t)$ (with premium rate c) starting with initial capital $s < b$ reaches b without leading to ruin before that event. Clearly $\tau^+(s,0,b)$ is a defective random variable. The quantity $B(s,b)$ will play a crucial role later on. It follows for instance from Gerber & Shiu [4], that $B(s,b)$ can be written as

$$B(s,b) = \frac{h(s)}{h(b)},$$

where the function $h(s)$ is the solution of the integro-differential equation

$$c\, h'(s) - (\lambda+\delta)\, h(s) + \lambda \int_0^s h(s-y)\, dF(y) = 0,$$

which is uniquely determined up to a constant (in the sequel we assume w.l.o.g. that $h(s) \geq 0$ for all $s \geq 0$). For instance, $h(s)$ can be explicitly expressed as

$$h(s) = e^{\rho s} - q_\delta(s),$$

where $\rho > 0$ is the unique positive solution in R of the Lundberg fundamental equation

$$cR - \lambda - \delta + \lambda \int_0^\infty e^{-Ry}\, dF(y) = 0,$$

and $q_\delta(s) = \mathbb{E}(e^{-\delta\tau+\rho R_0(\tau)}\, 1_{\{\tau<\infty\}}|R_0(0) = s)$, with τ denoting the time of ruin in the classical risk model (cf. Gerber & Shiu [4]). The quantity $q_\delta(s)$ can be interpreted as the present value of a payment of 1 at the time of recovery after the event of ruin or alternatively as a discounted penalty function with penalty $w(x,y) = e^{-\rho y}$.
Let further $V(s,b)$ denote the expected discounted dividend payments in the classical Cramér-Lundberg model with premium rate c, horizontal barrier b, discount rate $\delta > 0$ and initial capital $s < b$. It follows from Bühlmann [2, p.172] that $V(s,b)$ can be written as

$$V(s,b) = \frac{h(s)}{h'(b)}, \quad 0 \le s \le b, \tag{7}$$

and hence we obtain the classical identity

$$B(s,b) = \frac{V(s,b)}{V(b,b)}. \tag{8}$$

Although it is not necessary to use representation (8) for $B(s,b)$ in the sequel, it turns out to make some relations more transparent. Moreover, since for $\gamma = 1$ (i.e. all incoming premia are paid out as taxes) the risk process with tax payments is identical to the risk process with horizontal barrier strategy with barrier $b = s$, it is somewhat natural to express the results for $v(s)$ in terms of the function V. Note that since $\lim_{u\to\infty} q_\delta(u) = 0$ and also $\lim_{u\to\infty} q'_\delta(u) = 0$ (cf. [4]), one immediately deduces from (7) that

$$\lim_{s\to\infty} V(s,s) = \frac{1}{\rho}$$

(an alternative derivation of this limiting result goes back to Bühlmann [2, p.176]).

Let us now turn to the derivation of an explicit expression for $v(s)$.

Theorem 2. *Using the notation above, the expected discounted tax payments with initial capital s are given by*

$$v(s) = \frac{\gamma}{1-\gamma}\, e^{\int_0^s \frac{d\xi}{V(\xi,\xi)\,(1-\gamma)}} \int_s^\infty e^{-\int_0^t \frac{d\xi}{V(\xi,\xi)\,(1-\gamma)}}\, dt$$

or equivalently

$$v(s) = \frac{\gamma}{1-\gamma}\, h(s)^{\frac{1}{1-\gamma}} \int_s^\infty h(t)^{-\frac{1}{1-\gamma}}\, dt. \tag{9}$$

Proof: Let us first derive simple bounds for $v(s)$. Assuming that no claim occurs at all we get $v(s) \le \int_0^\infty c\,\gamma\, e^{-\delta t}\, dt = \frac{c\gamma}{\delta}$. On the other hand, if the first claim leads to ruin already, tax can only be collected until the time of this claim implying $v(s) \ge \int_0^\infty \lambda\, e^{-\lambda t}\, dt \int_0^t c\gamma\, e^{-\delta\xi}\, d\xi = \frac{c\gamma}{\lambda+\delta}$. Hence

$$\frac{c\gamma}{\lambda+\delta} \le v(s) \le \frac{c\gamma}{\delta}, \qquad \forall s \ge 0. \tag{10}$$

