FISCAL LIMITS AND THE PRICING OF EUROBONDS

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ABSTRACT. This paper proposes a methodology to price bonds jointly issued by a group of countries—Eurobonds in the euro-area context. We consider two types of bonds: the first is backed by several and joint (SJG) guarantees, the second features several but not joint (SNJG) guarantees. The pricing of SJG and SNJG bonds reflects different assumptions regarding the pooling of debtors' fiscal resources. We estimate fiscal limits for the six largest euro-area economies over 2008-2021 and deduce counterfactual Eurobond prices. For the 5-year maturity, SNJG bond yield spreads would have been about three times larger than SJG ones over the estimation sample. Hence, issuing SJG bonds could result in gains at the aggregate level. Notwithstanding our model also predicts that gains may temporarily vanish in periods of acute fiscal stress. We finally envision post-issuance redistribution schemes whereby gains stemming from the issuance of SJG bonds could be shared among participating countries; we argue that these schemes may alleviate the reduction in market discipline resulting from common bonds issuance.

JEL: C58, E43, F34, G12.

Keywords: Sovereign Credit Risk, Eurobonds, Public Debt Management, Joint Liability, Term Structure of Interest Rates.

1. INTRODUCTION

Following the last financial crisis and the COVID-19 pandemic, sovereign debts across the euro area have risen to levels unprecedented since the Second World War. In this context, the sustainability of fiscal positions—especially in the peripheral Member States—has been called

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into question. Against this backdrop, numerous academics, policymakers, and analysts have discussed proposals for issuing common bonds—often referred to as Eurobonds. The rationale behind such common bonds is most often, and more or less explicitly, a debt service relief for peripheral member states (Beetsma and Mavromatis, 2014; Favero and Missale, 2012). An ulterior motive backing common issuances is to ensure financial stability, notably by addressing the demand of financial institutions for safe assets (Brunnermeier et al., 2017).[1] Moreover, if issued on a large scale, a joint debt instrument is advocated as a useful device to increase bond market liquidity in the euro area (Hellwig and Philippon, 2011).

Surprisingly, the different proposals for common debt issuance seldom come with pricing attempts.[2] Arguably, this shortage of quantitative analysis may have contributed to the lack of support for common bond issuances. This paper offers a way to explore the pricing of joint sovereign debt instruments.

Guarantees play a significant role in the pricing of joint debt instruments. Our analysis focuses on two polar cases: (a) the case of several and joint guarantees (SJG) whereby all countries are jointly liable for each other's default through the common debt instrument, and (b) the case of several but not joint guarantees (SNJG) whereby each debtor is responsible only for a percentage contribution to each redemption. In the former case, participating European countries are responsible not only for their own percentage contribution to the bond, but also for covering any other state's unpaid contributions. In the latter case (SNJG bonds), each participant is liable only for the debt service and principal redemption corresponding to its share of the bond. In both cases, the joint debt instrument would trade as a single bond; it could be issued

[1] Although Eurobonds may constitute a way to guide the euro area towards financial stability, the objectives of Eurobond proposals do not fully overlap with those of the European Financial Stability Facility's (EFSF) and the European Stability Mechanism's (ESM) programs. Typically, the objective of the ESM is to provide financial assistance to euro-area countries experiencing, or threatened by, severe financing problems; this would complement joint issuance in times of financial distress, but goes beyond the preventive intention of a common euro-area bond.

[2] The evaluation of price effects remains merely speculative in this literature (Claessens, Mody, and Vallée, 2012, end of Section IV.B).

by an independent debt agency, with funds raised, and obligations divided between participating issuers in fixed shares (see, e.g., De Grauwe and Moesen, 2009; European Commission, 2011; Delivorias and Stamegna, 2020).

In the present paper, we propose a multi-country credit-risk model where both standard and common sovereign bonds—featuring one of the two polar types of guarantees discussed above—can be priced. The model estimation relies on national bond prices; the sample covers the period from 2008Q2 to 2021Q2.[3] We focus on the six largest euro-area economies: Germany, France, Italy, Spain, Netherlands, and Belgium (which account for almost 90% of the total Eurozone GDP). Once the model is estimated, we compute counterfactual Eurobond prices over the same period.

In the model, the probability of default depends on the considered entity's fiscal space, which can be a single country or a group of countries. The fiscal space is the distance between public debt and the so-called fiscal limit; this limit, in turn, represents the maximum outstanding debt that can credibly be covered by future primary budget surpluses (Bi, 2012; Bi and Leeper, 2013). The probability of default gets strictly positive only if public debt breaks the fiscal limit, that is, if the fiscal space is negative. In this framework, a natural way to conceive a SJG bond is to consider that it is issued by an entity for which both underlying debtors' fiscal revenues and debts are pooled. By contrast, a SNJG bond is equivalent to a combination of national bonds weighted by their participation share in the debt instrument.

Estimating the model involves the estimation of both the model parameters and the time series of national fiscal limits. These two tasks are jointly carried out by resorting to an adaptation of the so-called "inversion technique" *à la* Chen and Scott (1993). For a given model parametrization, formulas for the sovereign bond yield spreads are inverted to get fiscal limit

[3]Some bonds issued by European institutions can be seen as proxies for Eurobonds (see end of Subsection 6.1, where we compare our model-implied SJG prices with the latter). However, for the time being, there are not enough of these bonds to determine constant-maturity interest rates on a sufficiently long sample.

estimates.[4] The maximum likelihood function can then be computed; it is maximized to yield the estimated model parametrization.

Our model features a good fit of the observed term-structure of bond yield spreads across all countries; this fit is comparable to the one obtained in term-structure studies where default intensities are purely latent and have no macro-finance interpretation. We also obtain sizeable estimates of sovereign credit risk premiums, defined as those components of sovereign spreads that would not exist if agents were risk-neutral. Moreover, to the best of our knowledge, this paper is the first to provide time-varying estimates of fiscal limits for euro-area countries.[5]

Our counterfactual analysis results highlight the importance of guarantees on Eurobond pricing. By design, yield spreads associated with Eurobonds featuring several but not joint guarantees (SNJG) are close to the (issuance-weighted) average of country-specific spreads. By contrast, common bonds with several and joint guarantees (SJG) benefit from fiscal diversification effects resulting in a sizeable credit spread compression: across the estimation sample and different maturities, the SNJG bond yield spread was about three times larger than the SJG one. However, the model also predicts that SJG advantages diminish when expected fiscal spaces reduce at the euro-area scale, up to potential inversion; this turned out to be the case for two quarters in our sample—2011Q4 and 2012Q1, the peak of the euro-area debt crisis—and for the longer maturity only. Hence, except for strongly adverse fiscal conditions, raising funds through a joint liability debt instrument—the SJG bond—may reduce *aggregate* debt service in the presence of heterogenous fiscal conditions. Interestingly, for shorter maturities, the yield

[4] We posit reduced-form dynamics for national debts and fiscal limits and derive resulting bond pricing. Our approach shares some similarities with the Black-Scholes-Merton model (Black and Scholes, 1973; Merton, 1974, and its numerous extensions) in that it also features a default threshold. As noted by Duffie and Singleton (2003, Subsection 3.2.2), the tractability of the Black-Scholes-Merton model rapidly declines as one allows for a time-varying default threshold. Although our framework features a time-varying debt threshold, tractability is preserved thanks to approximation formulas—presented in Appendix B—that build on the literature on shadow-rate term-structure models (see, e.g., Krippner, 2015; Wu and Xia, 2016).

[5] Pallara and Renne (2021) provide time-varying estimates of fiscal limits for eight non-euro-area economies; therein, each country is considered independently from the others.

spread associated with the SJG bond is, at times, lower than the German bond one. (The German bonds, called Bunds, are considered the safest bond in the euro area.) Even when this is not the case, i.e. when SJG yields are higher than those of the bonds issued by the best-rated countries, one can envision post-issuance redistribution schemes under which all countries eventually benefit from common issuances. One such scheme is to distribute the overall gains in such a way as to achieve a reduction in "post-redistribution yields" that is the same in all countries (w.r.t. the yield on their respective national bonds). For the 10-year maturity, this reduction would have been about 30 basis points on average over the estimation sample.

The main concern associated with common debt issuance usually pertains to moral hazard. Under several and joint guarantees issuance schemes, some countries might be tempted to issue more debt given that the interest rate on jointly-guaranteed debt is less sensitive to an individual debt increase than non-guaranteed debt. Although our reduced-form modeling framework does not deal with moral hazard in a structural way, our findings remain valid under the conditions that (i) the amount of debt issued under the SJG scheme is relatively small or that (ii) some form of ex-post redistribution of the yield gains applies. First, as long as a sizable share of countries' funding needs are met with the issuance of national bonds, the overall debt service remains sensitive to countries' indebtedness. Thus, in the absence of redistribution schemes (case (i)), a necessary condition for market discipline to remain effective is to limit the issuance of Eurobonds. In our calculation, we typically envision that jointly-issued debt does not exceed 5% of total consolidated GDP. Second, we show that some ex-post yield gains' redistribution schemes may dampen moral hazard effects (case (ii)). For instance, considering the above-mentioned scheme—in which the issuance of SJG bonds ultimately translates into the same yield reduction for all involved countries—the funding costs of the different countries remain sensitive to the national fiscal conditions, thereby alleviating concerns of reduced fiscal discipline stemming from the issuance of common bonds. More precisely, for this scheme, we obtain that the slope of the curve relating post-redistribution yields to indebtedness is similar to that associated with national bonds (but, for each country, the former curve is below the latter as long as the issuance of SJG bonds is associated with aggregate gains).

The rest of this paper is organized as follows. Section 2 reviews related literature. The model is developed in Sections 3 (stylized version) and 4 (full-fledged version). Section 5 describes the estimation strategy. Section 6 discusses the results. Section 7 summarizes our findings and makes concluding remarks. The appendix gathers technical results; an online appendix provides additional details, proofs and results.

2. Related Literature

This paper contributes to the growing literature on sovereign credit risk and its pricing. Specifically, this paper is among the first to provide a quantitative assessment of Eurobonds' pricing. To do so, we develop a novel credit risk model where default intensities explicitly depend on fiscal variables.

2.1. **Eurobonds.** Various policy-oriented papers discuss pros and cons of common bond issuance in the euro area, and propose different forms of common bonds. Several of these studies stress that, if issued in large scale, a joint debt instrument could reduce market fragmentation and compete, in terms of size and liquidity, with the US bond market (Giovannini, 2000; Hellwig and Philippon, 2011). De Grauwe and Moesen (2009) and Claessens et al. (2012) argue that joint debt issuance can reduce borrowing costs for stressed sovereigns, allowing for gains at the aggregate level. Following the Great Financial Crisis and the euro-debt crisis, common debt issuance has been advocated by several policy-oriented studies as a device to enhance financial stability, notably because such a safe asset could break the "bank-sovereign doom loop" (European Commission, 2011; Brunnermeier et al., 2017; Delivorias and Stamegna, 2020). The challenges associated with joint debt issuances include coordination issues, political hurdle in transferring sovereignty to the EU level, and the removal of incentives for sound budgetary policies under the current fiscal discipline methods (Claessens et al., 2012). According, among others, to Delpla and Von Weizsacker (2010) and Claessens et al. (2012), common debt issuance calls for enhanced institutional frameworks and ex-ante surveillance to strengthen fiscal discipline.

TABLE 1. Eurobond proposals: main features

Features	Joint bond denomination				
	Stability bonds[a]		Euro-bills[b]	Blue/Red bonds[c]	ESBies/EJBies[d]
	Approach no. 1	Approach no. 3			
Guarantees	SJG[e]	SNJG[f]	SJG (10% of GDP)	Only blue: SJG (60% of GDP)	
Tranching		✓		✓	✓
Pooling[g]	✓	✓	✓	✓	✓
New issuance[h]	✓	✓ (partial)	✓ (partial)	✓ (partial)	
Risk of moral hazard	✓	✓	✓	✓	
Coordinated revenue management	✓			✓	
Coordinated debt management	✓	✓	✓	✓	
Pricing attempt in the study					✓ (partial and incomplete)

Notes: This table shows key features of some prominent euro-area joint debt instrument proposals in the literature. *a*: European Commission (2011); *b*: Hellwig and Philippon (2011); *c*: Delpla and Von Weizsacker (2010); *d*: Brunnermeier et al. (2017); *e*: Joint and several guarantees; *f*: Several but not joint guarantees; *g*: with "Pooling" we mean the pooling or common issuance of sovereign debts (either *ex ante* or *ex post* via pooling a portfolio of sovereign debts); *h*: with "New Issuance" we mean the issuance of a new debt instrument replacing totally or partially national bond issuance.

In Table 1, we review the features of some prominent proposals for a European joint debt instrument. Three proposals involve joint guarantees, but with varying proportions: the "Stability bond" approach no. 1 of the European Commission (2011) considers a full replacement of standard national issuances by those of an SJG bond; only short-term debt instruments, amounting to 10% of GDP, would benefit from joint guarantees under the "Eurobills" scheme proposed by Hellwig and Philippon (2011); under the blue/red scheme of Delpla and Von Weizsacker (2010), European countries would pool their public debt up to the Maastricht Treaty threshold—60% of GDP—under joint and several liability as senior ("blue") debt, while debt above this threshold would be issued as junior ("red") debt.

Other schemes depart from joint liability and consist of the partial substitution of European Member States' national issuance with several but not joint guarantees (SNJG) bonds. This is for instance the case of the "Stabilitity bond" approach no. 3 of the European Commission (2011).[6] In this scheme, Member States would retain liability for their respective share of "Stabilitity bond" issuance—as well as for their national issuances, naturally.[7] Due to the several but not joint guarantees, moral hazard would be mitigated.

The absence of joint guarantees also underlies Brunnermeier et al. (2017) proposal. Differently from the "Stability bond" approach no. 3 (European Commission, 2011), their proposal does not imply any substitution of national issuance. In their scheme, two synthetic tranches would be created out of a portfolio of (standard) national sovereign bonds, the senior and the junior tranche being respectively dubbed "European Safe Bonds" (ESBies) and "European junior bonds" (EJBies). As safe and liquid assets, ESBies would help limit financial institutions' exposure to sovereign credit risk, and thereby break the so-called sovereign-bank doom loop. Brunnermeier et al. (2017) simulate the loss given default of ESBies and EJBies under different tranching scenarios, thereby providing a partial pricing attempt for their instruments.

A few theoretical studies focus on Eurobonds. Tirole (2015) studies the effect of common bonds' issuance, focusing on the moral hazard implications. He distinguishes between two

[6] Issuance of bonds with several but not joint guarantees can be centralized (e.g., joint debt agency, European Commission, 2011; Delivorias and Stamegna, 2020) or left decentralized (De Grauwe and Moesen, 2009).

[7] The credit quality of a "Stabilitity bond" underpinned by several but not joint guarantees would be close to the weighted average of the credit qualities of the euro-area Member States.

forms of solidarity in a finite-horizon two-country setup: ex-post (spontaneous), e.g., bailouts, and ex-ante (contractual), e.g., joint-bond issuance. Given that one country's default imposes collateral damage on the other country, Tirole (2015) finds that ex-ante (respectively ex-post) solidarity is optimal when both countries exhibit a similar (resp. different) risk profile. Tsiropoulos (2019) builds a two-country general-equilibrium model of sovereign default and finds that welfare consequences of introducing SJG bonds hinge critically on the timing of their introduction. Lastly, Dávila and Weymuller (2016) study the optimal design of flexible joint borrowing agreements between a safe and a risky country; they find gains under joint liability schemes.

2.2. **Reduced-form approaches and sovereign risk premiums.** The present study draws extensively from the reduced-form approaches for pricing sovereign credit risk. Ang and Longstaff (2013) consider multi-factor affine models allowing for both systemic and sovereign-specific credit shocks to price the term structures of US states and Eurozone Member States. Estimating the default intensities for 26 countries, Longstaff, Pan, Pedersen, and Singleton (2011) find that the risk premium represents about a third of credit spreads on average. Monfort and Renne (2014) also estimate substantial sovereign risk premiums in euro-area sovereign spreads, employing a model allowing for both credit and liquidity effects. These studies show a close fit of sovereign bond yields/spreads and provide useful estimates of sovereign risk premiums. However, they do not explicitly account for the economic forces driving the movements of the sovereign default probabilities. By contrast, Borgy, Laubach, Mésonnier, and Renne (2011) and Hördahl and Tristani (2013) propose sovereign credit risk frameworks where default intensities explicitly depend on fiscal variables, and demonstrate that the fiscal environment is able to capture part of the fluctuations of sovereign credit spreads.

2.3. **Theory of fiscal limits.** Our paper relates to the literature studying the concept of *fiscal limit*, namely the maximum outstanding debt that a country could credibly sustain. In Bi (2012), Leeper (2013), Bi and Leeper (2013), Bi and Traum (2012), the concept of fiscal limit corresponds to the net present value of future maximum primary surpluses.[8] These maximum surpluses represent those surpluses implicit in the peak of the Laffer curve (Trabandt and Uhlig, 2011). After having introduced an estimated parametric reaction function of primary surpluses in a model of debt accumulation, Ghosh et al. (2013) show that there is a point—akin to the fiscal limit—where the primary balance cannot keep pace with the rising interest burden as debt increases. Beyond this point, debt dynamics becomes explosive and the government becomes unable to fully meet its obligations. Collard, Habib, and Rochet (2015) also exploit the idea of a maximum primary surplus to derive a measure of debt limit. More recently, Mehrotra and Sergeyev (2020) combine disaster risk and fiscal fatigue. In their framework, as in Lorenzoni and Werning (2013), debt dynamics are subject to a tipping point situation: in some instances, the public debt can be on an unsustainable path without immediately triggering default.

In the present paper, we do not make the maximum surplus explicit and we rely on a reduced-form approach. Assuming that the default intensity becomes strictly positive when the effective (observed) debt is higher than the (unobserved) fiscal limit, we infer the latter from bond prices.

3. Stylized model

As mentioned above, a crucial ingredient of our modelling framework is the relationship between the fiscal space—the difference between the fiscal limit and debt—and the sovereign probability of default. The parametric function we retain to model this relationship is presented in Subsection 3.1. Before incorporating this ingredient in a standard asset pricing model (in Section 4), we present a stylized model in Subsection 3.2. In Subsection 3.3, we elaborate on the pricing of SJG and SNJG common bonds in this simplified framework; and we discuss resulting asset-pricing mechanisms in Subsection 3.4.

[8] We refer to Aguiar and Amador (2014) or Yue and Wei (2019) for a general presentation of the theory of sovereign debt.

3.1. **Sovereign default probability.** On each date t, we assume that the default probability of country j ($j = A, B$) is given by

$$1 - \exp(-\underline{\lambda}_{j,t}), \tag{1}$$

where the default intensity $\underline{\lambda}_{j,t}$ is assumed to negatively depend on the fiscal space, defined as the distance between fiscal limit-to-GDP ($\ell_{j,t}$) and debt-to-GDP ($d_{j,t}$). Specifically:[9]

$$\underline{\lambda}_{j,t} = \alpha \max(0, d_{j,t} - \ell_{j,t}). \tag{2}$$

The previous formulation implies that the probability of default is strictly positive only if the fiscal space is negative, i.e. if debt stands above the fiscal limit. Parameter α characterizes the nature of the fiscal limit: if α is large, the fiscal limit is "strict", as the probability of default becomes large as soon as debt breaches the fiscal limit; for lower values of α, the fiscal limit is "soft", as negative fiscal spaces then do not necessarily trigger default.

The notion of soft fiscal limit is consistent with the widespread idea that it is difficult to assess sovereign debt sustainability (e.g., Warmedinger et al., 2017; Debrun et al., 2019), which gives rise to "grey areas" where default becomes likely but can also be avoided. The World Bank and the IMF themselves reckon that, alongside quantitative approaches, the use of judgment is needed to assess sovereign debt sustainability (IMF and World Bank, 2021).

In the rest of the present section, we consider the case of $\alpha = 1$, implying a relatively soft concept of fiscal limit: for a fiscal space of -1% of GDP, the probability of default is 1%.[10]

[9] It can be seen that we have $\underline{\lambda}_{j,t} = \max(0, \lambda_{j,t})$, with $\lambda_{j,t} = \alpha \times (d_{j,t} - \ell_{j,t})$. Using the vocabulary introduced by Black (1995), $\lambda_{j,t}$ can be interpreted as a "shadow default intensity." Alternatively, to have a non-negative intensity, $\underline{\lambda}_{j,t}$ could be modeled as a quadratic function of the fiscal space (see, e.g., Doshi et al., 2013). However, it is impossible to have a monotonous relationship between the (non-negative) default intensity and the covariates in a quadratic framework (while such a monotonous relationship is expected to hold in the present context). Coroneo and Pastorello (2020) also employ the shadow-rate approach to price sovereign bonds issued by different countries; contrary to the present paper though, sovereign default probabilities (or default intensities) are not explicitly modeled in their yields-only reduced-form framework. Therefore, the framework of Coroneo and Pastorello (2020) does not allow to recover sovereign probabilities of default, and cannot preclude negative default probabilities.

[10] Low values of α allow for approximate pricing formulas (Appendix B) that are intensively used in our empirical analysis (Section 4). As shown by Footnote 11, these approximate formulas are not needed in the context of the stylized model.

3.2. **Assumptions of the stylized model.** Investors are risk-neutral and risk-free interest rates are zero. In this context, the date-t price of a one-period zero-coupon zero-recovery-rate bond issued by j is simply given by:

$$P_{t,1}^{(j)} = \mathbb{E}_t \exp(-\max[0, d_{j,t+1} - \ell_{j,t+1}]), \quad (3)$$

where $\mathbb{E}_t$ denotes the expectation conditional on the information available to the investor as of date t.

For each country, the fiscal limit-to-GDP ($\ell_{j,t}$) is constant, fixed at $\bar{\ell}_j$, and the debt-to-GDP ratios are i.i.d. Gaussian:

$$\begin{bmatrix} d_{A,t} \\ d_{B,t} \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \bar{d}_A \\ \bar{d}_B \end{bmatrix}, \begin{bmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{bmatrix}\right). \quad (4)$$

In this context, the prices of zero-coupon bonds (see eq. 3) admit closed-form solutions deduced from standard results on truncated normal distributions.[11]

3.3. **Common bonds.** We consider two types of common bonds: the first is backed by several and joint (SJG) guarantees, whereby each issuing country guarantees the totality of the obligations, and the second features several but not joint (SNJG) guarantees, whereby each issuing country guarantees only its share of the joint instrument.

A natural way to conceive the SJG bond is to consider that it is issued by a synthetic area where both fiscal revenues and debts are pooled, and to assume that this area also features a probability of default of the form of (1). Denoting by ω the vector of GDP weights, the price of SJG bond is given by:[12]

$$P_{t,1}^{(SJG)} = \mathbb{E}_t \exp(-\max[0, \omega \cdot d_{t+1} - \omega \cdot \bar{\ell}]), \quad (5)$$

[11]Formally, $P_{t,1}^{(j)}$ is given by:

$$\Phi\left(\frac{\bar{\ell}_j - \bar{d}_j}{\sigma}\right) + \left(1 - \Phi\left(\frac{\bar{\ell}_j - \bar{d}_j}{\sigma}\right)\right) \exp\left(\alpha(\bar{\ell}_j - \bar{d}_j) + \frac{\alpha^2\sigma^2}{2}\right)\left\{1 - \Phi\left(\frac{\bar{\ell}_j - \bar{d}_j}{\sigma} + \alpha\sigma\right)\right\} \Big/ \left\{1 - \Phi\left(\frac{\bar{\ell}_j - \bar{d}_j}{\sigma}\right)\right\}.$$

[12]ω is such that $\omega = [\omega_A, 1 - \omega_A]$, with $\omega_A = Y_A/(Y_A + Y_B)$, where Y_j is country j's GDP.

where $\omega \cdot d_{t+1} = \omega_A d_{A,t+1} + \omega_B d_{B,t+1}$ and $\omega \cdot \overline{\ell} = \omega_A \overline{\ell}_{A,t+1} + \omega_B \overline{\ell}_{B,t+1}$ are, respectively, the GDP-weighted debt-to-GDP ratio and the GDP-weighted fiscal limit.

Regarding the SNJG bond, the absence of joint guarantee implies that the payoff of this bond is of the form $\omega \cdot (1 - \mathcal{D}_{t+1})$, where $\mathcal{D}_{t+1} = [\mathcal{D}_{A,t+1}, \mathcal{D}_{B,t+1}]$ is the vector of default indicators—a default indicator being equal to 1 in the case of default, and to 0 otherwise.[13] In other words, the payoff is equal to 1 if none of the countries default on date $t+1$, ω_A (respectively ω_B) if only B (resp. A) defaults on date $t+1$, and 0 if both countries default. This implies that the price of a SNJG bond is given by:

$$P_{t,1}^{(SNJG)} = \omega_A \mathbb{E}_t(1 - \mathcal{D}_{A,t+1}) + \omega_B \mathbb{E}_t(1 - \mathcal{D}_{B,t+1}) = \omega \cdot P_{t,1}, \quad (6)$$

with $P_{t,1} = [P_{t,1}^{(A)}, P_{t,1}^{(B)}]$.

3.4. **Calibration and resulting yields.** The different calibrations used in this section are summarized in Table 2. In our baseline case, we set the average fiscal spaces of both countries to 20% ($= \overline{\ell}_j - \overline{d}_j = 100\% - 80\%$), and the two countries are alike in all respects. In particular, they have the same (GDP) size, i.e. $\omega_A = \omega_B = 50\%$, and the correlation between debts is set to 50%. In this baseline case, the yields on one-year national bonds are equal to 28 basis points.[14] In this baseline context, where both countries are similar, it also comes that SNJG bond prices are equal to those of country-specific bonds (see eq. 6, with $P_{t,1}^{(A)} = P_{t,1}^{(B)}$); the SNJG bond yield is therefore also equal to 28 basis points. By contrast, the price of the SJG bond is higher, the SJG bond yield being of 13 basis points. This results from the fact that, for the synthetic "pooled" area, the probability to have an (average) debt-to-GDP larger than the (average) fiscal limit is

[13] We conceive state 1 (default) as an absorbing case. Given that the default state is a stopping time—in the sense that, in a case of default, the last payoff is on the default date—we can make this assumption without loss of generality (even when we will consider longer-term bonds).

