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# Analyzing the interest rate risk of equity-indexed annuities via scenario matrices ☆

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ABSTRACT

The financial return of equity-indexed annuities depends on an underlying fund or investment portfolio complemented by an investment guarantee. We discuss a so-called cliquet-style or ratchet-type guarantee granting a minimum annual return. Its path-dependent payoff complicates valuation and risk management, especially if interest rates are modelled stochastically. We develop a novel scenario-matrix (SM) method. In the example of a Vasicek-Black-Scholes model, we derive closed-form expressions for the value and moment-generating function of the final payoff in terms of the scenario matrix. This allows efficient evaluation of values and various risk measures, avoiding Monte-Carlo simulation or numerical Fourier inversion. In numerical tests, this procedure proves to converge quickly and outperforms the existing approaches in the literature in terms of computation time and accuracy.

## 1. Introduction

We discuss equity-indexed life insurance policies that guarantee a minimum annual return and allow to participate in returns of an underlying fund or investment portfolio. (see, e.g., Briys and Varenne, 1994; Grosen and Jørgensen, 2000; Graf et al., 2011; Bacinello et al., 2011). The payoff assures the policyholder a protected investment with potential additional gains while the insurance company takes a share of the surplus to cover expenses and reach profit targets. Variants that are particularly common in central Europe are cliquet-style guarantees that provide a constant minimum return on the contract's investment. In the literature, these type of guarantees are also referred to as ratchet-type. The path-dependent nature of this product makes its valuation and risk management a complex task.

Commonly, these policies are settled with a long-term maturity. Due to the lack or high cost of long-term bonds, insurance companies can typically not fully eliminate their interest rate risk. The value of long-term guarantees depends substantially on interest rate risk and cliquet-options are prone to change in the interest rate environment. In the past, we have observed periods of low interest rates as well as rapid increases in the interest rate level. That is why it is advisable to consider interest rate risk for these types of products. Technically, modelling interest rates stochastically leads to a dependence between consecutive annual returns, complicating the analysis of cliquet-style guarantees. We contribute to the tractability of these products by introducing a novel scenario-matrix (SM) method.

The valuation and contract design of these types of guarantees is a frequently discussed problem. In Lévy settings, that is, under stationary and independent increments, the contract's value can be expressed in terms of integral equations or Fourier algorithms, see Kassberger et al. (2008), Alonso-García et al. (2017) or, for a non-compounding variant of cliquet-guarantees, Bernard and Li (2013). Korn et al. (2017) use a central-limit type argument to approximate guarantee values in Heston's stochastic volatility model. Miltersen and Persson (2003) consider a compounding type and investigate how to set up a mechanism for sharing the surplus between customer and insurer. Bacinello (2001) and Barbarin and Devolder (2005) outline strategies for choosing contract parameters such as the annual guarantee level and the participation rate.

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The literature on cliquet guarantees in a stochastic interest rate environment is rather scarce, probably due to the technical challenges. While the case of a maturity guarantee (e.g., Lin and Tan, 2003; Barbarin and Devolder, 2005; Bernard et al., 2005) can be tackled by standard option pricing theory, the cliquet-style variant requires more elaborate techniques. Existing results mostly rely on simulation (e.g., Zaglauer and Bauer, 2008; Deelstra and Rayée, 2013; Hieber et al., 2019, and many others). The case of a regime switching interest rate where the valuation can use Fourier techniques (e.g., Fan et al., 2015; Ignatieva et al., 2016; Hieber, 2017; Cui et al., 2017) or Erlangization techniques (e.g., Deelstra and Hieber, 2023) is analytically rather tractable. For the special case of two subperiods, Persson and Aase (1997) and Miltersen and Persson (1999) derive premiums in closed-form in a generalization of the Vasicek-Black-Scholes model. For the multi-period case, Kijima and Wong (2007) represent the value of a cliquet option in terms of a multi-dimensional normal integral. Further, a computationally more efficient approach is the PROJ method by Kirkby (2015), a frame projection approach that relies on density approximations through fast Fourier transformations. This idea is applied by Cui et al. (2017) to a cliquet option where the underlying can incorporate stochastic volatility models with jumps that are approximated by Markov chain processes. Kirkby (2023) uses the same technique for stochastic interest rate models.

In contrast to most of the previous articles, we do not only focus on the valuation problem, but also analyse the (interest rate) risk of cliquet-style guarantees. The 2009-2022 low interest rate period has revealed that this risk can be substantial. Further, the SM method relies neither on simulations nor on Fourier transformation algorithms. To achieve this goal, we focus on a financial market that is modelled by a Vasicek-Black-Scholes model, being less general than some of the related literature. Conditioning on the annual interest rates, this framework allows us to derive many quantities like the conditional moment-generating function of the logarithmic return in closed-form. This allows us to get values and higher moments of the payoff of the cliquet-option in terms of the scenario matrix (SM). In combination with techniques from Carr and Madan (1999) we obtain quantiles and the Value-at-Risk of cliquet-style guarantees. This proves to conveniently well approximate the true prices and risk measures. The final approximation requires the computation of (scenario) matrix multiplications only and turns out to be faster than benchmark techniques like the PROJ method.

The remainder of this article is organized as follows: In Section 2, we describe the underlying dynamics of the financial market and the payoff of the cliquet option that we want to consider. The SM method for efficiently estimating the moment-generating function and quantiles of the log-payoff is introduced in Section 3. A brief introduction to techniques like Fourier pricing or the PROJ method and its application to cliquet options are given in Section 4. These serve as our benchmarks for accuracy and time efficiency, which we consider in numerical experiments in Section 5. We also investigate the sensitivity of risk metrics like the expected discounted payoff or the Value-at-Risk with respect to different parameters.

## 2. Equity-indexed annuities and financial market

We look at equity-indexed annuities with a single premium $P_0$ paid at time 0. Their payoff depends on the financial market. To describe the market, we consider a probability space $(\Omega, \mathscr{A}, \mathbb{P}, \mathscr{F}_t)$, where $\mathbb{P}$ is the real world measure and $\mathscr{F}_t = \sigma(W_s^{(1)}, W_s^{(2)}, s \leqslant t)$ is the natural filtration for two independent Brownian motions $W_t^{(1)}$ and $W_t^{(2)}$. The stochastic interest rate process follows

$$\mathrm{d}r_t = \kappa\left(\theta^* - r_t\right)\mathrm{d}t + \sigma_r\,\mathrm{d}W_t^{(1)}, \qquad r_0 \in \mathbb{R}, \tag{1}$$

under the real-world measure $\mathbb{P}$ with mean-reversion speed $\kappa \in \mathbb{R}$, long-term mean interest rate level $\theta^* \in \mathbb{R}$ and volatility $\sigma_r > 0$. The financial market contains a risky asset $\{S_t\}_{t\geq 0}$ with dynamics under the real-world measure $\mathbb{P}$

$$\mathrm{d}S_t = S_t\left(\mu_t\,\mathrm{d}t + \sigma_S\left(\rho\mathrm{d}W_t^{(1)} + \sqrt{1-\rho^2}\,\mathrm{d}W_t^{(2)}\right)\right), \qquad S_0 = 1, \tag{2}$$

where $\mu_t$ is the drift term, $\sigma_S > 0$ the volatility and $\rho \in [-1, 1]$ the correlation parameter.

Starting from the single premium $P_0$, every year, the policyholder account is continuously compounded by a rate of return. This rate is determined by the maximum of a participation share $\alpha \in (0,1)$ of the log-returns of the underlying $\ln(S_t/S_{t-1})$ and the guaranteed rate $g$. The maturity of the contract is $T > 0$. The compounding nature of the cliquet option leads to the time-$T$ payoff to the policyholders

$$Y_T = P_0 \cdot \prod_{t=1}^{T} \max\left(\mathrm{e}^g, \left(\frac{S_t}{S_{t-1}}\right)^{\alpha}\right). \tag{3}$$

We also introduce the risk-free discounted payoff as

$$Z_T = \mathrm{e}^{-\int_0^T r_s\,\mathrm{d}s} \cdot Y_T .$$

For the valuation of equity-indexed annuities, we assume that there exist constant risk premiums $\lambda_r, \lambda_S \in \mathbb{R}$ for both the interest rate (1) and asset process (2) (see also Barbarin and Devolder, 2005; Graf et al., 2011; Hieber et al., 2019). More specifically, we suppose $\theta^* = \theta + \frac{\lambda_r \sigma_r}{\kappa}$ and $\mu_t = r_t + \lambda_S$. Analogously to Graf et al. (2011), this uniquely determines the valuation measure via the change of measure:

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathscr{F}_t} = \mathrm{e}^{-\lambda_r W_t^{(1)} - \frac{1}{2}\lambda_r^2 t - \frac{\lambda_S - \rho\lambda_r\sigma_S}{\sqrt{1-\rho^2}\sigma_S} W_t^{(2)} - \frac{1}{2}\frac{(\lambda_S - \rho\lambda_r\sigma_S)^2}{(1-\rho^2)\sigma_S^2} t}. \tag{4}$$

Note that, given this assumption, the short rate process follows a Vasicek model under both $\mathbb{P}$ and $\mathbb{Q}$. Additionally, the dynamics for the risky asset under $\mathbb{Q}$ are as in (2) with drift term $\mu_t$ replaced by the interest rate $r_t$. The long-term mean interest rate under $\mathbb{Q}$ is $\theta$.

To analyse the interest rate risk of equity-indexed annuities, we want to efficiently compute the expected discounted payoff $\mathbb{E}_{\mathbb{Q}}[Z_T]$, the variance of the discounted payoff $\mathrm{Var}_{\mathbb{P}}[Z_T]$ and the probability that the terminal payoff exceeds certain thresholds $G > 0$, that is $\mathbb{P}(Y_T > G)$. The latter probability allows us to analyse contracts with an additional maturity guarantee $G$, that is payoffs of the form $\max(Y_T, G)$. We can also derive Value-at-Risk and downside-risk related quantities of interest.

### 3. Moment-generating function

To study these products more closely, we first derive the moment-generating function of the logarithm of the discounted payoff random variable $Z_T$. To simplify the notation, define the logarithm of the discounted annual returns $L_{t-1,t}$ as

$$L_{t-1,t} := \ln\left(\mathrm{e}^{-\int_{t-1}^{t} r_s\,\mathrm{d}s}\cdot\max\left(\mathrm{e}^{g},\left(\frac{S_t}{S_{t-1}}\right)^{\alpha}\right)\right) = \max\left(g,\,\alpha\cdot\ln\left(\frac{S_t}{S_{t-1}}\right)\right) - \int_{t-1}^{t} r_s\,\mathrm{d}s\,,$$

so that

$$Z_T = P_0\cdot\prod_{t=1}^{T}\mathrm{e}^{L_{t-1,t}}\,.$$

The moment-generating function $\mathscr{L}(u) := \mathbb{E}\left[\mathrm{e}^{uX}\right]$ of a random variable $X$ allows us to make inferences about its distribution. The moment-generating function of the logarithm of $Z_T$ can be expressed as

$$\mathscr{L}_T(u) = \mathbb{E}\left[\mathrm{e}^{u\cdot\ln(Z_T)}\right] = \mathbb{E}\left[(Z_T)^u\right] = \mathbb{E}\left[(P_0)^u\prod_{t=1}^{T}\mathbb{E}\left[\mathrm{e}^{u\cdot L_{t-1,t}}\,\middle|\,r_{t-1},r_t\right]\right]\,. \tag{5}$$

Theorem 3.1 provides closed-form expressions for the conditional moment-generating function $\phi_{r_{t-1},r_t}(u) := \mathbb{E}\left[\mathrm{e}^{u\cdot L_{t-1,t}}\,|\,r_{t-1},r_t\right]$ in equation (5).