By conditioning on the occurrence time and size of the first claim we get

$$v(s) = \int_0^\infty \lambda e^{-\lambda t}\, dt \left(\int_0^t c\,\gamma e^{-\delta\xi}\, d\xi \right.$$
$$\left. + e^{-\delta t} \int_0^{s+c(1-\gamma)t} B(s+c(1-\gamma)t-y, s+c(1-\gamma)t)\; v(s+c(1-\gamma)t)\, dF(y) \right). \tag{11}$$

Changing variables $w = s + c(1-\gamma)t$ leads to

$$v(s) = \frac{c\gamma}{\lambda+\delta} + \int_s^\infty \lambda e^{-\frac{(\lambda+\delta)}{c(1-\gamma)}(w-s)} \frac{dw}{c(1-\gamma)} \int_0^w B(w-y,w)\, v(w)\, dF(y)$$

or equivalently

$$v(s) = \frac{c\gamma}{\lambda+\delta} + e^{\frac{(\lambda+\delta)\,s}{c(1-\gamma)}} \int_s^\infty \frac{\lambda}{c(1-\gamma)} e^{-\frac{(\lambda+\delta)}{c(1-\gamma)}\,w}\, v(w)\, dw \int_0^w B(w-y,w)\, dF(y).$$

We now substitute the identity (8) into the above equation to get

$$v(s) = \frac{c\gamma}{\lambda+\delta} + e^{\frac{(\lambda+\delta)\,s}{c(1-\gamma)}} \int_s^\infty \frac{\lambda}{c(1-\gamma)} e^{-\frac{(\lambda+\delta)}{c(1-\gamma)}\,w} \frac{v(w)}{V(w,w)}\, dw \int_0^w V(w-y,w)\, dF(y).$$

But from classical risk theory we know that

$$\lambda \int_0^w V(w-y,w)\, dF(y) = (\lambda+\delta)\, V(w,w) - c$$

and hence

$$v(s) = \frac{c\gamma}{\lambda+\delta} + e^{\frac{(\lambda+\delta)\,s}{c(1-\gamma)}} \int_s^\infty \frac{e^{-\frac{(\lambda+\delta)}{c(1-\gamma)}\,w}}{c(1-\gamma)}\, v(w) \left(\lambda+\delta - \frac{c}{V(w,w)}\right) dw. \tag{12}$$

Due to (10), $v(s)$ is bounded for all $s \ge 0$ and hence in particular the limit $v(\infty)$ is finite. Taking the limit $s \to \infty$ in (12) we obtain, using de'l Hopital's rule

$$v(\infty) = \frac{c\gamma}{\lambda+\delta} + v(\infty)\left(1 - \frac{c}{(\lambda+\delta)\, \lim_{s\to\infty} V(s,s)}\right)$$

or equivalently

$$v(\infty) = \gamma \lim_{s\to\infty} V(s,s) = \frac{\gamma}{\rho}. \tag{13}$$

Now we can differentiate (12) with respect to s, yielding

$$v'(s) = \frac{\lambda+\delta}{c(1-\gamma)}\left(v(s) - \frac{c\gamma}{\lambda+\delta}\right) - \frac{v(s)}{c(1-\gamma)}\left(\lambda+\delta - \frac{c}{V(s,s)}\right)$$

and further

$$v'(s) = \frac{v(s)}{(1-\gamma)V(s,s)} - \frac{\gamma}{1-\gamma} \tag{14}$$

with initial condition (13). The solution of this ordinary differential equation of first order is given by

$$v(s) = \left(C - \frac{\gamma}{1-\gamma} U_1(s)\right) e^{\frac{1}{1-\gamma}U(s)},$$

where

$$U(s) = \int_0^s \frac{d\xi}{V(\xi,\xi)}, \quad U_1(s) = \int_0^s e^{-\frac{U(t)}{1-\gamma}}\, dt,$$

and C is some constant. The latter can now be determined using condition (13). Since $V(s,s) > 0$ for $s \ge 0$ and $\lim_{s\to\infty} V(s,s) = \rho > 0$, the function $U(s)$ is unbounded in

s and correspondingly (13) forces $U_1(s)$ to converge to $\frac{1-\gamma}{\gamma}C$ for $s \to \infty$ (and indeed it can easily be checked that $\lim_{s\to\infty} U_1(s)$ is finite). After some algebraic manipulations we hence obtain

$$v(s) = \frac{\gamma}{1-\gamma}\, e^{\int_0^s \frac{d\xi}{V(\xi,\xi)\,(1-\gamma)}} \int_s^\infty e^{-\int_0^t \frac{d\xi}{V(\xi,\xi)\,(1-\gamma)}}\, dt.$$

