Maxima and Minima of Complete and Incomplete Stationary Sequences

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Abstract: In the seminal contribution [7] the joint weak convergence of maxima and minima of weakly dependent stationary sequences is derived under some mild asymptotic conditions. In this paper we address additionally the case of incomplete samples assuming that the average proportion of incompleteness converges in probability to some random variable $\mathcal{P}$. We show the joint weak convergence of the maxima and the minima of both complete and incomplete samples. It turns out that the maxima and the minima are asymptotically independent when $\mathcal{P}$ is a deterministic constant.

Key words: maxima; minima; incomplete sample; joint limit distribution; stationary sequences; Berman condition; Gaussian ARMA processes.

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1 Introduction

The asymptotic behaviour of extremes of random samples is a topic of interest in many theoretical and applied research fields. As for the case of sample mean, under certain assumptions the limit distribution of maxima of random samples can be shown to converge weakly after a linear transformation to a random variable (rv) which is either Gumbel, Fréchet or Weibull, see the classical monographs [8, 9, 16, 22]. In many cases, for instance if we consider an independent random sample $X_1, \ldots, X_n$ with underlying $N(0,1)$ distribution, both maxima $M_n = \max_{1 \le i \le n} X_i$ and minima $m_n = \min_{1 \le i \le n} X_i$ converge weakly to a Gumbel rv, see below for technical details. For any fixed n both m_n and M_n are dependent rvs, however, maxima and minima are asymptotically independent. Surprisingly, as shown in [7, 19, 24] under mild conditions this is the case also if $\{X_n, n \ge 1\}$ is a strictly stationary random sequence. The asymptotic independence of minima and maxima is crucial in statistical application, see for instance [4, 17].

When dealing with real data, missing or censored observations are very common. Results for the joint asymptotic behaviour of maxima of complete and incomplete samples were initially derived in [14] and [18]; several authors followed these contributions see e.g., [6, 12, 20, 21, 23].

In our context the random sample $X_1, \ldots, X_n$ becomes incomplete if observations are missing. The probabilistic model governing the missing of the observations studied in this paper is that of [18], i.e., we shall consider independent Bernoulli rvs $\varepsilon_n, n \ge 1$ independent of X_i's so that ε_i is the indicator of the event that X_i is observed. Thus $S_n = \sum_{i=1}^n \varepsilon_i$ is just the number of observed rvs from $\{X_1, \ldots, X_n\}$. The main restriction on ε_i's imposed in this paper is that

$$\frac{S_n}{n} := \frac{\sum_{i=1}^n \varepsilon_i}{n} \to \mathcal{P} \in [0,1] \tag{1.1}$$

holds in probability as $n \to \infty$. For the incomplete sample define the maxima $M_n(\boldsymbol{\varepsilon})$ as

$$M_n(\boldsymbol{\varepsilon}) = \begin{cases} \max\{X_i, \varepsilon_i = 1, i \le n\}, & \text{if } \sum_{i=1}^n \varepsilon_i \ge 1, \\ \inf\{x | F(x) > 0\}, & \text{otherwise}, \end{cases}$$

where F shall denote the common distribution of X_i's and define similarly the minima $m_n(\boldsymbol{\varepsilon})$.
The seminal article [18] derived the joint asymptotic behaviour of M_n and $M_n(\boldsymbol{\varepsilon})$ considering $\mathcal{P}$ to be non-random, see (2.2) below for details. The more general case that $\mathcal{P}$ is random is established in [14]. Based on the latter contribution, in this paper we consider additionally the sample minima deriving the joint asymptotic behaviour of $(m_n, m_n(\boldsymbol{\varepsilon}), M_n, M_n(\boldsymbol{\varepsilon}))$ when (1.1) holds with some random $\mathcal{P}$ imposing some mild conditions on the strictly stationary random sequence $\{X_n, n \ge 1\}$.
Our main result shows that $(M_n(\boldsymbol{\varepsilon}), M_n)$ and $(m_n(\boldsymbol{\varepsilon}), m_n)$ are asymptotically independent if $\mathcal{P}$ is a deterministic constant. This fact is interesting and somewhat expected since the incompleteness of the data influences both maxima and minima, and therefore the asymptotic independence is not always possible.
Brief organisation of the paper. In the next section we present our main result and then apply it to some interesting cases of stationary sequences in Section 3. Proofs together with auxiliary results are displayed in Section 4.

2 Main Result

We shall consider below a strictly stationary random sequence $\{X_n, n \ge 1\}$ with marginal distribution F, i.e., all X_i's have the same distribution F, and $(X_{n+1}, \ldots, X_{n+j})$ has the same distribution as $(X_{n+k+1}, \ldots, X_{n+k+j})$ for any $j, k, n \in I\!N$. Suppose that there exist sequences $a_n > 0,\ c_n > 0,\ b_n,\ d_n \in I\!R$ and non-degenerate distributions G and H such that (write next $\overline{F} = 1 - F, \overline{H} = 1 - H$)

$$\lim_{n\to\infty} F^n(u_n(x)) = G(x), \qquad \lim_{n\to\infty} (\overline{F}(v_n(y)))^n = \overline{H}(y), \tag{2.1}$$

where $u_n(x) = a_n x + b_n$ and $v_n(y) = c_n y + d_n,\ x, y \in I\!R$. Under the well-known asymptotic conditions $D(u_n, v_n)$ and $D'(u_n)$ in [18] it was shown that (2.1) implies