[14] Since, in the stylized model described in this section, risk-free yields are taken equal to zero, bond yields essentially correspond to credit spreads. In addition, since the recovery rate is also zero, yields here coincide with probabilities of default. These restrictions are relaxed in the extended model (Section 4).

lower than for a single country. Formally:

$$\mathbb{P}(\omega_A d_{A,t} + \omega_B d_{B,t} \geq \bar{\ell}) < \mathbb{P}(d_{j,t} \geq \bar{\ell}), \quad j = A, B, \tag{7}$$

which is true as long as the correlation between the two debt-to-GDP ratios is strictly lower than 1. The fact that the SJG bond yield is lower than national bond yields implies that both countries would reduce their debt service through the issuance of joint-liability bonds.

TABLE 2. Calibrations of stylized models

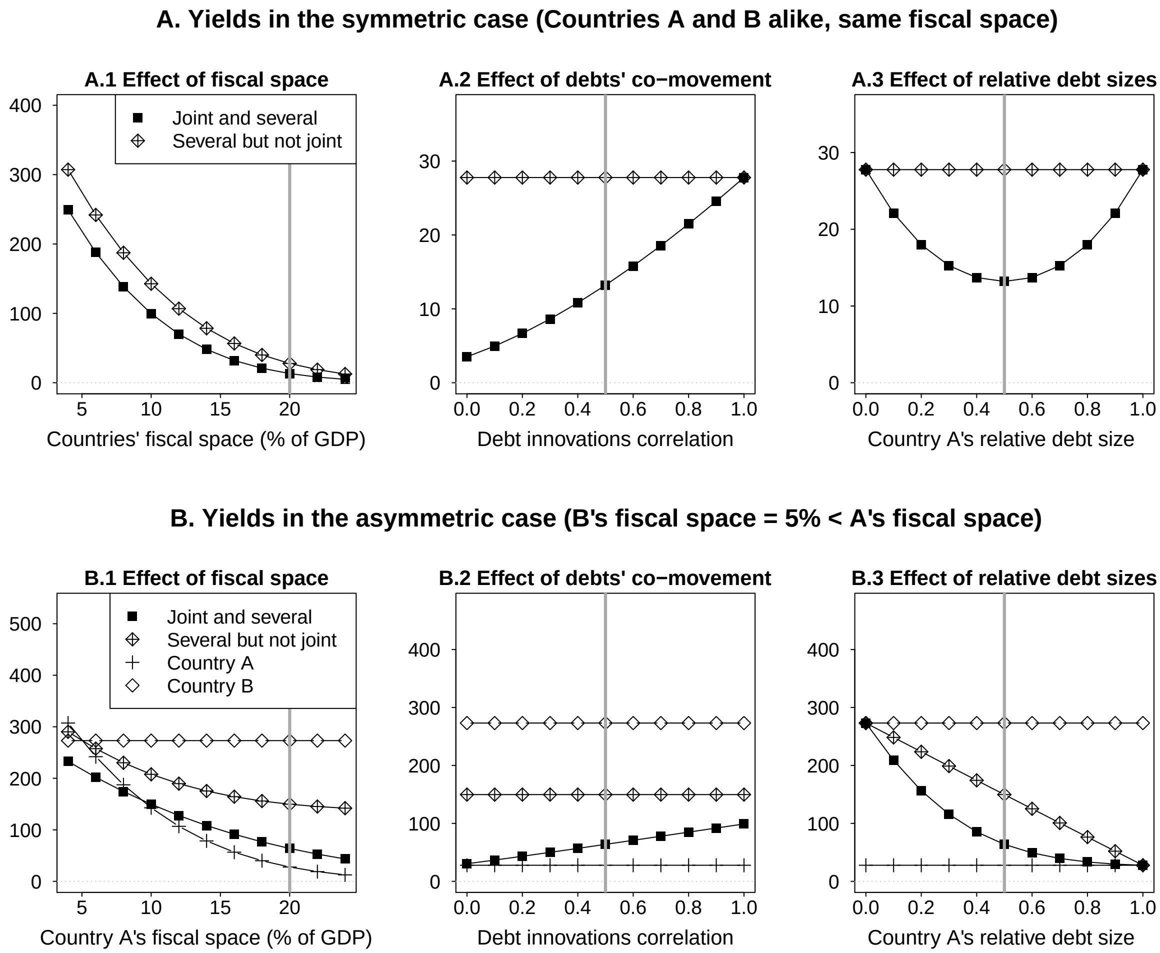
Baseline		Symmetric case (A and B are alike)			Asymmetric case (B's fiscal space ≤ A's fiscal space)		
		A.1	**A.2**	**A.3**	**B.1**	**B.2**	**B.3**
σ	12.5%						
$\bar{\ell}_{A,B}$	100%						
$\bar{d}_B$	80%	75%→95%			95%	95%	95%
$\bar{d}_A$	80%	75%→95%			75%→95%		
ρ	50%		0%→100%			0%→100%	
ω_A	50%			0%→100%			0%→100%

Notes: This table summarizes the calibrations used in our stylized model. The first column shows the calibration of the baseline case (represented by a vertical grey line in the first row of plots of Figure 1). The average fiscal space of country j corresponds to $\bar{\ell}_j - \bar{d}_j$, and ω_A denotes the relative GDP size of country A (such that $\omega_B = 1 - \omega_A$).

The baseline situation discussed above is represented by a vertical grey bar in the first row of plots in Figure 1. These plots further show how the SNJG and SJG yields are affected with respect to: (Panel A.1) changes in the fiscal spaces of the two countries, (Panel A.2) changes in the correlation across debts, and (Panel A.3) changes in the relative size of country A (in terms of GDP).

Panel A.1 shows that both SJG and SNJG bond yields nonlinearly decrease when fiscal spaces increase. It also shows that SJG bond yields are consistently lower than those of SNJG bonds. Panel A.2 illustrates the importance of debt co-movements to account for the yield reduction resulting from joint guarantees: while the SNJG bond yield is not affected by changes in debts' correlations, the yield of a SJG bond is reduced by a factor of 8 when the correlation decreases from 100%—in which case all bonds are equivalent—to 0%. Panel A.3 focuses on the effect of the two countries' relative sizes. In the extremes, when the relative size of country A is either 0

FIGURE 1. Two-country stylized model mechanisms



Notes: These plots show the yields-to-maturity, expressed in basis points, of different types of one-period bonds; it also shows how these yields are affected by changes in the calibration of the stylized model (see Table 2 for details regarding the baseline calibration and the alternative calibrations underlying Panels A.1 to B.3 of this figure). Three types of bonds are considered: national, or country-specific, bonds issued by countries A and B; a bond with several and joint guarantees (SJG); and a bond with several but not joint guarantees (SNJG). See Subsection 3.4 for more details. On each row of plots, the vertical grey line represents the same situation—the "baseline" case of Table 2.

or 1, there is no difference between SJG and SNJG bonds. As in the case of debt co-movement, and because we consider two equally-risky countries for the time being, the relative size of country A has no effect on the SNJG yield. But it has on the SJG yield; the effect is maximum when the two countries are equally large, corresponding to a situation where diversification effects are maximum.[15]

[15]Formally, this is because the variance of the aggregate debt-to-GDP ratio—that is $\sigma^2(\omega_A^2 + (1-\omega_A^2) + 2\rho\omega_A(1-\omega_A))$—admits a minimum for $\omega_A = \frac{1}{2}$.

The second row of plots in Figure 1 displays results obtained in an asymmetric situation, where country B is riskier than country A. We fix the fiscal space of country B to 5%, keeping A's one at 20%. National bond yields are now different for the two countries, and we add them to each plot. Up to small convexity effects, it can be checked that SNJG yields are equal to the GDP-weighted averages of the two national bond yields. In particular, in Panel B.3, where we modify the relative size of country A from 0 to 1, the SNJG bond yield goes from the (higher) country-B yield to the (lower) country-A yield. Regarding the difference between SNJG and SJG yields, an interesting situation is captured by Panel B.1: for low values of country A's fiscal space, not only is the SJG bond yield below the SNJG one (i.e. the average of the two national bond yields), it is also lower than the safer country's bond yields. Finally, Panel B.2 shows that when the two countries do not have the same average fiscal space, a correlation of 1 across debts does not imply that the SJG and the SNJG bonds are equivalent. In this extreme case, and contrary to the symmetric case, diversification effects are still at play in the SJG bond pricing: the SJG bond yield is 1.5 times lower than the SNJG one.

To end with, it is interesting to note that, in this framework, diversification mechanisms can have adverse effects on SJG bonds prices when expected fiscal spaces are negative enough. Intuitively, when this is the case, the distribution of the joint fiscal space (and therefore of the default intensity, see eq. 2) turns out to be more concentrated on the "wrong side" of zero, yielding to lower prices for SJG bonds. We discuss this situation in greater details in Online Appendix I.

4. Model

In this section, we enrich the stylized model to make it amenable to the data. We consider N countries. While the conditional probabilities of default remain as in Subsection 3.1—with default intensities that depend on fiscal spaces—debt-to-GDP ratios and fiscal limits are now time-varying; in addition, the state vector is augmented with a stochastic short term interest rate (Subsection 4.1). The representative investor is now risk averse, her risk preferences being captured by a reduced-form stochastic discount factor (Subsection 4.2). After having derived prices of zero-coupon risk-free bonds, we discuss the pricing of zero-coupon bonds with non-zero recovery rates and bond yield spreads (Subsection 4.3). The ability to swiftly price risk-free bonds and yield spreads is crucial to estimate the model (Section 5).

4.1. **Dynamics of the state vector.** Fiscal limits follow autoregressive processes of order one. For country j:

$$\ell_{j,t} \quad = \quad (1-\rho_\ell)\overline{\ell}_j + \rho_\ell \ell_{j,t-1} + \varepsilon_{\ell,j,t}, \tag{8}$$

where the $\varepsilon_{\ell,j,t}$'s are Gaussian white noise shocks.

The formulation of debt-to-GDP dynamics is inspired by standard debt accumulation processes, where debt-to-GDP depends on its first lag and on the budget surplus. Specifically:[16]

$$d_{j,t} \quad = \quad \rho_d d_{j,t-1} + \{\overline{\gamma}_j + \gamma_{j,t}\}, \tag{9}$$

where $\overline{\gamma}_j + \gamma_{j,t}$ proxies for country j's primary deficit (expressed as a fraction of GDP). The cyclical part of the deficit, $\gamma_{j,t}$, is assumed to follow an autoregressive process of order one:

$$\gamma_{j,t} \quad = \quad \rho_\gamma \gamma_{j,t-1} + \varepsilon_{d,j,t}. \tag{10}$$

Three remarks are in order. First, since $\gamma_{j,t}$ is of mean zero, eq. (9) implies that the unconditional mean of $d_{j,t}$ is given by $\overline{d}_j = \overline{\gamma}_j/(1-\rho_d)$. Second, from eqs. (9) and (10), it comes that $d_{j,t}$ follows an autoregressive process of order two; one can indeed easily check that $d_t = (1-\rho_\gamma)(1-\rho_d)\overline{d}_j + (\rho_d+\rho_\gamma)d_{t-1} - (\rho_d\rho_\gamma)d_{t-2} + \varepsilon_{d,j,t}$. Third, considering that both ρ_d and ρ_γ are in $[0,1[$, the debt process is stationary. Since investors use the previous processes to price government bonds, our framework implicitly excludes snowball effects and related multiple

[16]Standard debt-to-GDP accumulation processes read $d_t = \frac{1+r_t}{1+g_t}d_{t-1} + \tilde{\gamma}_t$, where r_t denotes the apparent interest rate (i.e., debt service over debt outstanding), g_t denotes GDP growth, and $\tilde{\gamma}_t$ is the primary deficit.

equilibria (that would give rise to non-stationary processes). This can be attributed to investors' limited rationality.

Fiscal-limit shocks ($\varepsilon_{\ell,j,t}$) and debt shocks ($\varepsilon_{d,j,t}$) can be correlated. Formally, using the notations $\varepsilon_{d,t} = [\varepsilon_{d,1,t}, \dots, \varepsilon_{d,N,t}]'$ and $\varepsilon_{\ell,t} = [\varepsilon_{\ell,1,t}, \dots, \varepsilon_{\ell,N,t}]'$, we set:

$$\begin{bmatrix} \varepsilon_{d,t} \\ \varepsilon_{\ell,t} \end{bmatrix} \sim i.i.d.\mathcal{N}(0, \Omega), \quad \text{with } \Omega = \begin{bmatrix} \Omega_d & \Omega'_{\ell,d} \\ \Omega_{\ell,d} & \Omega_\ell \end{bmatrix}.$$

The structures of Ω_d, Ω_ℓ, and $\Omega_{\ell,d}$ will be explained below, in Subsection 5.3, which details the estimation strategy and the parameter constraints.

The short-term risk-free interest rate also follows an auto-regressive process:[17]

$$i_t = (1 - \rho_i)\bar{i} + \rho_i i_{t-1} + \sigma_i \eta_{i,t}, \quad \eta_{i,t} \sim i.i.d.\mathcal{N}(0, 1). \tag{11}$$

Let us denote by d_t and ℓ_t two N-dimensional vectors gathering countries' debt-to-GDP ratios and fiscal limits, respectively. Under the previous assumptions, it is easily seen that the state vector $X_t = [i_t, i_{t-1}, d'_t, d'_{t-1}, \ell'_t]'$ follows a vector autoregressive process of order one.[18] That is:

$$X_t = \mu + \Phi X_{t-1} + \Sigma \eta_t, \tag{12}$$

where $\eta_t \sim i.i.d.\mathcal{N}(0, I)$, and where μ, Φ, and Σ (with $\Omega = \Sigma\Sigma'$) are detailed in Appendix A.

4.2. **Stochastic discount factor and the term structure of risk-free rates.** We assume that arbitrage opportunities do not exist, which ensures the existence of a positive stochastic discount factor (s.d.f.). Following Ang and Piazzesi (2003), we posit a reduced-form exponential affine s.d.f. between dates t and $t+1$:

$$\mathcal{M}_{t,t+1} = \exp(-i_t)\frac{\xi_{t+1}}{\xi_t}, \tag{13}$$

[17] This Gaussian process does not exclude negative nominal interest rates. Hence, this model is not consistent with the existence of the Zero Lower Bound (ZLB). This simple model however conveniently accommodates the period during which risk-free European nominal rates have been negative (indicating that the Effective Lower Bound, ELB, is lower than zero).

[18] We include the lagged short-term interest rate, i_{t-1}, in the state vector because the defaultable-bond pricing formulas are easier to derive if i_{t-1} can be expressed as a linear combination of X_t (see the notation below eq. a.7). Moreover, X_t includes d_{t-1} because, as mentioned above, d_t follows an auto-regressive process of order two under eqs. (9) and (10).

where ξ_{t+1} follows:

$$\xi_{t+1} = \xi_t \exp\left(-\frac{1}{2}\psi_t'\psi_t - \psi_t'\eta_{t+1}\right), \tag{14}$$

ψ_t being a vector of prices of risk that linearly depends on X_t:

$$\psi_t = \psi_0 + \psi_1 X_t. \tag{15}$$

In this context, it is well-known that risk-free bond prices admit closed-form recursive solutions. Specifically, the date-t price of a risk-free zero-coupon bond of maturity h is given by (proof in Online Appendix III):

$$B_{t,h} = \exp(A_h + B_h X_t), \tag{16}$$

where $A_1 = 0$ and $B_1 = [-1, 0, \dots]'$ and, for $h > 1$:

$$\left\{\begin{array}{lcl} A_h & = & A_{h-1} + B_{h-1}'(\mu - \Sigma\psi) + \frac{1}{2}B_{h-1}'\Sigma\Sigma' B_{h-1} \\ B_h & = & B_1 + \Phi' B_{h-1}. \end{array}\right. \tag{17}$$

Equivalently, the yield of a risk-free zero-coupon bond of maturity h is given by:

$$i_{t,h}^0 = -1/h(A_h - B_h' X_t). \tag{18}$$

4.3. **Zero-coupon bonds with non-zero recovery rates and sovereign bond yield spreads.** Consider a zero-coupon bond of maturity h issued by country j. Our recovery payoff assumption is based on the "Recovery of Treasury" (RT) convention of Duffie and Singleton (1999): on date $t + k$, with $0 < k \leq h$, the payoff of the considered bond is zero, unless the country defaults on date $t + k$, in which case the bond payoff is assumed to be the fraction RR (recovery rate) of the price of a risk-free zero-coupon bond of equivalent residual maturity, i.e. $\exp[-(h-k)i_{t+k,h-k}^0]$. Formally, the payoffs of this bond are of the form:[19]

$$\left\{\begin{array}{ll} RR \times \exp(-(h-k)i_{t+k,h-k}^0) \times (\mathcal{D}_{j,t+k} - \mathcal{D}_{j,t+k-1}) & \text{if} \quad 0 < k < h, \\ 1 - \mathcal{D}_{j,t+k} + RR \times (\mathcal{D}_{j,t+k} - \mathcal{D}_{j,t+k-1}) & \text{if} \quad k = h. \end{array}\right.$$

[19] Note that $\mathcal{D}_{j,t}$ is valued in $\{0, 1\}$, state 1 being the default state, which is absorbing. As a result, $\mathcal{D}_{j,t+k} - \mathcal{D}_{j,t+k-1}$ is equal to zero, except once, on the default date, where it is equal to 1. In reality, the default state is not absorbing. However, given that the default state is a stopping time—in the sense that, in a case of default, the last payoff is on the default date—we can make this assumption without loss of generality.

Denoting by $\mathcal{M}_{t,t+k}$ the stochastic discount factor between dates t and $t+k$ (i.e., $\mathcal{M}_{t,t+k} = \mathcal{M}_{t,t+1} \times \cdots \times \mathcal{M}_{t+k-1,t+k}$) and after some algebra (Online Appendix VI), one obtains the following expression for the price of this bond:

$$\mathcal{P}^{(j)}_{t,h} = (1 - RR) \times \mathbb{E}_t \left(\mathcal{M}_{t,t+h}(1 - \mathcal{D}_{j,t+h}) \right) + RR \times B_{t,h}, \tag{19}$$

where $B_{t,h}$, the price of the risk-free bond (Subsection 4.2), is equal to $\mathbb{E}_t(\mathcal{M}_{t,t+h})$, and the conditional expectation $\mathbb{E}_t \left(\mathcal{M}_{t,t+h}(1 - \mathcal{D}_{j,t+h}) \right)$ corresponds to the date-t price of a zero-coupon zero-recovery-rate bond of maturity h providing a payoff of 1 on date $t+h$ if country j has not defaulted before $t+h$, and zero otherwise. Online Appendix B details the computation of the latter conditional expectation.

Sovereign bond yields for country j are given by:

$$i^{(j)}_{t,h} = -\log(\mathcal{P}^{(j)}_{t,h})/h, \tag{20}$$

and sovereign spreads are computed as follows ($i^{0}_{t,h}$ being given by eq. 18):

$$s^{(j)}_{t,h} = i^{(j)}_{t,h} - i^{0}_{t,h}. \tag{21}$$

5. Estimation

5.1. **Data.** We consider six European countries: Germany, France, Italy, Spain, Netherlands, and Belgium. These countries' GDPs account for close to 90% of euro-area's GDP. The data are quarterly and span the period from 2008Q2 to 2021Q2. Sovereign yields and the 3-month Overnight Indexed Swap (OIS) interest rate—our short-term risk-free rate—are extracted from Thomson Reuters Datastream. Following Monfort and Renne (2014), risk-free yields of maturities of 2, 3, 5, and 10 years are proxied for by the difference between German bond yields and German CDSs of matching maturities. Observations of sovereign spreads ($s^{(j)}_{t,h}$'s in eq. 21) are computed as the difference between national bond yields and these risk-free yields. We consider three maturities of bond yield spreads: 3, 5, and 10 years. Time series of gross government debts and GDPs are collected from the Eurostat ESA2010 database.

5.2. **Estimation approach.** The model can be cast into a state-space form, with (i) transition equations describing the dynamics of the state variables (this is eq. 12) and (ii) measurement equations describing the relationships between observed financial market data—prices and yield spreads—and the state vector. Let us denote by Θ the set of model parameters,[20] the state-space model is of the form:

$$\begin{aligned} (i) \quad X_t &= \mathcal{F}(X_{t-1}, \eta_t; \Theta), \quad \text{(reformulation of eq. 12)} \\ (ii) \quad Y_t &= \mathcal{G}(X_t; \Theta) + \xi_t, \end{aligned}$$

where $X_t = [i_t, i_{t-1}, d_t', d_{t-1}', \ell_t']'$ is the state vector, Y_t denotes the vector of financial market data (gathering risk-free yields and sovereign spreads), and ξ_t is a vector of i.i.d. Gaussian measurement errors. Function $\mathcal{G}$ stands for pricing formulas, associating the state X_t to risk-free yields and sovereign spreads. While the risk-free rates are affine in X_t (see eq. 18), this is not the case for sovereign spreads because of the nonlinearity of the default intensity (resulting from the "max" operator in eq. 2). The vector of state variables X_t is only partially observed by the econometrician since the N national fiscal limits (ℓ_t) are latent. We therefore face two types of unknowns: the model parameters and the fiscal limits. We address this problem by employing "inversion techniques". These techniques, originally introduced by Chen and Scott (1993) in the term structure literature, consist in estimating the latent pricing factors by inverting a non-singular system relating prices to latent factors. This system results from the assumption that some of the observed prices are modeled without errors. In the present case, we assume that, for each country, the averages of the three sovereign spreads (with maturities 3, 5, and 10 years) are perfectly priced. Under this assumption, we can recover the fiscal limits and, simultaneously, compute the likelihood function associated with the considered model parametrization.[21] This

[20]We have $\Theta = \{\bar{i}, \rho_i, \sigma_i, \bar{d}_1, \dots, \bar{d}_N, \rho_d, \bar{\ell}_1, \dots, \bar{\ell}_N, \rho_\ell, \rho_\gamma, \Omega_d, \Omega_\ell, \Omega_{\ell,d}, \psi_0, \psi_1\}$.

[21]The likelihood then involves an adjustment term corresponding to the determinant of the Jacobian matrix associated with the non-singular system; this adjustment results from the transformation of the observables to the latent components (see e.g. Ang and Piazzesi, 2003, Appendix B). Computational details are given in the Online Appendix VIII.

opens the door to maximum-likelihood estimation. Online Appendix VIII provides computational details.

5.3. **Parameter constraints and estimates.** To facilitate the estimation and ensure plausible fiscal limit estimates, some parameters are calibrated or restricted to lay in pre-specified intervals.

For all countries, we set the stationary debt-to-GDP ratio $\bar{d}_j$ and the unconditional mean of the short term rate $\bar{i}$ to their respective sample averages. We restrict the unconditional standard deviation of the short-term rate to be larger than 0.8%, which is slightly lower than the sample standard deviation. To favor numerical stability, we impose upper bounds, of 0.999, to all autoregressive parameters. We set the bounds for the unconditional mean of the fiscal limit ($\bar{\ell}$ in Table 3) to lie in between 0% and 300% of GDP.[22] We restrict the autoregressive parameter of the $\gamma_{j,t}$'s—the proxies for the cyclical components of primary surpluses—to be larger than 0.7, given that the cross-country average of the autocorrelations of primary balance is 0.8. The maximum Sharpe ratio, that characterizes the pricing of risk, is supposed to lie between 0.5 and 1.5 (it is set to one in Cochrane and Saa-Requejo, 2000).[23] The standard deviations of the measurement errors associated with yields and sovereign spreads are respectively set to 10 basis points and to 10% of the country-wise sample standard errors of sovereign spreads.

The model parameters include the conditional covariance matrix of X_t, i.e., $\Omega = \Sigma\Sigma'$ (see eq. 12). Freely estimating all the parameters of this matrix would be numerically challenging. Instead, we design an approach that, while capturing the sample correlation structure of debt shocks, remains parsimonious. It works as follows:

[22]The bounds on the unconditional mean of fiscal limits is based on the observation that the average of estimates obtained by Ghosh et al. (2013) and Collard et al. (2015) fall within the same interval (and are never above 220% of GDP for the set of countries here analyzed). These (static) estimates are reported in Table 7 in Appendix XII.

[23]Setting bounds on the maximum Sharpe ratio is an approach that is employed by, e.g., Jiang et al. (2019). A maximum Sharpe ratio below 0.5 would be inconsistent with the empirical evidence, as Sharpe ratios above 0.5 are frequent (e.g., Lettau and Ludvigson, 2010; Hong and Linton, 2020). Note that, in our framework, the maximum Sharpe ratio is not constant since it depends on the short term rate i_t dynamics (see Appendix C). The maximum Sharpe ratio that we effectively constrain is the one evaluated at the average of the state vector, which is close to its sample average given that i_t is fairly constant over the sample. More details are provided in Appendix C.

(i) For a considered parametrization of ρ_d, ρ_γ, and of the $\overline{d}_j$'s (see eqs. 9 and 10), one can recover estimates of the $\varepsilon_{d,j,t}$'s. We perform a PCA analysis of the resulting shocks, and we denote the resulting standardized PCAs by $\eta_{d,k,t}$'s ($k =, 1, \dots, N$). At that stage, we have:

$$\varepsilon_{d,t} = \Gamma_d \eta_{d,t}, \tag{22}$$

where Γ_d is the matrix of PCA weights, that is such that $\mathbb{V}ar(\varepsilon_{d,t}) = \Gamma_d \Gamma_d'$ (using that $\mathbb{V}ar(\eta_{d,t}) = I$). Note that the parameters of the first column of Γ_d are larger as they correspond to the first PCA. In other words, $\eta_{d,1,t}$ accounts for the largest common variance of the $\varepsilon_{d,j,t}$'s ($k =, 1, \dots, N$).