Now, if we evaluate (7) at $u = b$, we see that $V(b,b) = h(b)/h'(b)$, so that $U(s)$ simplifies to $U(s) = \log \frac{h(s)}{h(0)}$ and we finally arrive at

$$v(s) = \frac{\gamma}{1-\gamma}\, h(s)^{\frac{1}{1-\gamma}} \int_s^\infty h(t)^{-\frac{1}{1-\gamma}}\, dt.$$

□

Remark: In terms of the function h, the differential equation (14) for $v(s)$ is given by

$$v'(s) = \frac{v(s)\,h'(s)}{(1-\gamma)h(s)} - \frac{\gamma}{1-\gamma}$$

with initial condition (13). For numerical purposes, when the evaluation of the exact solution (9) is not feasible, it may be helpful to start directly from the above differential equation.

On the other hand, in several cases explicit expressions for $V(b,b)$ are known which by the above theorem translate into explicit formulas the expected discounted sum of tax payments. For instance, whenever the claim size distribution has rational Laplace transform, the function $q_\delta(s)$ and subsequently $V(b,b)$ can be expressed analytically (see formula (6.54) in [4]). For illustration let us consider the case of exponential claim sizes.

Example 1: Let $F(y) = 1 - e^{-\alpha y}$. Then it is well known that

$$h(s) = (\alpha+\rho)\, e^{\rho s}\,(1-\eta(s)) \qquad \text{with} \quad \eta(s) = \frac{\alpha + r_2}{\alpha+\rho}\, e^{(r_2-\rho)s},$$

where $\rho > 0$ and $r_2 < 0$ are the two solutions of the Lundberg fundamental equation

$$c\,R^2 + (c\,\alpha - \lambda - \delta)\,R - \alpha\,\delta = 0$$

(see for instance Bühlmann [2] or Dickson & Waters [3]). Hence in this case we arrive at

$$v(s) = \frac{\gamma}{1-\gamma}\left(e^{\rho s}\,(1-\eta(s))\right)^{\frac{1}{1-\gamma}} \int_s^\infty \frac{dt}{\left(e^{\rho t}\,(1-\eta(t))\right)^{\frac{1}{1-\gamma}}}.$$

The change of variables $w = e^{(r_2-\rho)\,(t-s)}$ allows to translate the latter expression into a Gauss hypergeometric series yielding

$$v(s) = \frac{\gamma}{\rho}\left(1-\eta(s)\right)^{\frac{1}{1-\gamma}}\ {}_2F_1\left(\frac{1}{1-\gamma}, \frac{\rho}{(\rho-r_2)(1-\gamma)}, \frac{\rho}{(\rho-r_2)(1-\gamma)}+1; \eta(s)\right), \quad (15)$$

where

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt$$

with $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$ (cf. for instance Abramawitz & Stegun [1]).
In Section 3, numerical illustrations will be given for a particular set of parameters.

Remark: In fact, equation (6) can be interpreted as a special homogeneous case of (11), since for $\delta = 0$ we have $B(s, b) = g(s, b)$. The absence of the inhomogeneous term in (6) causes the problem not to rule out the solution $\phi_\gamma(s) = 0$ (as discussed in the Remark after Theorem 1).

2.3 Paying tax from surplus M onwards only

It may be better for the tax authority to collect tax only when the surplus has exceeded a threshold $M > s$. On the one hand this will decrease the ruin probability:

$$\phi_{\gamma,M}(s) = \frac{\phi_0(s)}{\phi_0(M)}\,\phi_\gamma(M) = \frac{\phi_0(s)}{\phi_0(M)}\,\phi_0(M)^{1/(1-\gamma)} = \phi_0(s)\,\phi_0(M)^{\gamma/(1-\gamma)} > \phi_\gamma(s).$$

On the other hand, taxation starts later. So a priori it is not clear which effect will dominate, and thus one needs to investigate the quantity $v_M(s)$, which is defined as the expected discounted sum of tax payments until ruin, given that tax payments only start at a surplus level $M > s$. Clearly,

$$v_M(s) = B(s, M)\,v(M) = \frac{V(s, M)}{V(M, M)}\,v(M) = \frac{h(s)}{h(M)}\,v(M).$$

Now let us look for the optimal value $M = M^*$ that maximizes $v_M(s)$:

Theorem 3. *If $v(0) > \frac{c}{\lambda+\delta}$, then for all $s \geq 0$ the optimal height M^* is the positive solution of the equation*

$$v'(M) = 1 \tag{16}$$

and the corresponding optimal expected discounted tax payment is given by

$$v_{M^*}(s) = \begin{cases} V(s, M^*), & \text{if } s < M^*, \\ v(s), & \text{if } s \geq M^*. \end{cases} \tag{17}$$

If, on the other hand, $v(0) \leq \frac{c}{\lambda+\delta}$, then $M^* = 0$ for all $s \geq 0$ and the optimal expected discounted tax payment is given by $v_{M^*}(s) = v(s)$.

Proof: Due to the regularity of $h(s)$ and consequently of $v(s)$, a necessary condition for a positive maximal value $M = M^*$ is

$$\frac{\partial v_M(s)}{\partial M} = h(s)\,\frac{v'(M)\,h(M) - v(M)\,h'(M)}{h^2(M)} = 0$$

which implies

$$\frac{v'(M^*)}{v(M^*)} = \frac{h'(M^*)}{h(M^*)} = \frac{1}{V(M^*, M^*)}.$$

But using the differential equation (14) this translates into

$$\frac{1}{(1-\gamma)\,V(M^*, M^*)} - \frac{\gamma}{(1-\gamma)\,v(M^*)} = \frac{1}{V(M^*, M^*)}$$

and hence

$$v(M^*) = V(M^*, M^*), \tag{18}$$

which, again in view of (14), is equivalent to $v'(M^*) = 1$. Recall that

$$\lim_{s\to\infty} v(s) = \frac{\gamma}{\rho} < \frac{1}{\rho} = \lim_{s\to\infty} V(s,s).$$

Hence, if $v(0) > \frac{c}{\lambda+\delta} = V(0,0)$, the continuity of both $v(s)$ and $V(s,s)$ imply the existence of a positive solution M^* of (18) or, equivalently, of (16). In order to ensure that the extremum M^* is in fact a maximum for $v_M(s)$, we look at the sign of the second derivative:

$$\begin{aligned}\operatorname{sgn}\left(\left.\frac{\partial^2 v_M(s)}{\partial M^2}\right|_{M=M^*}\right) &= \operatorname{sgn}\Big(h^2(M^*)\big(v''(M^*)h(M^*) - v(M^*)h''(M^*)\big)\\ &\qquad - 2\,h(M^*)\,h'(M^*)\underbrace{\big(v'(M^*)h(M^*) - v(M^*)h'(M^*)\big)}_{=0}\Big)\\ &= \operatorname{sgn}\big(v''(M^*)h(M^*) - v(M^*)h''(M^*)\big).\end{aligned}$$

By virtue of (14) and

$$V'(s,s) = 1 - \frac{h(s)\,h''(s)}{h'^2(s)} \quad \forall\, s \geq 0, \tag{19}$$

one easily derives

$$v''(M^*) = \frac{v(M^*)\,h''(M^*)}{(1-\gamma)\,h(M^*)},$$

so that the above can further be simplified to give

$$\operatorname{sgn}\left(\left.\frac{\partial^2 v_M(s)}{\partial M^2}\right|_{M=M^*}\right) = \operatorname{sgn}\Big(\frac{\gamma}{1-\gamma}\,v(M^*)\,h''(M^*)\Big) = \operatorname{sgn}\big(h''(M^*)\big).$$

Now, the optimal horizontal barrier b^* in the classical risk model without tax is defined as the value that minimizes $h'(b)$ $(b \geq 0)$ and from [4] it follows that if $b^* > 0$, then it is the unique positive solution of $h''(b) = 0$. Since $h(s) > 0\ \forall\, s \geq 0$, one can deduce from (19) that

$$V'(b,b) = 1 \quad \text{if and only if} \quad b = b^* > 0.$$

From $v(0) > V(0,0)$ it follows that

$$V'(M^*, M^*) > v'(M^*) = 1,$$

which together with the observation $\lim_{b\to\infty} V'(b,b) = 0$ implies $M^* < b^*$. Consequently, from (19) we finally find $h''(M^*) < 0$ and M^* indeed represents a maximum. For $s < M^*$ the resulting optimal expected discounted tax payment is thus given by

$$v_{M^*}(s) = \frac{V(s, M^*)}{V(M^*, M^*)}\,v(M^*) = V(s, M^*).$$

On the other hand, $s \geq M^*$ implies that taxes have to be paid right from the start, so that the tax payment strategy is equivalent to the case $M = 0$.