$$\lim_{n\to\infty} \mathbb{P}\{M_n(\boldsymbol{\varepsilon}) \le u_n(x), M_n \le u_n(y)\} = \mathcal{F}(x, y; \mathcal{P}) =: G^{\mathcal{P}}(x) G^{1-\mathcal{P}}(y) \tag{2.2}$$

for any $x < y$, provided that (1.1) holds with $\mathcal{P}$ a deterministic constant. In [14] it was shown that (2.2) still holds if $\mathcal{P}$ is a rv with $\mathcal{F}(x, y; \mathcal{P}) = \mathbb{E}\left\{G^{\mathcal{P}}(x) G^{1-\mathcal{P}}(y)\right\}$. Since we shall consider also the minima, both dependence conditions $D(u_n, v_n)$ and $D'(u_n)$ assumed in the aforementioned references are not sufficient for our investigation. Therefore, we shall impose below the stronger dependence conditions introduced by Davis [7]. Throughout in the sequel $z_n(\boldsymbol{x}, \boldsymbol{y}) := (u_n(x_1), u_n(x_2), v_n(y_1), v_n(y_2))$ are given constants.

Definition: Condition $D(z_n(\boldsymbol{x}, \boldsymbol{y}))$ is satisfied, if for any n and all $A_1,\ A_2,\ B_1,\ B_2 \subset \{1, \ldots, n\}$, such that $A_1 \cap A_2 = \emptyset$, $B_1 \cap B_2 = \emptyset$ and $b - a \ge l_n$, where $a \in A_1 \cup A_2$ and $b \in B_1 \cup B_2$ we have that

$$\Bigg| \mathbb{P}\left\{ \bigcap_{j \in A_1 \cup B_1} \{v_n(\widetilde{y}_2) < X_j \le u_n(\widetilde{x}_2)\} \cap \bigcap_{j \in A_2 \cup B_2} \{v_n(\widetilde{y}_1) < X_j \le u_n(\widetilde{x}_1)\} \right\}$$
$$- \mathbb{P}\left\{ \bigcap_{j \in A_1} \{v_n(\widetilde{y}_2) < X_j \le u_n(\widetilde{x}_2)\} \cap \bigcap_{j \in A_2} \{v_n(\widetilde{y}_1) < X_j \le u_n(\widetilde{x}_1)\} \right\}$$

$$\times \mathbb{P}\left\{\bigcap_{j\in B_1}\{v_n(\widetilde{y}_2)<X_j\le u_n(\widetilde{x}_2)\}\cap\bigcap_{j\in B_2}\{v_n(\widetilde{y}_1)\le X_j\le u_n(\widetilde{x}_1)\}\right\}\Bigg|\le \alpha_{n,l_n},$$

where $\lim_{n\to\infty}\alpha_{n,l_n}=0$ for some sequence $l_n\to\infty$ with $l_n/n\to 0$ and $\widetilde{x}_i=x_i+I\cdot\infty, \widetilde{y}_i=-(1-I)J\cdot\infty+y_i, i=1,2, I,J\in\{0,1\}$ (set $0\cdot\infty:=0$).

Definition: *Condition $D'(u_n(x),v_n(y))$ is satisfied for a real sequence $\{u_n(x),v_n(y),n\ge 1\}$ if*

$$\limsup_{n\to\infty} n\sum_{j=1}^{[n/k]}\Big[\mathbb{P}\{X_1>u_n(x),X_{j+1}>u_n(x)\}+\mathbb{P}\{X_1>u_n(x),X_{j+1}\le v_n(y)\}$$
$$+\mathbb{P}\{X_1\le v_n(y),X_{j+1}>u_n(x)\}+\mathbb{P}\{X_1\le v_n(y),X_{j+1}\le v_n(y)\}\Big]=o(1)$$

as $k\to\infty$.

We state next our main result.

Theorem 2.1 *Let $\{X_n,n\ge 1\}$ be a strictly stationary random sequence with underlying distribution F. Suppose that* (2.1) *holds for $u_n(x),v_n(y),\ x,y\in\mathbb{R}$. Assume further that both $D'(u_n(x),v_n(y))$ and $D(u_n(x_1),v_n(y_1),u_n(x_2),v_n(y_2))$ hold for $x_2<x_1,\ y_1<y_2$. If the indicator random sequence $\boldsymbol{\varepsilon}=\{\varepsilon_n,n\ge 1\}$ is independent of $\{X_n,n\ge 1\}$ and further* (1.1) *is satisfied, then*

$$\lim_{n\to\infty}\mathbb{P}\{v_n(y_2)<m_n(\boldsymbol{\varepsilon})\le M_n(\boldsymbol{\varepsilon})\le u_n(x_2),v_n(y_1)<m_n\le M_n\le u_n(x_1)\}$$
$$=\mathbb{E}\left\{G^{\mathcal{P}}(x_2)(\overline{H}(y_2))^{\mathcal{P}}G^{1-\mathcal{P}}(x_1)(\overline{H}(y_1))^{1-\mathcal{P}}\right\}.$$

Remarks: a) Under the conditions of Theorem 2.1 for $y_1<y_2$ we have

$$\lim_{n\to\infty}\mathbb{P}\{m_n(\boldsymbol{\varepsilon})>v_n(y_2),m_n>v_n(y_1)\}=\mathbb{E}\left\{(\overline{H}(y_2))^{\mathcal{P}}(\overline{H}(y_1))^{1-\mathcal{P}}\right\}.$$

Further,

$$\lim_{n\to\infty}\mathbb{P}\{M_n(\boldsymbol{\varepsilon})\le u_n(x_2),M_n\le u_n(x_1)\}=\mathbb{E}\left\{G^{\mathcal{P}}(x_2)G^{1-\mathcal{P}}(x_1)\right\}$$

holds with $x_2<x_1$.