**Theorem 3.1** *(Conditional moment-generating function). Assume that the financial market is described by* (1) *and* (2)*. Define*

$$g_\kappa(t) := \frac{1-\mathrm{e}^{-\kappa t}}{\kappa}.$$

*The moment-generating function of the discounted annual log-returns $L_{t-1,t}$, conditional on $r_{t-1}, r_t$, is given by*

$$\begin{aligned}\phi_{r_{t-1},r_t}(u) &:= \mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{u\cdot L_{t-1,t}}\,\middle|\,r_{t-1},r_t\right]\\ &= \mathrm{e}^{-u\mu(r_{t-1},r_t)+u^2\frac{\Sigma^2}{2}}\left(\mathrm{e}^{ug}\Phi\left(\frac{\frac{g}{\alpha}-\widetilde{\mu}(0,u,r_{t-1},r_t)}{\widetilde{\Sigma}}\right)+\mathrm{e}^{u\widetilde{\mu}(0,u,r_{t-1},r_t)\alpha+\frac{1}{2}u^2\alpha^2\widetilde{\Sigma}^2}\Phi\left(\frac{-\frac{g}{\alpha}+\widetilde{\mu}(0,u,r_{t-1},r_t)+u\alpha\widetilde{\Sigma}^2}{\widetilde{\Sigma}}\right)\right),\end{aligned} \tag{6}$$

*where $\Phi$ describes the cumulative distribution function of a standard normal random variable and*

$$\begin{aligned}\mu(r_{t-1},r_t) &= \frac{g_\kappa(1)}{1+\mathrm{e}^{-\kappa}}\cdot(r_{t-1}+r_t)+\left(1-\frac{2g_\kappa(1)}{1+\mathrm{e}^{-\kappa}}\right)\cdot\theta\,,\\ \Sigma^2 &= \sigma_r^2\left(\frac{1-g_\kappa(1)}{\kappa^2}-\frac{g_\kappa(1)^2}{\kappa(1+\mathrm{e}^{-\kappa})}\right),\\ c &= \sigma_r^2\left(\frac{1-g_\kappa(1)}{\kappa^2}-\frac{g_\kappa(1)^2}{2\kappa}-\frac{g_\kappa(1)^3}{2(1+e^{-\kappa})}\right)-\frac{\rho\sigma_S\sigma_r}{\kappa}\left(\frac{2g_\kappa(1)}{1+\mathrm{e}^{-\kappa}}-1\right),\\ \widetilde{\mu}(\lambda,u,r_{t-1},r_t) &= \lambda+\frac{g_\kappa(1)-2\rho\frac{\sigma_S}{\sigma_r}\mathrm{e}^{-\kappa}}{1+\mathrm{e}^{-\kappa}}\cdot r_{t-1}+\frac{g_\kappa(1)+2\rho\frac{\sigma_S}{\sigma_r}}{1+\mathrm{e}^{-\kappa}}\cdot r_t+\left(1-2\frac{g_\kappa(1)+\rho\frac{\sigma_S}{\sigma_r}(1-\mathrm{e}^{-\kappa})}{1+\mathrm{e}^{-\kappa}}\right)\cdot\theta-c\cdot u-\frac{1}{2}\sigma_S^2\,,\\ \widetilde{\Sigma}^2 &= c+\sigma_S^2-2\rho^2\sigma_S^2\frac{g_\kappa(1)}{1+\mathrm{e}^{-\kappa}}-\frac{\rho\sigma_S\sigma_r}{\kappa}\left(\frac{2g_\kappa(1)}{1+\mathrm{e}^{-\kappa}}-1\right).\end{aligned}$$

**Proof.** See the Appendix A. □

**Remark 3.2.** (a) In (6), we determined the conditional moment-generating function under the risk neutral measure $\mathbb{Q}$. We can also derive an expression under $\mathbb{P}$ since the financial market follows a Vasicek-Black-Scholes model under both measures. Considering the change of measure as in (4), we only have to set $\lambda = \lambda_S$ and change $\theta$ to $\theta^* = \theta + \frac{\lambda_r\sigma_r}{\kappa}$ to obtain the result under $\mathbb{P}$.
(b) Note that (6) can be modified to calculate the moment-generating function of the undiscounted annual log-returns $L^*_{t-1,t} := \max(g,\,\alpha\cdot\ln(S_t/S_{t-1}))$. For this derivation, we do not change the measure as in Appendix A and we thus find

$$\mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{u\cdot L^*_{t-1,t}}\,\middle|\,r_{t-1},r_t\right] = \mathrm{e}^{ug}\Phi\left(\frac{\frac{g}{\alpha}-\widetilde{\mu}^*(0,r_{t-1},r_t)}{\widetilde{\Sigma}}\right)+\mathrm{e}^{u\widetilde{\mu}^*(0,r_{t-1},r_t)\alpha+\frac{1}{2}u^2\alpha^2\widetilde{\Sigma}^2}\Phi\left(\frac{-\frac{g}{\alpha}+\widetilde{\mu}^*(0,r_{t-1},r_t)+u\alpha\widetilde{\Sigma}^2}{\widetilde{\Sigma}}\right),$$

where $\widetilde{\mu}^*(\lambda,r_{t-1},r_t) = \widetilde{\mu}(\lambda,u,r_{t-1},r_t)+c\cdot u$. Therefore, we are also able to describe the distribution of the undiscounted payoff $Y_T$. This is, for example, needed to include caps on the ultimate payoff $Y_T$ at maturity. However, for simplicity, we continue to derive our approximation for the discounted payoff $Z_T$.
(c) We are interested in estimating quantiles of the solvency ratio $Y_T/S_T$ since this provides insightful risk measures. We have

$$\mathbb{E}\left[\mathrm{e}^{u\cdot\ln(Y_T/S_T)}\right] = \mathbb{E}\left[\left(\frac{Y_T}{S_T}\right)^u\right] = \mathbb{E}\left[\prod_{t=1}^{T}\mathbb{E}\left[\mathrm{e}^{u\cdot L^*_{t-1,t}-u\cdot\ln\left(\frac{S_t}{S_{t-1}}\right)}\,\middle|\,r_{t-1},r_t\right]\right]\,.$$

It is straightforward to derive from the proof in Appendix A that

$$\mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{u\cdot L^{*}_{t-1,t}-u\cdot\ln\left(\frac{S_t}{S_{t-1}}\right)}\,\middle|\, r_{t-1},r_t\right] = \mathrm{e}^{ug-u\widetilde{\mu}^{*}(0,r_{t-1},r_t)+\frac{1}{2}u^2\widetilde{\Sigma}^2}\,\Phi\left(\frac{\frac{g}{\alpha}-\widetilde{\mu}^{*}(0,r_{t-1},r_t)+u\widetilde{\Sigma}^2}{\widetilde{\Sigma}}\right) + \mathrm{e}^{u\widetilde{\mu}^{*}(0,r_{t-1},r_t)(\alpha-1)+\frac{1}{2}u^2(\alpha-1)^2\widetilde{\Sigma}^2}\,\Phi\left(\frac{-\frac{g}{\alpha}+\widetilde{\mu}^{*}(0,r_{t-1},r_t)+u(\alpha-1)\widetilde{\Sigma}^2}{\widetilde{\Sigma}}\right).$$

### 3.1. Approximation on a grid

Given the conditional moment-generating function (6), we still need to determine the outer expectation in (5), that is to integrate over the annual interest rates $(r_1, r_2, \ldots, r_T)$. We approximate this integral by choosing discrete grid points $-\infty < r^{(1)} < \cdots < r^{(K+1)} = \infty$, $K \in \mathbb{N}$, for the annual interest rates $r_1, \ldots, r_T$. We further set $r^{(0)} = -\infty$ and $r^{(K+1)} = \infty$. We want to choose a grid that contains $r_0$ and set $r^{(i_0)} = r_0$. A possibility on how to choose this grid is Example 3.3.

**Example 3.3** *(Choice of discretization).* In a Vasicek model, the interest rate $r_t$ is normally distributed with variance $\hat{\sigma}_t^2 = \frac{\sigma_r^2}{2\kappa}(1-\mathrm{e}^{-2\kappa t})$. We may just choose an equidistant grid around $r_0$ whose upper and lower bounds exceed the 99.9% and 0.01% quantiles of $r_T$, for example $r^{(j)} = r_0 + 8\left(j - \frac{K}{2}\right)/K \cdot \hat{\sigma}_T$, for $j = 1, 2, \ldots, K$ and uneven $K$. In this case, $i_0 = (K+1)/2$. Note, that centering around $r_0$ in this way is only reasonable if it is not too far from the long-term mean $\theta$.

We denote the interval around $r^{(j)}$, $1 \le j \le K$, by

$$\left[r^{(j)}\right] := \left(\frac{r^{(j-1)}+r^{(j)}}{2}, \frac{r^{(j)}+r^{(j+1)}}{2}\right]. \tag{7}$$

We proceed as follows: Using the same grid for each rate $r_1, \ldots, r_T$ allows us to speed up calculations significantly. For subsequent rates $(r_{t-1}, r_t)$, we apply a two-dimensional trapezoidal rule based on rectangles $r_{t-1} \times r_t = \left[r^{(i)}\right] \times \left[r^{(j)}\right]$, $i, j = 1, 2, \ldots, K$. On each of these rectangles, we assume that the conditional moment-generating function is constant, approximated by $\phi_{r^{(i)},r^{(j)}}(u)$, that is:

$$\mathbb{E}\left[\mathrm{e}^{u\cdot L_{t-1,t}}\,\middle|\, r_{t-1}\in\left[r^{(i)}\right], r_t\in\left[r^{(j)}\right]\right] \approx \mathbb{E}\left[\mathrm{e}^{u\cdot L_{t-1,t}}\,\middle|\, r_{t-1}=r^{(i)}, r_t=r^{(j)}\right] = \phi_{r^{(i)},r^{(j)}}(u). \tag{8}$$

The probability mass over the rectangle $r_{t-1} \times r_t \in \left[r^{(i)}\right] \times \left[r^{(j)}\right]$ is approximately the same as the integrated density over $r_{t-1} \times r_t \in r^{(i)} \times \left[r^{(j)}\right]$ and thus we find the probabilities

$$\mathbb{P}\left(r_t\in\left[r^{(j)}\right]\,\middle|\, r_{t-1}\in\left[r^{(i)}\right]\right) \approx \mathbb{P}\left(\frac{r^{(j-1)}+r^{(j)}}{2} < r_t \le \frac{r^{(j)}+r^{(j+1)}}{2}\,\middle|\, r_{t-1}=r^{(i)}\right) = \Phi\left(\frac{\frac{r^{(j)}+r^{(j+1)}}{2}-\mu_i}{\hat{\sigma}_1}\right) - \Phi\left(\frac{\frac{r^{(j-1)}+r^{(j)}}{2}-\mu_i}{\hat{\sigma}_1}\right) =: p_{ij}, \tag{9}$$

for any combination $i \times j \in \{1, 2, \ldots, K\} \times \{1, 2, \ldots, K\}$, where $\mu_i = \mathrm{e}^{-\kappa}\cdot r^{(i)} + \theta(1-\mathrm{e}^{-\kappa})$ and $\hat{\sigma}_1^2 = \frac{\sigma_r^2}{2\kappa}(1-\mathrm{e}^{-2\kappa})$. These probabilities together with the conditional moment-generating functions define the scenario matrix $\left(\mathcal{Q}(u)\right)_{i,j} \approx \left(\phi_{r^{(i)},r^{(j)}}(u)\cdot p_{ij}\right)_{i,j}$. The scenario probabilities $p_{ij}$ in each row of this matrix sum up to 1, that is $\sum_{i=1}^{K} p_{ij} = 1$. The moment-generating function of the logarithm of the discounted payoff variable $Z_T$ can be expressed in terms of this matrix, see Theorem 3.4.