In the second case $v(0) < \frac{c}{\lambda+\delta} = V(0,0)$, it is clear that at a possible intersection point M^* of (18) we have to have $V'(M^*, M^*) < v'(M^*) = 1$, which in view of (19) would imply $h''(M^*) > 0$ so that M^* can not be a maximum. As $v_M(s) \to 0$ for $M \to \infty$, the absence of a maximal value of $v_M(s)$ for $0 < M < \infty$ implies that in this case the maximum is attained at $M^* = 0$. □

Remark: Note in particular that the optimal taxation level M^* does not depend on the initial capital s. From (9), the criterion (16) translates into finding M^* such that

$$\frac{1-\gamma}{\gamma}\, h(M^*)^{-\frac{\gamma}{1-\gamma}} = h'(M^*) \int_{M^*}^{\infty} h(t)^{-\frac{1}{1-\gamma}}\, dt.$$

3 Numerical Example

Let us consider the case of an exponential claim size distribution with parameter $\alpha = 1$ and choose $c = 2$, $\lambda = 1$, $\delta = 0.04$. Let furthermore $\gamma = 0.5$. Then we obtain from (15)

$$v(s) = 12.019\, e^{-0.077s}(-0.481\, e^{-0.519s} + 1.039\, e^{0.0386s})^2 \;\, {}_2F_1(2, 0.139; 1.139; 0.464\, e^{-0.557s}),$$

where all numbers are rounded to their last digit. Figure 2 depicts the expected discounted tax payments as a function of initial capital s. Note that the trivial bounds (10) are in this case $0.9615 \leq v(s) \leq 25\ \forall s \geq 0$, whereas the exact value at zero is $v(0) = 4.4252$ and the limit is given by $v(\infty) = \gamma/\rho = 12.9642$.

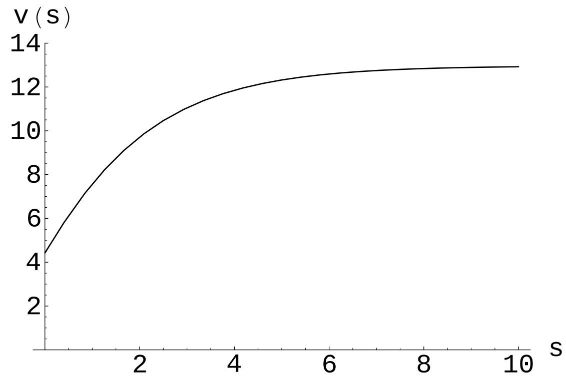
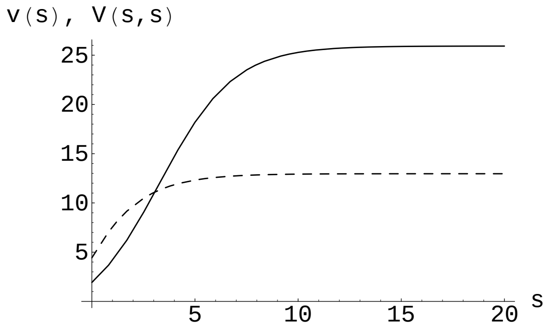
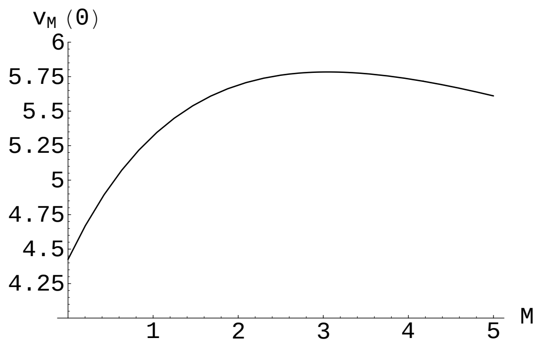


Figure 2: $v(s)$ for $Exp(\alpha)$ claim amounts with $\lambda = 1$, $\alpha = 1$, $c = 2$, $\delta = 0.04$ and $\gamma = 0.5$