b) Theorem 2.1 implies for any $x,y\in\mathbb{R}$

$$\lim_{n\to\infty}\mathbb{P}\{v_n(y)<m_n(\boldsymbol{\varepsilon})\le M_n(\boldsymbol{\varepsilon})\le u_n(x)\}=\mathbb{E}\left\{G^{\mathcal{P}}(x)(\overline{H}(y))^{\mathcal{P}}\right\}.$$

Hence, if $\mathcal{P}$ is a constant, then the maxima and the minima are asymptotically independent.

c) Our result shows in particular the joint asymptotic convergence of $(m_n(\boldsymbol{\varepsilon}),m_n)$ (and similarly for $(M_n(\boldsymbol{\varepsilon}),M_n)$). We have thus

$$\left(\frac{m_n(\boldsymbol{\varepsilon})-d_n}{c_n},\frac{m_n-d_n}{c_n}\right)\xrightarrow{d}(\mathcal{M}^*,\mathcal{M}),\quad n\to\infty$$

and consequently,

$$\left(\frac{m_n(\boldsymbol{\varepsilon})-m_n}{c_n},\frac{m_n-d_n}{c_n}\right)\xrightarrow{d}(\mathcal{M}^*-\mathcal{M},\mathcal{M}),\quad n\to\infty.$$

A similar result is given in [15] for the case that $\mathcal{P}$ is a deterministic constant.

3 Examples

In this section we present four illustrating examples.

Example 1. (Gaussian sequence) We consider the case that $\{X_n, n \geq 1\}$ is a centered stationary Gaussian sequence with correlations $\rho_n = \mathbb{E}\{X_1 X_{n+1}\} < 1, n \geq 1$ such that $\mathbb{E}\left\{X_1^2\right\} = 1$. With the choice of constants

$$a_n = c_n = \frac{1}{\sqrt{2\ln n}}, \quad b_n = d_n = \sqrt{2\ln n} - \frac{\ln\ln n + \ln 4\pi}{2\sqrt{2\ln n}} \tag{3.1}$$

condition (2.1) holds where $u_n(x) = a_n x + b_n$, $v_n(y) = -c_n y - d_n$ and $\overline{H}(x) = G(x) = \exp(-\exp(-x))$ if further the Berman condition

$$\lim_{n\to\infty} \rho_n \ln n = 0 \tag{3.2}$$

is valid, see e.g., [2, 3]. Note in passing that (2.1) also holds if

$$\sum_{n\geq 1} |\rho_n|^p < \infty \tag{3.3}$$

for some $p > 1$, see [7]. In view of [16], both $D(z_n(\boldsymbol{x}, \boldsymbol{y}))$ and $D'(u_n(x), v_n(y))$ are satisfied under (3.2) or (3.3), and hence the claim of Theorem 2.1 holds for such stationary Gaussian sequences.

Example 2. (Gaussian linear processes) An important class of stationary sequences is that of the linear processes (see e.g., [5, 8]), which have an infinite moving average representation

$$X_n = \sum_{j=-\infty}^{\infty} \psi_j Z_{n-j},$$

where $\{Z_n, n \geq 1\}$ is an iid sequence and $\sum_j \psi_j^2 < \infty$. We also assume that $\{Z_n, n \geq 1\}$ have mean zero and finite variance σ_Z^2. If $\{Z_n, n \geq 1\}$ is Gaussian, so is $\{X_n, n \geq 1\}$. In particular, for autoregressive-moving average (ARMA) processes, the coefficients ψ_j decrease to zero at an exponential rate. As a consequence, for such sequences the Berman condition (3.2) holds. Therefore we conclude that Theorem 2.1 is applicable to Gaussian ARMA processes.

Example 3. (Scaled Gaussian sequence) Define rvs $X_n^* = T_n X_n, n \geq 1$ where X_n is as in Example 1 or Example 2; here T_n is a positive rv which scales X_n. Our assumption is that $T, T_n, n \geq 1$ are independent rvs with a common distribution Q being further independent of the stationary Gaussian sequence $\{X_n, n \geq 1\}$. Suppose that the distribution Q has upper endpoint equal to 1 and for any $u \in (\nu, 1)$ with $\nu \in (0,1)$

$$\mathbb{P}\{\mathcal{T}_\tau > u\} \geq \mathbb{P}\{T > u\} \geq \mathbb{P}\{\mathcal{T}_\gamma > u\}$$

holds with $\mathcal{T}_\gamma$, $\mathcal{T}_\tau$ two non-negative rvs. Let Q_1^{-1} be the generalised inverse of the distribution Q_1 of $T_1 X_1$. With the choice of constants

$$b_n = d_n = Q_1^{-1}\left(1 - \frac{1}{n}\right), \qquad a_n = c_n = \frac{1}{b_n}$$

if further $\mathcal{T}_\gamma$ and $\mathcal{T}_\tau$ have regularly varying tails at 1 with non-negative indexes γ and τ, respectively and the modified Berman condition

$$\lim_{n\to\infty} \rho_n (\ln n)^{1+\Delta} = 0 \tag{3.4}$$

holds for some $\Delta > 2(\gamma - \tau)$, then in view of Theorem 1.2 in [13] the assumptions of Theorem 2.1 hold for such scaled Gaussian sequence. We note in passing that if for some α, c positive

$$\mathbb{P}\{T > 1 - 1/u\} \sim cu^{-\alpha}$$

as $u \to \infty$, then instead of (3.4) we can assume the Berman condition (which is weaker).
Similarly, if T has a Weibullian-type tail as defined in [1], according to Theorem 2.1 in [11] the claim of Theorem 2.1 holds under the Berman condition.