(ii) We further assume that the fiscal-limit shocks admit the same structure, up to a multiplicative factor. More precisely, we assume that:

$$\varepsilon_{\ell,t} = \Gamma_\ell \eta_{\ell,t}, \tag{23}$$

with $\Gamma_\ell = \zeta \Gamma_d$ and $\mathbb{V}ar(\eta_{\ell,t}) = I$. Again, by construction, the first column of Γ_ℓ contains larger parameters. That is, $\eta_{\ell,1,t}$ is the main common driver of the fiscal-limit shocks. In the estimation, we restrict ζ to be between 0.5 and 1.5.

(iii) In order to allow for correlation between debts and fiscal limits in a parsimonious way, we assume that the two "main common shocks," namely $\eta_{d,1,t}$ and $\eta_{\ell,1,t}$, are correlated. Specifically, we assume that these shocks admit the following decomposition:

$$\eta_{d,1,t} = \sqrt{1-\rho_{d,\ell}}\tilde{\eta}_{d,1,t} + \sqrt{\rho_{d,\ell}}\tilde{\eta}_{d,\ell,t} \tag{24}$$

$$\eta_{\ell,1,t} = \sqrt{1-\rho_{d,\ell}}\tilde{\eta}_{\ell,1,t} + \sqrt{\rho_{d,\ell}}\tilde{\eta}_{d,\ell,t}, \tag{25}$$

where $\tilde{\eta}_{d,\ell,t}$, $\tilde{\eta}_{d,1,t}$, and $\tilde{\eta}_{\ell,1,t}$ are independent standard Gaussian shocks. Together, eqs. (24) and (25) imply that $\rho_{d,\ell}$ is the correlation between $\eta_{d,1,t}$ and $\eta_{\ell,1,t}$.

Hence, the complete vector of independent shocks affecting the system (eq. 12) is:

$$\eta_t = \left[\begin{array}{cc} \eta_{i,t} & \tilde{\eta}_t' \end{array}\right]' \sim i.i.d. \mathcal{N}(0, I), \quad \text{with } \tilde{\eta}_t = \left[\begin{array}{ccc} \tilde{\eta}_{d,\ell,t} & \tilde{\eta}_{d,t}' & \tilde{\eta}_{\ell,t}' \end{array}\right]', \tag{26}$$

where $\tilde{\eta}_{d,t} = [\tilde{\eta}_{d,1,t}, \eta_{d,2,t}, \dots, \eta_{d,N,t}]'$ and $\tilde{\eta}_{\ell,t} = [\tilde{\eta}_{\ell,1,t}, \eta_{\ell,2,t}, \dots, \eta_{\ell,N,t}]'$. Appendix A details the shape of matrices Σ and Ω (with $\Omega = \Sigma\Sigma'$) that results from these assumptions.

To discipline the estimation further, we adopt a parsimonious specification for the prices of risk (ψ_t in eqs 14 and 15). First, we assume that only the first entry of ψ_t—that corresponds to interest-rate risk—is time-varying. Specifically, we have: $\psi_{1,t} = \psi_{i,0} + \psi_{i,1} i_t$. As a result, matrix ψ_1 (eq. 15) is filled with zeros, except its $(1,1)$ entry, which is equal to $\psi_{i,1}$.[24] Second, as regards debt and fiscal-limit shocks ($\tilde{\eta}_t$), we assume that the s.d.f. depends only on the main common shocks, namely $\tilde{\eta}_{d,1,t}$ and $\tilde{\eta}_{\ell,1,t}$. Formally, we posit:

$$\psi_0 = [\psi_{i,0}, 0, -\nu, \mathbf{0}_{1\times(N-1)}, \nu, \mathbf{0}_{1\times(N-1)}]'. \tag{27}$$

If $\nu > 0$, this specification ensures that the s.d.f.—that can be seen as the ratio of marginal utilities—goes up when there is an increase in the main common debt shock $\tilde{\eta}_{d,1,t}$ or a decrease in the main common fiscal-limit shock $\tilde{\eta}_{\ell,1,t}$ (see eqs. 13 and 14).

The resulting model parametrization is given in Table 3. Several of the restrictions described above turn out to be binding, which we indicate by "†" in the table. We find, in particular, that the unconditional average of the fiscal limit, namely $\bar{\ell}$, hits the upper bound of 300%. However, as we find that ρ_ℓ is close to one, this implies that ℓ_t almost follows a random walk process and, thus, $\bar{\ell}$ is only weakly identified.

5.4. **Sovereign spreads fit and credit risk premiums.** Figure 2 shows the fit of sovereign spreads. The fit is comparable to the one obtained in term-structure studies where default intensities are purely latent and have no macro-finance interpretation.

On Figure 2, model-implied spreads (dotted black lines) result from eq. (21), which involves formulas using the stochastic discount factor $\mathcal{M}_{t,t+1}$ that itself depends on prices of risk ψ (eq. 14). The black solid lines represent the (model-implied) spreads that would be observed if

[24] This specification for $\psi_{1,t}$ implies that the risk-neutral dynamics of the short-term rate i_t is $i_t = (1-\rho_i\bar{i} - \sigma_i\psi_{i,0}) + (\rho_i - \sigma_i\psi_{i,1})i_{t-1} + \sigma_i\eta^*_{i,t}$, where $\eta^*_{i,t} \sim i.i.d.\mathcal{N}(0,1)$ under the risk-neutral measure (see Online Appendix II).

TABLE 3. Model parametrization

Param.	Value	Param.	Value
ρ_d	0.970	$\rho_{d,\ell}$	0.745
ρ_ℓ	0.997	$\overline{\ell}$	$3.000^\dagger$
ρ_γ	$0.700^\dagger$	ζ	$1.500^\dagger$
ρ_i	0.938	$\sqrt{\mathbb{V}ar(i_t)}$	$0.008^\dagger$
$\rho_i^{\mathbb{Q}}$	$0.999^\dagger$	$maxSR$	$0.500^\dagger$
α	0.083	ν	0.321
$\overline{d}_{DE}$	$0.714^\ddagger$	$\overline{d}_{FR}$	$0.938^\ddagger$
$\overline{d}_{IT}$	$1.311^\ddagger$	$\overline{d}_{ES}$	$0.862^\ddagger$
$\overline{d}_{NL}$	$0.591^\ddagger$	$\overline{d}_{BE}$	$1.050^\ddagger$
$\overline{i}$	$27.11^\ddagger$		

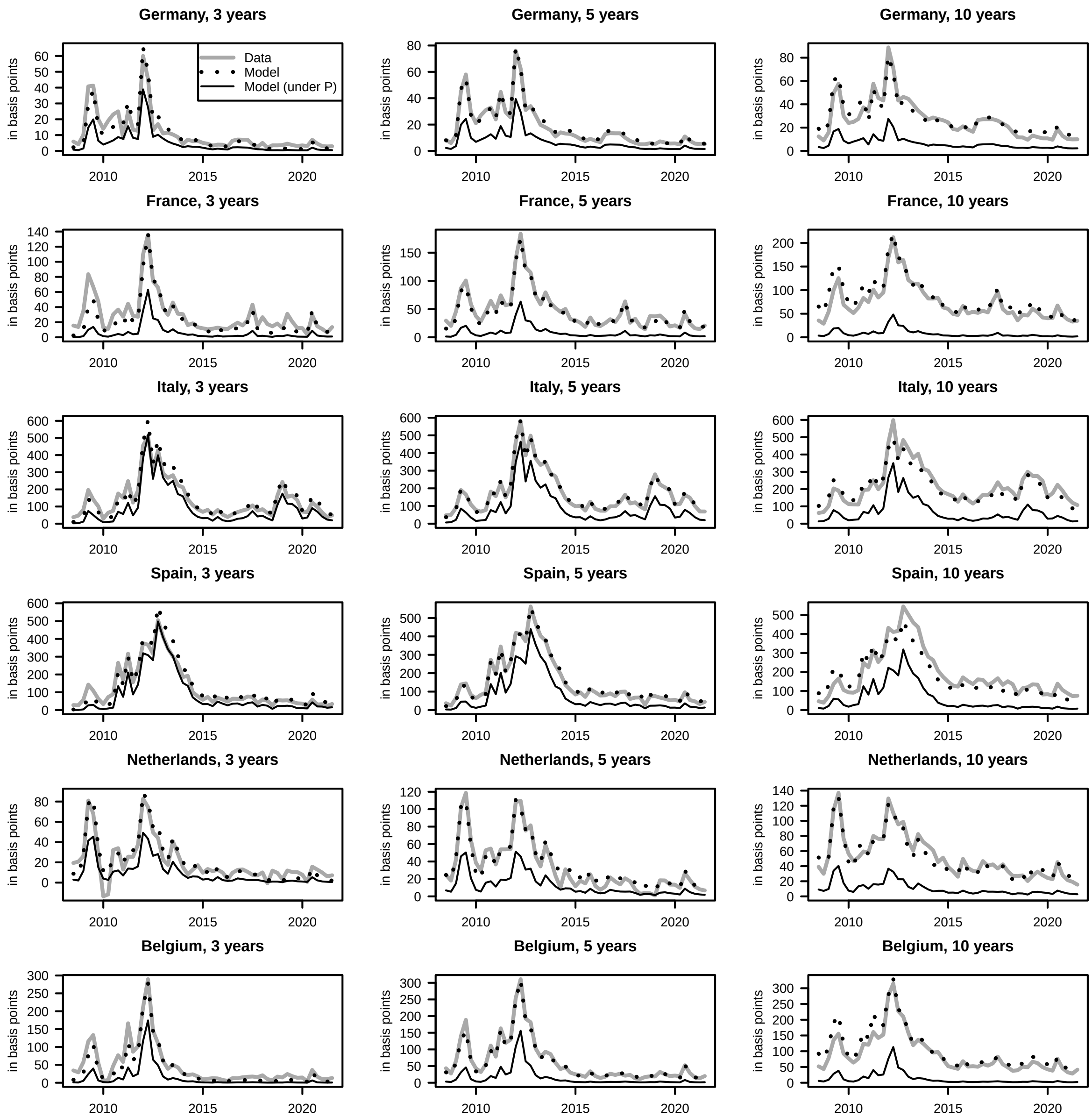
Notes: The subscript † indicates parameters for which the restrictions described in 5.3 turn out to be binding in the context of the constrained maximum likelihood estimation. The subscript ‡ indicates parameters that are calibrated: $\overline{d_j}$ is set to the observed sample mean of debt-to-GDP of country j; $\overline{i}$ is set to the sample mean of the short-term rate (3-month OIS rate), it is expressed in basis points (annualized). $maxSR$ and ν determine the vector of prices of risk ψ (see eq. 14 and Appendix C). We have the following relationship between ρ_i and $\rho_i^{\mathbb{Q}}$: $\rho_i^{\mathbb{Q}} = \rho_i - \sigma_i \psi_{i,1}$, where $\psi_{i,1}$ is the $(1,1)$ entry of matrix ψ_1 (see Online Appendix II). Parameter $\rho_{d,\ell}$ is the correlation between the two "main common components" of debts and fiscal limits (see eqs. 24 and 25). Parameter ζ determines the covariance matrix of fiscal-limit shocks (see eq. 23). Parameter α is the elasticity of the probability of default to the fiscal space (see eq. 2).

agents were not risk averse; these spreads are obtained by implementing the formulas implicit in eq. (21) after having set the prices of risk to zero. The differences between the two types of model-implied spreads correspond to credit-risk premiums. Our results indicate that these risk premiums are sizeable. The ratio between the two types of spreads, which reflects the importance of risk premiums, is broadly comparable to the ones found in sovereign credit-risk studies based on reduced-form intensity approaches (e.g. Pan and Singleton, 2008; Longstaff et al., 2011; Monfort and Renne, 2014; Monfort et al., 2020).

Let us stress that, in the present model, credit risk premium are time-varying even if the prices of risk associated with debt and fiscal-limit shocks are constant. This stems from the conditional heteroskedasticity of the default intensity inherent to our model.[25]

[25]Intuitively, a risk premium can be seen as the product of a price of risk times a risk quantity. Hence, the risk premium is time-varying if at least one of its two multiplicative constituents (prices of risk or the risk quantity) also is. In standard Gaussian affine term-structure models, prices of risk are time-varying but the risk quantity—that is the conditional variance of the factors—is constant. The opposite is true In the present model: prices of risk are constant (except for the short-term risk-free interest rate), but the conditional variance of the default intensity, i.e.,

FIGURE 2. Model fit of sovereign bond yield spreads

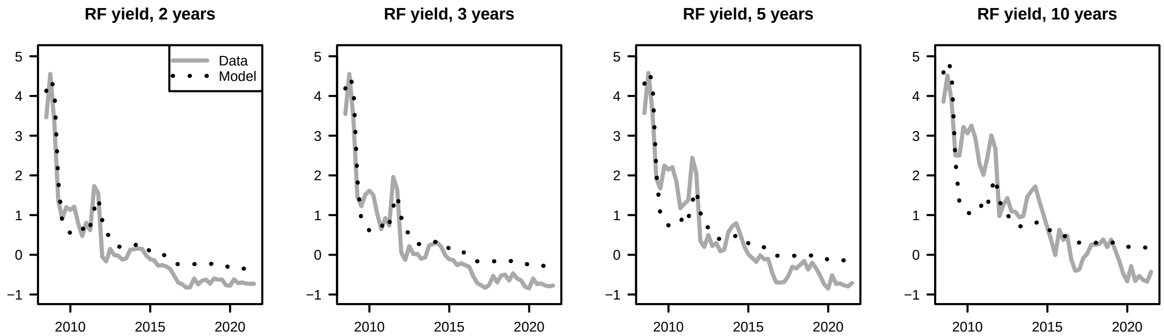


Notes: Model-implied sovereign spreads result from eq. (21). Dashed lines represent the (model-implied) spreads that would be observed if agents were not risk averse (obtained also by eq. 21, but after having set the prices of risk, that are the components of ψ, to zero). The differences between the two types of model-implied spreads (dotted and solid lines) correspond to credit-risk premiums.

Lastly, Figure 3 shows that the model captures a substantial share of the fluctuations of risk-free rates across all maturities.

$\alpha \max[0, (d_t - \ell_t)]$ (see eq. 2) is time-varying (because of the max operator). See Online Appendix VII for additional explanation.

FIGURE 3. Model fit of risk-free yields



Notes: The model implied risk-free yields (grey solid line) result from eq. (18). Interest rates are annualized, and expressed in percentage points.

5.5. **Fiscal limit estimates.** To the best of our knowledge, the present paper is the first to propose time-varying estimates of fiscal limits (together with Pallara and Renne, 2021). These estimates, expressed in percent of GDP, are displayed in Figure 4. On a given quarter, if debt-to-GDP ($d_{j,t}$, black solid line) is higher than the fiscal limit ($\ell_{j,t}$, grey solid line), then, the probability of default is strictly positive (see eq. 1). Everything else equal, if debt-to-GDP stays above the black dotted line (respectively in the grey-shaded area) for four quarters in a row, then the annual default probability of the considered country would be larger than 10% (respectively in $]0\%, 10\%]$). For what follows, and unless differently specified, our numbers refer to the threshold fiscal limit estimates, namely the grey solid lines in Figure 4. According to our estimates, the global financial crisis of 2008 translated into a decrease of the fiscal limits. On average, fiscal limits decreased by 10 percent of GDP between 2008 and 2009.[26] From the beginning of 2010 to early 2012, amid the European sovereign debt crisis, fiscal limits recorded an average decrease close to 20 percent of GDP. Notably, the "*whatever it takes*" statement by Draghi (2012, July) and the European Central Bank (ECB) announcement of the Outright Monetary Transactions (OMT)

[26]This may be seen as a consequence of transfers from private to public debts through explicit channels (bank bailouts) or implicit ones (debt and deposit guarantees), along the logic of the so-called sovereign-bank nexus (see e.g. Jordà, Schularick, and Taylor, 2016).

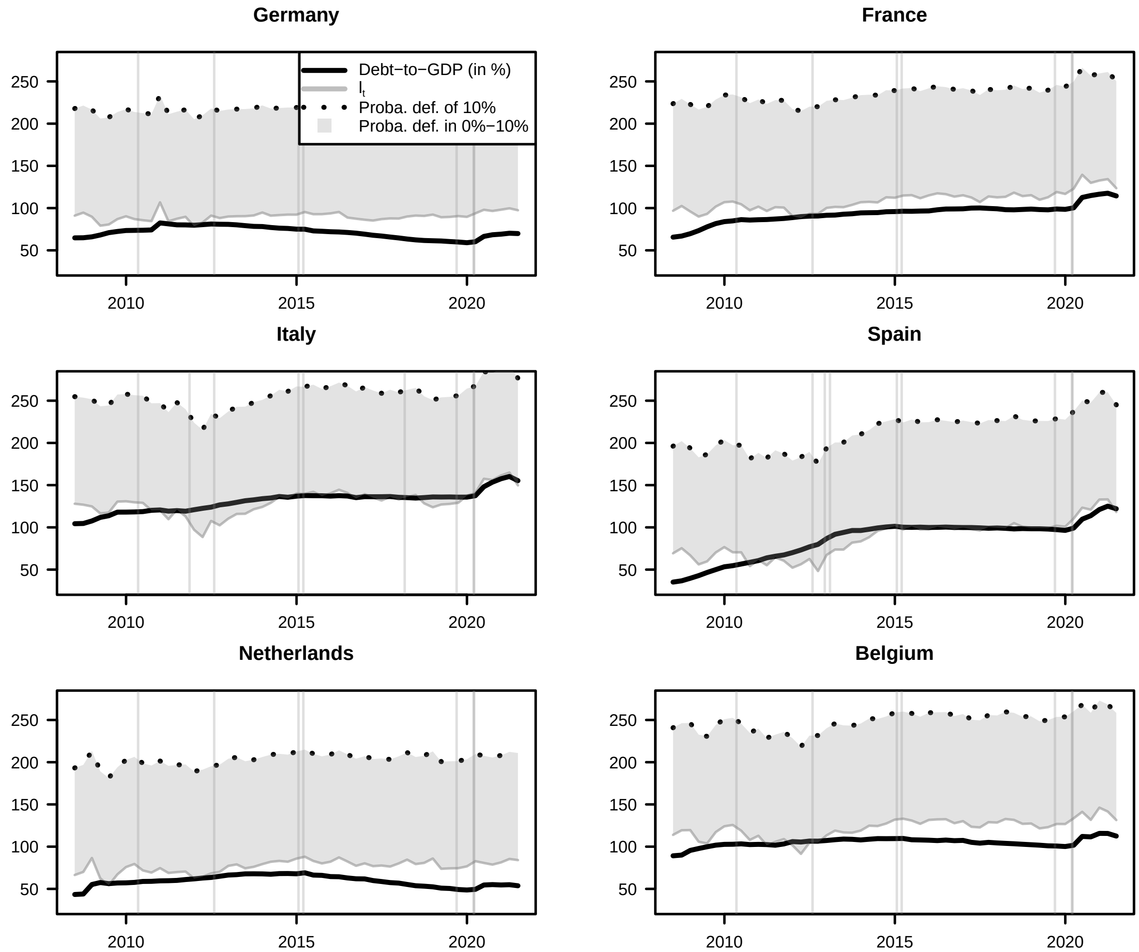
were followed by a 5 p.p. jump in the average fiscal limit (from 2012$Q2$ to 2012$Q4$).[27] After the euro-debt crisis, and until the onset of the COVID-19 pandemic, fiscal limits across countries show an increasing trend on average, translating into a widening of fiscal space in Europe. Fiscal limits decrease by 5 p.p. on average across countries during the pandemic, from mid-2020 until the end of the estimation sample. In Online Appendix XIV, we report a set of sensitivity analyses based on varying key parameters in the model. Specifically, in Figure XIV.5 (Online Appendix XIV), we show the fiscal limit estimates across different model specifications.

6. Results

6.1. **Pricing Eurobonds.** In Figure 5, we compare counterfactual yield spreads associated with common bonds benefitting from several and joint guarantees (SJG) and bonds with several but not joint guarantees (SNJG). By design, the latter is close to the debt-weighted average of country-specific observed sovereign spreads. The difference between SNJG and SJG is positive and sizeable across the estimation sample. This result suggests that raising funds through a joint liability debt instrument—the SJG bond—may substantially reduce debt service in the presence of heterogenous fiscal conditions. This is due to the associated diversification of fiscal risks across countries: as long as the fiscal positions across countries are not perfectly correlated, one can expect gains from common bond issuance in the presence of joint and several guarantees (SJG) w.r.t. several but not joint guarantees (SNJG). On average across the estimation sample, the ratio of SNJG bond yield spread on the SJG one is approximately equal to 7, 3.5 and 2 for the 3-, 5- and 10-year maturities, respectively. Over the estimation sample and maturities, the wedge between SJG and SNJG bond yields is equal, on average, to approximately 35 basis points. Notably, for the 10-year maturity, the SJG bond yield spread is higher than the SNJG one by roughly 10 basis points between 2011Q4 and 2012Q1. This implies that aggregate

[27]The OMT represents a mechanism aimed to "safeguard an appropriate monetary policy transmission and the singleness of the monetary policy" (2012, August).

FIGURE 4. Fiscal limits



Notes: These plots display estimated fiscal limits ($\ell_{j,t}$) and observed public debts ($d_{j,t}$), both expressed in % of GDP. On a given quarter, if debt-to-GDP is higher than the fiscal limit (grey solid line), then the probability of default is strictly positive (see eq. 1). Everything else equal, if debt-to-GDP (black solid line) stayed above the black dotted line (respectively in the grey-shaded area) for four quarters in a row, then the annual default probability of the considered country is larger than 10% (respectively in $]0\%, 10\%]$). On each plot, the vertical bars indicate important dates (monetary-policy decisions and/or noteworthy pivotal economic events): **All countries**—10/05/2010: Announcement of Securities Market Program (SMP); 02/08/2012: ECB announces it may undertake outright transactions in sovereign bond markets (OMT); 22/01/2015: ECB announces expanded asset purchase programme to include bonds issued by euro area central governments, agencies and European institutions (combined monthly asset purchases to amount to €60bn); 04/03/2015: Announcement of the Public Sector Purchase Programme (PSPP); 12/09/2019: Announcement that net purchases will be restarted under the Governing Council's asset purchase programme (APP) at a monthly pace of €20bn as from 1 November 2019. **Italy**—12/11/2011: Berlusconi resigns from office (BTP/Bund spread is over 550 bps); 04/03/2018: Populist parties (M5S and Lega) win the majority of votes in Italian government elections. **Spain**—11/12/2012: ESM (European Stability Mechanism) disburses €39.5bn for recapitalisation of banking sector; 05/03/2013: ESM disburses €1.9bn.

gains would have been slightly negative under issuance with joint and several guarantees during these two quarters. This finding parallels the discussion presented at the end of Section 3 concerning the possibility of a reversion of gains arising from SJG bond issuance compared to

SNJG one. Given the turmoil jointly faced by European member states during the euro-debt crisis, several debt-to-GDP ratios prove larger than fiscal limits (see Figure 4 in Subsection 5.5), this leads to detrimental diversification effects. Such effects revert the probability of default that is larger in the SJG bond case compared to the national bond cases causing negative yield gains (see Online Appendix I for further discussion on the detrimental diversification effects).

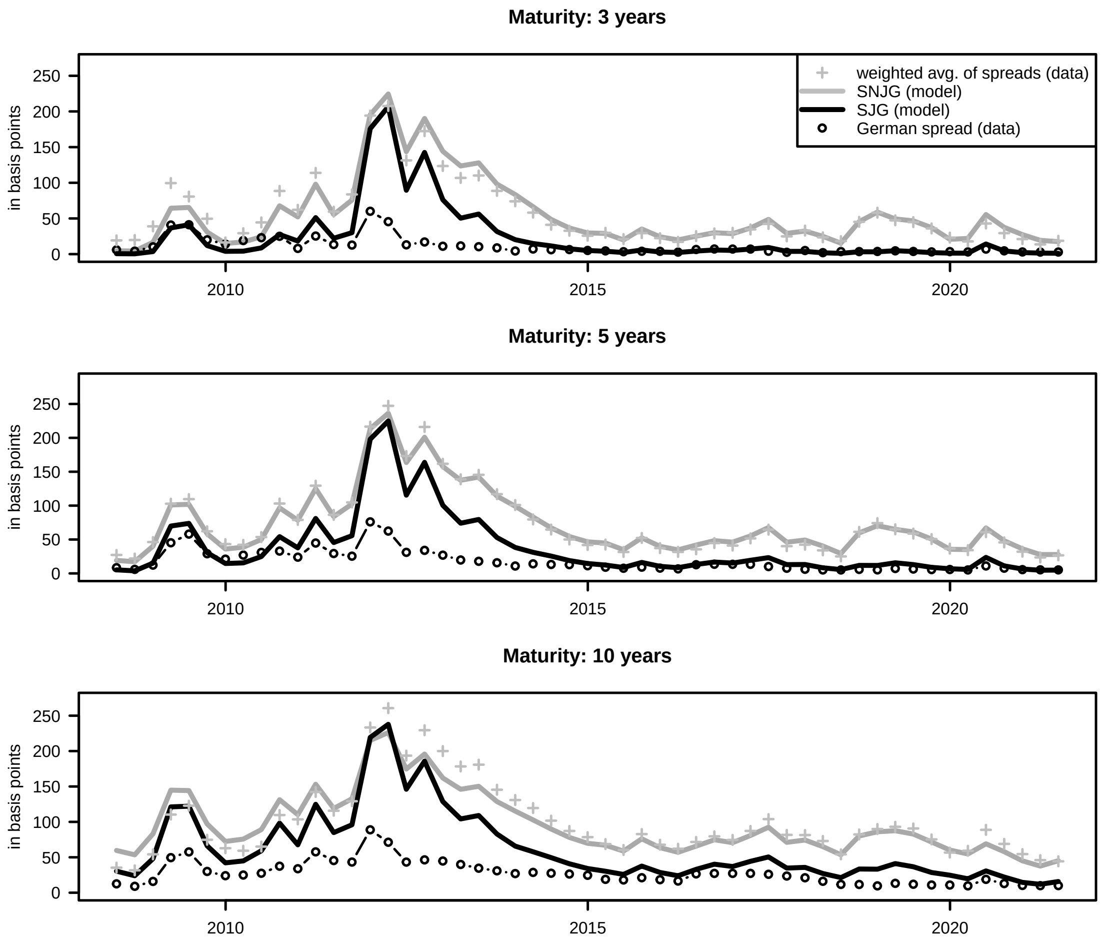
For the sake of comparison, we add the German bond yield spreads in Figure 5 (black circles). Interestingly, during the great financial crisis, the baseline SJG spread lays below the German yield spread for the 3-year and 5-year maturity. Hence, diversification effects underyling the SJG bond pricing might, at times, prove beneficial also for fiscally virtuous countries in the euro area—and not only for the peripheral Member States. Notwithstanding, even in the scenarios under which SJG bond yields are higher than Bunds' ones, one can design post-issuance redistribution schemes translating into gains to all countries. This is discussed in Subsections 6.2 and 6.3.