**Theorem 3.4** *(Moment-generating function $\mathcal{L}_T(u)$). We approximate the moment-generating function $\mathcal{L}_T(u)$ of the logarithm of the time-0 value of the payoff $Y_T$ by $\hat{\mathcal{L}}_T(u)$ through a matrix iteration*

$$\mathcal{L}_T(u) = (P_0)^u \cdot \mathbf{1}'_{i_0}\,\mathcal{Q}(u)^T\mathbf{1} \approx (P_0)^u \cdot \mathbf{1}'_{i_0}\,\hat{\mathcal{Q}}(u)^T\mathbf{1} = \hat{\mathcal{L}}_T(u), \tag{10}$$

*where $\mathbf{1} = (1, \ldots, 1)'$ and $'$ denotes the transposition of a vector. $\mathbf{1}_{i_0}$ is a unit vector that has an entry of 1 in the $i_0$-th position. For a matrix $\mathcal{Q} \in \mathbb{R}^{K\times K}$, note that $\left(\mathcal{Q}\right)^T := \mathcal{Q}\cdot\mathcal{Q}\cdots\mathcal{Q}$. The scenario matrix $\mathcal{Q}(u)$ has entries*

$$\left(\mathcal{Q}(u)\right)_{i,j} = \mathbb{E}\left[\mathrm{e}^{u\cdot L_{t-1,t}}\,\middle|\, r_{t-1}\in\left[r^{(i)}\right], r_t\in\left[r^{(j)}\right]\right]\cdot\mathbb{P}\left(r_t\in\left[r^{(j)}\right]\,\middle|\, r_{t-1}\in\left[r^{(i)}\right]\right). \tag{11}$$

*Using (8), (9), these entries can be approximated as $\left(\mathcal{Q}(u)\right)_{i,j} \approx \left(\hat{\mathcal{Q}}(u)\right)_{i,j} = \phi_{r^{(i)},r^{(j)}}(u)\cdot p_{ij}$, where $p_{ij}$ is as in (9) and $\phi_{r^{(i)},r^{(j)}}(u)$ was defined in Theorem 3.1.*

**Proof.** We use the fact that, conditional on the annual interest rates $r_1, \ldots, r_T$, the annual increments of our underlying are independent. Adapting a conditioning idea from Hieber (2017) for regime switching models to our setting yields

$$\begin{aligned}
\hat{\mathcal{L}}_T(u) =\ & (P_0)^u \cdot \mathbb{E}\left[\mathrm{e}^{-u\cdot\int_0^T r_s\,\mathrm{d}s}\mathrm{e}^{u\cdot\sum_{t=1}^T \max\left(g,\alpha\ln\left(\frac{S_t}{S_{t-1}}\right)\right)}\right] = (P_0)^u\cdot\mathbb{E}\left[\mathbb{E}\left[\prod_{t=1}^T \mathrm{e}^{u\cdot L_{t-1,t}}\,\middle|\, r_1\right]\right] \\
=\ & (P_0)^u \cdot \sum_{i=1}^K \mathbb{E}\left[\mathbb{1}_{\left\{r_1\in\left[r^{(i)}\right]\right\}}\mathrm{e}^{u\cdot L_{0,1}}\right]\mathbb{E}\left[\prod_{t=2}^T \mathrm{e}^{u\cdot L_{t-1,t}}\,\middle|\, r_1\in\left[r^{(i)}\right]\right] \\
=\ & (P_0)^u \cdot \left(\mathbb{E}\left[\mathbb{1}_{\left\{r_1\in\left[r^{(1)}\right]\right\}}\mathrm{e}^{u\cdot L_{0,1}}\right]\cdots\mathbb{E}\left[\mathbb{1}_{\left\{r_1\in\left[r^{(K)}\right]\right\}}\mathrm{e}^{u\cdot L_{t-1,t}}\right]\right)\cdot\begin{pmatrix}\mathbb{E}\left[\prod_{t=2}^T \mathrm{e}^{u\cdot L_{t-1,t}}\,\middle|\, r_1\in\left[r^{(1)}\right]\right] \\ \vdots \\ \mathbb{E}\left[\prod_{t=2}^T \mathrm{e}^{u\cdot L_{t-1,t}}\,\middle|\, r_1\in\left[r^{(K)}\right]\right]\end{pmatrix}
\end{aligned}$$

$$= (P_0)^u \cdot \mathbf{1}'_{i_0} \cdot \mathcal{Q}(u) \begin{pmatrix} \sum_{j=1}^K \mathbb{E}\left[\mathbb{1}_{\left\{r_2 \in [r^{(j)}]\right\}} \mathrm{e}^{u \cdot L_{1,2}} \,\middle|\, r_1 \in \left[r^{(1)}\right]\right] \mathbb{E}\left[\prod_{t=3}^T \mathrm{e}^{u\, L_{t-1,t}} \,\middle|\, r_2 \in \left[r^{(j)}\right]\right] \\ \vdots \\ \sum_{j=1}^K \mathbb{E}\left[\mathbb{1}_{\left\{r_2 \in [r^{(j)}]\right\}} \mathrm{e}^{u \cdot L_{1,2}} \,\middle|\, r_1 \in \left[r^{(K)}\right]\right] \mathbb{E}\left[\prod_{t=3}^T \mathrm{e}^{u \cdot L_{t-1,t}} \,\middle|\, r_2 \in \left[r^{(j)}\right]\right] \end{pmatrix}$$

$$= (P_0)^u \cdot \mathbf{1}'_{i_0} \left(\mathcal{Q}(u)\right)^2 \cdot \begin{pmatrix} \mathbb{E}\left[\prod_{t=3}^T \mathrm{e}^{u \cdot L_{t-1,t}} \,\middle|\, r_2 \in \left[r^{(1)}\right]\right] \\ \vdots \\ \mathbb{E}\left[\prod_{t=3}^T \mathrm{e}^{u \cdot L_{t-1,t}} \,\middle|\, r_2 \in \left[r^{(K)}\right]\right] \end{pmatrix} = \cdots = (P_0)^u \cdot \mathbf{1}'_{i_0}\, \mathcal{Q}(u)^T \mathbf{1}\,.$$

The entries of $\mathcal{Q}(u)$ are

$$\left(\mathcal{Q}(u)\right)_{i,j} = \mathbb{E}\left[\mathrm{e}^{u \cdot L_{t-1,t}} \,\middle|\, r_{t-1} \in \left[r^{(i)}\right], r_t \in \left[r^{(j)}\right]\right] \cdot \mathbb{P}\left(r_t \in \left[r^{(j)}\right] \,\middle|\, r_{t-1} \in \left[r^{(i)}\right]\right).$$

Note that these do not depend on the time $t$ since, conditioned on the same preceding and succeeding interest rates $r^{(i)}$ and $r^{(j)}$, the annual returns and discount factors are identically distributed. □

**Remark 3.5.** In practice, we avoid computing $T$ matrix-matrix products in (10) by iteratively computing $T$ matrix-vector products $\hat{\mathcal{Q}}(u)^T \mathbf{1} = \hat{\mathcal{Q}}(u)(\hat{\mathcal{Q}}(u)^{T-1}\mathbf{1})$. Therefore, our algorithm implicitly computes the moment-generating function for the same policy with all maturities in one run. This allows to incorporate mortality and cancellation probabilties that are independent of the financial market prior to maturity $T$.

Letting $K \to \infty$ and thus reducing the size of the intervals, our approximation of the interest rates converges weakly to the true process, see Theorem 3.6. In Section 5, we discuss the choice of the number of gridpoints $K$.

**Theorem 3.6** *(Weak convergence). Let $(\bar{r}_t^K)_{t\in\mathbb{N}}$ be a discrete time process that uses the transition probabilities described in (9) and let the grid points $r^{(j)}$ be chosen so that $\mathbb{P}(r_1 \in [r^{(j)}]) \in \mathcal{O}(1/K)$, $j = 1, \dots, K$. Then, for all $t \in \mathbb{N}$, $\bar{r}_t$ converges weakly to the continuous process $r_t$ in (1).*

**Proof.** See the Appendix B. □

### 3.2. Quantile and tail probability estimation

We are interested in quantiles of the undiscounted payoff variable $Y_T$. For our cliquet option with maturity $T$ and a constant $G \in \mathbb{R}$, we have:

$$P_T(G) := \mathbb{P}\left(\ln(Y_T) > G\right) = \int_G^\infty p_T(s)\mathrm{d}s\,,$$

where $p_T$ denotes the density of the logarithm of $Y_T$. To compute these tail probabilities given the moment-generating function of the logarithm of $Y_T$, we apply the fast Fourier algorithm, see, for example, Carr and Madan (1999).

We introduce a dampening coefficient $\gamma \in (0,\infty)$ to define a dampened probability $d_T(G)$ and its Fourier transformation $\psi_T(v)$ as

$$d_T(G) := \mathrm{e}^{\gamma G} P_T(G), \qquad \psi_T(v) = \int_{-\infty}^\infty \mathrm{e}^{\mathrm{i}vG} d_T(G)\,\mathrm{d}G\,.$$

Then, we can relate these two quantities via

$$P_T(G) = \frac{\mathrm{e}^{-\gamma G}}{2\pi} \int_{-\infty}^\infty \mathrm{e}^{-\mathrm{i}vG} \psi_T(v)\,\mathrm{d}v = \frac{\mathrm{e}^{-\gamma G}}{\pi} \int_0^\infty \mathrm{e}^{-\mathrm{i}vG} \psi_T(v)\,\mathrm{d}v\,, \tag{12}$$

where we can express

$$\psi_T(v) = \int_{-\infty}^\infty \mathrm{e}^{\mathrm{i}vG} \int_G^\infty \mathrm{e}^{\gamma G} p_T(s)\,\mathrm{d}s\,\mathrm{d}G = \int_{-\infty}^\infty p_T(s) \int_{-\infty}^s \mathrm{e}^{(\mathrm{i}v+\gamma)G}\,\mathrm{d}G\,\mathrm{d}s = \int_{-\infty}^\infty p_T(s) \frac{\mathrm{e}^{(\mathrm{i}v+\gamma)s}}{\gamma + \mathrm{i}v}\,\mathrm{d}s = \frac{\mathcal{L}_T(\gamma + \mathrm{i}v)}{\gamma + \mathrm{i}v}\,.$$

It is possible to simultaneously calculate these probabilities for various thresholds $G_k$ via fast Fourier transformation.
To this end, let $v_j = \eta(j-1)$ and $G_k = b + \omega(k-1)$ for $\eta, \omega > 0$ and $j, k = 1, \dots, N$, $N \in \mathbb{N}$, such that $\omega\eta = \frac{2\pi}{N}$. A standard Simpson's rule applied to (12) gives

$$P_T(G_k) \approx \frac{\mathrm{e}^{-\gamma G_k}}{\pi} \sum_{j=1}^N \mathrm{e}^{-\mathrm{i}v_j G_k} \psi(v_j) \cdot \frac{\eta}{3}\left(3 + (-1)^j - \delta_{j-1}\right) = \frac{\mathrm{e}^{-\gamma G_k}}{\pi} \sum_{j=1}^N \mathrm{e}^{-\mathrm{i}\frac{2\pi}{N}(j-1)(u-1)} \mathrm{e}^{-\mathrm{i}bv_j} \frac{\mathcal{L}_T(\gamma + \mathrm{i}v_j)}{\gamma + \mathrm{i}v_j} \cdot \frac{\eta}{3}\left(3 + (-1)^j - \delta_{j-1}\right),$$

where $\delta_n$ is equal to 1 for $n = 0$ and zero otherwise. This immediately allows to apply fast Fourier transformation.

An application of this technique for calculating exceedance probabilities is to consider a capped index participation that limits the maximum reward for the policyholder at time $T$ at some upper barrier $G_k$.