For the determination of the optimal starting level M^*, Figure 3(a) plots the functions $V(s,s)$ and $v(s)$ simultaneously; the intersection point $s = M^*$ occurs at $M^* = 3.0529$. Figure 3(b) illustrates for $s = 0$ that $M^* = 3.0529$ indeed maximizes the expected discounted tax payouts $v_M(s)$ w.r.t. the choice of M.
Alternatively, for $\gamma = 0.1$ (but otherwise identical parameters), we get $v(0) = 1.3640 < \frac{c}{\lambda+\delta} = 1.9231$ and hence, due to Theorem 3 the optimal value for M is given by $M^* = 0$. Indeed, Figure 4(a) shows that the functions $V(s,s)$ and $v(s)$ do not have a positive



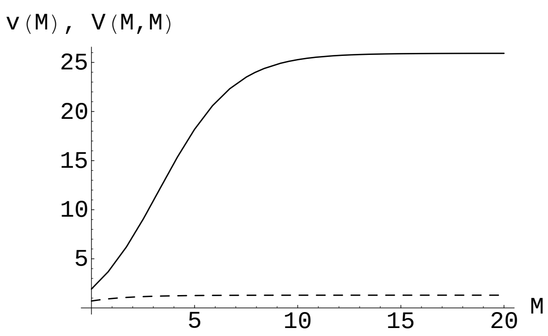
(a) $v(s)$ (dashed line) versus $V(s,s)$ (solid line)



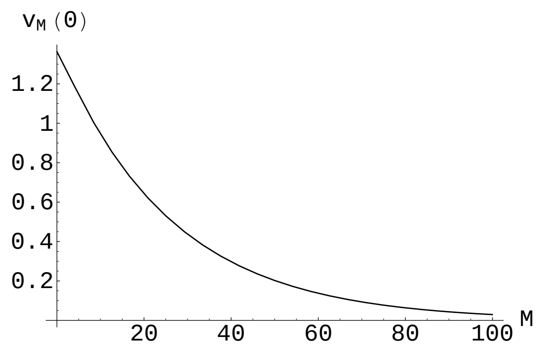
(b) The expected discounted tax payments $v_M(0)$ as a function of M

Figure 3: $Exp(\alpha)$ claim amounts with $\lambda = 1$, $\alpha = 1$, $c = 2$, $\delta = 0.04$ and $\gamma = 0.5$

intersection point and Figure 4(b) illustrates for $s = 0$ that in this case $M^* = 0$ maximizes the expected discounted tax payouts $v_M(s)$ w.r.t. the choice of M.
Figure 5(a) depicts the value of $v(0)$ for γ ranging from 0 to 1 together with the value of $V(0,0) = 1.923$, which does not depend on γ. The optimal value M^* is positive for $\gamma > 0.1454$. For $\gamma \to 1$, the tax payment strategy converges to the dividend barrier strategy with horizontal dividend barrier and correspondingly $M^* \to b^*$. Figure 5(b) depicts the value of M^* as a function of γ for the numerical example given above (note that $b^* = 7.9487$ in this case).

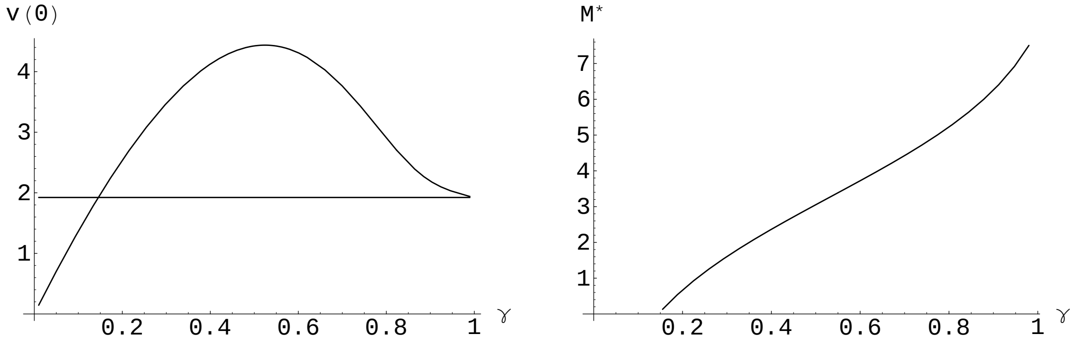


(a) $v(s)$ (dashed line) versus $V(s,s)$ (solid line)



(b) The expected discounted tax payments $v_M(0)$ as a function of M

Figure 4: $Exp(\alpha)$ claim amounts with $\lambda = 1$, $\alpha = 1$, $c = 2$, $\delta = 0.04$ and $\gamma = 0.1$



(a) $v(0)$ (together with $V(0,0) = 1.923$) as a function of γ

(b) M^* as a function of γ

Figure 5: $Exp(\alpha)$ claim amounts with $\lambda = 1$, $\alpha = 1$, $c = 2$, $\delta = 0.04$

Appendix

Here we collect some known results which are reproduced to make the paper more self contained.