The magnitudes of our model-implied SJG and SNJG bond spreads are broadly in line with those pertaining to observed proxies of (SJG) Eurobonds. We consider as Eurobond proxies those bonds issued by the following European institutions: the European Investment Bank (EIB), the European Financial Stability Facility (EFSF), the European Stability Mechanism (ESM), and the European Commission itself, which, against the backdrop of the COVID-19 crisis, has initiated large-scale issuance programs.[28] These bonds benefit from various types of guarantees, which makes them close to SJG bonds.[29] Figure 6 shows the spreads between such bonds

[28]These programs notably include the SURE program (for "Support to mitigate Unemployment Risks in an Emergency") and the Next-Generation-EU program. See, e.g., the investor presentation of the European Commission (12 March 2021), available at https://ec.europa.eu/info/sites/default/files/about_the_european_commission/eu_budget/ip_07.2021.pdf. The EU already had issued some bonds before 2020, in particular in the context of the Euratom loans.

[29]To justify Moody's top rating (Aaa) for the EU's bond programs, the rating agency points out, for example, that "the multiple layers of debt service protection, including explicit recourse to extraordinary support [...] creates the equivalent of a joint and several undertaking and obligation on the part of EU member states to provide financial support to the EU" (https://www.moodys.com/research/Moodys-affirms-the-European-Unions-Aaa-rating-outlook-stable--PR_430731).

FIGURE 5. Counterfactual bond yield spreads

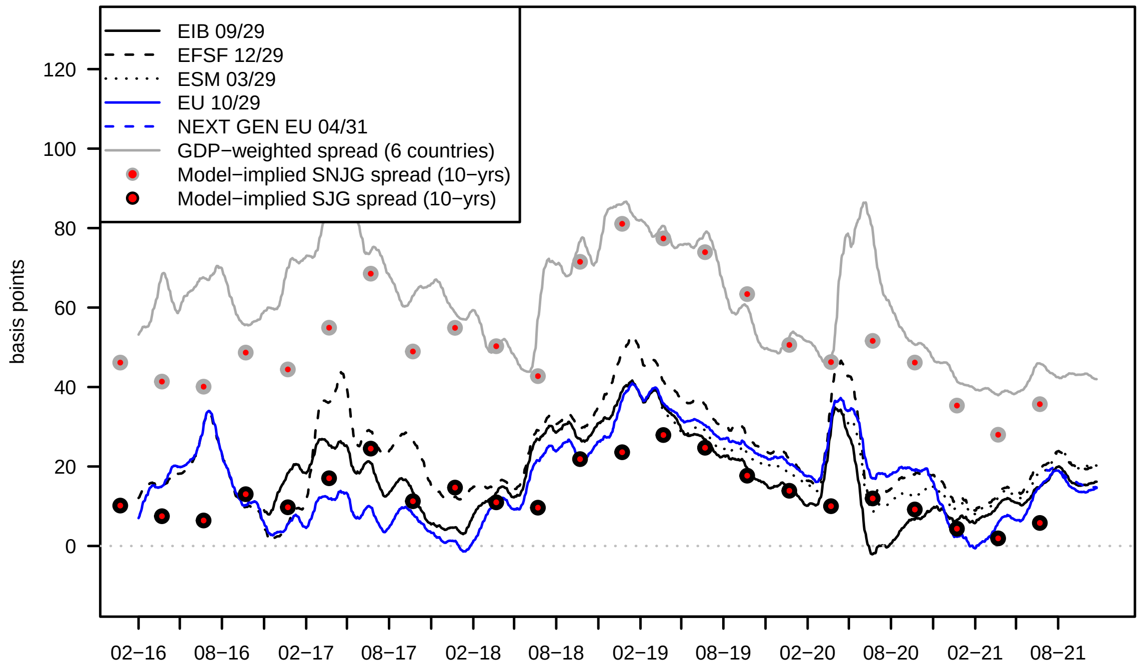


Notes: This figure compares counterfactual yield spreads (versus risk-free interest rates) associated with common bonds benefitting from several and joint guarantees (SJG) and bonds with several but not joint guarantees (SNJG). For the sake of comparison, we also add German bond yield spreads (circles).

and the German benchmark bond (the Bund) of equivalent residual maturity. It also displays, in grey, proxies of SNJG spreads, computed as GDP-weighted averages of 10-year national spreads versus the Bund. It appears that the prices of the different SJG Eurobond proxies are close to each other. The red dots indicate the model-implied SJG and SNJG bond spreads (versus Germany). The plot shows that the model captures a substantial amount of the fluctuations of observed spreads.

Our framework also offers the possibility to consider "partial" SNJG and SJG bonds whose emission is circumscribed to a smaller set of countries excluding, for instance, either "super"

FIGURE 6. Observed proxies of common bond spreads versus 10-year German benchmark



Notes: This figure shows bond yield spreads w.r.t. the German 10-year benchmark bond. Black and blue (respectively grey) lines correspond to proxies for SJG bonds (resp. SNJG bonds). We consider bonds issued by the European Investment Bank (EIB), the European Financial Stability Facility (EFSF), the European Stability Mechanism (ESM), the European Union (EU, NEXT GEN EU). The SNJG proxy (grey lines) is computed as a GDP-weighted average of national-bond spreads (versus Germany). The data are at the daily frequency (20-day moving averages); they span the period from February, 9 2016 to November 1, 2021. The dates reported in the legend of the figure correspond to maturity dates (2029 or 2031) of the specific bonds. The spreads are computed as the differences in asset swap spreads w.r.t. to the Bund; (see Online Appendix XI for more details). As of November 2021, the credit ratings of the considered European institutions were as follows (Moody's/S&P/Fitch): EIB (Aaa/AAA/AAA), EFSF (Aa1/AA/AA+), ESM (Aa1, AAA/AAA), and EU (Aaa/AA/AAA).

core member states (Germany and Netherlands) or peripheral ones (Italy and Spain). The results of such counterfactual exercises are reported in Online Appendix XIII. The main finding is that the wedge between "partial" SJG and SNJG bonds is smaller compared to the baseline scenario (under which all countries participate in the joint issuance), which reflects enhanced diversification effects in the latter case.

Online Appendix XIV reports the results of analyses where we study the sensitivity of SJG bond yield spreads to changes in several important parameters, or in the way these parameters are constrained within the estimation (see Subsection 5.3). The order of magnitude of the spread between SJG and SNJG bonds appear to be robust to these changes.

6.2. **Aggregate gains and redistribution.** In Subsection 6.1, we have seen that the price of a common debt instrument might be lower than the German one (equivalently, Eurobond yields are higher than Bund ones). However SJG bond prices are usually higher than SNJG ones. Since the latter correspond to a weighted average of national bond prices, replacing national bonds with SJG bonds results in aggregate gains. These gains could be redistributed ex-post—i.e. after issuance—across all countries. In that case, and considering only strictly positive redistribution weights, the issuance of SJG bonds would eventually result in a reduction in funding costs for all countries (w.r.t. the issuance of national bonds).[30] Naturally, the number of redistribution schemes is infinite. In this subsection, we focus on three situations. In the first one (Scheme A), countries pay the same yield (i.e., there is no redistribution); in the second one (Scheme B), gains are distributed in proportion to GDP; in the third one (Scheme C), gains are distributed in such a way that the interest rate reduction—relative to the respective national bond rates—is the same for all countries. Formulas used to perform these exercises are detailed in Online Appendix IX.[31]

Table 4 shows the results of these counterfactual exercises. We focus on 5-year bonds (5 years roughly being to the average issuance maturity in the euro area), and three periods: beginning of the estimation sample (2008Q2), midst of the euro debt crisis (2011Q4), and end of the estimation sample (2021Q2). The three upper panels (A, B and C) of Table 4 correspond to the three SJG-based schemes described above. For the sake of comparison, the lower panel (Panel D) shows results for the SNJG case, for which there are no aggregated gains. For this latter case (Scheme D), we consider only the situation in which all countries pay the same interest rate (i.e. the SNJG issuance yield). Table 4 also reports post-redistribution yields, which are the differences between national bond yields and reductions in the funding costs (or "yield

[30] In some sense, any scheme involving strictly positive weights can be seen as Pareto-improving.

[31] This online appendix also reports results of schemes where the funding costs of Germany and France are left unchanged (see Online Appendix IX.5).

gains") resulting from the considered schemes. In addition, we show redistribution weights; these weights indicate how aggregate gains are shared across countries.

Let us stress that the reported reduction in the funding cost (or yield gain) pertains to one given bond, and not to the whole debt outstanding. To be sure: a yield gain of 100 basis points (say) would effectively translate into a reduction of yearly aggregate funding costs of €1bn if an outstanding amount of €100bn of SJG bonds was issued. This being said, to give an idea of the amounts potentially involved, the top part of Table 4 indicates the aggregate gains that would have resulted from the issuance of the equivalent of approximately 5% of the euro-area GDP (€500bn) during the three considered quarters. For instance, for the same face value (€500bn), issuing SJG bonds instead of SNJG bonds in 2008$Q2$ would have increased the issuance proceeds by €2.78bn. For 2011$Q4$ and 2021$Q2$, the gains would have been €3.3bn and €5.73bn, respectively.

Panel A of Table 4 characterizes the scheme where there is no redistribution of the aggregated gains (Scheme A). As illustrated by our results, this scheme can result in negative "gains" for some countries: funding costs for Germany, France and Netherlands get considerably higher in 2011$Q4$. Italy and Spain are the countries that benefit the most out of the SJG issuance scheme in 2011$Q4$: the spread between post-redistribution and national yields is equal to 280 basis points for Italy and 224 basis points for Spain.

By contrast, Schemes B and C are such that all countries mechanically benefit from the issuance of SJG bonds. These two schemes deliver similar results (see Panels B and C of Table 4). While yield reductions are modest before and during the sovereign debt crisis period (around 15 basis points), they become more sizeable at the end of the estimation sample (about 25 basis points in 2021$Q2$).

Figure 7 displays the time series of yield gains associated with Scheme C. We consider three maturities: 3, 5 and 10 years. For the 3- and 5-year maturity, yield gains peak at the end of the euro-debt crisis, between 2012Q4 and 2013Q1, reaching approximately 75 and 65 basis points, respectively. As regards the 10-year maturity, yield gains associated with scheme C revolve around 35 basis points before and after the euro-debt crisis, while, between 2011Q4 and 2012Q1, they turn out to be negative (around -10 basis points). For further details on this finding concerning negative yield gains, we reference to the previous subsection and the discussion at the end of Subsection 3.

In Online Appendix XIV (Figure XIV.7), we show the yield gains associated with Scheme C across different sensitivity exercises. The order of magnitude of these gains appears to be fairly robust to the considered changes in the model parametrization.

It is important to mention that our results do not take into account potential higher-order effects. The mechanisms underlying such effects would be as follows: if the average funding cost of a government decreases—because part of its funding needs are met with Eurobonds—then expected future debt would decrease because of lower debt service. (For this to hold, one has to assume, however, that the decrease in future debt service will not be compensated by higher primary deficits.) If agents effectively expect lower future debt levels, then bond prices move. That is, the initial funding cost effects are followed by second-order ones. This, in turn, reduces future debt service, and so on. This issue is complicated to handle, especially in the context of a reduced-form approach like ours. Nevertheless, in Online Appendix X, we propose an iterative approach aimed at gauging the potential impacts of such higher-order effects. For moderate levels of SJG bond issuance—we consider, therein, that 20% of the euro-area debt is issued in the form of SJG bonds—our results point to relatively mild higher-order effects.

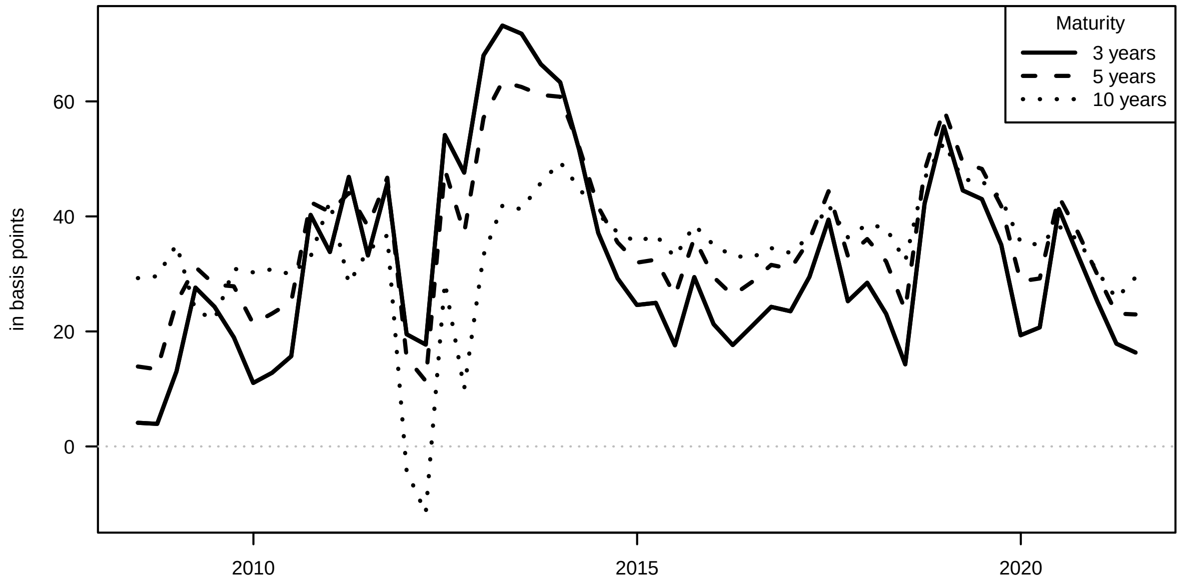
6.3. **Moral hazard and redistribution schemes.** Usual concerns associated with common debt issuance pertains to moral hazard (see, e.g., Claessens et al., 2012; Favero and Missale, 2012; Tirole, 2015; Dávila and Weymuller, 2016): knowing that part of their debt is guaranteed by other countries, some countries may be tempted to increase their spending—and start issuing

TABLE 4. Effect of redistribution schemes on funding costs

	2008-06-30			2011-12-31			2021-06-30		
SJG									
A.G.[a]	€2.78 bn			€3.3 bn			€5.73 bn		
Panel A: SJG, Same funding costs (i.e. no ex-post redistribution)									
	redist. weigth	post redist. yield	yield gain	redist. weigth	post redist. yield	yield gain	redist. weigth	post redist. yield	yield gain
DE	7%	436	3	-262%	306	-119	1%	-10	1
FR	18%	436	10	-99%	306	-62	16%	-10	15
IT	43%	436	31	325%	306	280	55%	-10	67
ES	14%	436	16	165%	306	224	25%	-10	47
NL	11%	436	19	-44%	306	-87	1%	-10	3
BE	8%	436	26	15%	306	52	2%	-10	13
Panel B: SJG, Redistribution based on GDP weights									
	redist. weigth	post redist. yield	yield gain	redist. weigth	post redist. yield	yield gain	redist. weigth	post redist. yield	yield gain
DE	33%	425	14	33%	172	14	33%	-32	23
FR	24%	433	14	24%	228	15	24%	-18	23
IT	19%	453	14	19%	568	18	19%	34	23
ES	12%	438	14	12%	512	17	12%	14	23
NL	8%	442	14	8%	204	15	8%	-29	23
BE	4%	448	14	4%	342	16	4%	-19	23
Panel C: SJG, Same yield gains across countries									
	redist. weigth	post redist. yield	yield gain	redist. weigth	post redist. yield	yield gain	redist. weigth	post redist. yield	yield gain
DE	33%	425	14	35%	171	15	33%	-32	23
FR	24%	433	14	25%	228	15	24%	-18	23
IT	19%	453	14	17%	570	15	19%	35	23
ES	12%	438	14	11%	514	15	12%	15	23
NL	8%	442	14	8%	203	15	8%	-29	23
BE	4%	448	14	4%	342	15	4%	-20	23
SNJG									
A.G.	€0 bn			€0 bn			€0 bn		
Panel D: SNJG, Same funding costs									
	redist. weigth	post redist. yield	yield gain	redist. weigth	post redist. yield	yield gain	redist. weigth	post redist. yield	yield gain
DE	–	450	-11	–	321	-135	–	13	-22
FR	–	450	-4	–	321	-78	–	13	-8
IT	–	450	17	–	321	264	–	13	44
ES	–	450	2	–	321	208	–	13	24
NL	–	450	5	–	321	-103	–	13	-20
BE	–	450	12	–	321	36	–	13	-10

[a]: Aggregate gains. *Notes*: This table compares post-redistribution funding costs across countries under the two issuance schemes (SJG and SNJG) and under different redistribution schemes described in Subsection 6.2. We focus on the 5-year maturity and on three periods: beginning of the estimation sample (2008*Q*2), midst of the euro debt crisis (2011*Q*4) and end of the estimation sample (2021*Q*2). Yields are expressed in basis points. Aggregate gains (reported at the top of the table) are computed under the assumption that total issuance is equal to 5% of aggregate GDP. In each panel, for all countries and dates, we show the redistribution weights, the post-redistribution yields, and the spread between national yields and the post-redistribution yields (that are the yield gains). Under SNJG (Panel D), redistribution weights are unnecessary since there are no aggregated gains. See Online Appendix IX for computational details.

FIGURE 7. Yield gains associated with redistribution scheme with same yield gains across countries



Notes: This figure shows yield gains associated with redistribution Scheme C (same yield gains across countries) throughout the whole estimation sample and for different maturities. See Subsection 6.2 for details regarding this redistribution scheme. Yield gains are expressed in basis points.

more debt—since the interest rate on jointly-guaranteed debt is less sensitive to an individual debt increase than non-guaranteed debt.

Although our reduced-form modeling framework does not allow to explore such mechanisms in a structural way, it can illustrate how market discipline would be impaired by massive issuance of SJG bonds. Specifically, we perform counterfactual exercises in which Italy and Spain decide to deviate from their current debt level, all else being equal. We then observe the changes in spreads induced by these modifications. We consider two dates: 2011$Q4$ (euro-area debt crisis) and 2021$Q2$ (end of the estimation sample). Figure 8 shows the results. For each date and each country, large increases in the debt-to-GDP ratio result in modest increases in SJG and SNJG Eurobond spreads (see, respectively, the grey and black solid lines).[32] These increases are far lower than those of national bond yields (grey dashed line). This illustrates

[32] The SJG bond yield proves higher than the SNJG one under a sizeable rise in Italian indebtedness in 2011Q4 (top left panel of Figure 8). This stems from the fact that diversification effects become detrimental when expected joint fiscal limits are overcome by pooled debts (see discussion at the end of Section 3 and, also, Subsection 6.1). In this scenario, the probability of default is reversed (larger for SJG than for national bonds, on average), causing negative yield gains (see Online Appendix I for further details on detrimental diversification effects).

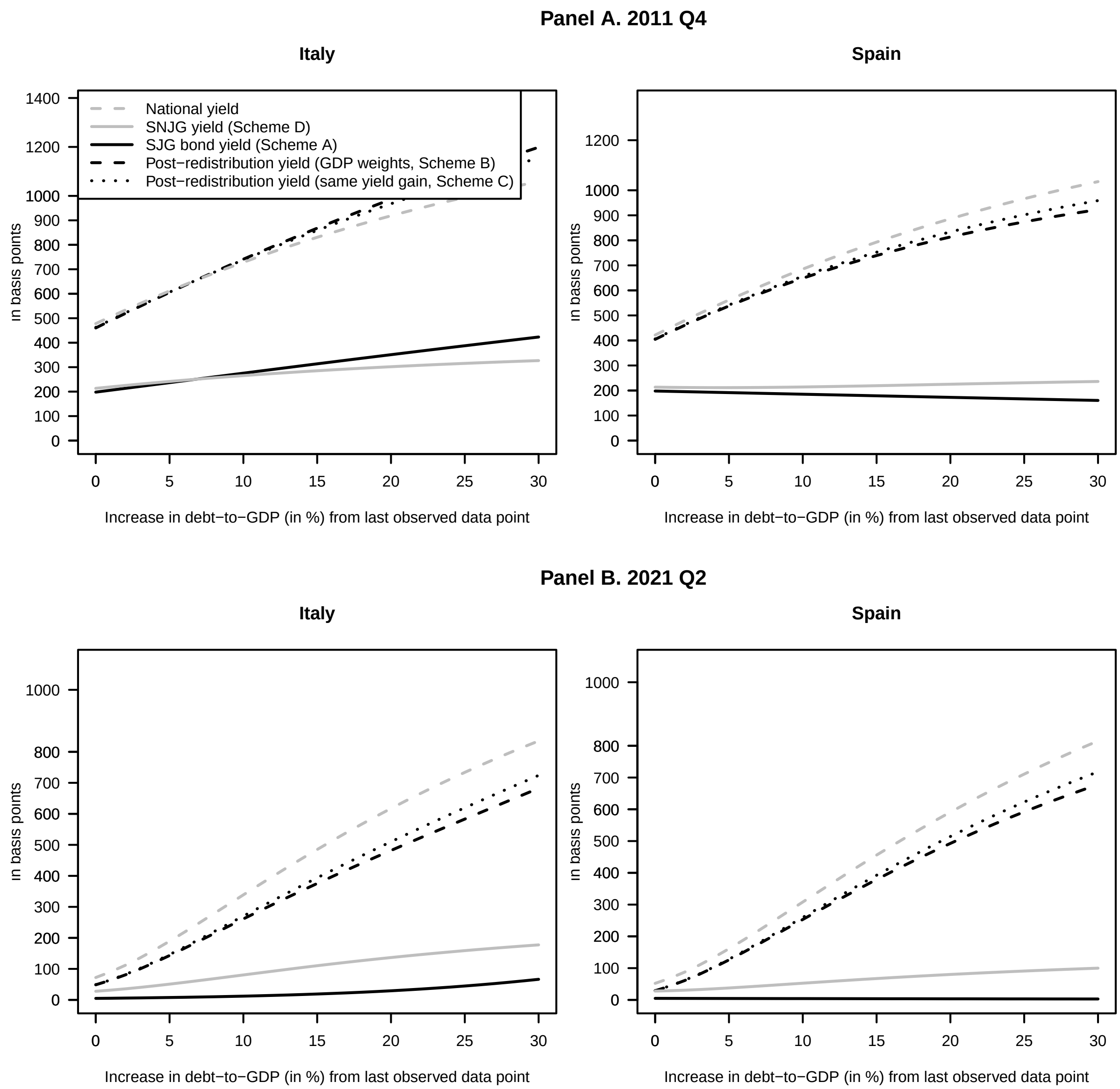
the moral hazard issue: under the issuance of common bonds, and if the each country pays the issuance SJG/SNJG yield (i.e., under Schemes A or D), then the ability of financial markets to restore fiscal discipline via rising interest rates is hampered. Let us stress that the strength of this hampering effect depends on the extent to which national issuances would be replaced with eurobonds: as long as a sizable share of countries' funding needs are met with the issuance of national bonds, the overall debt service remains sensitive to countries' indebtedness. In other words, under Schemes A or D, a necessary condition for market discipline to remain effective is to limit the issuance of eurobonds (as suggested by Delpla and Weizsacker, 2010; Hellwig and Philippon, 2011). The simulation results suggest that moral hazard effects are dampened under Schemes B and C (see black dashed lines in Figure 8); these schemes indeed imply that post-redistribution funding costs remain sensitive to countries' indebtedness showing a similar slope as national bond yields (grey dashed line). Moreover, these post-redistribution yields remain lower than national bond yields as long as aggregate gains are positive.[33]

7. Concluding remarks

This paper aims at pricing bonds jointly issued by a group of countries. Our focus is on Eurobonds, which are debt instruments jointly issued by euro-area countries. We consider two types of common bonds: the first features joint and several guarantees (SJG bond); the second is characterized by several but not joint guarantees (SNJG bond). To price these two types of common bonds, we develop a novel multi-country sovereign credit risk framework. Our model captures the joint dynamics of national bond prices, sovereign debt, and the fiscal limit—the level of debt beyond which the risk of default is no longer zero.

[33]Post-redistribution yields under Scheme B and C are above the Italian national bond yield in 2011Q4 when Italian debt-to-GDP ratio considerably grows (top left panel in Figure 8). As mentioned in Footnote 32, this finding of negative aggregate yield gains arise from the reversal of diversification effects under periods of particular turmoil.

FIGURE 8. Moral hazard risk and redistribution: counterfactual exercise



Notes: This figure shows the increase in different bond spreads (w.r.t. to risk-free rates) resulting from counterfactual increases in Italian indebtedness (left column of plots) or Spanish indebtedness (right column of plots), all else being equal. The two rows correspond to different periods, namely 2011Q4 (euro-area sovereign debt crisis) and 2021Q2 (end of the estimation sample). The different schemes (A to D) are described in Subsection 6.2.