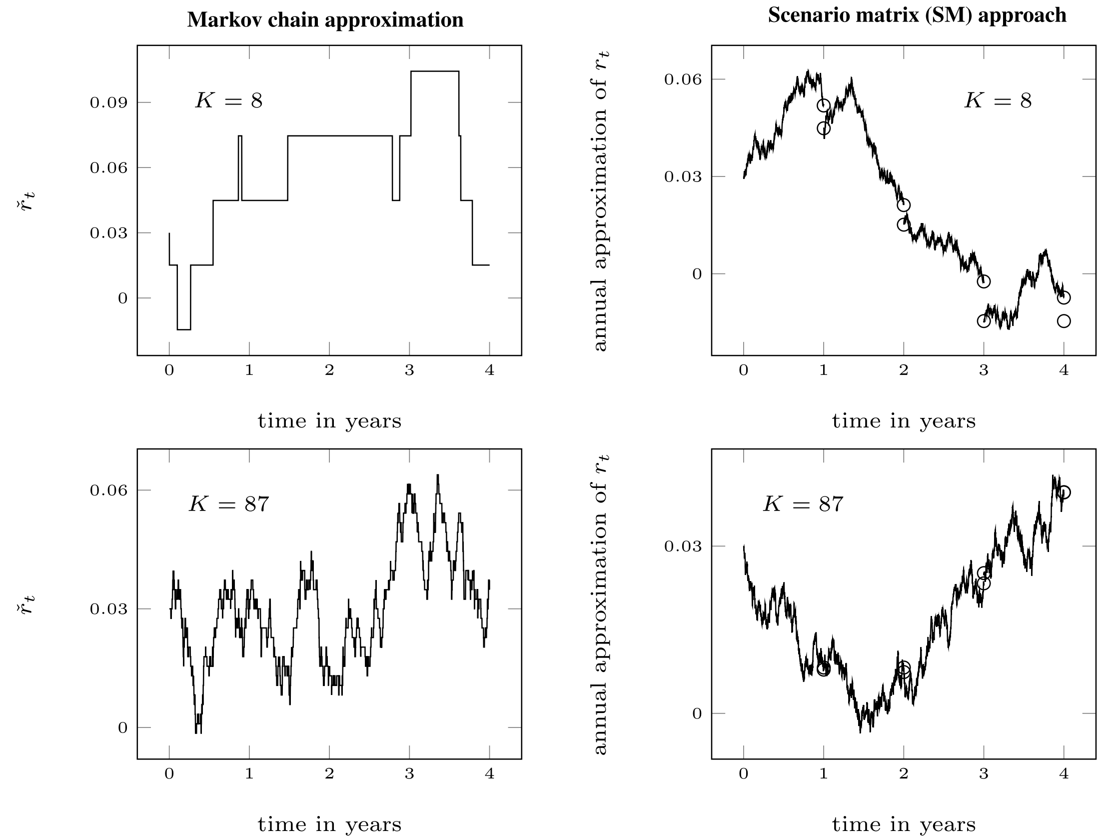


**Fig. 1.** Sample paths of Markov chain approximations $\check{r}$ (left column) and annual rounding to the nearest grid point as in the SM method (right column) for $K = 8$ (first row) and $K = 87$ grid points (second row). The parameters have been chosen as in Table 1.

## 4. Benchmarks: Markov chain approximation, Fourier pricing and PROJ

Different from our approach, several authors approximate the interest rate process by a Markov chain. Using the discrete grid introduced earlier, this means that the interest rate can only take discrete values $r^{(1)}$, $r^{(2)}$, …, $r^{(K)}$, summarized in the vector $\boldsymbol{r} := (r^{(1)}, r^{(2)}, \dots, r^{(K)})' \in \mathbb{R}^K$. The Markov chain approximation $(\check{r}_t)_{t\geq 0}$ is defined by a suitable choice of the intensity matrix $\boldsymbol{Q} = (q_{k,j})_{k,j}$ that defines the transition between the $K$ states. For $k, j = 1, 2, \dots, K$, an example is

$$q_{k,j} = \begin{cases} \frac{\sigma_r^2 - \kappa(\theta - r^{(k)})c_k}{c_{k-1}(c_{k-1}+c_k)} & \text{if } j = k-1\,, \\ \frac{\sigma_r^2 + \kappa(\theta - r^{(k)})c_{k-1}}{c_k(c_{k-1}+c_k)} & \text{if } j = k+1\,, \\ -q_{k,k-1} - q_{k,k+1} & \text{if } j = k\,, \end{cases} \tag{13}$$

where $c_k := r^{(k+1)} - r^{(k)}$. The approximation (13) was introduced by Lo and Skindilias (2014) as a generalization of Kushner and Dupuis (2001) and Chourdakis (2004). It is designed to match the first two moments of the original interest rate process. However, this formula can only be applied for sufficiently small $c_k$ that is for large enough $K$. Otherwise, if we get negative entries off the main-diagonal of $\boldsymbol{Q}$, we can resort to an approximation by Piccioni (1987) which only matches the first moment of the original process.

This approximation of the interest rate process is fundamentally different from our approach, since $\check{r}_t$ does not behave like a Vasicek model within one year. This is in contrast to the SM method where the interest rate process between consecutive annual time points follows a Vasicek model. A visualization of the two approximations is given in Fig. 1 for $K = 8$ and $K = 87$.

### 4.1. Fourier pricing

Given the intensity matrix $Q$, the discounted characteristic function $\boldsymbol{\varphi}_t(u, \boldsymbol{r}) := \left\{\varphi_t^{(k,j)}(u, \boldsymbol{r})\right\}_{k,j}$ is given explicitly in terms of a matrix exponential:

$$\varphi_t^{(k,j)}(u) := \mathbb{E}\Big[\mathbb{1}_{\{\check{r}_t = r^{(j)}\}}\, \mathrm{e}^{-\int_0^t \check{r}_s\,\mathrm{d}s + \mathrm{i}u\ln(S_t/S_0)}\Big|\, r_0 = r^{(k)}\Big] = \mathbf{1}_k' \,\exp\Big(\boldsymbol{Q}t + \mathrm{diag}\Big(-\boldsymbol{r} + \mathrm{i}u\Big(\boldsymbol{r} + \lambda - \frac{1}{2}\sigma_S^2\Big) - \frac{1}{2}\sigma_S^2 u^2\Big)t\Big)\,\mathbf{1}_j\,. \tag{14}$$

Given (14), let us introduce the matrices:

$$\big\{\mathscr{D}\big\}_{k,j=1,2,\dots,n} := \mathbb{E}\Big[\mathbb{1}_{\{\check{r}_t = r^{(j)}\}}\, \mathrm{e}^{-\int_{t-1}^t \check{r}_s\,\mathrm{d}s}\Big|\, r_{t-1} = r^{(k)}\Big] = \exp\Big(\boldsymbol{Q} - \mathrm{diag}(\boldsymbol{r})\Big)\,,$$

$$\big\{\mathscr{C}(b)\big\}_{k,j=1,2,\dots,n} := \mathbb{E}\Big[\mathbb{1}_{\{\check{r}_t = r^{(j)}\}}\, \mathrm{e}^{-\int_{t-1}^t \check{r}_s\,\mathrm{d}s}\Big[\big(S_T/S_0\big)^\alpha - \mathrm{e}^g\Big]^+\Big|\, r_{t-1} = r^{(k)}\Big] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \boldsymbol{\varphi}_T\big(v - \mathrm{i}\delta\big)\cdot \hat{f}_g\big(\mathrm{i}\delta - v\big)\,\mathrm{d}v\,, \tag{15}$$

 

where $\delta \in (0,\infty)$ is a dampening coefficient. To compute the expectation, we derive

$$\hat{f}_g(u) := \int_{\mathbb{R}} e^{iuy}\left[e^{\alpha y} - e^{g}\right]^{+} dy = \int_{\frac{g}{\alpha}}^{\infty} e^{(\alpha+iu)y}\,dy - e^{g}\int_{\frac{g}{\alpha}}^{\infty} e^{iuy}\,dy = e^{g(1+\frac{iu}{\alpha})}\frac{\alpha}{iu(\alpha+iu)}\,.$$

From Hieber (2017), we then obtain:

$$\mathbb{E}[Z_T] = P_0 \cdot \mathbf{1}_k'\left(e^{g}\mathscr{D}(1) + \mathscr{C}(g)\right)^{T}\mathbf{1}\,. \tag{16}$$

This requires to compute the Fourier integral (15).

### 4.2. PROJ method

An alternative to the SM method is the projection method (PROJ) introduced by Kirkby (2015), Cui et al. (2017). The method approximates numerical integrals using a B-spline basis, is very general and has a variety of applications. We apply PROJ to compute the expectation:

$$\mathbb{E}\left[e^{L_{t-1,t}}\,\middle|\,\check{r}_{t-1} = r^{(k)},\,\check{r}_t = r^{(j)}\right] = \int_0^{\infty} h_{k,j}(y)\cdot f(y)\,dy\,, \tag{17}$$

where $f(y)$ is the density of the integrated interest rate $y = \int_{t-1}^{t}\check{r}_s\,ds$ and $h_{k,j}(y)$ the expectation given $y = \int_{t-1}^{t}\check{r}_s\,ds$. PROJ projects the density $f(y)$ on a basis of transformed hat functions $\varphi^{[1]}(y) = (1-|x|)\mathbb{1}_{[-1,1]}(y)$. More specifically, PROJ uses the linear B-spline basis $\{\varphi_{a,n}(y)\}_{n=1,2,\ldots,N} := \left\{\sqrt{a}\cdot\varphi^{[1]}\big(a(x-x_n)\big)\right\}_{n=1,2,\ldots,N}$ with resolution $a>0$ and grid points $x_n = x_1 + (n-1)/a$, $n = 1,2,\ldots,N$.
The coefficients $\beta_{a,n}$ of the approximation $f(y) \approx \sum_{n=1}^{N}\beta_{a,n}\cdot\varphi_{a,n}(y)$ by $N\in\mathbb{N}$ basis functions are available as an integral

$$\beta_{a,n} = \left\langle f(y), \varphi_{a,n}(y)\right\rangle = \frac{12a^{\frac{3}{2}}}{\pi}\int_0^{\infty}\hat{f}(\xi)\cdot e^{-ix_n\xi}\cdot\frac{\sin^2\left(\frac{\xi}{2a}\right)}{\xi^2\left(2+\cos\frac{\xi}{a}\right)}\,d\xi\,, \tag{18}$$

see Kirkby (2015), Cui et al. (2017) or the Appendix C for details. The integral (18) can efficiently be solved by discrete Fourier transformation (DFT). From (14), we obtain the Fourier transform $\hat{f}(\xi) := \int_{-\infty}^{\infty} e^{-i\xi y}f(y)\,dy$ of the integrated interest rate process $y = \int_{t-1}^{t}\check{r}_s\,ds$:

$$\hat{f}(\xi) = \mathbb{E}\left[e^{-i\xi\int_{t-1}^{t}\check{r}_s\,ds}\,\middle|\,\check{r}_{t-1} = r^{(k)}\,\check{r}_t = r^{(j)}\right] = \frac{\mathbb{E}\left[\mathbb{1}_{\{\check{r}_t = r^{(j)}\}}e^{-i\xi\int_{t-1}^{t}\check{r}_s\,ds}\,\middle|\,\check{r}_{t-1} = r^{(k)}\right]}{\mathbb{P}\left(\check{r}_t = r^{(j)}\,\middle|\,\check{r}_{t-1} = r^{(k)}\right)} = \frac{\mathbf{1}_k'\exp\left(\boldsymbol{Q} - i\xi\,\mathrm{diag}(\boldsymbol{r})\right)\mathbf{1}_j}{\mathbf{1}_k'\exp\left(\boldsymbol{Q}\right)\mathbf{1}_j}\,, \tag{19}$$

with $\boldsymbol{r} := (r^{(1)}, r^{(2)},\ldots,r^{(K)})$ and $\boldsymbol{Q} = (q_{k,j})_{k,j}$ as in (13). We finally obtain:

$$\int_0^{\infty} h_{k,j}(y)\cdot f(y)\,dy \approx \sum_{n=1}^{N}\beta_{a,n}\cdot\int_{x_{n-1}}^{x_n} h_{k,j}(y)\cdot\varphi_{a,n}(y)\,dy =: \sum_{n=1}^{N}\beta_{a,n}\cdot\theta_{a,n}^{k,j}\,.$$

For a small resolution $a$, the terms $\theta_{a,n}^{k,j}$ are computed by a composite Simpson‘s rule:

$$\theta_{a,n}^{k,j} \approx \frac{\sqrt{a}}{3}\Big(h_{k,j}(x_n - a/2) + h_{k,j}(x_n) + h_{k,j}(x_n + a/2)\Big)\,. \tag{20}$$

In Appendix C, we derive the function $h_{k,j}$ explicitly as:

$$h_{k,j}(y) := \mathbb{E}\left[e^{L_{t-1,t}}\,\middle|\,\check{r}_{t-1} = r^{(k)},\,\check{r}_t = r^{(j)},\,\int_{t-1}^{t}\check{r}_s\,ds = y\right] = e^{g-y}\Phi\left(\frac{\frac{g}{\alpha} - \mu_{k,j}(y)}{\sigma_{k,j}}\right) + e^{\alpha\mu_{k,j}(y) + \frac{\alpha\sigma_{k,j}^2}{2} - y}\Phi\left(\frac{\alpha\sigma_{k,j}^2 + \mu_{k,j}(y) - \frac{g}{\alpha}}{\sigma_{k,j}}\right)\,, \tag{21}$$

where $\mu_{k,j}(y) = \kappa\frac{\sigma_S}{\sigma_r}\left(r^{(k)} - r^{(j)}\right) + y\left(1 - \rho\frac{\sigma_S}{\sigma_r}\theta\right) - \left(\rho\frac{\sigma_S}{\sigma_r}\theta + \frac{\sigma_S^2}{2}\right)$ and $\sigma_{k,j}^2 = \sigma_S^2(1-\rho^2)$.