Example 4. (FGM random sequence) A stationary Farlie-Gumbel-Morgenstern (FGM) random sequence is defined by the common univariate marginal F, and a symmetric function $a(j,l)$ which depends on j,l only through their difference, i.e., $a(j,l) =: \alpha(|j-l|)$, for all $j \neq l$, such that the joint distribution $H_{i_1,\ldots,i_n}$ of $X_{i_1}, \ldots, X_{i_n}$ is given by the FGM distribution

$$H_{i_1,\ldots,i_n}(x_1, \ldots, x_n) = \prod_{h=1}^{n} F(x_h) \left(1 + \sum_{1 \le j < l \le n} \alpha(|i_j - i_l|)\bar{F}(x_j)\bar{F}(x_l)\right).$$

The function $\alpha(\cdot)$ is admissible if for every $n \ge 1$ and indices $\{i_1, \ldots, i_n\}$ the inequalities

$$1 + \sum_{1 \le j < l \le n} \varsigma_{i_j}\varsigma_{i_l}\alpha(|i_j - i_l|) \ge 0$$

hold for all ς_{i_j} taking values ± 1.
If we assume that for the normalizations $u_n(x)$ and $v_n(y)$ the condition (2.1) is satisfied, then it follows that the condition $D'(u_n(x), v_n(y))$ is valid. If further

$$\lim_{n\to\infty} \sup_{l>n} |\alpha(l)| = 0, \tag{3.5}$$

then the condition $D(z_n(\boldsymbol{x}, \boldsymbol{y}))$ is also satisfied (for a detailed proof see Appendix). Hence the claim of Theorem 2.1 holds for such stationary FGM random sequence.

4 Further Results and Proofs

In order to prove the main theorem, we need some auxiliary results. Let $\boldsymbol{\beta} = \{\beta_n, n \ge 1\}$ be a non-random sequence taking values in $\{0, 1\}$. Given an index set $I \subset \{1, \ldots, n\}$ we shall define

$$M(I, \boldsymbol{\beta}) = \begin{cases} \max\{X_i, \beta_i = 1, i \in I\}, & \text{if } \sum_{i\in I} \beta_i \ge 1; \\ \inf\{x | F(x) > 0\}, & \text{otherwise} \end{cases}$$

and similarly for $m(I, \boldsymbol{\beta})$ where we consider instead of the maximum, the minimum of X_i's. If J is another index set we shall put $\tilde{d}(I, J) := \min_{i\in I, j\in J} |i - j|$. Let k be a fixed positive integer, $t = [n/k]$ and define

$$K_s = \{j : (s-1)t + 1 \le j \le st\}$$

for $1 \le s \le k$. For a rv $\mathcal{P} \in [0, 1]$ write we shall write

$$B_{r,k} = \left\{\omega : \mathcal{P}(\omega) \in \begin{cases} [0, \frac{1}{2^k}], & r = 0, \\ (\frac{r}{2^k}, \frac{r+1}{2^k}], & 0 < r \le 2^k - 1 \end{cases}\right\}$$

and then set

$$B_{r,k,\boldsymbol{\beta},n} = \{\omega : \varepsilon_i(\omega) = \beta_i, 1 \le i \le n\} \cap B_{r,k}.$$

Lemma 4.1 *If condition $D(u_n(x_1), v_n(y_1), u_n(x_2), v_n(y_2))$ holds for $x_2 < x_1$, $y_1 < y_2$, then for $I_1, \ldots, I_k$ non-empty subsets of $\{1, \ldots, n\}$ we have*

$$\begin{aligned} &\Bigg|\mathbb{P}\left\{\bigcap_{s=1}^{k} v_n(y_2) < m(I_s, \boldsymbol{\beta}) \le M(I_s, \boldsymbol{\beta}) \le u_n(x_2), v_n(y_1) < m(I_s) \le M(I_s) \le u_n(x_1)\right\} \\ &\quad - \prod_{s=1}^{k} \mathbb{P}\left\{v_n(y_2) < m(I_s, \boldsymbol{\beta}) \le M(I_s, \boldsymbol{\beta}) \le u_n(x_2), v_n(y_1) < m(I_s) \le M(I_s) \le u_n(x_1)\right\}\Bigg| \\ \le\ & (k-1)\alpha_{n,l_n}, \end{aligned} \tag{4.1}$$

provided that $\min_{1 \le i < j \le k} \tilde{d}(I_i, I_j) \ge l_n$.

Proof of Lemma 4.1 For $k = 2$, the inequality (4.1) is just the condition $D(u_n(x_1), v_n(y_1), u_n(x_2), v_n(y_2))$. Suppose that (4.1) holds for arbitrary index sets $I_1, \ldots, I_{k-1}$ such that the distance between any two index sets is not less then l_n. Define

$$\mathcal{A}(I) = \{v_n(y_2) < m(I, \boldsymbol{\beta}) \le M(I, \boldsymbol{\beta}) \le u_n(x_2), v_n(y_1) < m(I) \le M(I) \le u_n(x_1)\}$$

for any interval $I \in \{1, \ldots, n\}$. By induction and the condition $D(u_n(x_1), v_n(y_1), u_n(x_2), v_n(y_2))$ we have that