Estimating the model involves both determining the model parameterization and countries' fiscal limits. Thanks to the tractability of our asset-pricing framework, these two tasks are operated jointly. Our estimation sample comprises data associated with the sixth largest euro-area economies over the period 2008-2021. The estimated model fits observed sovereign spreads across maturities and countries. To the best of our knowledge, this paper is the first to provide time-varying estimates of fiscal limits for the euro area.

The estimated model is exploited to examine the pricing of (counterfactual) SJG and SNJG bonds. In most instances, yields associated with SNJG bonds are higher than those associated with their SJG equivalents. Notably, across the estimation sample and maturities, the SNJG bond yield spread, w.r.t. a risk-free rate, is three times larger than the SJG one. Interestingly, our model shows also that aggregate gains associated with SJG bond issuance might considerably decrease when expected fiscal spaces reduce at the euro-area scale, up to potential inversion. Therefore, in the presence of heterogenous and not too adverse fiscal conditions, raising funds through SJG bonds may lower aggregate debt service (w.r.t. situations where only national bonds and/or SNJG bonds are issued). We discuss potential ex-post redistributions of such aggregate gains, and we show that some of these redistribution schemes may alleviate the reduction in market discipline resulting from joint bond issuances.

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APPENDIX A. X_t'S DYNAMICS

Denote by $\bar{d}$ and $\bar{\ell}$ the unconditional means of vectors d_t and ℓ_t, respectively. The state vector $X_t = [i_t, i_{t-1}, 0, d_t', d_{t-1}', \ell_t']'$, follows the vector autoregressive process of order one given in eq. (12), with:

$$\mu = \begin{bmatrix} (1-\rho_i)\bar{i} \\ 0 \\ (1-\rho_d)\bar{d} \\ \mathbf{0}_{N\times 1} \\ (1-\rho_\ell)\bar{\ell} \end{bmatrix}, \Phi = \begin{bmatrix} \rho_i & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_d \mathbf{Id}_{N\times N} & 0 & 0 \\ 0 & 0 & \mathbf{Id}_{N\times N} & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho_\ell \mathbf{Id}_{N\times N} \end{bmatrix}, \Sigma\Sigma' = \begin{bmatrix} \sigma_i^2 & 0 & \cdots & & \\ 0 & 0 & \cdots & & \\ \vdots & \vdots & \Omega_d & \mathbf{0}_{N\times N} & \Omega_{\ell,d}' \\ & & \mathbf{0}_{N\times N} & \mathbf{0}_{N\times N} & \mathbf{0}_{N\times N} \\ & & \Omega_{\ell,d} & \mathbf{0}_{N\times N} & \Omega_\ell \end{bmatrix}.$$

Let us detail the parametrization of matrices Ω_d, Ω_ℓ, and $\Omega_{\ell,d}$. The structure exposed in Subsection 5.3 implies that we have, for shocks $\varepsilon_{d,t}$ and $\varepsilon_{\ell,t}$ (appearing in eqs. 8 and 10):[34]

$$\varepsilon_{d,t} = \Gamma_d \widetilde{\Gamma}_d \tilde{\eta}_t \quad \text{(a.1)}$$

$$\varepsilon_{\ell,t} = \Gamma_\ell \widetilde{\Gamma}_\ell \tilde{\eta}_t, \quad \text{(a.2)}$$

[34] Note that $\eta_t = [\eta_{i,t}, \tilde{\eta}_t']'$ (see eq. (26)).

where (i) $\tilde{\eta}_t = [\tilde{\eta}_{d,\ell,t}, \tilde{\eta}'_{d,t}, \tilde{\eta}'_{\ell,t}]' \sim i.i.d.\mathcal{N}(0, I)$ (hence, the dimension of $\tilde{\eta}_t$ is $(1+2N) \times 1)$, (ii) Γ_d and Γ_ℓ are based on PCA (see Subsection 5.3), and (iii) with

$$\widetilde{\Gamma}_d = \begin{bmatrix} \sqrt{\rho_{d,\ell}} & \sqrt{1-\rho_{d,\ell}} & 0_{1\times(N-1)} & 0_{1\times N} \\ 0_{(N-1)\times 1} & 0_{(N-1)\times 1} & I_{N-1} & 0_{(N-1)\times N} \end{bmatrix} \tag{a.3}$$

$$\widetilde{\Gamma}_\ell = \begin{bmatrix} \sqrt{\rho_{d,\ell}} & 0_{1\times N} & \sqrt{1-\rho_{d,\ell}} & 0_{1\times(N-1)} \\ 0_{(N-1)\times 1} & 0_{(N-1)\times N} & 0_{(N-1)\times 1} & I_{N-1} \end{bmatrix}. \tag{a.4}$$

With these notations, we have:

$$\Sigma = \begin{bmatrix} \sigma_i & \mathbf{0}_{1\times(2N+1)} \\ 0 & \mathbf{0}_{1\times(2N+1)} \\ \mathbf{0}_{N\times 1} & \Gamma_d\widetilde{\Gamma}_d \\ \mathbf{0}_{N\times 1} & \mathbf{0}_{N\times(2N+1)} \\ \mathbf{0}_{N\times 1} & \Gamma_\ell\widetilde{\Gamma}_\ell \end{bmatrix}.$$

Therefore, noting that $\widetilde{\Gamma}_\ell\widetilde{\Gamma}'_\ell = \widetilde{\Gamma}_d\widetilde{\Gamma}'_d = I$, we have $\Omega_d = \Gamma_d\Gamma'_d$, $\Omega_\ell = \Gamma_\ell\Gamma'_\ell$, and $\Omega_{\ell,d} = \Gamma_\ell\widetilde{\Gamma}_\ell(\Gamma_d\widetilde{\Gamma}_d)'$.

Appendix B. Pricing of zero-coupon zero-recovery risky bonds

Denote by $P^{(j)}_{t,h}$ the date-t price of a zero-coupon bond providing a payoff of 1 on date $t+h$ if country j has not defaulted before $t+h$, and zero otherwise. We have:

$$\begin{aligned} P^{(j)}_{t,h} &= \mathbb{E}^{\mathbb{Q}}_t(\Lambda_{t,t+h}(1-\mathcal{D}_{j,t+h})) = \mathbb{E}^{\mathbb{Q}}_t\left\{\exp(-i_t - \cdots - i_{t+h-1})(1-\mathcal{D}_{j,t+h})\right\} \\ &= \mathbb{E}^{\mathbb{Q}}_t\left\{\mathbb{E}^{\mathbb{Q}}_t\left\{\exp(-i_t - \cdots - i_{t+h-1})(1-\mathcal{D}_{j,t+h})|X_{t+h}, X_{t+h-1}, \dots\right\}\right\} \\ &= \mathbb{E}^{\mathbb{Q}}_t\left\{\exp(-i_t - \cdots - i_{t+h-1} - \underline{\lambda}_{j,t+1} - \cdots - \underline{\lambda}_{j,t+h})\right\}, \end{aligned} \tag{a.5}$$

where the last equality is obtained under the assumption that $\mathcal{D}_t$ does not cause X_t in the Granger's or Sims' sense (Monfort and Renne, 2013, Proposition 3). Note here that the risk-neutral dynamics of X_t ($\mathbb{Q}$) is easily deduced from the physical one, characterized by eq. 12 (eq. II.2 in Online Appendix II).

Because the default intensities $\underline{\lambda}_{j,t}$ involve a max operator (see eq. 2), eq. (a.5) does not admit closed-form solutions. We follow Wu and Xia (2016) and look for an approximation for the following "forward" rate:

$$p_{j,h-1,h} \quad = \quad -\log(P^{(j)}_{t,h}) + \log(P^{(j)}_{t,h-1}). \tag{a.6}$$

Then, we get an approximation to $P^{(j)}_{t,h}$ by taking the exponential of the cumulated forward rates. The approximation is essentially based on $\log \mathbb{E}[\exp(Z)] \approx \mathbb{E}(Z) + \frac{1}{2}\mathbb{V}(Z)$, which is exact when Z is Gaussian, but not if it is truncated Gaussian, as is the case here.

As detailed in the Online Appendix IV, we get:

$$\begin{aligned} p_{j,k-1,k} \quad \approx \quad & \delta' \mu^{\mathbb{Q}}_{t,k} + \Phi\left(\frac{\mu^{\mathbb{Q}}_{j,t,k}}{\sigma^{\mathbb{Q}}_{j,k}}\right)\mu^{\mathbb{Q}}_{j,t,k} + \phi\left(-\frac{\mu^{\mathbb{Q}}_{j,t,k}}{\sigma^{\mathbb{Q}}_{j,k}}\right)\sigma^{\mathbb{Q}}_{j,k} - \frac{1}{2}\Big(\mathfrak{q}_{j,t,k}(\delta+\delta_j)'\Gamma^{\mathbb{Q}}_{k,0}(\delta+a_j) + (1-\mathfrak{q}_{j,t,k})\delta'\Gamma^{\mathbb{Q}}_{k,0}\delta\Big) \\ & - \sum_{i=1}^{k-1}\Big(\mathfrak{q}_{j,t,k-i}(\delta+\delta_j)'\Gamma^{\mathbb{Q}}_{k,i}(\delta+\delta_j) + (1-\mathfrak{q}_{j,t,k-i})\delta'\Gamma^{\mathbb{Q}}_{k,i}\delta\Big), \end{aligned} \tag{a.7}$$

where $\delta = [0, 1, 0, \ldots]'$ (in such a way that $i_{t-1} = \delta' X_t$), and where $\delta_j = [0, 0, \alpha e_j', \mathbf{0}_{1\times N}, -\alpha e_j']'$ (e_j denoting the j^{th} column vector of the $N \times N$ identity matrix), $\mathfrak{q}_{j,t,k} = \Phi\left(\mu^{\mathbb{Q}}_{t,k}/\sigma^{\mathbb{Q}}_{j,k}\right)$, and

$$\left\{\begin{array}{lclcl} \mu^{\mathbb{Q}}_{t,k} & = & \mathbb{E}^{\mathbb{Q}}_t(X_{t+k}) & = & (Id - \Phi^{\mathbb{Q}})^{-1}(Id - \Phi^{\mathbb{Q}^k})\mu^{\mathbb{Q}} + \Phi^{\mathbb{Q}^k} X_t, \\ \Gamma^{\mathbb{Q}}_{k,0} & = & \mathbb{V}^{\mathbb{Q}}_t(X_{t+k}) & = & \Omega + \Phi^{\mathbb{Q}}\Gamma^{\mathbb{Q}}_{k-1,0}\Phi^{\mathbb{Q}'}, \quad \text{with} \quad \Gamma^{\mathbb{Q}}_{1,0} = \Omega \\ & & & = & \Omega + \Phi^{\mathbb{Q}}\Omega\Phi^{\mathbb{Q}'} + \cdots + \Phi^{\mathbb{Q}^{k-1}}\Omega\Phi^{\mathbb{Q}^{k-1}\prime}, \\ \Gamma^{\mathbb{Q}}_{k,i} & = & \mathbb{C}ov^{\mathbb{Q}}_t(X_{t+k}, X_{t+k-i}) & = & \Phi^{\mathbb{Q}^i}\Gamma^{\mathbb{Q}}_{k-i,0} \quad \text{if} \quad k-i>0, \end{array}\right.$$

where $\mu^{\mathbb{Q}} = \mu - \Sigma\psi_0$ and $\Phi^{\mathbb{Q}} = \Phi - \Sigma\psi_1$ (see Online Appendix II).

Appendix C. Maximum Sharpe Ratio

The maximum Sharpe ratio for a one-period investment is given by (Hansen and Jagannathan, 1991):

$$maxSR_t \quad = \quad \frac{\sqrt{\mathbb{V}ar_t(\mathcal{M}_{t,t+1})}}{\mathbb{E}_t(\mathcal{M}_{t,t+1})}.$$

In the present context, the exponential affine form of our s.d.f. (13) implies that:

$$maxSR_t = \frac{\sqrt{\mathbb{V}ar_t \exp(-\psi'\varepsilon_{t+1})}}{\mathbb{E}_t \exp(-\psi'\varepsilon_{t+1})} = \sqrt{\exp(\psi_t'\psi_t) - 1}.$$

Since ψ_1 is a matrix of zeros, except the (1,1) entry that is equal to $\psi_{i,1}$, and using the specification of ψ_0 given in eq. (27), we obtain $maxSR_t = \sqrt{\exp([\psi_{i,0} + \psi_{i,1} i_t]^2 + 2\nu^2) - 1}$. Denoting by $maxSR$ the value of $maxSR_t$ obtained when the state vector is at its average value, we get:

$$maxSR = \sqrt{\exp([\psi_{i,0} + \psi_{i,1}\bar{i}]^2 + 2\nu^2) - 1}. \tag{a.8}$$

The short-term interest rate i_t remained constant for most of our estimation sample (from 2012 to 2021), it was therefore often close to $\bar{i}$ ($\approx$ 0.3%). As a result, we have $maxSR_t \approx maxSR$ for most of the sample.

When calibrating the model, it is convenient to set constraints on $maxSR$, rather than on $\psi_{i,0}$, say, because the literature provides us with priors regarding $maxSR$. Accordingly, we choose to put $maxSR$ among the parameters and to use (a.8) to get $\psi_{i,0}$ (that is therefore removed from the list of degrees of freedom). Specifically, (a.8) gives:

$$\psi_{i,0} = -\psi_{i,1}\bar{i} \pm \sqrt{\log(maxSR^2 + 1) - 2\nu^2}.$$

We keep the solution that gives a positive average slope of the risk-free yield curve (that is the one for which $\pm$ is replaced by the minus sign in the previous expression).

—Online Appendix –

Fiscal Limits and the Pricing of Eurobonds

Kevin PALLARA and Jean-Paul RENNE

APPENDIX I. NEGATIVE EXPECTED FISCAL SPACES IN THE STYLIZED MODEL

This appendix discusses the effects of negative fiscal spaces on the prices of jointly-issued bonds in our framework. For that, we use the stylized situation described in Section 3. For the sake of simplicity, we focus on the symmetrical situation (Countries A and B are alike). Moreover, we set to zero the correlation between the fiscal spaces of A and B, i.e., $\rho = 0$ in eq. 4. (Mechanisms are more evident in this case.)

The three panels of Figure I.1 correspond to three situations. In the top panel, expected fiscal spaces are positive; expected fiscal spaces are null in the middle panel; they are negative in the third one. In all three cases, the distribution associated with the joint area (blue line) is narrower than the national ones (red line). But the implications of this diversification effects, in terms of default intensities, are different. The distributions of the default intensities are represented by the shaded areas: bluish for the joint area (or SJG bond) and reddish for the single countries (or, approximately, SNJG bonds). Note that, in addition to these shaded areas, the distributions of the default intensities also include (unrepresented) Dirac masses located at zero.

In the first situation (top panel), we see that the default probability is far smaller for the SJG bond (bluish area) than for the national bonds (reddish area). In the second case, where the average fiscal spaces are zero, we see that diversification effects are still at play: the bluish area is more concentrated towards zero. Finally, the third plot shows that when debts are large enough compared to fiscal limits, then diversification effects are reversed: the probability of default is larger in the SJG bond case than in the national bond cases (the reddish area is relatively more concentrated towards zero).

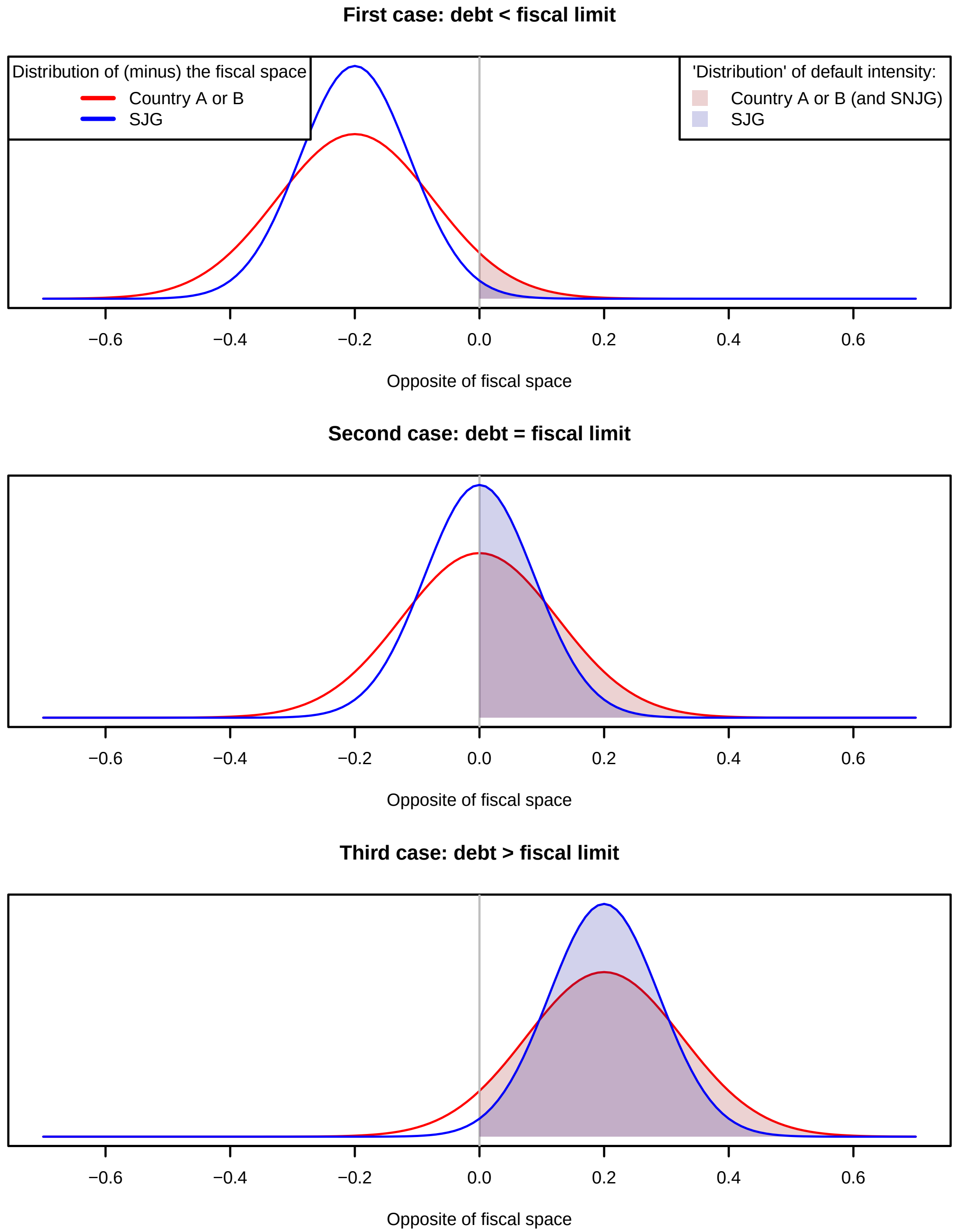
This is further illustrated by Figure I.2, that shows how the yields of SJG and SNJG bonds behave when the average debt is larger than the average fiscal limit. In the left-hand side panel of the figure, we plot SJG and SNJG bond yields when negative fiscal space (% of GDP) varies between -1% and -40%. The right-hand side panel of the figure shows the difference between the two yields. It appears that when fiscal space is large and negative, the spread between SJG and SNJG bond yields turns positive.

APPENDIX II. $\mathbb{P}$ TO $\mathbb{Q}$ DYNAMICS

The risk-neutral measure is defined with respect to the physical measure through the following Radon-Nikodym derivative:

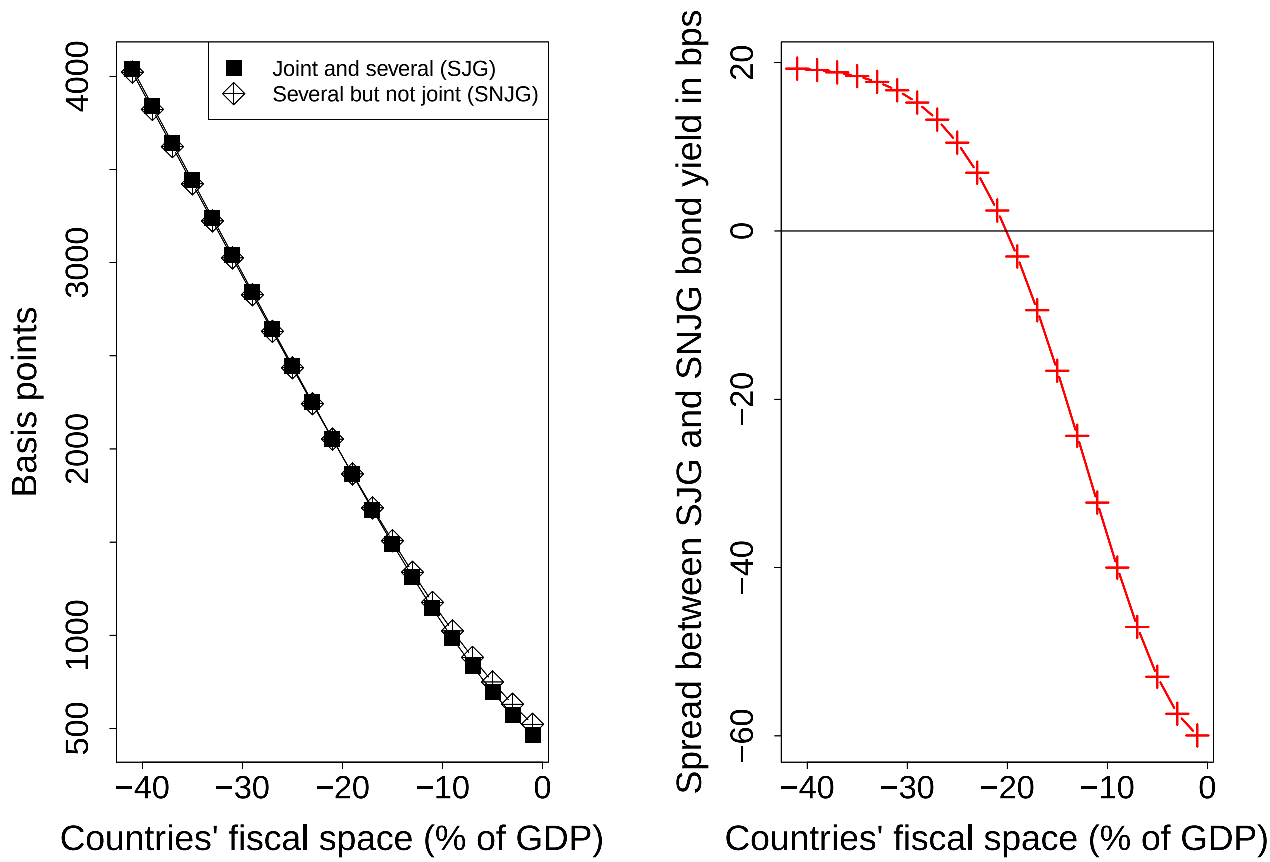
$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{t,t+1} = \frac{\mathcal{M}_{t,t+1}}{\mathbb{E}_t(\mathcal{M}_{t,t+1})} = \exp\left(-\frac{1}{2}\psi_t'\psi_t - \psi_t'\eta_{t+1}\right),$$

FIGURE I.1. SJG and SNJG Bond yields under negative fiscal space ($\bar{d} > \bar{\ell}$)



This figure illustrates the influence of the sign of expected fiscal spaces on diversification effects. The context is the one of the stylized model presented in Section 3. Blue (respectively red) elements correspond to the joint area (respectively to single countries, or to SNJG bonds). The first panel corresponds to the conventional situation, where the expected fiscal space is positive. In that case, the default intensity associated with the SJG bond (joint area) is more concentrated towards zero than for the national default intensities. (To see that, compare the bluish and reddish areas, which represent the distributions of the default intensities—excluding the Dirac mass at zero.) The diversification effect is still at play when the fiscal space expectation is zero (middle plot); and it reverts when the expectation of the fiscal space is negative (third and last plot). In the latter case, the distribution of the joint-area default intensity is relatively more concentrated on the right-hand side of zero than for national default intensities. The calibration is as in the baseline situation described in Section 3 (stylized model), except that the correlation between debts (ρ in eq. 4) is set to zero.

FIGURE I.2. SJG and SNJG Bond yields under negative fiscal space ($\bar{d} > \bar{\ell}$)



This figure shows how the yields of SJG and SNJG bonds behave when the average debt is larger than the average fiscal limit. The left-hand side panel of the figure displays SJG and SNJG bond yields. The right-hand side panel reports the difference between the two types of yields. The calibration is as in the baseline situation described in Section 3 of the paper (stylized model), except that, for expository purpose, the correlation between debts is set to zero.

where the vector of prices of risk ψ_t is given in eq. (15). Under the physical measure, the conditional Laplace transform of X_t is given by:

$$\mathbb{E}_t(\exp(u'X_{t+1})) = \exp\left(u'\mu + u'\Phi X_t + \frac{1}{2}u'\Sigma\Sigma' u\right). \tag{II.1}$$

Let us now compute the conditional Laplace transform of X_t under the risk-neutral measure:

$$
\begin{aligned}
\mathbb{E}_t^{\mathbb{Q}}(\exp(u'X_{t+1})) &= \mathbb{E}_t\Big(\exp\left(-\frac{1}{2}\psi_t'\psi_t - \psi_t'\eta_{t+1}\right) \exp(u'X_{t+1}) \Big) \\
&= \mathbb{E}_t\Big(\exp\Big(-\frac{1}{2}(\psi_0 + \psi_1 X_t)'(\psi_0 + \psi_1 X_t) + u'\mu + u'\Phi X_t + (\Sigma' u - \psi_0 - \psi_1 X_t)'\eta_{t+1} \Big)\Big) \\
&= \mathbb{E}_t\Big(\exp\Big(-\frac{1}{2}(\psi_0 + \psi_1 X_t)'(\psi_0 + \psi_1 X_t) + u'\mu + u'\Phi X_t + \\
&\qquad \frac{1}{2}(\Sigma' u - \psi_0 - \psi_1 X_t)'(\Sigma' u - \psi_0 - \psi_1 X_t) \Big)\Big) \\
&= \exp\left(u'(\mu - \Sigma\psi_0) + u'(\Phi - \Sigma\psi_1)X_t + \frac{1}{2}u'\Sigma\Sigma' u \right).
\end{aligned}
$$

By analogy with (II.1), it comes that the risk-neutral dynamics of X_t reads:

$$
X_t = \mu^{\mathbb{Q}} + \Phi^{\mathbb{Q}} X_{t-1} + \Sigma \eta_t^{\mathbb{Q}}, \quad \eta_t^{\mathbb{Q}} \sim i.i.d.\mathcal{N}(0, I), \tag{II.2}
$$

where $\mu^{\mathbb{Q}} = \mu - \Sigma\psi_0$, and $\Phi^{\mathbb{Q}} = \Phi - \Sigma\psi_1$.