**Table 1**
Choice of parameters for the contract and the financial market.

| | | | |
|---|---|---|---|
| contract | initial capital | $P_0$ | 1 |
| | guarantee level | $g$ | 0.015 |
| | participation rate | $\alpha$ | 0.422 |
| | maturity in years | $T$ | 25 |
| financial market | initial interest rate | $r_0$ | 0.03 |
| | long-term mean interest rate level | $\theta$ | 0.03 |
| | mean reversion speed | $\kappa$ | 0.3 |
| | volatility of the interest rate process | $\sigma_r$ | 0.015 |
| | constant risk premium of interest rate | $\lambda_r$ | −0.23 |
| | correlation parameter | $\rho$ | 0.15 |
| | volatility of the asset process | $\sigma_S$ | 0.1 |
| | constant risk premium of asset process | $\lambda_S$ | 0.03 |

## 5. Numerical results

For numerical experiments, we implemented the methods described in Sections 3 and 4 in R on a personal computer with 16 GB RAM and 11th Gen Intel(R) Core(TM) i7-1185G7 @ 3.00 GHz. Generally, we looked at a contract maturing after $T = 25$ years. All parameters for this implementation are listed in Table 1. They have been chosen in accordance with the existing literature (e.g. Barbarin and Devolder, 2005; Graf et al., 2011; Hieber et al., 2019). However, we adjusted the initial interest rate to be in line with the EURIBOR 1-week rates in May 2023. We also reduced the volatility $\sigma_S$ to account for the fact that the investment strategy of the insurer will usually focus on less risky investments than the overall stock market. Furthermore, we chose the participation rate $\alpha$ by creating a fair policy assuming a constant interest rate equal to the long-term mean $\theta$. That is, it is designed by the premium equivalence $\mathbb{E}_{\mathbb{Q}}[Z_T] = P_0$ that is assumed to be true for a constant interest rate scenario $r_t = \theta = r_0$ for all $t$.

### 5.1. Convergence

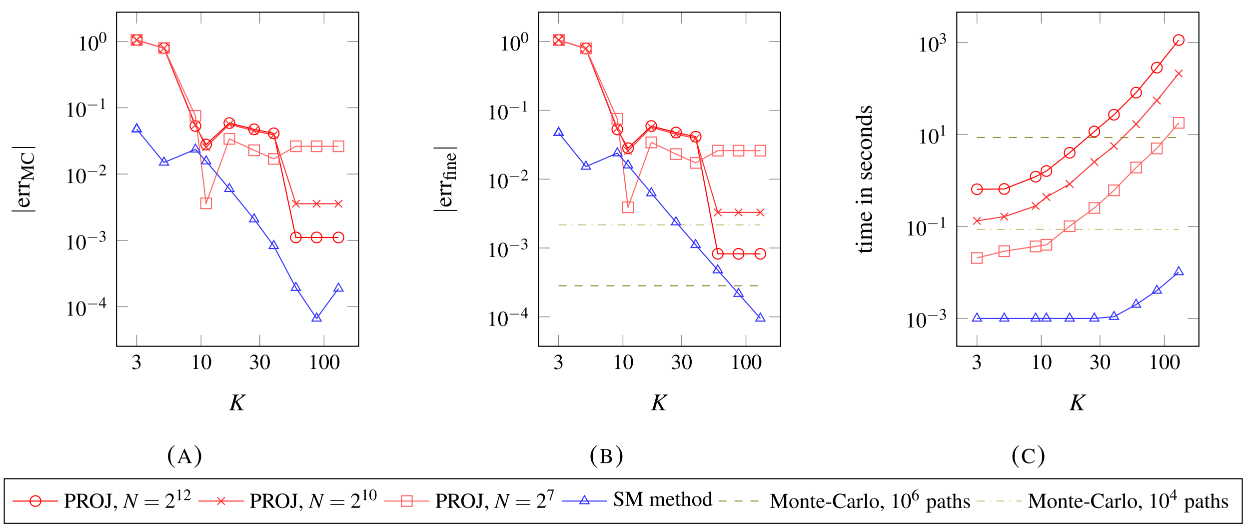


**Fig. 2.** Comparison of absolute error and computation time between the SM method, the PROJ method for different values of $N$, and Monte-Carlo simulations. The parameters have been chosen as in Table 1.
Panel (A): absolute error compared to the Monte-Carlo simulation with $10^6$ scenarios in relation to the number of discretization points $K$. Panel (B): absolute error of all methods compared to true solution (obtained by a fine approximation with $K = 10\,001$ in the SM method). Panel (C): time spent on calculations for all methods.

The left image in Fig. 2 shows the error of the PROJ method and the SM method in relation to a Monte-Carlo-simulation with $10^6$ scenarios. We can see that all estimations reach a plateau after using 50 to 100 discretization points for the interest rate. At this point, our approach entered the 95% confidence interval of the Monte-Carlo simulations and thus outperforms its accuracy afterwards. The middle image shows the absolute error of all methods for different amounts of grid points in comparison to a very fine approximation by our method with 10 001 points. The estimations with the SM method almost perfectly exhibit a quadratic convergence. Note that the sudden drop for the SM method at $K = 5$ and for the PROJ method at $K = 11$ are random. In these instances, the grid approximation is by chance very close to the true solution. We can also see that the precision of the PROJ method is not only limited by the number of discretization points $K$, but also by the number of terms $N$ used in the fast Fourier transform, since for $N = 2^7$ and $N = 2^{10}$, it is not as close to the finest approximation as a Monte-Carlo simulation with $10^4$ paths.

For $K = 87$ grid points, the SM method first outperforms a Monte-Carlo simulation with $10^6$ scenarios and has a relative error of 0.013%. The calculations for a maturity of 25 years took approximately 0.004 seconds. Generally, we can see on the right side of Fig. 2 that both methods have a complexity of $\mathcal{O}(K^2)$, while the computation of every matrix entry in the PROJ method requires $N$ times as many calculations for determining the elements of the fast Fourier algorithm compared to one calculation in the SM method.

### 5.2. Analysis of interest rate risk

**Table 2**
Expected discounted payoff, 99%-quantile of the ratio between the terminal payoff and the value of the underlying at maturity $Q_{99\%}(Y_T/S_T)$ and the probability of being underfunded, that is $\mathbb{P}(Y_T > S_T)$, for different values of $\sigma_r$. The remaining parameters are chosen as in Table 1.

| $\sigma_r$ | 0 | 0.005 | 0.010 | 0.015 | 0.020 | 0.025 | 0.030 |
|---|---|---|---|---|---|---|---|
| $\mathbb{E}_{\mathbb{Q}}(Z_T)$ | 0.999 | 1.002 | 1.011 | 1.024 | 1.043 | 1.068 | 1.098 |
| $Q_{99\%}(Y_T/S_T)$ | 1.586 | 1.757 | 1.989 | 2.291 | 2.691 | 3.207 | 3.864 |
| $\mathbb{P}(Y_T > S_T)$ | 11.4% | 15.7% | 21.1% | 27.3% | 33.7% | 39.9% | 45.5% |

Expected discounted payoffs for different levels of the mean reversion speed $\kappa$, the interest rate volatility $\sigma_r$ and the correlation parameter $\rho$ are shown in Tables 2 to 4. Note that a Vasicek model with $\sigma_r = 0$ and $\theta = r_0$ is equivalent to assuming a constant interest rate $r_0$. These values are

**Table 3**
Expected discounted payoff, 99%-quantile of the ratio between the terminal payoff and the value of the underlying at maturity $Q_{99\%}(Y_T/S_T)$ and the probability of being underfunded, that is $\mathbb{P}(Y_T > S_T)$, for different values of $\kappa$. The remaining parameters are chosen as in Table 1.

| $\kappa$ | 0.10 | 0.25 | 0.40 | 0.55 | 0.70 | 0.85 | 1.00 |
|---|---|---|---|---|---|---|---|
| $\mathbb{E}_{\mathbb{Q}}(Z_T)$ | 1.108 | 1.031 | 1.016 | 1.011 | 1.008 | 1.006 | 1.005 |
| $Q_{99\%}(Y_T/S_T)$ | 4.277 | 2.478 | 2.083 | 1.929 | 1.846 | 1.795 | 1.761 |
| $\mathbb{P}(Y_T > S_T)$ | 47.4% | 30.3% | 23.2% | 19.8% | 17.9% | 16.7% | 15.8% |

**Table 4**
Expected discounted payoff, 99%-quantile of the ratio between the terminal payoff and the value of the underlying at maturity $Q_{99\%}(Y_T/S_T)$ and the probability of being underfunded, that is $\mathbb{P}(Y_T > S_T)$, for different values of $\rho$. The remaining parameters are chosen as in Table 1.

| $\rho$ | −0.9 | −0.6 | −0.3 | 0.0 | 0.3 | 0.6 | 0.9 |
|---|---|---|---|---|---|---|---|
| $\mathbb{E}_{\mathbb{Q}}(Z_T)$ | 1.011 | 1.015 | 1.019 | 1.022 | 1.026 | 1.029 | 1.032 |
| $Q_{99\%}(Y_T/S_T)$ | 1.376 | 1.640 | 1.897 | 2.160 | 2.430 | 2.714 | 3.009 |
| $\mathbb{P}(Y_T > S_T)$ | 12.7% | 18.7% | 22.9% | 26.0% | 28.4% | 30.3% | 31.9% |

computed under the risk-neutral measure $\mathbb{Q}$. Low values of $\kappa$ increase the likelihood for longer periods of interest rates deviating from the long-term mean $\theta$. Particularly low interest rate periods reduce the discounting effect while the payoff is still bounded from below by the guaranteed level, whereas the stronger discounting effect of high interest rate periods is partially compensated by larger returns from the underlying fund. For all values, this effect is even more pronounced on the 99%-quantile of the ratio between the undiscounted payoff $Y_T$ and the investment in the underlying $S_T$ since these predominantly include extreme scenarios with long low-interest rate periods. This means, for example, that for an extreme value of $\kappa = 0.1$, the investment in the underlying covers less than $\frac{1}{4.277} \approx 24\%$ of the guaranteed payoff to the policyholder. The quantiles, that is $Q_{99\%}(Y_T/S_T)$, are determined under the real-world measure $\mathbb{P}$. Also, we can see that the probability for $Y_T > S_T$, i.e. of being underfunded, is affected considerably.

A similar effect can be identified on varying values of $\sigma_r$ where an increased volatility results in larger deviations from the long-term mean $\theta$ and the effects of low and high interest rate periods are the same as described before. However, increasing the volatility does not affect the length of these periods, but their gap to the long-term mean.