$$\begin{aligned} \left|\mathbb{P}\left\{\bigcap_{s=1}^{k} \mathcal{A}(I_s)\right\} - \prod_{s=1}^{k} \mathbb{P}\{\mathcal{A}(I_s)\}\right| &\le \left|\mathbb{P}\left\{\bigcap_{s=1}^{k} \mathcal{A}(I_s)\right\} - \mathbb{P}\left\{\bigcap_{s=1}^{k-1} \mathcal{A}(I_s)\right\}\mathbb{P}\{\mathcal{A}(I_k)\}\right| \\ &\quad + \left|\mathbb{P}\left\{\bigcap_{s=1}^{k-1} \mathcal{A}(I_s)\right\} - \prod_{s=1}^{k-1} \mathbb{P}\{\mathcal{A}(I_s)\}\right| \mathbb{P}\{\mathcal{A}(I_k)\} \\ &\le \alpha_{n,l_n} + (k-2)\alpha_{n,l_n} \\ &= (k-1)\alpha_{n,l_n}, \end{aligned}$$

establishing the proof. □

Lemma 4.2 *Under the assumptions of Lemma 4.1 we have*

$$\begin{aligned} &\Big|\mathbb{P}\left\{v_n(y_2) < m_n(\boldsymbol{\beta}) \le M_n(\boldsymbol{\beta}) \le u_n(x_2), v_n(y_1) < m_n \le M_n \le u_n(x_1)\right\} \\ &\quad - \prod_{s=1}^{k} \mathbb{P}\left\{v_n(y_2) < m(K_s, \boldsymbol{\beta}) \le M(K_s, \boldsymbol{\beta}) \le u_n(x_2), v_n(y_1) < m(K_s) \le M(K_s) \le u_n(x_1)\right\}\Big| \\ \le\ & (k-1)\alpha_{n,l_n} + (4k+2)l_n(\overline{F}(u_n(x_2)) + F(v_n(y_2))). \end{aligned}$$

Proof of Lemma 4.2 Define $N_n = \{1, \ldots, n\}$ for any positive integer n. For large n we can choose a positive integer l_n such that $k < l_n < t$. Let

$$N_{tk} = \bigcup_{s=1}^{k} K_s \qquad \text{and} \qquad K_s = I_s \cup J_s,$$

where $I_s = \{(s-1)t + 1, \ldots, st - l_n\}$ and $J_s = \{st - l_n + 1, \ldots, st\}$ for $s = 1, \ldots, k$.
Since $tk \le n < (t+1)k < tk + l_n$, we get $|N_n \backslash N_{tk}| < k < l_n$. Define sets I_{k+1} and J_{k+1} as

$$\begin{aligned} I_{k+1} &= \{tk - t + l_n + 1, \ldots, tk - 1, tk\}, \\ J_{k+1} &= \{tk + 1, \ldots, tk + l_n - 1, tk + l_n\}. \end{aligned}$$

Then $|I_{k+1}| = t - l_n$ and $|J_{k+1}| = l_n$. The set I_{k+1} is a subset of N_{tk} and the set J_{k+1} contains the set $N_n \backslash N_{tk}$. The maxima (minima) on the sets $I_1, I_2, \ldots, I_k$ are weakly dependent, and the small intervals $J_1, J_2, \ldots, J_k, J_{k+1}$ can be essentially neglected.
Let

$$\Delta_1 = \mathbb{P}\left\{\bigcap_{s=1}^{k} \mathcal{A}(I_s)\right\} - \mathbb{P}\left\{\mathcal{A}(N_n)\right\}, \quad \Delta_2 = \mathbb{P}\left\{\bigcap_{s=1}^{k} \mathcal{A}(I_s)\right\} - \prod_{s=1}^{k} \mathbb{P}\left\{\mathcal{A}(I_s)\right\}$$

and

$$\Delta_3 = \prod_{s=1}^{k} \mathbb{P}\left\{\mathcal{A}(I_s)\right\} - \prod_{s=1}^{k} \mathbb{P}\left\{\mathcal{A}(K_s)\right\},$$

where $\mathcal{A}(I)$ is defined as in Lemma 4.1. The first term Δ_1 is non-negative and further

$$\begin{aligned}
\Delta_1 \quad &\le \quad \sum_{s=1}^{k+1} \left(\mathbb{P}\left\{m(J_s, \boldsymbol{\beta}) \le v_n(y_2)\right\} + \mathbb{P}\left\{u_n(x_2) < M(J_s, \boldsymbol{\beta})\right\}\right) \\
&\quad\quad + (k+1)\left(\mathbb{P}\left\{m(J_1) \le v_n(y_1)\right\} + \mathbb{P}\left\{u_n(x_1) < M(J_1)\right\}\right) \\
&\le \quad 2(k+1) l_n \Big(F(v_n(y_2)) + \overline{F}(u_n(x_2))\Big).
\end{aligned}$$