Appendix III. Pricing of risk-free bonds

By definition of the state vector $X_t = [i_t, i_{t-1}, d_t, d_{t-1}, \ell_t]$, eq. (18) is satisfied for $h = 1$, with:

$$
A_1 = 0, \quad \text{and} \quad B_1 = -[1, 0, \dots]'.
$$

Let us assume that eq. (18) holds for a maturity $h-1$, with $h > 1$ (and for any date t). Then, the price of a risk-free zero-coupon bond of maturity $h-1$ is given by

$$
P_{t,h-1} = \exp(A_{h-1} + B_{h-1}' X_t). \tag{III.3}
$$

Let us then express the price of a risk-free zero-coupon bond of maturity h:

$$
\begin{aligned}
P_{t,h} &= \mathbb{E}_t(\mathcal{M}_{t,t+1} P_{t,h-1}) = \exp(-i_t)\mathbb{E}_t^{\mathbb{Q}}(\exp(A_{h-1} + B_{h-1}' X_{t+1})) \quad \text{using (III.3)} \\
&= \exp(B_1' X_t)\mathbb{E}_t^{\mathbb{Q}}(\exp(A_{h-1} + B_{h-1}'[\mu^{\mathbb{Q}} + \Phi^{\mathbb{Q}} X_t + \Sigma\eta_{t+1}]))) \\
&= \exp\left(A_{h-1} + B_{h-1}'\mu^{\mathbb{Q}} + \frac{1}{2}B_{h-1}'\Sigma\Sigma' B_{h-1} + [B_1 + \Phi^{\mathbb{Q}\prime} B_{h-1}]' X_t \right),
\end{aligned}
$$

which leads to eq. (17), using the definitions of $\mu^{\mathbb{Q}}$ and $\Phi^{\mathbb{Q}}$ given in (II.2).

Appendix IV. Approximate formula for zero-coupon risky bond

This appendix details the approximation to the price $P_{t,h}^{(j)}$ (this price being defined though eq. a.5); the resulting formula is given in Appendix B.

Since $X_t = [i_t, i_{t-1}, d_{1,t}, \cdots, d_{n,t}, \ell_{1,t}, \cdots, \ell_{n,t}]'$, we have

$$
i_{t-1} = \delta' X_t, \tag{IV.4}
$$

where $\delta = [0, 1, 0, \dots, 0]'$. Moreover, we also introduce the following notation:

$$
\lambda_{j,t} = \delta_j' X_t,
$$

where $\delta_j = [0, 0, \alpha e_j', \mathbf{0}_{1\times N}, -\alpha e_j']'$, e_j denoting the j^{th} column vector of the $N \times N$ identity matrix. With these notations, eq. (2) rewrites:

$$\underline{\lambda}_{j,t} = \max(0, \lambda_{j,t}),$$

that is, $\lambda_{j,t}$ can be seen as a "shadow intensity". With these notations, we can rewrite eq. (a.5) as:

$$P_{t,h}^{(j)} = \mathbb{E}_t^{\mathbb{Q}}[\exp(-\delta' X_{t+1} - \max(0, \lambda_{j,t+1}) - \cdots - \delta' X_{t+h} - \max(0, \lambda_{j,t+h}))]. \quad \text{(IV.5)}$$

Let us recall the notation introduced in Appendix B:

$$p_{j,h-1,h} = -\log(P_{t,h}^{(j)}) + \log(P_{t,h-1}^{(j)}). \quad \text{(IV.6)}$$

In the spirit of Wu and Xia (2016), we determine approximations to $p_{j,h-1,h}$ that we further use to get approximations to $P_{t,h}^{(j)}$, using:

$$P_{t,h}^{(j)} = \exp(p_{j,0,1} + p_{j,1,2} + \cdots + p_{j,h-1,h}). \quad \text{(IV.7)}$$

The approximation to $p_{j,h-1,h}$ is essentially based on $\log \mathbb{E}[\exp(Z)] \approx \mathbb{E}(Z) + \frac{1}{2}\mathbb{V}(Z)$, which is exact when Z is Gaussian, but not if it is truncated Gaussian, as is the case here. This gives:

$$\begin{aligned} p_{j,k-1,k} &= \mathbb{E}_t^{\mathbb{Q}}(\delta' X_{t+k} + \underline{\lambda}_{j,t+k}) - \frac{1}{2}\mathbb{V}_t^{\mathbb{Q}}(\delta' X_{t+k} + \underline{\lambda}_{j,t+k}) - \\ &-\mathbb{C}ov_t^{\mathbb{Q}}\left(\delta' X_{t+k} + \underline{\lambda}_{j,t+k}, \sum_{i=1}^{k-1}(\delta' X_{t+i} + \underline{\lambda}_{j,t+i})\right) \end{aligned} \quad \text{(IV.8)}$$

Following Wu and Xia (2016), considering that $\lambda_{j,t}$ is a persistent process and introducing the following notation:

$$\mathfrak{q}_{j,t,k} = \mathbb{P}_t^{\mathbb{Q}}(d_{j,t+k} > \ell_{j,t+k}),$$

we have, for $k > 0$ and $0 \le i \le k$:

$$\mathbb{C}ov_t^{\mathbb{Q}}(\underbrace{i_{t-1+k}}_{\delta' X_{t+k}}, \underline{\lambda}_{j,t+k-i}) \approx \mathfrak{q}_{j,t,k-i}\mathbb{C}ov_t^{\mathbb{Q}}(\underbrace{i_{t-1+k}}_{\delta' X_{t+k}}, \lambda_{j,t+k-i}) \quad \text{(IV.9)}$$

$$\mathbb{C}ov_t^{\mathbb{Q}}(\underline{\lambda}_{j,t+k}, \underline{\lambda}_{j,t+k-i}) \approx \mathfrak{q}_{j,t,k-i}\mathbb{C}ov_t^{\mathbb{Q}}(\lambda_{j,t+k}, \lambda_{j,t+k-i}) \quad \text{(IV.10)}$$

Using the last two equations, we can rewrite eq. (IV.8) as follows:

$$\begin{aligned} p_{j,k-1,k} \approx\ & \mathbb{E}_t^{\mathbb{Q}}(\delta' X_{t+k} + \underline{\lambda}_{j,t+k}) - \\ & -\frac{1}{2}\left(\mathfrak{q}_{j,t,k}\mathbb{V}_t^{\mathbb{Q}}(\delta' X_{t+k} + \lambda_{j,t+k}) + (1 - \mathfrak{q}_{j,t,k})\mathbb{V}_t^{\mathbb{Q}}(\delta' X_{t+k})\right) - \\ & -\sum_{i=1}^{k-1}\left(\mathfrak{q}_{j,t,i}\mathbb{C}ov_t^{\mathbb{Q}}\left(\delta' X_{t+k} + \lambda_{j,t+k}, \delta' X_{t+i} + \lambda_{j,t+i}\right) + \right. \\ & \left. +(1 - \mathfrak{q}_{j,t,i})\mathbb{C}ov_t^{\mathbb{Q}}\left(\delta' X_{t+k}, \delta' X_{t+i}\right)\right). \end{aligned} \quad \text{(IV.11)}$$

Posing

$$\begin{aligned} \mu_{t,k}^{\mathbb{Q}} &= \mathbb{E}_t^{\mathbb{Q}}(X_{t+k}), & \mu_{j,t,k}^{\mathbb{Q}} &= \mathbb{E}_t^{\mathbb{Q}}(\lambda_{j,t+k}), \\ \sigma_{j,k}^{\mathbb{Q}} &= \sqrt{\mathbb{V}_t^{\mathbb{Q}}(\lambda_{j,t+k})}, & \Gamma_{k,i}^{\mathbb{Q}} &= \mathbb{C}ov_t^{\mathbb{Q}}(X_{t+k}, X_{t+k-i}), \end{aligned}$$

and using $\lambda_{j,t} = \delta_j' X_t$, we finally obtain

$$
\begin{aligned}
p_{j,k-1,k} \approx \; & \delta' \mu^{\mathbb{Q}}_{t,k} + \Phi(\mu^{\mathbb{Q}}_{j,t,k}/\sigma^{\mathbb{Q}}_{j,k})\mu^{\mathbb{Q}}_{j,t,k} + \phi(-\mu^{\mathbb{Q}}_{j,t,k}/\sigma^{\mathbb{Q}}_{j,k})\sigma^{\mathbb{Q}}_{j,k} - \\
& -\frac{1}{2}\left(\mathfrak{q}_{j,t,k}(\delta+\delta_j)'\Gamma^{\mathbb{Q}}_{k,0}(\delta + a_j) + (1-\mathfrak{q}_{j,t,k})\delta'\Gamma^{\mathbb{Q}}_{k,0}\delta \right) - \\
& - \sum_{i=1}^{k-1} \left(\mathfrak{q}_{j,t,k-i}(\delta+\delta_j)'\Gamma^{\mathbb{Q}}_{k,i}(\delta+\delta_j) + (1-\mathfrak{q}_{j,t,k-i})\delta'\Gamma^{\mathbb{Q}}_{k,i}\delta \right),
\end{aligned} \tag{IV.12}
$$

with

$$
\mathfrak{q}_{j,t,k} = \Phi\left(\mu^{\mathbb{Q}}_{t,k}/\sigma^{\mathbb{Q}}_{j,k} \right).
$$

The next appendix details a fast (coding-oriented) approach to compute the $\mu^{\mathbb{Q}}_{t,k}$s and $\Gamma^{\mathbb{Q}}_{k,j}$s.

Appendix V. Computation of $\mu^{\mathbb{Q}}_{t,k}$ and $\Gamma^{\mathbb{Q}}_{k,j}$

Recall X_t's law of motion (eq. 12):

$$
X_t = \mu^{\mathbb{Q}} + \Phi^{\mathbb{Q}} x^{\mathbb{Q}}_{t-1} + \Sigma \varepsilon^{\mathbb{Q}}_{x,t}, \quad \varepsilon_{x,t} \sim i.i.d.\,\mathcal{N}(0, Id).
$$

Using the notation $\Omega = \Sigma\Sigma'$, we have:

$$
\left\{
\begin{array}{lllll}
\mu^{\mathbb{Q}}_{t,k} & = & \mathbb{E}^{\mathbb{Q}}_t(X_{t+k}) & = & (Id - \Phi^{\mathbb{Q}})^{-1}(Id - \Phi^{\mathbb{Q}^k})\mu^{\mathbb{Q}} + \Phi^{\mathbb{Q}^k} X_t, \\
\Gamma^{\mathbb{Q}}_{k,0} & = & \mathbb{V}^{\mathbb{Q}}_t(X_{t+k}) & = & \Omega + \Phi^{\mathbb{Q}}\Gamma^{\mathbb{Q}}_{k-1,0}\Phi^{\mathbb{Q}'}, \quad \text{with} \quad \Gamma^{\mathbb{Q}}_{1,0} = \Omega \\
 & & & = & \Omega + \Phi^{\mathbb{Q}}\Omega\Phi^{\mathbb{Q}'} + \dots + \Phi^{\mathbb{Q}^{k-1}}\Omega\Phi^{\mathbb{Q}^{k-1'}}, \\
\Gamma^{\mathbb{Q}}_{k,i} & = & \mathbb{C}ov^{\mathbb{Q}}_t(X_{t+k}, X_{t+k-i}) & = & \Phi^{\mathbb{Q}^i}\Gamma^{\mathbb{Q}}_{k-i,0} \quad \text{if} \quad k-i>0.
\end{array}
\right.
$$

The estimation involves a large number of computations of the $\Gamma^{\mathbb{Q}}_{k,j}$'s. In order to speed up the computation, one can employ the following approach.

Consider a vector β of dimension n_x, that is the dimension of X_t, and let us denote by ξ^{β}_i the vector defined by $\xi^{\beta}_i = (\Phi^{\mathbb{Q}^i}_x)'\beta$ (β will typically be δ_j, or $(\delta_j + \delta)$). Because we have $\Gamma^{\mathbb{Q}}_{k,i} = \Phi^{\mathbb{Q}^i}_x\Omega + \Phi^{\mathbb{Q}^{i+1}}_x\Omega\Phi_x' + \dots + \Phi^{k-1}_x\Omega\Phi^{k-1-i'}_x$, it comes that:

$$
\beta'\Gamma_{k,j}\beta = \xi^{\beta'}_i\Omega\xi^{\beta}_0 + \xi^{\beta}_{i+1}{}'\Omega\xi^{\beta}_1 + \dots + \xi^{\beta}_{k-1}{}'\Omega\xi^{\beta}_{k-1-i}. \tag{V.13}
$$

Let us consider a maximal value for k, say H, and let us denote by Ξ_β the $n_x \times (H+1)$ matrix whose w^{th} column is ξ^{β}_{w-1}. It can then be seen that the (i,k) entry of $\Psi^{\beta} := \Xi_\beta{}'\Omega\Xi_\beta$ – which is a matrix of dimension $(H+1)\times(H+1)$ – is equal to $\xi^{\beta}_{i-1}{}'\Omega\xi^{\beta}_{k-1}$. The sum of the entries $(i+1,1)$, $(i+2,2)$, ..., $(i+k,k)$ of Ψ^{β} therefore is

$$
\xi^{\beta'}_j\Omega\xi^{\beta}_0 + \xi^{\beta}_{i+1}{}'\Omega\xi^{\beta}_1 + \dots + \xi^{\beta}_{i+k-1}{}'\Omega\xi^{\beta}_{k-1},
$$

which is equal to $\beta'\Gamma^{\mathbb{Q}}_{i+k,i}\beta$ according to (V.13). Equivalently, $\beta'\Gamma^{\mathbb{Q}}_{k,i}\beta$ is the sum of the entries $(i+1,1)$, $(i+2,2)$, ..., $(k,k-i)$ of Ψ^{β}.

In particular, the entry $(1,1)$ of Ψ^{β} is equal to $\beta'\Gamma_{1,0}\beta$, the sum of the entries $(1,1)$ and $(2,2)$ is equal to $\beta'\Omega\beta + \beta'\Phi_x\Omega\Phi_x'\beta = \beta'\Gamma_{2,0}\beta$, and, more generally, the sum of the entries $(1,1),\dots,(n-1,n-1)$ of Ψ^{β} is equal to $\beta'\Gamma_{n,0}\beta$.

Appendix VI. Pricing zero-coupon bonds with non-zero recovery rates

Consider a zero-coupon bond of maturity h issued by country j. Assume the recovery rate is RR. On date $t+k$, with $0<k\leq h$, the payoff of this bond is zero, unless the country defaults on date $t+k$, in which case the bond payoff is assumed to be the fraction RR of the price of a risk-free zero-coupon bond of equivalent residual maturity, i.e. $\exp[-(h-k)i_{t+k,h-k}]$ (this is the Recovery of Treasury convention—RT—of Duffie and Singleton, 1999). Hence, the payoffs of this bond are of the form:

$$\begin{cases} RR\times\exp(-(h-k)i_{t+k,h-k})\times(\mathcal{D}_{j,t+k}-\mathcal{D}_{j,t+k-1}) & \text{if} \quad 0<k<h, \\ 1-\mathcal{D}_{j,t+k}+RR\times(\mathcal{D}_{j,t+k}-\mathcal{D}_{j,t+k-1}) & \text{if} \quad k=h. \end{cases}$$

As a result, denoting by $\Lambda_{t,t+k}$ the (non stochastic) discount factor $\exp(-i_t-\cdots-i_{t+k-1})$, the price of this bond is given by:

$$\begin{aligned} \mathcal{P}^{(j)}_{t,h} &= \mathbb{E}^{\mathbb{Q}}_t\left(\Lambda_{t,t+h}(1-\mathcal{D}_{j,t+h})+RR\sum_{k=1}^{h}\Lambda_{t,t+k}\exp(-(h-k)i_{t+k,h-k})(\mathcal{D}_{j,t+k}-\mathcal{D}_{j,t+k-1})\right) \\ &= \mathbb{E}^{\mathbb{Q}}_t(\Lambda_{t,t+h}(1-\mathcal{D}_{j,t+h}))+RR\sum_{k=1}^{h}\mathbb{E}^{\mathbb{Q}}_t\left[\Lambda_{t,t+k}\mathbb{E}^{\mathbb{Q}}_{t+k}\left\{\exp(-i_{t+k}-\cdots-i_{t+h-1})\right\}(\mathcal{D}_{j,t+k}-\mathcal{D}_{j,t+k-1})\right] \\ &= \mathbb{E}^{\mathbb{Q}}_t(\Lambda_{t,t+h}(1-\mathcal{D}_{j,t+h}))+RR\sum_{k=1}^{h}\mathbb{E}^{\mathbb{Q}}_t\left[\Lambda_{t,t+h}(\mathcal{D}_{j,t+k}-\mathcal{D}_{j,t+k-1})\right] \quad \text{(by the law of iterated expectations)} \\ &= \mathbb{E}^{\mathbb{Q}}_t(\Lambda_{t,t+h}(1-\mathcal{D}_{j,t+h}))+RR\mathbb{E}^{\mathbb{Q}}_t(\Lambda_{t,t+h})\sum_{k=1}^{h}\mathbb{E}^{\mathbb{Q}}_t\left[(\mathcal{D}_{j,t+k}-\mathcal{D}_{j,t+k-1})\right], \end{aligned}$$

where the conditional expectation $\mathbb{E}^{\mathbb{Q}}_t(\Lambda_{t,t+h}(1-\mathcal{D}_{j,t+h}))$ represents the date-t price of a zero-coupon zero-recovery risky bond of maturity h providing a payoff of 1 on date $t+h$ if country j has not defaulted before $t+h$, and zero otherwise (see Appendices B for an approximation of this price). Moreover, $\mathbb{E}^{\mathbb{Q}}_t(\Lambda_{t,t+h}\mathcal{D}_{j,t+k}) = \mathbb{E}^{\mathbb{Q}}_t(\Lambda_{t,t+h})\mathbb{E}^{\mathbb{Q}}_t(\mathcal{D}_{j,t+h})$ results from the fact that, under our assumptions regarding the s.d.f., $\mathcal{D}_t$ and i_t are independent under the risk-neutral measure $\mathbb{Q}$ (as they are under $\mathbb{P}$). Therefore:

$$\begin{aligned} \mathcal{P}^{(j)}_{t,h} &= \mathbb{E}^{\mathbb{Q}}_t(\Lambda_{t,t+h}(1-\mathcal{D}_{j,t+h}))+RR\mathbb{E}^{\mathbb{Q}}_t(\Lambda_{t,t+h}\mathcal{D}_{j,t+h}) \\ &= \mathbb{E}^{\mathbb{Q}}_t(\Lambda_{t,t+h}(1-\mathcal{D}_{j,t+h}))-RR\mathbb{E}^{\mathbb{Q}}_t(\Lambda_{t,t+h}(1-\mathcal{D}_{j,t+h}))+RR\mathbb{E}^{\mathbb{Q}}_t(\Lambda_{t,t+h}) \\ &= (1-RR)P^{(j)}_{t,h}+RR\exp(-hi^0_{t,h}), \end{aligned}$$

where approximation formulas for $P^{(j)}_{t,h}$ are given in Appendix B (computation details are given in Online Appendices IV and V).

Appendix VII. Time variability of credit risk premiums

This appendix explains why the present framework accommodates time-varying credit risk premiums in spite of featuring constant prices of risk associated with debt and fiscal-limit shocks (that drive default risk).

Loosely speaking, risk premiums can be seen as the product of a (a) price of risk (ψ_t in our framework, see eq. 15) and (b) a quantity of risk, characterized by the amount of randomness in the system, and measured by conditional variances. We obtain time-varying risk premiums as soon as (a) or (b) varies.

In our model, the conditional variance associated with the default intensity varies through time because of the non-linearity implied by the max operator, as detailed below.

To simplify, consider a situation where risk-free interest rates are null. The conditional probability of default is given by

$$\mathbb{P}(\mathcal{D}_t = 1|\mathcal{D}_{t-1} = 0, s_t) = 1 - \exp(-\max(-s_t, 0)), \qquad \text{(VII.14)}$$

where s_t is the fiscal space. (Hence, the probability of default is null if $s_t \geq 0$, and strictly positive otherwise.) The fiscal space follows a random walk:

$$s_t = s_{t-1} + \sigma\eta_t,$$

where $\eta_t \sim$ *i.i.d.* $\mathcal{N}(0,1)$.

The s.d.f. is given by:

$$\mathcal{M}_{t,t+1} = \exp\left(-\nu\eta_{t+1} - \frac{1}{2}\nu^2\right).$$

With $\nu > 0$, this model implies that the s.d.f. is higher when η_{t+1} is negative, i.e., when the fiscal space diminishes.

In this context, and with a recovery rate, the price of a one-period defaultable bond is:

$$\begin{aligned} P_{t,1} &= \mathbb{E}_t^{\mathbb{Q}}\{(1-\mathcal{D}_{t+1})\} = \mathbb{E}_t^{\mathbb{Q}}\{\exp(-\max[-s_{t+1},0])\} \\ &= \mathbb{E}_t\left\{\exp(-\max[-s_{t+1},0])\exp\left(-\nu\eta_{t+1} - \frac{1}{2}\nu^2\right)\right\}. \end{aligned}$$

Denoting by $P^*_{t,1}$ the price of the bond that would be observed under the expectation hypothesis ($\nu = 0$), the credit risk premium is given by:

$$-\log P_{t,1} + \log P^*_{t,1} = -\log\left(\frac{\mathbb{E}_t^{\mathbb{Q}}\{(1-\mathcal{D}_{t+1})\}}{\mathbb{E}_t\{(1-\mathcal{D}_{t+1})\}}\right),$$

with

$$\left\{\begin{aligned} P_{t,1} &= \mathbb{E}_t\left\{\exp\left(-\max[-s_t - \sigma\eta_{t+1}, 0] - \nu\eta_{t+1} - \frac{1}{2}\nu^2\right)\right\} \\ P^*_{t,1} &= \mathbb{E}_t\left\{\exp(-\max[-s_t - \sigma\eta_{t+1}, 0])\right\}. \end{aligned}\right.$$

Consider two polar cases:

- When s_t is large and positive (e.g., $s_t > 4\sigma$), it is extremely likely that $\max[-s_t - \sigma\eta_{t+1}, 0]$ will be equal to zero and, accordingly, we will have $P_{t,1} \approx P^*_{t,1} = 1$. The credit risk premium is therefore essentially zero.
- When s_t is large and negative, it is extremely likely that $\max[-s_t - \sigma\eta_{t+1}, 0] = -s_t - \sigma\eta_{t+1}$, and, as a result:

  $$\left\{\begin{aligned} P_{t,1} &\approx \mathbb{E}_t\left\{\exp\left(s_t + \sigma\eta_{t+1} - \nu\eta_{t+1} - \tfrac{1}{2}\nu^2\right)\right\} \\ P^*_{t,1} &\approx \mathbb{E}_t\left\{\exp(s_t + \sigma\eta_{t+1})\right\}, \end{aligned}\right.$$

  which leads, after simple algebra, to:

  $$-\log P_{t,1} + \log P^*_{t,1} = \nu\sigma.$$

For intermediate values of s_t, the credit risk premium will vary between 0 and $\nu\sigma$.

Without the non-linearity stemming from the max() operator—and keeping the ψ_1 entries associated with those shocks affecting the fiscal space (i.e., the η_d's and η_ℓ's) at zero—then the credit risk premiums would be constant. In the previous example, this would correspond to the situation where we would remove the max() operator in eq. (VII.14) above.[35] (The credit risk premium would then be equal to $\nu\sigma$ for any value of the only state variable considered in the present example, namely s_t.)

Appendix VIII. Inversion technique

This appendix describes the computation of the likelihood function (see Subsection 5.2 for a general description of our estimation approach).

We consider the following decomposition of the state vector $X_t = [i_t, i_{t-1}, d_t', \ell_t']'$:

$$\underbrace{X_t}_{m\times 1} = \left[\begin{array}{c} \underbrace{\widetilde{X}_t}_{(m-N)\times 1} \\ \underbrace{\ell_t}_{N\times 1} \end{array}\right],$$

where $\widetilde{X}_t$ are the observable components of X_t.

The state vector follows a vector autoregressive process of order one (eq. 12).

The vector of observed financial data is organized as follows:

$$Y_t = \left[\begin{array}{c} \underbrace{Y_t^{(y)}}_{n_y\times 1} \\ \underbrace{Y_{1,t}^{(YS)}}_{n_1\times 1} \\ \underbrace{Y_{2,t}^{(YS)}}_{N\times 1} \end{array}\right],$$

where $Y_t^{(y)}$ is a vector of risk-free yields (of maturities 2, 3, 5 and 10 years), $Y_{1,t}^{(YS)}$ is a vector of imperfectly-fitted bond yield spreads (e.g. maturities 2 and 10 yrs) and $Y_{2,t}^{(YS)}$ is a $N \times 1$ vector of perfectly-fitted bond yield spreads (in our case, the average of bond yield spreads of maturities 2, 5 and 10 years). These yields and spreads are given by:

$$\left\{\begin{array}{lcl} Y_t^{(y)} & = & A_y + B_y'\widetilde{X}_t + \xi_t^{(y)} \\ Y_{1,t}^{(YS)} & = & f_1(\widetilde{X}_t, \ell_t) + \xi_t^{(YS)} \\ Y_{2,t}^{(YS)} & = & f_2(\widetilde{X}_t, \ell_t) \quad \text{(these spreads are perfectly fitted).} \end{array}\right. \tag{VIII.15}$$

We assume that the components of $\xi_t^{(y)}$ and $\xi_t^{(YS)}$ are i.i.d. normally-distributed measurement errors. The variance of each component of $\xi_t^{(y)}$ is σ_y^2. The variance of the i^{th} component of $\xi_t^{(YS)}$ is $\sigma_{YS,i}^2$.