Lemma A.1 *Let $W_1, W_2, ...$ be exponentially distributed with parameter λ and $f(x)$ non-negative and integrable. Then*

(i) $E\left[\sum_{n=1}^{\infty} f(W_1 + ... + W_n)\right] = \lambda \int_0^{\infty} f(x)dx$;

(ii) $E\left[\prod_{n=1}^{\infty}(1 - f(W_1 + ... + W_n))\right] = \exp\left(-\lambda \int_0^{\infty} f(x)dx\right)$.

Both statements can be found, e.g., in [5] (statement (i) on p. 127 and statement (ii) on p. 130, Prop. 3.6., in the language of Laplace functionals).

Proof: We first consider simple functions $f(x) = \sum_{i=1}^{\infty} \alpha_i 1_{A_i}(x)$. Since all terms in the following finite sums are non-negative, we can interchange the order of summation:

$$\sum_{n=1}^{\infty} f(W_1 + ... + W_n) = \sum_{i=1}^{\infty} \alpha_i N(A_i) \tag{20}$$

$$E\left[\sum_{n=1}^{\infty} f(W_1 + ... + W_n)\right] = \sum_{i=1}^{\infty} \alpha_i E[N(A_i)] = \lambda \int_0^{\infty} f(x)dx \tag{21}$$

where in (21) we have used that $N(A_i) = \#\{n : W_1 + ... + W_n \in A_i\}$ is Poisson distributed with parameter $\lambda|A_i|$ with $|A|$ the Lebesgue measure of the set A. For the product we want to argue similarly:

$$\prod_{n=1}^{\infty}(1 - f(W_1 + ... + W_n)) = \prod_{i=1}^{\infty}(1 - \alpha_i)^{N(A_i)} \tag{22}$$

$$
\begin{aligned}
E\left[\prod_{n=1}^{\infty}(1-f(W_1+...+W_n))\right] &= \prod_{i=1}^{\infty}E[(1-\alpha_i)^{N(A_i)}] \quad (23)\\
=\prod_{i=1}^{\infty}\exp(-\lambda\alpha_i|A_i|) &= \exp\left(-\lambda\int_0^{\infty}f(x)dx\right) \quad (24)
\end{aligned}
$$

where in (23) we used that $N(A_1), N(A_2), ...$ are stochastically independent. For equation (22) some care is needed since we have changed the order in the product which is allowed only if the product on the left hand side of equation (22) is converging in the following sense: either one term in the product is zero or the sum of logarithms of the terms in the product is converging. We may exclude sets of measure zero since they do not contribute to the expectation. Since the function $f(x)$ is integrable, we know from our first assertion that $f(W_1 + ... + W_n) \to 0$ almost surely. Assume that $f(W_1 + ... + W_n) < 1/2$ for all $n \geq k$. Then for these n

$$-\log(1-f(W_1+...+W_n)) \leq \frac{7}{5}f(W_1+...+W_n).$$

Furthermore, $\sum_{n=1}^{\infty} f(W_1 + ... + W_n) < \infty$ almost surely. This implies that the above product is converging and thus equation (22) is true.
For general functions $f(x)$ we approximate with simple functions $f_-(x) \leq f(x) \leq f_+(x)$ for which $\int_0^{\infty} |f_+(x) - f_-(x)|dx$ is small. Then the sum $S = \sum_{n=1}^{\infty} f(W_1 + ... + W_n)$ and the product $P = \prod_{n=1}^{\infty}(1-f(W_1+...+W_n))$ are bounded by the corresponding quantities with $f_-(x)$ and $f_+(x)$, respectively:

$$
\begin{aligned}
\sum_{n=1}^{\infty} f_-(W_1+...+W_n) &\leq S \leq \sum_{n=1}^{\infty} f_+(W_1+...+W_n)\\
\prod_{n=1}^{\infty}(1-f_-(W_1+...+W_n)) &\geq P \geq \prod_{n=1}^{\infty}(1-f_+(W_1+...+W_n)),
\end{aligned}
$$

and this proves the assertions for general functions $f(x)$. □

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