When we vary the correlation parameter $\rho$, the effect is less distinct, but still considerable. Note, however, that values of $\rho = \pm 0.9$ are extreme choices for the correlation parameter. We can observe that negative correlations lead to less risky contracts and lower prices. This is due to the fact that negative values for the correlation parameter $\rho$ result in an overall lower volatility of the underlying asset process in (2) and vice-versa. Thus, we can conclude that modelling interest rates stochastically strongly influences the expected disounted payoff and will usually reveal a higher value than constant interest rates.

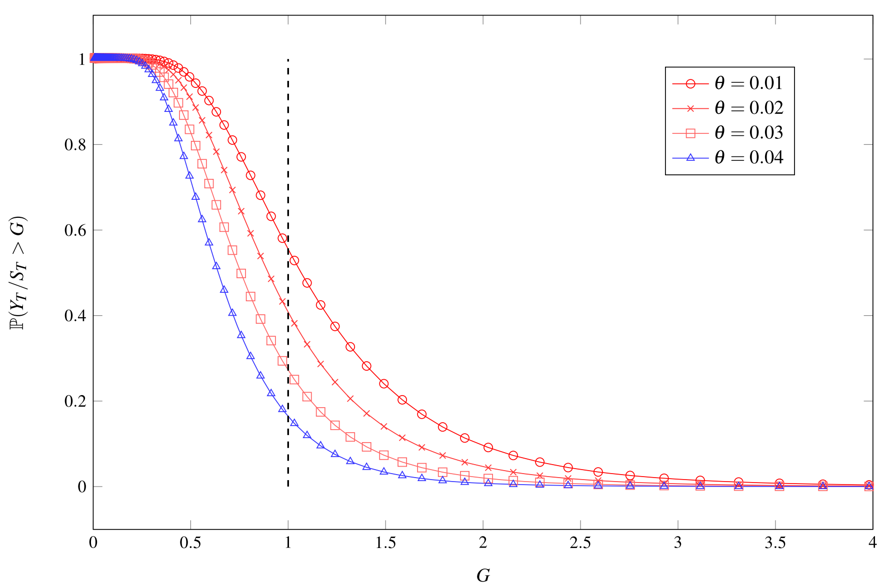


**Fig. 3.** Probability that the ratio between the terminal payoff and the value of the underlying at maturity $Y_T/S_T$ exceeds thresholds $G$ for different values of the long-term mean $\theta$. The dashed line at $G = 1$ marks the probability of being underfunded. The remaining parameters have been chosen as in Table 1.

To manage the risk of an equity-indexed annuity, it is necessary to evaluate the probability that the underlying fund can cover the guaranteed expenses at maturity, and quantiles of the potential loss, that is to describe the distribution of the solvency ratio $Y_T/S_T$. This relative Value-at-Risk is computed using the SM method as explained in Remark 3.2. In Fig. 3, we plot the tail function of the solvency ratio $Y_T/S_T$, that is the probability that $Y_T/S_T$ exceeds thresholds $G$. Note that the tail function at the threshold $G = 1$ describes the probability that $Y_T > S_T$ and thus that the initial investment can not cover the payment at maturity. Lowering the long-term mean interest rate $\theta$ from 0.03 to 0.02 increases this probability from 27.3% to 40.8%. The ratio $Y_T/S_T$ in the 1% least favourable scenarios, that is the 99%-quantile of this quotient, for $\theta = 0.03$ is above 2.76, compared to 1.59 under constant interest rates. In other words, with a probability of 1%, the underlying fund covers only around 1/2 of the payments to the policyholder. This imbalance intensifies to 3.35 for $\theta = 0.01$ showing how sensitive extreme events are to changes in the underlying assumptions and thus giving another reason for modelling interest rates stochastically.

## 6. Conclusion

This article introduces the scenario-matrix (SM) method for evaluating the risks of equity-indexed annuities with a cliquet-style payoff structure. This approach is fundamentally different from a Markov chain approximation, but has some analogy to the case of regime switching models (see, e.g. Hieber, 2017). In the case of a Vasicek-Black-Scholes model, it outperforms benchmark techniques such as the PROJ method in terms of speed and accuracy and is easy to implement. Approximating the moment-generating function of the log-payoff allows us to describe the whole distribution of the payoff. In contrast to the existing literature, we consider quantile estimation and offer an access to the relative Value-at-Risk.

In numerical studies, we are able to show convergence with quadratic order. The algorithm is computationally less expensive and provides more accurate results than existing approaches. The sensitivity analysis suggests that first, modelling interest rates stochastically considerably influences the risk assessment when compared to constant interest rates. Second, changes in the financial market should be considered in the management of these products.

We introduced the SM method adapted to the special case of a Vasicek-Black-Scholes model that leads to very convenient formulas in Theorem 3.1. This technique can, however, be applied and extended to much more general settings. First, it is straightforward to use a different stochastic interest rate model, for example a two-factor Hull-White model. Second, the scenarios might incorporate not only interest rate risk but also a stochastic volatility (see also Cui et al., 2017; Kirkby, 2023). Instead of conditioning on $r^{(k)}$ and $r^{(j)}$, the scenario matrix conditions on pairs $(r^{(k)}, \sigma^{(k)})$ and $(r^{(j)}, \sigma^{(j)})$ for an appropriately chosen volatility grid $\sigma^{(k)}$, $k = 1, 2, \dots, K$. A further possible research direction would be to estimate the scenario matrix immediately from data without specifying a model, see the recent results on data-driven estimations (e.g. Li and Forsyth, 2019).

In future research, it may be interesting to incorporate several additional contract features. First, instead of fixing the maturity time, we may consider early contract terminations and mortality. Second, it could be interesting to analyse the effect of the investment strategy of the underlying fund on the contract payoff and its risk, see also Chen and Hieber (2016) for the case of a maturity guarantee. Third, in this article, we focus on a single contract or a portfolio of homogeneous contracts. In practice, we may be interested in the risk management of a heterogeneous portfolio (see also Hieber et al., 2019). If the SM method cannot be adapted to such settings, the estimates can at least serve as control variates to speed up computations for solvency ratios or absolute Value-at-Risks.

### Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

### Data availability

No data was used for the research described in the article.

## Appendix A. Proof of Theorem 3.1

**Definition A.1.** Using the notation $P_u(t-1,t) = \mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{-u\int_{t-1}^t r_s\,\mathrm{d}s}\right]$, the $u$-scaled $t$-forward measure $\mathbb{Q}_u^t$ is defined via

$$\frac{\mathrm{d}\mathbb{Q}_u^t}{\mathrm{d}\mathbb{Q}} = \frac{\mathrm{e}^{-u\int_{t-1}^t r_s\,\mathrm{d}s}}{P_u(t-1,t)} = \exp\left(-u\sigma_r\int_{t-1}^{t} g_\kappa(t-s)\,\mathrm{d}W_s^{(1)} + \frac{1}{2}u^2\sigma_r^2\int_{t-1}^{t} g_\kappa(t-s)^2\,\mathrm{d}s\right). \tag{22}$$

The integrated interest $\int_{t-1}^t r_s\,\mathrm{d}s$ and the log return $\ln(S_t/S_{t-1})$, both conditional on the interest rates $r_t$ and $r_{t-1}$, under the pricing measure $\mathbb{Q}$ and the $u$-scaled $t$-forward measure $\mathbb{Q}_u^t$ respectively, are distributed as

$$\int_{t-1}^{t} r_s\,\mathrm{d}s\,\Big|\,r_t, r_{t-1} \sim \mathcal{N}_{\mathbb{Q}}(\mu, \Sigma^2),$$

$$\ln(S_t/S_{t-1})\,\Big|\,r_t, r_{t-1} \sim \mathcal{N}_{\mathbb{Q}_u^t}(\widetilde{\mu}, \widetilde{\Sigma}^2),$$

where $\mu, \widetilde{\mu}, \Sigma, \widetilde{\Sigma}$ are derived in the following. Using the t-forward measure $\mathbb{Q}_u^t$ as introduced in (22), we know that according to Girsanov's Theorem in the form of (Shreve, 2010, Theorem 5.4.1),

$$\widetilde{W}_t^{(1)} = W_t^{(1)} + u\cdot\sigma_r\int_{t-1}^{t} g_\kappa(t-s)\,\mathrm{d}s$$

and $W_t^{(2)}$ are Brownian motions under $\mathbb{Q}_u^t$.

Therefore, we can use this change of measure to decouple the integrated interest rate and the annual returns of the underlying stock as follows:

$$\phi_{r_{t-1},r_t}(u) = P_u(t-1,t)\cdot\mathbb{E}_{\mathbb{Q}}\left[\frac{\mathrm{e}^{-u\int_{t-1}^t r_s\,\mathrm{d}s}}{P_u(t-1,t)}\mathrm{e}^{u\cdot L_{t-1,t}}\,\Big|\,r_{t-1}, r_t\right] = P_u(t-1,t)\cdot\mathbb{E}_{\mathbb{Q}_u^t}\left[\mathrm{e}^{u\cdot L_{t-1,t}}\,\Big|\,r_{t-1}, r_t\right]$$

$$= P_u(t-1,t)\cdot\left(\mathbb{E}_{\mathbb{Q}_u^t}\left[e^{u\cdot g}\mathbb{1}_{\left\{\alpha\ln\left(\frac{S_t}{S_{t-1}}\right)\leqslant g\right\}}\middle|r_{t-1},r_t\right]+\mathbb{E}_{\mathbb{Q}_u^t}\left[e^{u\cdot\alpha\ln\left(\frac{S_t}{S_{t-1}}\right)}\mathbb{1}_{\left\{\alpha\ln\left(\frac{S_t}{S_{t-1}}\right)>g\right\}}\middle|r_{t-1},r_t\right]\right). \tag{23}$$

To determine the distribution of $\int_{t-1}^{t} r_s\,\mathrm{d}s$, given $r_{t-1}$, $r_t$, under the risk neutral measure $\mathbb{Q}$, we use that for a Vasicek interest rate model, we have (see (Shreve, 2010, Chapter 4.4))

$$r_t = e^{-\kappa}r_{t-1}+\theta(1-e^{-\kappa})+\sigma_r e^{-\kappa t}\int_{t-1}^{t} e^{\kappa s}\,\mathrm{d}W_s^{(1)},$$

$$\int_{t-1}^{t} r_s\,\mathrm{d}s = g_\kappa(1)r_{t-1}+\theta\big(1-g_\kappa(1)\big)+\sigma_r\int_{t-1}^{t} g_\kappa(t-s)\,\mathrm{d}W_s^{(1)},$$

$$\ln\left(\frac{S_t}{S_{t-1}}\right)=\int_{t-1}^{t} r_s\,\mathrm{d}s-\frac{1}{2}\sigma_S^2+\sigma_S\left(\rho(W_t^{(1)}-W_{t-1}^{(1)})+\sqrt{1-\rho^2}(W_t^{(2)}-W_{t-1}^{(2)})\right).$$

Using Itô's isometry, we can describe the distribution of $\int_{t-1}^{t} r_s\,\mathrm{d}s$ and $r_t$ under $\mathbb{Q}$ as a two-dimensional normal distribution with parameters

$$\mu_1 := \mathbb{E}_{\mathbb{Q}}\left[\int_{t-1}^{t} r_s\,\mathrm{d}s\right]=g_\kappa(1)r_{t-1}+\theta\big(1-g_\kappa(1)\big),$$

$$\mu_2 := \mathbb{E}_{\mathbb{Q}}\left[r_t\right]=e^{-\kappa}r_{t-1}+\theta(1-e^{-\kappa}),$$

$$\sigma_{11}^2 := \mathrm{Var}_{\mathbb{Q}}\left(\int_{t-1}^{t} r_s\,\mathrm{d}s\right)=\frac{\sigma_r^2}{\kappa^2}\left(1-g_\kappa(1)-\frac{\kappa}{2}g_\kappa(1)^2\right),$$

$$\sigma_{22}^2 := \mathrm{Var}_{\mathbb{Q}}(r_t)=\frac{\sigma_r^2}{2}g_\kappa(2),$$

$$\sigma_{12} := \mathrm{Cov}_{\mathbb{Q}}\left(r_t,\int_{t-1}^{t} r_s\,\mathrm{d}s\right)=\frac{\sigma_r^2}{2}g_\kappa(1)^2.$$