System (VIII.15) can be rewritten:

$$\left\{\begin{array}{lcl} Y_t^{(y)} & = & A_y + B_y'\widetilde{X}_t + \xi_t^{(y)} \\ Y_{1,t}^{(YS)} & = & f_1(\widetilde{X}_t, f_2^*(\widetilde{X}_t, Y_{2,t}^{(YS)})) + \xi_t^{(YS)}, \end{array}\right. \tag{VIII.16}$$

where function f_2^* represents the inversion of the pricing of $Y_{2,t}^{(YS)}$, i.e.:

$$Y_{2,t}^{(YS)} = f_2\left(\widetilde{X}_t, \ell_t\right) \Leftrightarrow \ell_t = f_2^*\left(\widetilde{X}_t, Y_{2,t}^{(YS)}\right).$$

[35]Such a purely-Gaussian specification would not be consistent with the fact that the probability of default connot be negative. It has however be sometimes used in the literature, e.g., by Liu, Longstaff, and Mandell (2006).

Let us use the following notations:

$$W_t = \begin{bmatrix} Y_t^{(y)} \\ Y_{1,t}^{(YS)} \\ \widetilde{X}_t \\ Y_{2,t}^{(YS)} \end{bmatrix} \quad \text{and} \quad Z_t = \begin{bmatrix} Y_t^{(y)} \\ Y_{1,t}^{(YS)} \\ \widetilde{X}_t \\ \ell_t \end{bmatrix} = \begin{bmatrix} Y_t^{(y)} \\ Y_{1,t}^{(YS)} \\ X_t \end{bmatrix}.$$

Under the assumption that $Y_{2,t}^{(YS)}$ is perfectly fitted by the model, the information contained in Z_t is the same as that contained in W_t. But the p.d.f. of Z_t, conditional on W_{t-1} (or, equivalently, conditional on Z_{t-1}), is easier tom derive than that of W_t.

Indeed, we have:

$$\begin{aligned}
&\log f_{Z_t|Z_{t-1}}(Z_t) = \\
&-\frac{n_y}{2}\log(2\pi) - n_y \log \sigma_y - \frac{1}{2\sigma_y^2}\left(Y_t^{(y)} - A_y - B_y'\widetilde{X}_t\right)'\left(Y_t^{(y)} - A_y - B_y'\widetilde{X}_t\right) \\
&-\frac{n_1}{2}\log(2\pi) - \sum_{i=1}^{n_1}\log\sigma_{YS,i} - \frac{1}{2}\left(Y_{1,t}^{(YS)} - f_1(\widetilde{X}_t,\ell_t)\right)' \operatorname{diag}(1/\sigma_{YS}^2)\left(Y_{1,t}^{(YS)} - f_1(\widetilde{X}_t,\ell_t)\right) \\
&-\frac{m}{2}\log(2\pi) - \frac{1}{2}\log(|\Sigma\Sigma'|) - \frac{1}{2}\left(X_t - \mu - \Phi X_{t-1}\right)'(\Sigma\Sigma')^{-1}\left(X_t - \mu - \Phi X_{t-1}\right), \qquad \text{(VIII.17)}
\end{aligned}$$

where $\operatorname{diag}(1/\sigma_{YS}^2)$ is a diagonal matrix whose i^{th} diagonal entry is $1/\sigma_{YS,i}^2$.

Remark that this does not provide us with the likelihood associated with observed data since ℓ_t is not directly observed.

We have:

$$W_t = g(Z_t),$$

with

$$g\left(\begin{bmatrix} Y_t^{(y)} \\ Y_{1,t}^{(YS)} \\ \widetilde{X}_t \\ \ell_t \end{bmatrix}\right) = \begin{bmatrix} Y_t^{(y)} \\ Y_{1,t}^{(YS)} \\ \widetilde{X}_t \\ Y_{2,t}^{(YS)} \end{bmatrix} = \begin{bmatrix} Y_t^{(y)} \\ Y_{1,t}^{(YS)} \\ \widetilde{X}_t \\ f_2(\widetilde{X}_t,\ell_t) \end{bmatrix}.$$

In general, we have:

$$f_{W_t|W_{t-1}}(W_t) = \left|\frac{\partial g^{-1}(W_t)}{\partial W'}\right| f_{Z_t|Z_{t-1}}(g^{-1}(W_t)), \qquad \text{(VIII.18)}$$

and, therefore:

$$\log f_{W_t|W_{t-1}}(W_t) = \underbrace{\log\left|\frac{\partial g^{-1}(W_t)}{\partial W'}\right|}_{\text{calculated using eq. (VIII.20)}} + \underbrace{\log f_{Z_t|Z_{t-1}}(g^{-1}(W_t))}_{\text{calculated using eq. (VIII.17)}}, \qquad \text{(VIII.19)}$$

where, using the inverse function theorem and the fact that $\left|\frac{\partial g^{-1}(W_t)}{\partial W'}\right|$ is diagonal:

$$\log\left|\frac{\partial g^{-1}(W_t)}{\partial W'}\right| = -\sum_{i=1}^{N}\log\frac{\partial f_2(\widetilde{X}_t,\ell_t)}{\partial \ell_{i,t}}. \qquad \text{(VIII.20)}$$

In practice, in (VIII.17) and (VIII.20), we replace ℓ_t by $f_2^*\left(\widetilde{X}_t, Y_{2,t}^{(YS)}\right)$ —that is the fiscal limit recovered by the inversion technique.

The vector of observed variables can be extended to include $\mathcal{D}_t$. Using the notation $W_t^* = [W_t', \mathcal{D}_t']'$ and exploiting the fact that $\mathcal{D}_t$ does not Granger-cause W_t, we have:

$$\log f_{W_t^*|W_{t-1}^*}(W_t, \mathcal{D}_t) = \underbrace{\log f_{W_t|W_{t-1}}(W_t)}_{\text{calculated using eq. (VIII.19)}} + \underbrace{\log f_{\mathcal{D}_t|W_t}(\mathcal{D}_t).}_{\text{calculated using eq. (VIII.22)}} \quad \text{(VIII.21)}$$

In particular, if all the components of $\mathcal{D}_t$ are zero (absence of default), we have:

$$\log f_{W_t^*|W_{t-1}^*}(W_t, \mathcal{D}_t = 0) = \log f_{W_t|W_{t-1}}(W_t) + \sum_{j=1}^{N} \log\left[1 - \mathcal{F}(d_{j,t} - \ell_{j,t})\right]. \quad \text{(VIII.22)}$$

APPENDIX IX. REDISTRIBUTION SCHEMES: FORMULAS AND ADDITIONAL SCHEMES

This appendix details the formulas underlying Section 6.2 of the paper.

IX.1. **General formulas.** Assume that, on date t, a European debt agency issues common bonds with maturity h and face value F (it repays F at date $t+h$). The proceeds of the issuance are $P_{t,h}^e F$, with $e \in \{SJG, SNJG\}$, depending on the type of common bond that is issued. The proceeds are allocated across countries proportionally to GDPs. Recalling that GDP weights are denoted by ω_j, country j receives $\omega_j P_{t,h}^e F$. If country j had issued national bonds with the same face value ($\omega_j F$), it would have obtained $P_{t,h}^{(j)} \omega_j F$ on date t. Therefore, at the euro-area level, the gains are:

$$G_{t,h} F = P_{t,h}^e F - \left(\omega' \mathbf{P}_{t,h} F\right), \quad \text{(IX.23)}$$

where $\mathbf{P}_{t,h}$ represent the N-dimensional vector of national prices and ω stands for the N-dimensional vector of GDP weights. (It can be seen from the previous formula that the aggregate gains are null when $e = SNJG$.)

Now, denote by ω_G the redistribution weights of the gains (with $\sum_j \omega_{G,j} = 1$). The after-gain-redistribution proceeds are:

$$\omega' \mathbf{P}_{t,h} F + G_{t,h} \omega_G F,$$

which is of the form $\omega' \mathbf{P}_{e,t,h}(\omega_G) F$, with

$$\mathbf{P}_{e,t,h}(\omega_G) = \mathbf{P}_{t,h} + G_{t,h} \frac{\omega_G}{\omega}, \quad \text{(IX.24)}$$

where, by abuse of notation, $\frac{\omega_G}{\omega}$ denotes the vector whose j^{th} entry is $\omega_{G,j}/\omega_j$. $\mathbf{P}_{e,t,h}(\omega_G)$ can be interpreted as the pseudo issuance N-dimensional vector of prices after redistribution. The post-redistribution yields faced by the different countries are given by the following N-dimensional vector:

$$\mathbf{i}_{e,t,h}(\omega_G) = -\frac{1}{h} \log \mathbf{P}_{e,t,h}(\omega_G), \quad \text{(IX.25)}$$

where, by abuse of notation, the log operator is applied element-wise.

Below, we describe the different after-gain redistribution schemes that we propose. Given that aggregate gains for the SNJG bond issuance scheme are nil, for the latter, we only focus on the scheme in which all countries face the same funding costs.

IX.2. **Scheme where countries face the same funding costs.** In this scheme, the after-redistribution issuance price faced by all countries is the eurobond price. That is:

$$\mathbf{P}_{e,t,h}(\omega_G) = P_{t,h}^e.$$

Using $G_{t,h} = P^e_{t,h} - \omega' \mathbf{P}_{t,h}$ together with (IX.24) then gives:

$$\omega_G = \omega \odot \frac{P^e_{t,h}\mathbf{1} - \mathbf{P}_{t,h}}{P^e_{t,h} - \omega'\mathbf{P}_{t,h}},$$

where $\odot$ is the element-wise product. Note that the sign of each country's redistribution weight $\omega_{G,j}$ depends on $P^e_{t,h} - P^{(j)}_{t,h}$. Therefore, this scheme implies negative "gains" for those countries j whose national bond prices are higher than that of the considered eurobond.

IX.3. **Scheme with GDP weights.** In this case, the redistribution weights (ω_G) are equal to the GDP weights (ω). Using $G_{t,h} = P^e_{t,h} - \omega'\mathbf{P}_{t,h}$ (i.e., Eq. (IX.23)), Eq. (IX.24) gives:

$$\mathbf{P}_{e,t,h}(\omega_G) = \mathbf{P}_{t,h} + (P^e_{t,h} - \omega'\mathbf{P}_{t,h})\mathbf{1}.$$

IX.4. **Scheme with the same yield gains across countries.** Under this scheme, all countries benefit from the same yield gain, denoted by Δi_t. Denote by $\mathbf{i}_{t,h}$ the N-dimensional vector of national bond yields. We want to have $\mathbf{P}_{e,t,h}(\omega_G) = \exp(-h(\mathbf{i}_{t,h} - \Delta i_t))$. Using (IX.24), we get:

$$\mathbf{P}_{t,h} + G_{t,h}\frac{\omega_G}{\omega} = \exp(-h(\mathbf{i}_{t,h} - \Delta i_t)),$$

where, by abuse of notation, $\frac{\omega_G}{\omega}$ denotes the vector whose j^{th} entry is $\omega_{G,j}/\omega_j$. This gives:

$$\omega_G = \frac{1}{G_{t,h}}\omega \odot \left[\exp(-h(\mathbf{i}_{t,h} - \Delta i_t)) - \mathbf{P}_{t,h}\right],$$

where $\odot$ is the element-wise product. Since the components of ω_G have to sum to one, we have:

$$1 = \mathbf{1}'\left(\frac{1}{G_{t,h}}\omega \odot \left[\exp(-h(\mathbf{i}_{t,h} - \Delta i_t)) - \mathbf{P}_{t,h}\right]\right),$$

or, using that $\exp(-h\mathbf{i}_{t,h}) = \mathbf{P}_{t,h}$:

$$G_{t,h} = (\exp(h\Delta i_t) - 1)\mathbf{1}'(\omega \odot \mathbf{P}_{t,h}).$$

This further gives:

$$1 + \frac{G_{t,h}}{\mathbf{1}'\omega \odot \mathbf{P}_{t,h}} = \exp(h\Delta i_t),$$

and, finally:

$$\Delta i_t = \frac{1}{h}\log\left(1 + \frac{G_{t,h}}{\mathbf{1}'(\omega \odot \mathbf{P}_{t,h})}\right).$$

IX.5. **Scheme with no change in funding costs for Germany and France.** Table 5 complements the analysis developed in Subsection 6.2 with two additional schemes. In the first scheme (respectively second scheme), Germany (resp. both Germany and France) faces the same funding costs it would have faced under national issuance. Moreover, the aggregate gains are shared among the other countries on the base of their relative GDP size.

Appendix X. Higher-order effects

This appendix proposes an analysis of potential higher-order effects associated with debt-service relief. The mechanisms underlying such effects would be as follows: if the average funding cost of a

TABLE 5. Effect of redistribution schemes on funding costs (additional schemes)

	2008-06-30			2011-12-31			2021-06-30		
	SJG								
Panel A: No change in German funding cost									
	redist. weigth	post redist. yield	yield gain	redist. weigth	post redist. yield	yield gain	redist. weigth	post redist. yield	yield gain
DE	0%	439	0	0%	186	0	0%	-9	0
FR	36%	426	21	36%	221	22	36%	-29	34
IT	28%	447	21	28%	559	26	28%	23	35
ES	18%	432	21	18%	504	25	18%	3	34
NL	11%	435	21	11%	197	22	11%	-40	34
BE	7%	442	21	7%	334	23	7%	-30	34
Panel B: No change in German and French funding cost									
	redist. weigth	post redist. yield	yield gain	redist. weigth	post redist. yield	yield gain	redist. weigth	post redist. yield	yield gain
DE	0%	439	0	0%	186	0	0%	-9	0
FR	0%	446	0	0%	243	0	0%	5	0
IT	44%	435	32	44%	545	41	44%	4	54
ES	28%	420	32	28%	490	39	28%	-16	53
NL	18%	423	32	18%	185	34	18%	-59	52
BE	10%	430	32	10%	321	36	10%	-49	52

Notes: This table compares post-redistribution funding costs across countries under the two issuance schemes (SJG and SNJG) and under the redistribution schemes described in IX.5. We focus on the 5-year maturity and on three periods: beginning of the estimation sample (2008Q2), midst of the euro debt crisis (2011Q4) and end of the estimation sample (2021Q2). Yields are expressed in basis points. Aggregate gains are computed under the assumption that total issuance is equal to 5% of aggregate GDP. In each panel, for all countries and dates, we show the redistribution weights $\omega_{G,j}$, the post-redistribution yields and the yield gains, that are the differences between national bond yields and post-redistribution yields.

government decreases—because part of its funding needs is met with Eurobonds—then expected future debt decreases (through lower debt service), which further decreases national bond yields (through lower credit spreads) which, in turn, reduces again future debt service, and so on.

Assume that the government of country j issues bonds of maturity h. For notational simplicity, let us drop the subscript h. That is, denote by $y_{j,t}$ the yields associated with these bonds, and by $y_{j,t}^{(SJG)}$ the post-redistribution yields associated with the issuance of SJG bonds.

Consider a change in the funding strategy: while the whole debt was only funded through national bonds before date t, a fraction θ_j of the debt gets funded by SJG bonds after that date. Note that θ_j depends on countries given that the proceeds of a Eurobond issuance are supposed to allocated according to GDPs (eq. 7). Specifically, we have:[36]

$$\theta_j = \theta \frac{\omega_j}{\omega_j^D}, \tag{X.26}$$

[36] Indeed, denoting by D_j the debt outstanding of country j (i.e., $D_j = d_j Y_j$), we must have $\theta_j D_j / (\theta \sum_i D_i) = \omega_j$. Hence, $\theta_j = \omega_j \theta (\sum_i D_i) / D_j$.

where the ω_j^Ds denotes debt weights (while the ω_js are GDP weights). The previous expression shows that, for countries whose indebtedness is larger than that prevailing at the euro-area aggregate level (which corresponds to $\omega_j^D > \omega_j$), then the share of debt issued in the form of Eurobonds is lower (since $\theta_j < \theta$).

For newly issued debt, the apparent yield then becomes $\theta_j y_{j,t}^{(SJG)} + (1-\theta_j)y_{j,t}$, which is lower than $y_{j,t}$ if $y_{j,t}^{(SJG)} < y_{j,t}$. All else being equal, this should give rise to a decrease in debt payments, and hence in debt. Using that the maturity of newly-issued debt duration is h, it comes that the total reduction in debt payments, after h years, will be of the order of magnitude of:[37]

$$h \times \left[y_{j,t} - \left(\theta_j y_{j,t}^{(SJG)} + (1-\theta_j) y_{j,t} \right) \right]. \tag{X.27}$$

On date t, investors may take into account that debt-reduction effect when pricing bonds. (If they consider, in particular, that this debt reduction will not be substituted with higher primary surpluses.) In that case, in terms of funding cost, the first-round effect:

$$\theta_j y_{j,t}^{(SJG)} + (1-\theta_j) y_{j,t} < y_{j,t}. \tag{X.28}$$

would be reinforced by second-round effects resulting from lower future debt payments. And, in turn, higher-round effects may follow. Investigating these effects, therefore, involves solving a fixed-point problem.

We proceed as follows. We start by computing national and SJG yields using our pricing formulas and a given value of the state vector X (that we take equal to its sample average value). Representing our pricing formulas by function f, this first step formally is:

$$\mathcal{Y} = f(X),$$

where $\mathcal{Y}$ is a vector gathering the relevant national yields and SJG yields. More precisely:

$$\mathcal{Y} = \begin{bmatrix} y_1 \\ \vdots \\ y_N \\ y_{SJG} \end{bmatrix}.$$

Next, we use these yields to compute the debt reduction (X.27). We then modifiy the state vector X in such a way that, over the next h periods, the average expected debt will indeed be reduced by this amount. (More precisely, we modify the debt trend, that is $\gamma = d - (1-\rho_d)\bar{d} - \rho_d d_{-1}$ (eq. 9), to achieve

[37] Note that this is only an approximation as the exact number would also depend on future yields (since the debt is completely renewed in h dates).

that.[38]) This provides us with $X^{(1)}$, which we use to compute new yields:

$$\mathcal{Y}^{(1)} = f(X^{(1)}).$$

The superscript (1) indicates that the resulting yields result from the first iteration. The yields in $\mathcal{Y}^{(1)}$ are going to be different from those in $\mathcal{Y}$ (because expected debt is lower). Hence, the debt reduction resulting from the partial issuance of SJG bonds is higher than what is suggested by (X.27). For each country, we then compute a novel debt reduction by using:

$$h \times \left[y_j - \left(\theta y_{SJG}^{(1)} + (1-\theta) y_j^{(1)}\right)\right],$$

which we use to construct a new state vector $X^{(2)}$, which gives a new vector of yields $\mathcal{Y}^{(2)}$, and so on until convergence.

Figure X.3 implements this approach, with a share of euro-area debt issued in the form of SJG bonds equal to $\theta = 20\%$. We consider two redistribution schemes (see Subsection 6.2), namely Scheme A (no redistribution after the issuance of SJG bonds) and Scheme B (where the redistribution of SJG aggregate gains is based on GDP weights). At "Order 0", the state vector X_t is set to its sample mean.

[38]The model described in Section 4 is such that, for each country:

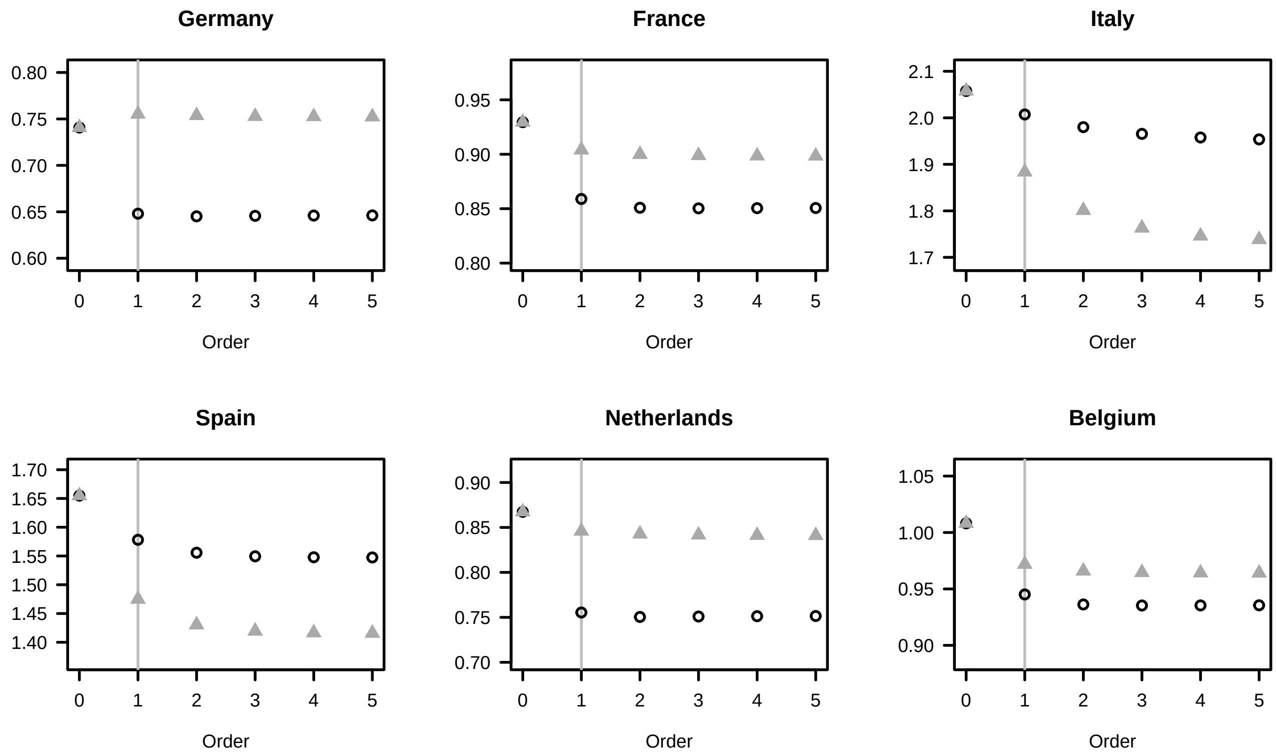
$$\left[\begin{array}{c} d_t \\ d_{t-1} \end{array}\right] = \underbrace{\left[\begin{array}{c} \bar{d}(1-\rho_d)(1-\rho_\gamma) \\ 0 \end{array}\right]}_{=c} + \underbrace{\left[\begin{array}{cc} \rho_d + \rho_\gamma & -\rho_\gamma \rho_d \\ 1 & 0 \end{array}\right]}_{=F} \left[\begin{array}{c} d_{t-1} \\ d_{t-2} \end{array}\right] + \left[\begin{array}{c} \varepsilon_{d,t} \\ 0 \end{array}\right].$$

This implies that:

$$\begin{aligned} & \mathbb{E}_t\left[\frac{1}{h}\left(d_{t+1} + \cdots + d_{t+h}\right)\right] \\ = \; & \frac{1}{h}(I-F)^{-1}\left[(h+1)I - (I-F)^{-1}(I-F^{h+1})\right] c + \frac{1}{h}[(I-F)^{-1}(I-F^{h+1}) - I]\left[\begin{array}{c} d_t \\ d_{t-1} \end{array}\right]. \end{aligned}$$

We use the previous formula to look for the value of d_{t-1} that results in the desired change in the expected average debt, i.e., $\mathbb{E}_t\left[\frac{1}{h}\left(d_{t+1} + \cdots + d_{t+h}\right)\right]$. (This amounts to a change in $\gamma_t = d_t - (1-\rho_d)\bar{d} - \rho_d d_{t-1}$.)

FIGURE X.3. Higher-order effects



Notes: These plots illustrate potential higher-order effects stemming from the (partial) issuance of SJG bonds. Specifically, we consider that $\theta = 20\%$ of the euro-area debt is issued in the form of SJG bonds. The grey triangles and black circles respectively correspond to Schemes A and B. (In Scheme A, there is no post-issuance redistribution when SJG bonds are issued; in Scheme B, aggregate gains associated with the issuance of SJG bonds are redistributed according to GDP weights). The first points of the plots ("Order" $= 0$) give the model-implied average 5-year yields associated with the different countries. If a fraction of the government funding needs is met by issuing Eurobonds, then the average funding cost is modified (eq. X.28). This gives the second point, of the charts ("Order" $= 1$), which is highlighted by a vertical grey line given that it represents the first-round effect of issuing Eurobonds (namely, the effects presented in the main findings in Section 6). Changes in funding costs affect expected debt trajectories, which, in turn, modify bond yields (for national and SJG bonds). The resulting funding costs are represented by the third set of points ("Order" $= 2$). The following points, for higher orders, are obtained by using the same steps, in an iterative fashion. Yields are annualized, and expressed in percentage points.

APPENDIX XI. BONDS USED IN FIGURE 6

TABLE 6. Bonds used in Figure 6

Issuer	Eikon ticker	Coupon (in percent)	Maturity date
France	FRGV	2.50	25-May-2030
Belgium	BEGV	0.55	04-Mar-2029
Portugal	PTGV	3.875	15-Feb-2030
ESM	ESM	0.50	05-Mar-2029
Spain	ESGV	1.95	30-Jul-2030
Netherlands	NLGV	0.25	15-Jul-2029
Germany	DEGV IO Str	0	04-Jul-2030
NEXTGENEU	EUUNI	0	04-Jul-2031
EFSF	EFSFC	2.75	03-Dec-2029
EU	EUUNI	1.375	04-Oct-2029
Italy	ITGV	3.50	01-Mar-2030
EIB	EIB	0.25	14-Sep-2029

Notes: This table lists the bonds used in Figure 6. Asset swap spreads (ASW) are computed by Refinitiv Eikon.