By Lemma 4.1 we have

$$|\Delta_2| \le (k-1)\alpha_{n,l_n}.$$

Next since $|\prod_{s=1}^{k} a_s - \prod_{s=1}^{k} b_s| \le \sum_{s=1}^{k} |a_s - b_s|$ holds for all $|a_s| \le 1$, $|b_s| \le 1$, $s = 1, \ldots, k$ we obtain

$$\begin{aligned}
0 \le \Delta_3 \quad &\le \quad \sum_{s=1}^{k} \left(\mathbb{P}\left\{m(J_s, \boldsymbol{\beta}) \le v_n(y_2)\right\} + \mathbb{P}\left\{u_n(x_2) < M(J_s, \boldsymbol{\beta})\right\}\right) \\
&\quad\quad + k\left(\mathbb{P}\left\{m(J_1) \le v_n(y_1)\right\} + \mathbb{P}\left\{u_n(x_1) < M(J_1)\right\}\right) \\
&\le \quad 2k l_n \Big(F(v_n(y_2)) + \overline{F}(u_n(x_2))\Big)
\end{aligned}$$

and thus the claim follows. □

Proof of Theorem 2.1 Define in the following $\Psi_n(z_1, z_2) = \overline{F}(u_n(z_1)) + F(v_n(z_2))$,

$$P(K_s, \boldsymbol{\varepsilon}) = \mathbb{P}\left\{v_n(y_2) < m(K_s, \boldsymbol{\varepsilon}) \le M(K_s, \boldsymbol{\varepsilon}) \le u_n(x_2), v_n(y_1) < m(K_s) \le M(K_s) \le u_n(x_1)\right\}$$

for $1 \le s \le k$ and

$$P(n, \boldsymbol{\varepsilon}) = \mathbb{P}\left\{v_n(y_2) < m_n(\boldsymbol{\varepsilon}) \le M_n(\boldsymbol{\varepsilon}) \le u_n(x_2), v_n(y_1) < m_n \le M_n \le u_n(x_1)\right\}.$$

We have the following upper bound

$$\begin{aligned}
&\left| P(n, \boldsymbol{\varepsilon}) - \mathbb{E}\left\{\prod_{s=1}^{k} \left[1 - \frac{\mathcal{P} n \Psi_n(x_2, y_2) + (1-\mathcal{P}) n \Psi_n(x_1, y_1)}{k}\right]\right\}\right| \\
\le \quad & \sum_{r=0}^{2^k - 1} \sum_{\boldsymbol{\beta} \in \{0,1\}^n} \mathbb{E}\left\{\left| P(n, \boldsymbol{\beta}) - \prod_{s=1}^{k} \left[1 - \frac{\mathcal{P} n \Psi_n(x_2, y_2) + (1-\mathcal{P}) n \Psi_n(x_1, y_1)}{k}\right]\right| \mathbb{I}(B_{r,k,\boldsymbol{\beta},n})\right\} \\
\le \quad & E_1 + E_2 + E_3,
\end{aligned}$$

where

$$E_1 = \sum_{r=0}^{2^k - 1} \sum_{\boldsymbol{\beta} \in \{0,1\}^n} \mathbb{E}\left\{\left| P(n, \boldsymbol{\beta}) - \prod_{s=1}^{k} P(K_s, \boldsymbol{\beta})\right| \mathbb{I}(B_{r,k,\boldsymbol{\beta},n})\right\},$$

$$E_2=\sum_{r=0}^{2^k-1}\sum_{\boldsymbol{\beta}\in\{0,1\}^n}\mathbb{E}\left\{\left|\prod_{s=1}^{k}P(K_s,\boldsymbol{\beta})-\prod_{s=1}^{k}\left[1-\frac{\frac{r}{2^k}n\Psi_n(x_2,y_2)+(1-\frac{r}{2^k})n\Psi_n(x_1,y_1)}{k}\right]\right|\mathbb{I}(B_{r,k,\boldsymbol{\beta},n})\right\}$$

and

$$\begin{aligned}E_3 \;=\; & \sum_{r=0}^{2^k-1}\sum_{\boldsymbol{\beta}\in\{0,1\}^n}\mathbb{E}\left\{\left|\prod_{s=1}^{k}\left[1-\frac{\frac{r}{2^k}n\Psi_n(x_2,y_2)+(1-\frac{r}{2^k})n\Psi_n(x_1,y_1)}{k}\right]\right.\right.\\ & \left.\left.-\prod_{s=1}^{k}\left[1-\frac{\mathcal{P}n\Psi_n(x_2,y_2)+(1-\mathcal{P})n\Psi_n(x_1,y_1)}{k}\right]\right|\mathbb{I}(B_{r,k,\boldsymbol{\beta},n})\right\}.\end{aligned}$$

Since by the assumptions

$$\lim_{n\to\infty}n\Psi_n(x_2,y_2)=-\ln G(x_2)-\ln\overline{H}(y_2),$$

and

$$\lim_{n\to\infty}\alpha_{n,l_n}=0,\quad \lim_{n\to\infty}l_n/n=0,$$

then Lemma 4.2 implies

$$E_1\le(k-1)\alpha_{n,l_n}+(4k+2)\frac{l_n}{n}n\Psi_n(x_2,y_2)\to 0 \tag{4.2}$$

as $n\to\infty$. Next, for $0\le r\le 2^k-1$

$$\begin{aligned}&\left[1-\frac{rt}{2^k}\Psi_n(x_2,y_2)+t\left(1-\frac{r}{2^k}\right)\Psi_n(x_1,y_1)\right]+\left[\frac{\sum_{j\in K_s}\beta_j}{t}-\frac{r}{2^k}\right]tF_n(\boldsymbol{x},\boldsymbol{y})\\ \le\;& P(K_s,\boldsymbol{\beta})\\ \le\;& \left[1-\frac{rt}{2^k}\Psi_n(x_2,y_2)+t\left(1-\frac{r}{2^k}\right)\Psi_n(x_1,y_1)\right]\\ &+t\sum_{j=2}^{t}\mathbb{P}\left\{A_{s1},A_{sj}\right\}+\left[\frac{\sum_{j\in K_s}\beta_j}{t}-\frac{r}{2^k}\right]tF_n(\boldsymbol{x},\boldsymbol{y}),\end{aligned}$$

where

$$F_n(\boldsymbol{x},\boldsymbol{y})=F(u_n(x_2))-F(u_n(x_1))-F(v_n(y_2))+F(v_n(y_1))$$

and

$$A_{sj}=\{X_{(s-1)t+j}>u_n(x_2)\}\cup\{X_{(s-1)t+j}\le v_n(y_2)\},\quad j\in\{1,\dots,t\}.$$