Therefore, under $\mathbb{Q}$, (Eaton, 2007, Proposition 3.13) shows that

$$\int_{t-1}^{t} r_s\,\mathrm{d}s\,\Big|\,r_t,r_{t-1}\sim\mathcal{N}(\mu,\Sigma^2), \tag{24}$$

with

$$\mu := \mu_1+\sigma_{12}\sigma_{22}^{-2}(r_t-\mu_2),\qquad \Sigma^2 := \sigma_{11}^2-\sigma_{12}^2\sigma_{22}^{-2}.$$

To express these values in the form of Theorem 3.1, it is helpful to write

$$\sigma_{12}\sigma_{22}^{-2}=\frac{g_\kappa(1)^2}{g_\kappa(2)}=\frac{g_\kappa(1)(1-e^{-\kappa})}{(1+e^{-\kappa})(1-e^{-\kappa})}=\frac{g_\kappa(1)}{1+e^{-\kappa}}. \tag{25}$$

To determine the distribution of $\ln\left(S_t/S_{t-1}\right)$, given $r_t, r_{t-1}$, under $\mathbb{Q}_u^t$, we calculate

$$r_t = e^{-\kappa}r_{t-1}+\theta(1-e^{-\kappa})+\sigma_r e^{-\kappa t}\left(\int_{t-1}^{t} e^{\kappa s}\,\mathrm{d}\widetilde{W}_s^{(1)}-\int_{t-1}^{t} e^{\kappa s}u\sigma_r g_\kappa(t-s)\,\mathrm{d}s\right)=e^{-\kappa}r_{t-1}+\theta(1-e^{-\kappa})-u\frac{\sigma_r^2}{2}g_\kappa(1)^2+\sigma_r e^{-\kappa t}\int_{t-1}^{t} e^{\kappa s}\,\mathrm{d}\widetilde{W}_s^{(1)},$$

$$\begin{aligned}\int_{t-1}^{t} r_s\,\mathrm{d}s &= g_\kappa(1)r_{t-1}+\theta\big(1-g_\kappa(1)\big)+\sigma_r\left(\int_{t-1}^{t} g_\kappa(t-s)\,\mathrm{d}\widetilde{W}_s^{(1)}-\int_{t-1}^{t} u\sigma_r g_\kappa(t-s)^2\,\mathrm{d}s\right)\\ &= g_\kappa(1)r_{t-1}+\theta\big(1-g_\kappa(1)\big)-u\frac{\sigma_r^2}{\kappa^2}\left(1-g_\kappa(1)-\frac{\kappa}{2}g_\kappa(1)^2\right)+\sigma_r\int_{t-1}^{t} g_\kappa(t-s)\,\mathrm{d}\widetilde{W}_s^{(1)},\end{aligned}$$

$$\begin{aligned}\ln\left(\frac{S_t}{S_{t-1}}\right) &= \int_{t-1}^{t} r_s\,\mathrm{d}s-\frac{1}{2}\sigma_S^2+\sigma_S\left(\int_{t-1}^{t}\rho\,\mathrm{d}\widetilde{W}_s^{(1)}-\int_{t-1}^{t}\rho u\sigma_r g_\kappa(t-s)\,\mathrm{d}s+\int_{t-1}^{t}\sqrt{1-\rho^2}\,\mathrm{d}W_s^{(2)}\right)\\ &= g_\kappa(1)r_{t-1}+\theta\big(1-g_\kappa(1)\big)-u\frac{\sigma_r^2}{\kappa^2}\left(1-g_\kappa(1)-\frac{\kappa}{2}g_\kappa(1)^2\right)-\frac{1}{2}\sigma_S^2-u\frac{\sigma_S\sigma_r\rho}{\kappa}\big(1-g_\kappa(1)\big)\\ &\quad+\sigma_r\int_{t-1}^{t} g_\kappa(t-s)\,\mathrm{d}\widetilde{W}_s^{(1)}+\sigma_S\int_{t-1}^{t}\rho\,\mathrm{d}\widetilde{W}_s^{(1)}+\sigma_S\int_{t-1}^{t}\sqrt{1-\rho^2}\,\mathrm{d}W_s^{(2)}.\end{aligned}$$

Again, we can describe the distribution of $\ln\left(S_t/S_{t-1}\right)$ and $r_t$ as a two-dimensional normal distribution with parameters

$$\begin{aligned}
\widetilde{\mu}_1 &:= \mathbb{E}_{\mathbb{Q}_u^t}\left[\ln\left(\frac{S_t}{S_{t-1}}\right)\right] = g_\kappa(1)r_{t-1} + \theta\big(1-g_\kappa(1)\big) - u\frac{\sigma_r^2}{\kappa^2}\left(1-g_\kappa(1)-\frac{\kappa}{2}g_\kappa(1)^2\right) - \frac{1}{2}\sigma_S^2 - u\frac{\sigma_S\sigma_r\rho}{\kappa}\big(1-g_\kappa(1)\big),\\
\widetilde{\mu}_2 &:= \mathbb{E}_{\mathbb{Q}_u^t}\left[r_t\right] = \mathrm{e}^{-\kappa}r_{t-1} + \theta(1-\mathrm{e}^{-\kappa}) - u\frac{\sigma_r^2}{2}g_\kappa(1)^2,\\
\widetilde{\sigma}_{11}^2 &:= \mathrm{Var}_{\mathbb{Q}_u^t}\left(\ln\left(\frac{S_t}{S_{t-1}}\right)\right) = \frac{\sigma_r^2}{\kappa^2}\left(1-g_\kappa(1)-\frac{\kappa}{2}g_\kappa(1)^2\right) + \frac{2\sigma_r\sigma_S\rho}{\kappa}\big(1-g_\kappa(1)\big) + \sigma_S^2,\\
\widetilde{\sigma}_{22}^2 &:= \mathrm{Var}_{\mathbb{Q}_u^t}(r_t) = \frac{\sigma_r^2}{2}g_\kappa(2),\\
\widetilde{\sigma}_{12} &:= \mathrm{Cov}_{\mathbb{Q}_u^t}\left(r_t, \ln\left(\frac{S_t}{S_{t-1}}\right)\right) = \frac{\sigma_r^2}{2}g_\kappa(1)^2 + \sigma_r\sigma_S\rho g_\kappa(1).
\end{aligned}$$

Therefore, we can conclude that, under $\mathbb{Q}_u^t$, we have

$$\ln(S_t/S_{t-1})\Big|\ r_t, r_{t-1} \sim \mathcal{N}(\widetilde{\mu}, \widetilde{\Sigma}^2), \tag{26}$$

with

$$\widetilde{\mu} := \widetilde{\mu}_1 + \widetilde{\sigma}_{12}\widetilde{\sigma}_{22}^{-2}(r_t - \widetilde{\mu}_2), \qquad \widetilde{\Sigma}^2 := \widetilde{\sigma}_{11}^2 - \widetilde{\sigma}_{12}^2\widetilde{\sigma}_{22}^{-2}.$$

Note that, similarly to equation (25), we find

$$\sigma_{12}\sigma_{22}^{-2} = \frac{g_k(1)^2 + s\frac{\sigma_s}{\sigma_r}\rho g_\kappa(1)}{g_\kappa(2)} = \frac{g_k(1) + s\frac{\sigma_s}{\sigma_r}\rho}{1+e^{-\kappa}}.$$

Using (24), we get

$$P_u(t,t-1) = \mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{-\int_{t-1}^t r_s\,\mathrm{d}s}\,\Big|\, r_{t-1}, r_t\right] = \exp\left(-u\mu + \frac{u^2\Sigma^2}{2}\right).$$

With (26), we can calculate

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}_u^t}\left[\mathrm{e}^{u\cdot g}\mathbb{1}_{\left\{\alpha\ln\left(\frac{S_t}{S_{t-1}}\right)\leqslant g\right\}}\,\Big|\, r_{t-1}, r_t\right] + \mathbb{E}_{\mathbb{Q}_u^t}\left[\mathrm{e}^{u\cdot\alpha\ln\left(\frac{S_t}{S_{t-1}}\right)}\mathbb{1}_{\left\{\alpha\ln\left(\frac{S_t}{S_{t-1}}\right)> g\right\}}\,\Big|\, r_{t-1}, r_t\right]\\
&= \mathrm{e}^{ug}\mathbb{Q}_u^t\left(\ln\left(\frac{S_t}{S_{t-1}}\right)\leqslant\frac{g}{\alpha}\,\Big|\, r_{t-1}, r_t\right) + \int_{\frac{g}{\alpha}}^{\infty}\mathrm{e}^{u\alpha x}\sqrt{2\pi\widetilde{\Sigma}^2}^{-1}\mathrm{e}^{-\frac{1}{2}\left(\frac{x-\widetilde{\mu}}{\widetilde{\Sigma}}\right)^2}\mathrm{d}x = \mathrm{e}^{ug}\Phi\left(\frac{\frac{g}{\alpha}-\widetilde{\mu}}{\widetilde{\Sigma}}\right) + \mathrm{e}^{u\alpha\widetilde{\mu}+\frac{1}{2}u^2\alpha^2\widetilde{\Sigma}^2}\Phi\left(-\frac{\frac{g}{\alpha}-\widetilde{\mu}-u\alpha\widetilde{\Sigma}^2}{\widetilde{\Sigma}}\right).
\end{aligned}$$

Inserting both results into (23), we arrive at

$$\phi_{r_{t-1},r_t}(u) = \mathrm{e}^{-u\mu+\frac{u^2\Sigma^2}{2}}\left(\mathrm{e}^{ug}\Phi\left(\frac{\frac{g}{\alpha}-\widetilde{\mu}}{\widetilde{\Sigma}}\right) + \mathrm{e}^{u\alpha\widetilde{\mu}+\frac{1}{2}u^2\alpha^2\widetilde{\Sigma}^2}\Phi\left(-\frac{\frac{g}{\alpha}-\widetilde{\mu}-u\alpha\widetilde{\Sigma}^2}{\widetilde{\Sigma}}\right)\right). \quad \square \tag{27}$$

## Appendix B. Proof of Theorem 3.6

We have to show that for all $r^* \in \mathbb{R}$ and all $t \in \mathbb{N}$

$$\lim_{K\to\infty}\left|\mathbb{P}\left(\bar{r}_t^K \leqslant r^*\right) - \mathbb{P}(r_t \leqslant r^*)\right| = 0,$$

which we will proof by induction. Let $r^* \in (r^{(j)}, r^{(j+1)})$. We have for $t = 1$, that

$$\left|\mathbb{P}(\bar{r}_1^K \leqslant r^*) - \mathbb{P}(r_1 \leqslant r^*)\right| = \left|\mathbb{P}\left(r_1 \leqslant \frac{1}{2}\left(r^{(j)} + r^{(j+1)}\right)\right) - \mathbb{P}\left(r_1 \leqslant r^*\right)\right| \leqslant \mathbb{P}\left(r_1 \in \left[r^{(j)}, r^{(j+1)}\right]\right) \in \mathcal{O}(1/K).$$