APPENDIX XII. ALTERNATIVE (STATIC) FISCAL LIMIT ESTIMATES

TABLE 7. Fiscal limit static estimates in the literature

Country	Ghosh et al. (2013)		Collard et al. (2015)					
	Hist.	**Proj.**	5% **MPS**	**MRR**	**TVR**	**CATA**	4% **MPS**	**hist. MPS**
DE	154.1	175.8	130.1	132.3	114.6	85.5	104.1	112.9
FR	170.9	176.1	146.6	148.6	119.8	97.8	117.2	40.0
IT	–	–	113.2	115.6	106.8	74.2	90.6	147.5
ES	218.3	153.9	144.2	146.2	119.3	95.8	115.3	115.6

Notes: All estimates are reported in percent of GDP. **Estimates of Ghosh et al. (2013)** – Debt limits (fiscal limits in our terminology) are statically estimated through the interest payment schedule for the period 1985-2007. **Hist.**: Estimates are based on the average interest rate / growth differential of 1998-2007, using the implied interest rate on public debt; **Proj.**: The interest rate / growth differential is based on the long term government bond yield (average for 2010-2014, IMF projections as of 2010). **Estimates of Collard et al. (2015)** – The computation of maximum sustainable debts (fiscal limits in our terminology) exploits the idea of a maximum primary surplus (MPS). In the model, there is a maximum amount that can be issued on each date (that itself depends on the MPS). 5% **MPS**: Case where the MPS is set to to 5%; **MRR**: The computation involves a maximum recovery rate; **TVR**: The model features a time-varying interest rate; **CATA**: The model features catastrophes; 4% **MPS**: The MPS is set to 5%; **hist. MPS**: The MPS is set to the historical peak of primary surplus-to-GDP.

Appendix XIII. Partial Eurobonds

Our framework can also be used to study "partial" SNJG and SJG bonds, defined as bonds jointly issued by a subset of countries. Specifically, we focus on the computation of SNJG and SJG bonds issued by four countries (out of the six we consider) either excluding "super" core member states (Germany and Netherlands), or excluding peripheral countries (Italy and Spain). These prices are computed under the baseline estimated model, the only parameters that have to be adjusted to perform this analysis are the weights (ω) defining the groups of issuing countries (see Subsection 3.3).

Figure XIII.4 shows the SNJG and SJG bond yield spreads across different maturities (i) under the baseline scenario, where all countries participate in the emission, as in the main results presented in Section 2.1 (solid lines); (ii) under the scenario in which Germany and Netherlands do not participate in the issuance (dashed lines); and (iii) under the scenario in which peripheral member states are excluded from the joint issuance program (dotted lines). Not surprisingly, when Italy and Spain are excluded from the joint issuance program, yield spreads are below those obtained in the baseline scenario; and when Germany and Netherlands do not participate, yield spreads are higher. The spread between "partial" SJG and SNJG bonds—which reflects the aggregate yield gains—is smaller in the "partial" scenarios than when all countries participate in the program, as diversification effects are magnified in the latter case.

Appendix XIV. Sensitivity analysis

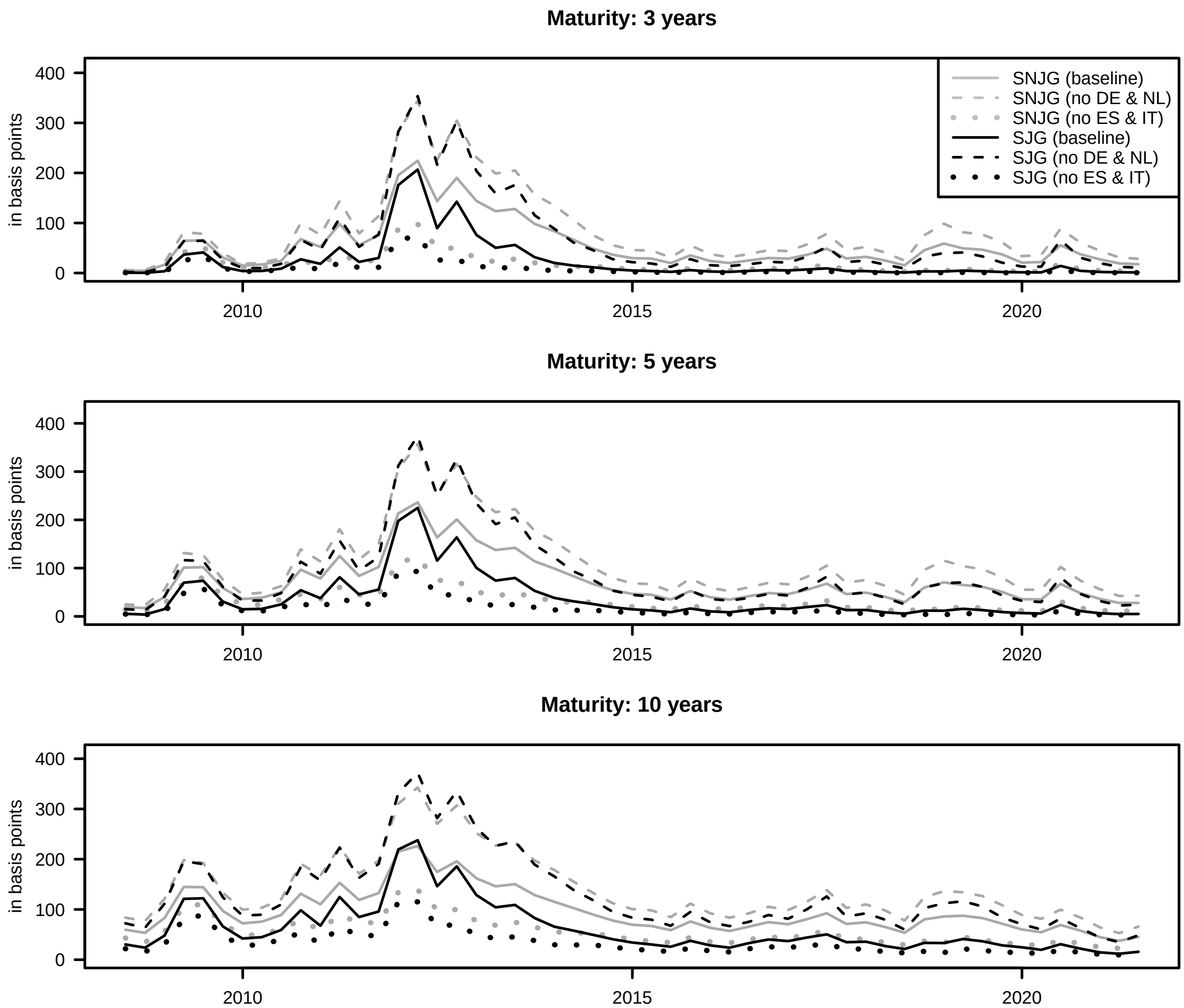
This appendix presents the results of sensitivity analyses performed to assess the robustness of our main baseline results. Specifically, we modify the model by changing the bounds or imposing a specific value on some key parameters, one at the time, and run the complete estimation. More precisely:

- We exclude the COVID period (after 2020Q1) from the PCA analysis of the recovered estimates for $\epsilon_{d,j,t}$'s so that also $\mathbb{V}ar(\varepsilon_{d,t}) = \Gamma_d \Gamma_d'$ is modified. (Γ_d represents the matrix of PCA weights, see Subsection 5.3 for details regarding Γ_d.)
- Considering that $maxSR$ is constrained at the lower bound, we relax such bound by reducing it to 0.25, instead of 0.5.
- Given that ρ_γ is constrained at the lower bound, we relax such bound by shifting it to 0.5, instead of 0.7.
- We impose a higher value on α (even if estimated), equal to 0.2, which corresponds to more than double the estimated value (see Table 3 in Sec. 5).
- We set $\rho_{d,\ell}$, the correlation between the two "main common shocks" ($\eta_{d,1,t}$ and $\eta_{\ell,1,t}$) to zero.
- Considering that the parameter ζ is constrained at its upper bound, we relax such bound by increasing it by 0.5 (from 1.5 to 2). Note that the parameter ζ is defined as the multiplicative factor disciplining the relation between Γ_d and Γ_ℓ ($\Gamma_\ell = \zeta \Gamma_d$) and, thus, it is pivotal for $\mathbb{V}ar(\varepsilon_{\ell,t}) = \Gamma_\ell \Gamma_\ell'$ (see Subsection 5.3).

Figure XIV.5 shows the fiscal limit estimates under the baseline parametrization (grey thick solid lines) and across the above-described sensitivity exercises, together with debt-to-GDP ratios (black solid lines). Units are in percent of GDP. While the different parametrizations tend to result in shifts in the estimated fiscal limits, it appears that the fluctuations are fairly consistent across the different specifications.

Figure XIV.6 displays the yield spreads of SJG bonds across the sensitivity exercises and for the baseline estimation (grey thick solid line). The three panels correspond to different maturities: 3, 5, and 10

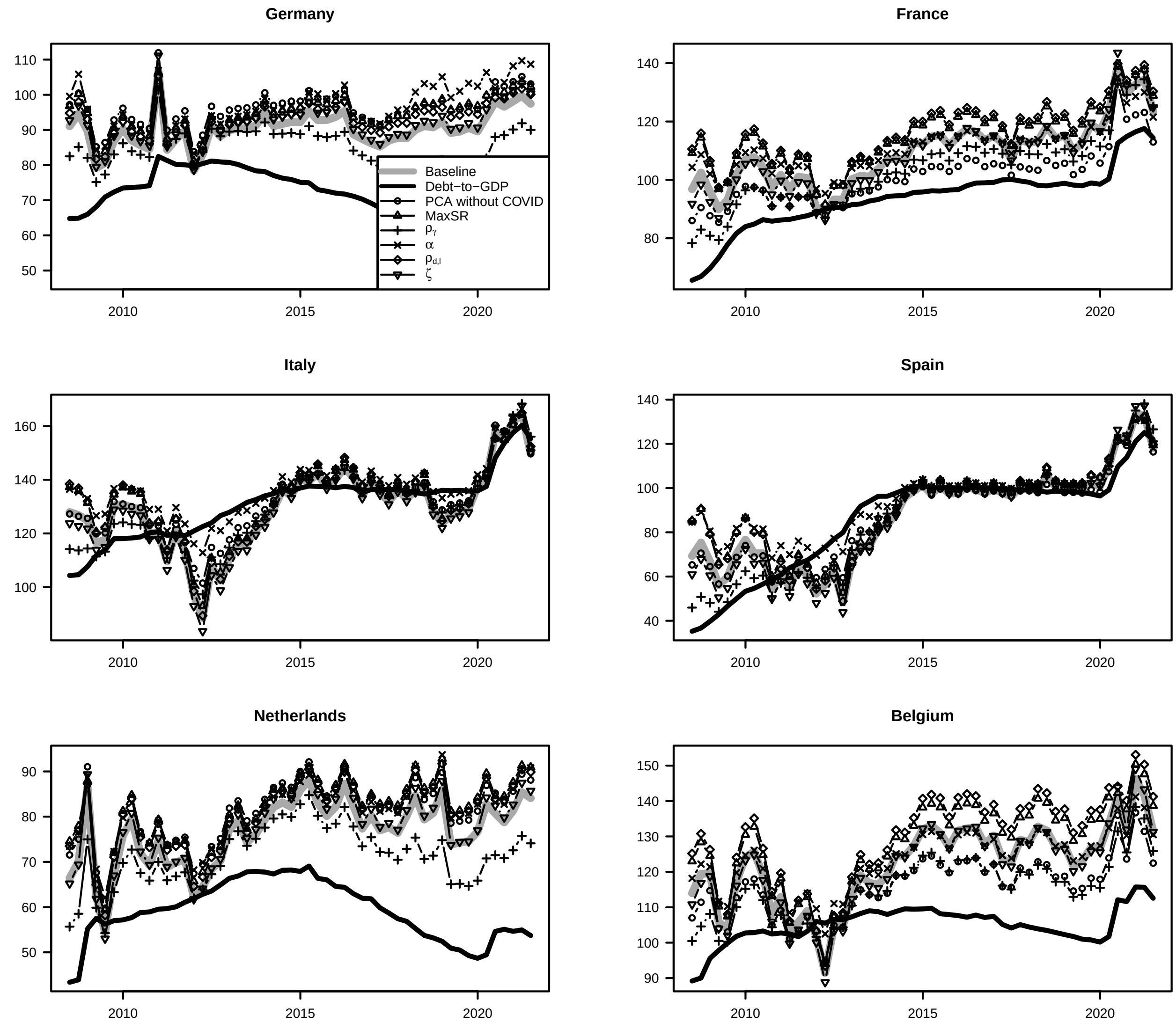
FIGURE XIII.4. Partial SJG Eurobonds: baseline, excluding "super" core (Germany and Netherlands) and excluding periphery (Italy and Spain).



This figure shows the yield spreads in basis points across different maturities (3-, 5- and 10-year maturity) associated with SNJG (grey lines) and SJG (black lines) bonds under three different scenarios: (i) under the baseline scenario, where all countries participate in the emission, as in the main results presented in Section 2.1 (solid lines); (ii) under the scenario in which Germany and Netherlands do not participate in the issuance (dashed lines); and (iii) under the scenario in which peripheral member states are excluded from the joint issuance program (dotted lines).

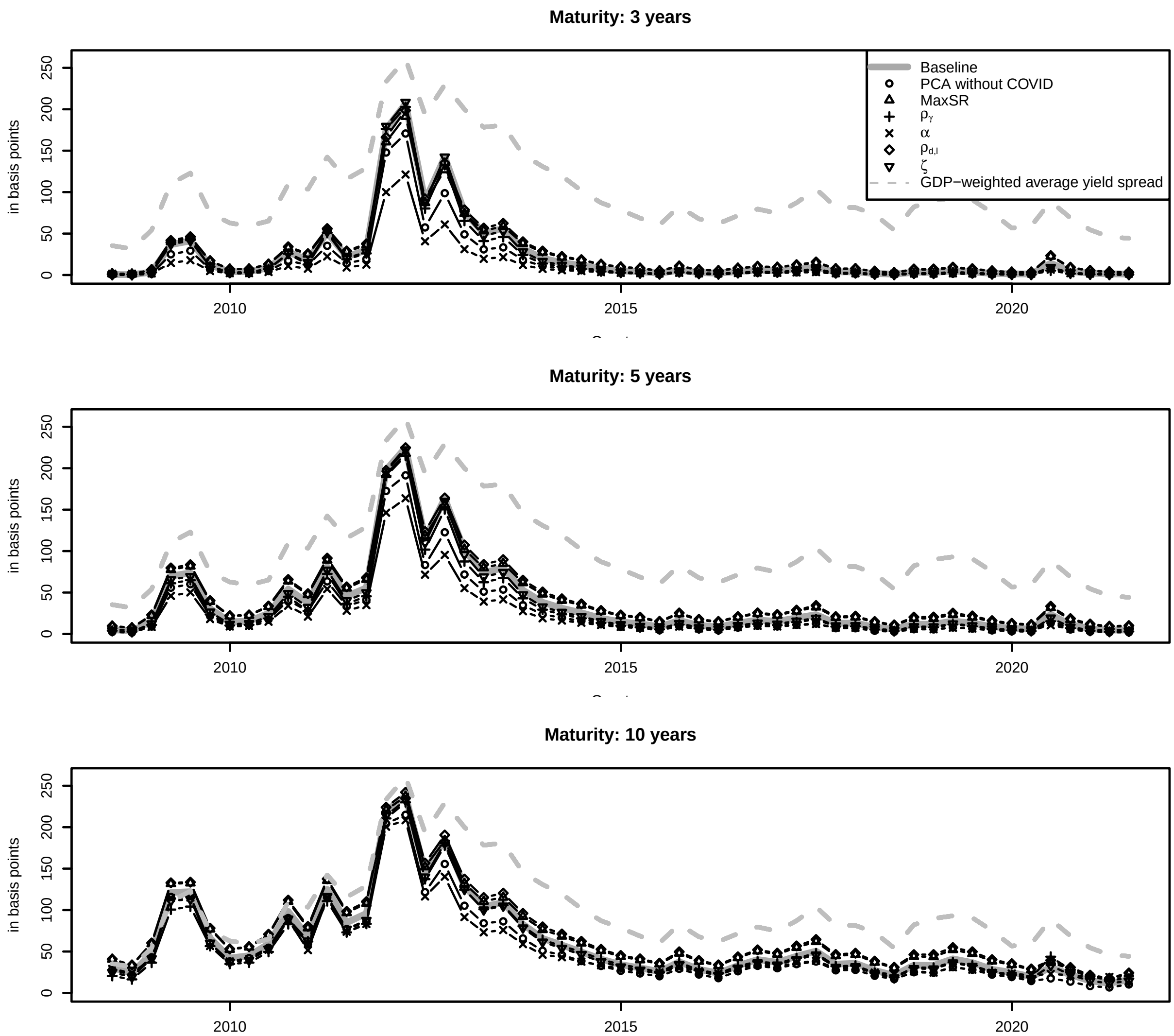
years. The blue line corresponds to a model-free approximation of the SNJG bond spread, computed as the GDP-weighted average of the national bond spreads. We observe that the order of magnitude of the SJG-vs-SNJG spreads is fairly robust under different model parametrizations. This is confirmed by Figure XIV.7, that shows the 3-, 5-, and 10-year maturity yield gains associated with redistribution Scheme C (same yield gain across countries, see Subsection 6.2).

FIGURE XIV.5. Fiscal limit estimates - Sensitivity analysis



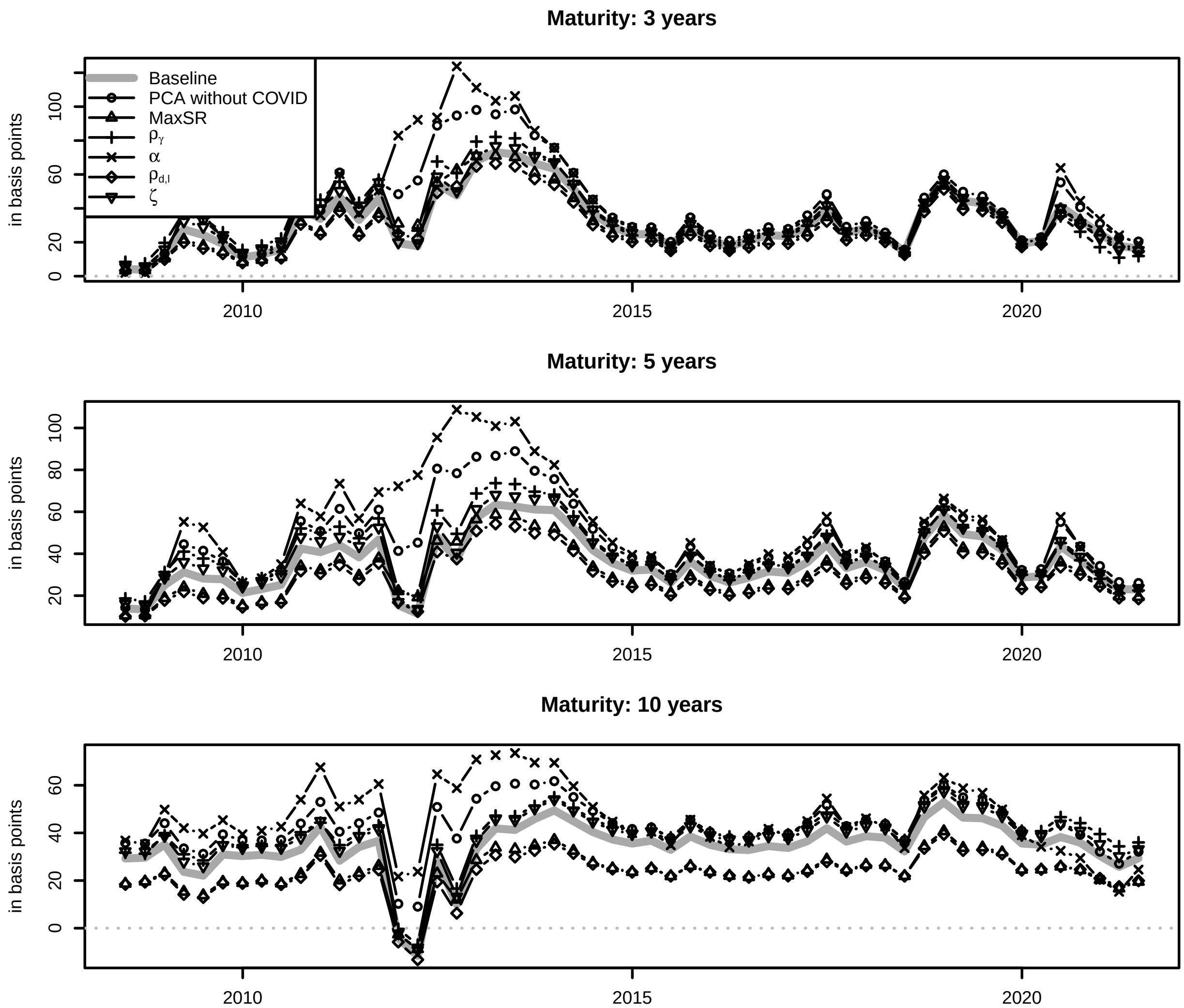
This figure shows the fiscal limit estimates under the baseline parametrization (grey thick solid lines) and across different sensitivity exercises: (i) exclusion of COVID period (after 2020Q1) from the PCA analysis of the estimated $\epsilon_{d,j,t}$'s (black line with circles) so that $\mathbb{V}ar(\varepsilon_{d,t}) = \Gamma_d \Gamma_d'$ is modified (Γ_d represents the matrix of PCA weights, see Subsection 5.3 for details regarding Γ_d.); (ii) the lower bound for $maxSR$ is set to 0.25 (black line with upward-facing triangles), considering that this bound is binding under the baseline model; (iii) the lower bound for ρ_γ is shifted to 0.5, instead of 0.7 (black crossed line), given that the lower bound is binding in the baseline parametrization; (iv) α is set to 0.2 (black line with "x" marks), which corresponds to more than double the estimated value (see Table 3 in Sec. 5); (v) the correlation between the two "main common shocks" ($\eta_{d,1,t}$ and $\eta_{\ell,1,t}$), $\rho_{d,\ell}$, is set to zero (black line with rhombuses); (vi) the upper bound for ζ is shifted from 1.5 to 2 (black line with downward-facing triangles) given that this parameter is constrained at its upper bound under the baseline parametrization (for more details on ζ, see Subsection 5.3). Debt-to-GDP ratios for each country are also plotted (black solid lines). Units are in percent of GDP.

FIGURE XIV.6. SJG bond yield spreads - Sensitivity analysis



This figure shows SJG bond yield spreads for the 3-, 5- and 10-year maturity under the baseline parametrization (grey thick solid lines) and across different sensitivity exercises: (i) exclusion of COVID period (after 2020Q1) from the PCA analysis of the estimated $\epsilon_{d,j,t}$'s (black line with circles) so that $\mathbb{V}ar(\varepsilon_{d,t}) = \Gamma_d \Gamma_d'$ is modified (Γ_d represents the matrix of PCA weights, see Subsection 5.3 for details regarding Γ_d.); (ii) the lower bound for $maxSR$ is set to 0.25 (black line with upward-facing triangles), considering that this bound is binding under the baseline model; (iii) the lower bound for ρ_γ is shifted to 0.5, instead of 0.7 (black crossed line), given that the lower bound is binding in the baseline parametrization; (iv) α is set to 0.2 (black line with "x" marks), which corresponds to more than double the estimated value (see Table 3 in Sec. 5); (v) the correlation between the two "main common shocks" ($\eta_{d,1,t}$ and $\eta_{\ell,1,t}$), $\rho_{d,\ell}$, is set to zero (black line with rhombuses); (vi) the upper bound for ζ is shifted from 1.5 to 2 (black line with downward-facing triangles) given that this parameter is constrained at its upper bound under the baseline parametrization (for more details on ζ, see Subsection 5.3). The figure also reports the GDP-weighted average of the observed yield spreads across maturities (grey dashed lines), which are close to SNJG bond yield spreads (see Figure 5 in Subsection 6.1). Units are in basis points.

FIGURE XIV.7. Yield gains associated with redistribution scheme with same yield gains across countries - Sensitivity analysis



This figure shows yield gains associated with redistribution scheme with same yield gains across countries for the 3-, 5- and 10-year maturity (redistribution scheme C as described in Subsection 6.2) under the baseline parametrization (grey thick solid lines) and across different sensitivity exercises: (i) exclusion of COVID period (after 2020Q1) from the PCA analysis of the estimated $\epsilon_{d,j,t}$'s (black line with circles) so that $\mathbb{V}ar(\varepsilon_{d,t}) = \Gamma_d \Gamma_d'$ is modified (Γ_d represents the matrix of PCA weights, see Subsection 5.3 for details regarding Γ_d.); (ii) the lower bound for $maxSR$ is set to 0.25 (black line with upward-facing triangles), considering that this bound is binding under the baseline model; (iii) the lower bound for ρ_γ is shifted to 0.5, instead of 0.7 (black crossed line), given that the lower bound is binding in the baseline parametrization; (iv) α is set to 0.2 (black line with "x" marks), which corresponds to more than double the estimated value (see Table 3 in Sec. 5); (v) the correlation between the two "main common shocks" ($\eta_{d,1,t}$ and $\eta_{\ell,1,t}$), $\rho_{d,\ell}$, is set to zero (black line with rhombuses); (vi) the upper bound for ζ is shifted from 1.5 to 2 (black line with downward-facing triangles) given that this parameter is constrained at its upper bound under the baseline parametrization (for more details on ζ, see Subsection 5.3). Units are in basis points.