Hence, Lemma 3 in [14] implies

$$\begin{aligned}E_2\;\le\;&\sum_{r=0}^{2^k-1}\sum_{\boldsymbol{\beta}\in\{0,1\}^n}\sum_{s=1}^{k}\mathbb{E}\left\{\left|P(K_s,\boldsymbol{\beta})-\left[1-\frac{\frac{r}{2^k}n\Psi_n(x_2,y_2)+(1-\frac{r}{2^k})n\Psi_n(x_1,y_1)}{k}\right]\right|\mathbb{I}(B_{r,k,\boldsymbol{\beta},n})\right\}\\ \le\;&\sum_{r=0}^{2^k-1}\sum_{\boldsymbol{\beta}\in\{0,1\}^n}\sum_{s=1}^{k}\mathbb{E}\left\{\left|\sum_{j\in K_s}\frac{\beta_j}{t}-\frac{r}{2^k}\right|\frac{n}{k}F_n(\boldsymbol{x},\boldsymbol{y})\,\mathbb{I}(B_{r,k,\boldsymbol{\beta},n})\right\}+n\sum_{j=2}^{t}\mathbb{P}\left\{A_{11},A_{1j}\right\}\\ \le\;&\sum_{r=0}^{2^k-1}\sum_{s=1}^{k}\mathbb{E}\left\{\left|\sum_{j\in K_s}\frac{\beta_j}{t}-\frac{r}{2^k}\right|\mathbb{I}(B_{r,k})\right\}\frac{n}{k}F_n(\boldsymbol{x},\boldsymbol{y})+n\sum_{j=2}^{t}\mathbb{P}\left\{A_{11},A_{1j}\right\}\\ \le\;&\sum_{s=1}^{k}\left[2(2s-1)\left(d\left(\frac{S_{ts}}{ts},\mathcal{P}\right)+d\left(\frac{S_{t(s-1)}}{t(s-1)},\mathcal{P}\right)\right)+\frac{1}{2^k}\right]\frac{n}{k}F_n(\boldsymbol{x},\boldsymbol{y})+n\sum_{j=2}^{t}\mathbb{P}\left\{A_{11},A_{1j}\right\},\end{aligned}$$

where $d(X,Y)=\inf\{\epsilon,\mathbb{P}\{|X-Y|>\epsilon\}<\epsilon\}$. Since $\lim_{t\to\infty}d\left(\frac{S_{ts}}{ts},\mathcal{P}\right)=0$ and

$$\lim_{n\to\infty}nF_n(\boldsymbol{x},\boldsymbol{y})=\ln G(x_2)-\ln G(x_1)+\ln\overline{H}(y_2)-\ln\overline{H}(y_1)$$

taking the limit as $n \to \infty$ and then as $t \to \infty$, we get

$$\limsup_{n\to\infty} E_2 \le \frac{1}{2^k}[\ln G(x_2) - \ln G(x_1) + \ln \overline{H}(y_2) - \ln \overline{H}(y_1)] + ko(\frac{1}{k}). \tag{4.3}$$

Further, as $n \to \infty$

$$\begin{aligned}
E_3 &\le \sum_{r=0}^{2^k-1} \sum_{\boldsymbol{\beta}\in\{0,1\}^n} \sum_{s=1}^{k} \mathbb{E}\left\{ \left| \frac{r}{2^k} - \mathcal{P} \right| \frac{n}{k}(2 - F(u_n(x_2)) + F(v_n(y_2)) - F(u_n(x_1)) + F(v_n(y_1)))\, \mathbb{I}(B_{r,k,\boldsymbol{\beta},n}) \right\} \\
&= \sum_{r=0}^{2^k-1} \mathbb{E}\left\{ \left| \frac{r}{2^k} - \mathcal{P} \right| \mathbb{I}(B_{r,k}) \right\} n(2 - F(u_n(x_2)) + F(v_n(y_2)) - F(u_n(x_1)) + F(v_n(y_1))) \\
&\le \frac{1}{2^k} n(2 - F(u_n(x_2)) + F(v_n(y_2)) - F(u_n(x_1)) + F(v_n(y_1))) \\
&\to \frac{1}{2^k}[-\ln G(x_2) - \ln G(x_1) - \ln \overline{H}(y_2) - \ln \overline{H}(y_1)],
\end{aligned}$$

which together with (4.2) and (4.3) imply

$$\begin{aligned}
&\left| \limsup_{n\to\infty} P(n, \boldsymbol{\varepsilon}) - \mathbb{E}\left(1 - \frac{-\ln G^{\mathcal{P}}(x_2) - \ln(\overline{H}(y_2))^{\mathcal{P}} - \ln G^{1-\mathcal{P}}(x_2) - \ln(\overline{H}(y_2))^{1-\mathcal{P}}}{k} \right)^k \right| \\
&\le \quad ko\left(\frac{1}{k}\right) + \frac{-\ln G(x_1) - \ln(\overline{H}(y_1))}{2^{k-1}}.
\end{aligned}$$

The proof is then established by letting $k \to \infty$. $\square$

5 Appendix

We show below that the stationary FGM random sequence in Example 4 satisfies both conditions $D(z_n(\boldsymbol{x}, \boldsymbol{y}))$ and $D'(u_n(x), v_n(y))$.