Assuming that the statement is valid for $t-1$, we find for $t > 1$ that

$$\begin{aligned}
&\left|\mathbb{P}(\bar{r}_t^K \leqslant r^*) - \mathbb{P}(r_t \leqslant r^*)\right|\\
&= \left|\sum_{i=1}^K \mathbb{P}\left(\bar{r}_t^K \leqslant r^*\,\Big|\,\bar{r}_{t-1}^K = r^{(i)}\right)\cdot\mathbb{P}\left(\bar{r}_{t-1}^K = r^{(i)}\right) - \mathbb{P}\left(r_t \leqslant r^*\,\Big|\, r_{t-1} \in \left[r^{(i)}\right]\right)\cdot\mathbb{P}\left(r_{t-1} \in \left[r^{(i)}\right]\right)\right|\\
&\leqslant \left|\sum_{i=1}^K \int_{\frac{r^{(i)}+r^{(i-1)}}{2}}^{\frac{r^{(i)}+r^{(i+1)}}{2}} \mathbb{P}\left(r_t \leqslant r^*\,\Big|\, r_{t-1} = x\right) - \mathbb{P}\left(r_t \leqslant \frac{1}{2}\left(r^{(j)} + r^{(j+1)}\right)\,\Big|\, r_{t-1} = x\right)\mathbb{P}\left(r_{t-1} \in \mathrm{d}x\right)\right|\\
&\qquad + \left|\sum_{i=1}^K \mathbb{P}\left(r_{t-1} \in [r^{(i)}]\right)\left(\mathbb{P}\left(r_t \leqslant \frac{1}{2}\left(r^{(j)} + r^{(j+1)}\right)\,\Big|\, r_{t-1} = r^{(i)}\right) - \mathbb{P}\left(r_t \leqslant \frac{1}{2}\left(r^{(j)} + r^{(j+1)}\right)\,\Big|\, r_{t-1} \in [r^{(i)}]\right)\right)\right|\\
&\qquad + \left|\sum_{i=1}^K \mathbb{P}\left(r_t \leqslant \frac{1}{2}\left(r^{(j)} + r^{(j+1)}\right)\,\Big|\, r_{t-1} = r^{(i)}\right)\left(\mathbb{P}\left(r_{t-1} \in [r^{(i)}]\right) - \mathbb{P}\left(\bar{r}_{t-1}^K = r^{(i)}\right)\right)\right|\\
&=: |S_1| + |S_2| + |S_3| \in \mathcal{O}(1/K).
\end{aligned}$$

Here, we used that

$$\mathbb{P}\left(\bar{r}_t^K \leqslant r^* \,\middle|\, \bar{r}_{t-1}^K = r^{(i)}\right) \cdot \mathbb{P}\left(\bar{r}_{t-1}^K = r^{(i)}\right) + S_2 + S_3 = \mathbb{P}\Big(r_t \leqslant \frac{1}{2}\left(r^{(j)} + r^{(j+1)}\right) \Big| r_{t-1} \in \left[r^{(i)}\right]\Big) \cdot \mathbb{P}\Big(r_{t-1} \in \left[r^{(i)}\right]\Big).$$

Thus, we are able to express the right hand side of the above equation as an integral in $S_1$.
Additionally, $S_1$ is a sum of $K$ integrals whose integrand is in $\mathcal{O}(1/K)$ by the same reasoning as for $t=1$. Furthermore, its integration area, $[r^{(i)}]$ has a probability weight of $\mathcal{O}(1/K)$ since the grid points were chosen such that $\mathbb{P}(r_1 \in [r^{(j)}]) \in \mathcal{O}(1/K)$. For finite $t$, this property transfers from $r_1$ to $r_t$. Combining these observations, $S_1$ is in $K \cdot \mathcal{O}(1/K^2) = \mathcal{O}(1/K)$.
By the same argument, in $S_2$, $\mathbb{P}(r_1 \in [r^{(j)}])$ is in $\mathcal{O}(1/K)$. Also $\mathbb{P}(r_t \leqslant \frac{1}{2}(r^{(j)} + r^{(j+1)}) \,|\, r_{t-1} = r^{(i)}) - \mathbb{P}(r_t \leqslant \frac{1}{2}(r^{(j)} + r^{(j+1)}) \,|\, r_{t-1} = [r^{(i)}])$ is in $\mathcal{O}(1/K)$ since it varies the starting condition $r_{t-1}$ by no more than $r^{(i-1)} - r^{(i+1)}$.
Finally, to see that $S_3$ is in $\mathcal{O}(1/K)$, we need to apply summation by parts and the induction assumption that $\mathbb{P}(\bar{r}_{t-1}^K \leqslant r^*) - \mathbb{P}(r_{t-1} \leqslant r^*) \in \mathcal{O}(1/K)$. □

## Appendix C. PROJ method: details

This follows Kirkby (2015), Cui et al. (2017). Given the Haar scaling function $\varphi^{[0]}(y) = \mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}(y)$, it is easy to show that $\varphi^{[1]}(y) = \varphi^{[0]}(y) \star \varphi^{[0]}(y)$.[1] The Fourier transform of the Haar scaling function is given by:

$$\hat{\varphi}^{[0]}(\xi) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\xi y} \cdot \varphi^{[0]}(y)\,\mathrm{d}y = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\xi y} \cdot \mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}(y)\,\mathrm{d}y = \frac{2 \cdot \sin\left(-\frac{\xi}{2}\right)}{-\xi} = \frac{\sin\left(\frac{\xi}{2}\right)}{\frac{\xi}{2}}.$$

This and $\varphi^{[1]}(y) = \varphi^{[0]}(y) \star \varphi^{[0]}(y)$ shows that the Fourier transform of the B-spline basis function is $\hat{\varphi}^{[1]}(\xi) = 4 \cdot \sin^2\left(\frac{\xi}{2}\right)/\xi^2$. Obviously the function $\hat{\varphi}^{[1]}(\xi)$ is symmetric and real-valued. Using the definition of scalar product for complex numbers and the Plancherel theorem, we obtain:

$$\begin{aligned}\beta_{a,n} &= \left\langle f(y), \varphi_{a,n}(y)\right\rangle = \int_{-\infty}^{\infty} f(y) \cdot \overline{\varphi_{a,n}(y)}\,\mathrm{d}y = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi) \cdot \overline{\hat{\varphi}_{a,n}(\xi)}\,\mathrm{d}\xi = \frac{\sqrt{a}}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi) \cdot \frac{\mathrm{e}^{-\mathrm{i}x_n\xi}}{a}\overline{\hat{\varphi}(\xi/a)}\,\mathrm{d}\xi \\ &= \frac{\sqrt{a}}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi) \cdot \frac{\mathrm{e}^{-\mathrm{i}x_n\xi}}{a}\overline{\hat{\varphi}^{[1]}(\xi/a)}\,\mathrm{d}\xi = \frac{12}{2\pi\sqrt{a}}\int_{-\infty}^{\infty} \hat{f}(\xi) \cdot \mathrm{e}^{-\mathrm{i}x_n\xi} \cdot \frac{a^2 \cdot \sin^2\left(\frac{\xi}{2a}\right)}{\xi^2\left(2 + \cos\frac{\xi}{a}\right)}\,\mathrm{d}\xi = \frac{12a^{\frac{3}{2}}}{\pi}\int_{0}^{\infty} \hat{f}(\xi) \cdot \mathrm{e}^{-\mathrm{i}x_n\xi} \cdot \frac{\sin^2\left(\frac{\xi}{2a}\right)}{\xi^2\left(2 + \cos\frac{\xi}{a}\right)}\,\mathrm{d}\xi.\end{aligned} \tag{28}$$

We evaluate the integral (28) among grid points $\xi_j = 2\pi(j-1)a/N$ for $j = 1, 2, \ldots, N$. This results in a discrete Fourier transform using the integration steps of length $\triangle\xi_j = 2\pi a/N$:

$$\beta_{a,n} \approx \frac{24a^{\frac{5}{2}}}{N} \cdot \mathscr{D}_n(\boldsymbol{H}), \qquad \mathscr{D}_n(\boldsymbol{H}) = \sum_{j=1}^{N} \mathrm{e}^{-\mathrm{i}\frac{2\pi}{N}(j-1)(n-1)} \cdot H_j,$$

where $H_1 = 1/24a^2$ and $H_j = \mathrm{e}^{-\mathrm{i}x_1\xi_j} \cdot \hat{f}(\xi_j) \cdot \sin^2(\xi_j/2a)/(\xi_j^2(2 + \cos(\xi_j/a)))$, $j = 2, 3, \ldots, N$.

Given the financial market (1) and (2), we can exploit the affine structure of the Vasicek interest model. Given the functions $\eta(r_t) := \rho\frac{\sigma_S}{\sigma_r}(\theta - r_t)$ and $\zeta(r_t, r_0) := \kappa\frac{\sigma_S}{\sigma_r}(r_t - r_0)$, it is straightforward to show that $\rho\sigma_S\,\mathrm{d}W_t^{(1)} = \mathrm{d}\zeta(r_t, r_0) - \rho \cdot \eta(r_t)\,\mathrm{d}t$. This yields:

$$\mathrm{d}\ln(S_t) = \left(r_t + \lambda - \frac{\sigma_S^2}{2}\right)\mathrm{d}t + \sqrt{1-\rho^2}\sigma_S\,\mathrm{d}W_t^{(2)} + \rho\sigma_S\mathrm{d}W_t^{(1)} = \left(r_t + \lambda - \frac{\sigma_S^2}{2}\right)\mathrm{d}t + \sqrt{1-\rho^2}\sigma_S\,\mathrm{d}W_t^{(2)} + \mathrm{d}\zeta(r_t, r_0) - \rho\eta(r_t)\,\mathrm{d}t$$

which can be integrated to:

$$\ln\left(\frac{S_t}{S_{t-1}}\right) = \zeta(r_t, r_{t-1}) + \lambda + \int_{t-1}^{t}\left(r_s + \lambda - \eta(r_s)\right)\mathrm{d}s - \frac{\sigma_S^2}{2} + \sqrt{1-\rho^2}\sigma_S\left(W_t^{(2)} - W_{t-1}^{(2)}\right).$$

We condition on $\left(\int_{t-1}^{t} r_s\,\mathrm{d}s, r_{t-1}, r_t\right) = (y, r^{(k)}, r^{(j)})$ and use the affinity of $\eta$ to show that the log-returns $\ln(S_t/S_{t-1})$ are normally distributed with mean and variance:

$$\mu_{k,j}(y) := \mathbb{E}\left[\ln(S_t/S_{t-1})\right] = \zeta(r^{(k)}, r^{(j)}) + y\left(1 - \rho\frac{\sigma_S}{\sigma_r}\theta\right) - \left(\rho\frac{\sigma_S}{\sigma_r}\theta + \frac{\sigma_s^2}{2}\right), \qquad \sigma_{k,j}^2 := \mathrm{Var}\left(\ln(S_t/S_{t-1})\right) = \sigma_S^2\left(1 - \rho^2\right). \tag{29}$$

We represent the value of an equity-indexed annuity in the integral form (17) over the integrated interest rate $y = \int_{t-1}^{t} r_s\,\mathrm{d}s$, where the expected payoff function $h(y)$ is conditioned on $r_{t-1} = r^{(k)}$ and $r_t = r^{(j)}$, denoted by $h_{k,j}(y)$. Similarly, the density of $y = \int_{t-1}^{t} r_s\,\mathrm{d}s$ conditional on $r_{t-1} = r^{(k)}$ and $r_t = r^{(j)}$ is denoted $f_{k,j}(y)$. Analogous to our proof of Theorem 3.1, we can show that:

$$h_{k,j}(y) := \mathbb{E}_{\mathbb{P}}\left[\mathrm{e}^{L_{t-1,t}} \,\middle|\, r_{t-1} = r^{(k)}, r_t = r^{(j)}, \int_{t-1}^{t} r_s\,\mathrm{d}s = y\right] = \mathrm{e}^{g-y}\Phi\left(\frac{\frac{g}{\alpha} - \mu_{k,j}(y)}{\sigma_{k,j}}\right) + \mathrm{e}^{\alpha\mu_{k,j}(y) + \frac{\alpha\sigma_{k,j}^2}{2} - y}\Phi\left(\frac{\alpha\sigma_{k,j}^2 + \mu_{k,j}(y) - \frac{g}{\alpha}}{\sigma_{k,j}}\right). \tag{30}$$

[1] Note that:

$$\varphi^{[1]}(y) = \varphi^{[0]}(y) \star \varphi^{[0]}(y) = \int_{-\infty}^{\infty} \mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}(y-x) \cdot \mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}(x)\,\mathrm{d}x = \begin{cases}\int_{-\frac{1}{2}}^{y+\frac{1}{2}}\mathrm{d}x = 1 + y, & y \in [-1, 0] \\ \int_{y-\frac{1}{2}}^{\frac{1}{2}}\mathrm{d}x = 1 - y, & y \in [0, 1]\end{cases}.$$

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