Since for the normalization $u_n(x)$ and $v_n(y)$ the condition (2.1) holds, we have

$$\lim_{n\to\infty} n\overline{F}(u_n(x)) = -\ln G(x), \qquad \lim_{n\to\infty} nF(v_n(y)) = -\ln \overline{H}(y) \tag{5.1}$$

and further

$$\lim_{n\to\infty} \overline{F}(u_n(x)) = 0, \qquad \lim_{n\to\infty} F(v_n(y)) = 0. \tag{5.2}$$

For any $I \subset \{1, \dots, n\}$ with m elements

$$\mathbb{P}\{X_i \le u_n(x), i \in I\} = F^m(u_n(x)) \left(1 + \sum_{j<l\in I} \alpha(l-j)\overline{F}^2(u_n(x)) \right),$$

$$\mathbb{P}\{X_i > v_n(y), i \in I\} = \overline{F}^m(v_n(y)) \left(1 + \sum_{j<l\in I} \alpha(l-j)F^2(v_n(y)) \right). \tag{5.3}$$

By some tedious (but straightforward calculations) we establish also the following

$$\mathbb{P}\{v_n(y) < X_i \le u_n(x), i \in I\} = (F(u_n(x)) - F(v_n(y)))^m \left(1 + \sum_{j<l\in I} \alpha(l-j)(\Delta_n x, y)^2 \right), \tag{5.4}$$

with $\Delta_n(x, y) = \overline{F}(u_n(x)) - F(v_n(y)$. By Lemma 1 in [10] for any $I, J \subset \{1, \dots, n\}$ we obtain

$$|\mathbb{P}\{X_i \le u_n(x), i \in I \cup J\} - \mathbb{P}\{X_i \le u_n(x), i \in I\}\, \mathbb{P}\{X_i \le u_n(x), i \in J\}| \to 0 \tag{5.5}$$

as $n \to \infty$. Using similar arguments as Lemma 1 in [10], since (3.5) holds, for any $\epsilon > 0$ and with suitable l_0 such that $|\alpha(l-j)| < \epsilon$ for $l - j > l_0$, the absolute value of the double sum in (5.3) can be bounded by

$$\begin{aligned} & \sum_{j<l\in I, l-j\le l_0} |\alpha(l-j)| F^2(v_n(y)) + \sum_{j<l\in I, l-j>l_0} \epsilon F^2(v_n(y)) \\ = \; & O\left(l_0 n F^2(v_n(y))\right) + O\left(\epsilon n^2 F^2(v_n(y))\right). \end{aligned}$$

Using (5.1) and (5.2) this double sum converges to 0 as $n \to \infty$. Thus

$$\begin{aligned} & |\mathbb{P}\{X_i > v_n(y), i \in I \cup J\} - \mathbb{P}\{X_i > v_n(y), i \in I\} \mathbb{P}\{X_i > v_n(y), i \in J\}| \\ \le \; & \prod_{i \in I \cup J} \overline{F}(v_n(x))(1 + o(1) - (1 + o(1))^2) \to 0 \end{aligned} \tag{5.6}$$

as $n \to \infty$. Similarly, the absolute value of the double sum in (5.4) can be bounded by

$$\begin{aligned} & \sum_{j<l\in I, l-j\le l_0} |\alpha(l-j)| (\Delta_n(x,y))^2 + \sum_{j<l\in I, l-j>l_0} \epsilon (\Delta_n(x,y))^2 \\ = \; & O\left(l_0 n (\Delta_n(x,y))^2\right) + O\left(\epsilon n^2 (\Delta_n(x,y))^2\right). \end{aligned}$$

Using (5.1) and (5.2) again, we have

$$\begin{aligned} & |\mathbb{P}\{v_n(y) < X_i \le u_n(x), i \in I \cup J\} - \mathbb{P}\{v_n(y) < X_i \le u_n(x), i \in I\} \mathbb{P}\{v_n(y) < X_i \le u_n(x), i \in J\}| \\ \le \; & \prod_{i \in I \cup J} (F(u_n(x)) - F(v_n(y)))(1 + o(1) - (1 + o(1))^2) \to 0 \end{aligned} \tag{5.7}$$

as $n \to \infty$. Consequently, (5.5)-(5.7) establish condition $D(z_n(\boldsymbol{x}, \boldsymbol{y}))$.

Next we need to prove that for such stationary FGM random sequence the condition $D'(u_n(x), v_n(y))$ holds. We only prove the first sum in the condition $D'(u_n(x), v_n(y))$ tending to 0, the proof of the other sums tending to 0 are the same. By (5.1)

$$\begin{aligned} & \limsup_{n\to\infty} n \sum_{j=1}^{[n/k]} \mathbb{P}\{X_1 > u_n(x), X_{j+1} > u_n(x)\} \\ = \; & \limsup_{n\to\infty} n \sum_{j=1}^{[n/k]} \overline{F}^2(u_n(x))(1 + \alpha(j) F^2(u_n(x))) \\ \le \; & \limsup_{n\to\infty} \frac{n^2}{k} \overline{F}^2(u_n(x))(1 + \alpha^* F^2(u_n(x))) \\ = \; & \frac{1}{k} (\ln G(x))^2 (1 + \alpha^*), \end{aligned}$$

where $\alpha^* = \max_{j\ge1} \alpha(j)$. Letting $k \to \infty$, this limit tends to 0. Hence, the condition $D'(u_n(x), v_n(y))$ holds.

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