RANDOMIZED OBSERVATION PERIODS FOR THE COMPOUND POISSON RISK MODEL: DIVIDENDS

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Abstract

In the framework of the classical compound Poisson process in collective risk theory, we study a modification of the horizontal dividend barrier strategy by introducing random observation times at which dividends can be paid and ruin can be observed. This model contains both the continuous-time and the discrete-time risk model as a limit and represents a certain type of bridge between them which still enables the explicit calculation of moments of total discounted dividend payments until ruin. Numerical illustrations for several sets of parameters are given and the effect of random observation times on the performance of the dividend strategy is studied.

Keywords: Compound Poisson risk model; horizontal dividend barrier strategy; Erlangization

1 Introduction

In the classical compound Poisson risk model, the surplus process $\{C(t)\}_{t\geq 0}$ of an insurance company is described by

$$C(t) := x + c\,t - S(t) = x + c\,t - \sum_{i=1}^{N(t)} Y_i, \qquad t \geq 0, \tag{1}$$

where $x = C(0) \geq 0$ is the initial surplus level, $\{S(t)\}_{t\geq 0}$ is the aggregate claims process and $c > \mathbb{E}[S(1)]$ is the (constant) premium income per unit time. More specifically, $\{N(t)\}_{t\geq 0}$ is assumed to be a homogeneous Poisson process with rate $\lambda > 0$, and the claim sizes $Y_1, Y_2, \ldots$ form a sequence of independent and identically distributed (i.i.d.) positive random variables (r.v.'s), independent of $\{N(t)\}_{t\geq 0}$, and with generic continuous r.v. Y, c.d.f. $F_Y(\cdot)$, p.d.f. $f_Y(\cdot)$ and Laplace transform $\widetilde{f}_Y(\cdot)$. If $C(t) < 0$ for some $t > 0$, then this event is called ruin of the risk process (see e.g. Asmussen & Albrecher [4] for a recent survey of risk models).

Under the horizontal dividend barrier strategy, any excess of the surplus over a pre-defined barrier level $b \geq 0$ is immediately paid out as dividends to the shareholders of the company as long as ruin has

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not yet occurred for this modified process. The effect of this strategy on the risk process and on the resulting total discounted dividend payments (where the discount rate is usually assumed to be a constant $\delta \geq 0$) is extensively studied in the literature. In particular, it turns out that in certain situations the above model assumptions lead to pleasant and explicit expressions for some quantities of interest, such as the moments of the total discounted dividend payments until ruin (see for instance Dickson & Waters [7] and Gerber & Shiu [9]).

However, in a continuous-time model the horizontal dividend strategy implies a continuous dividend payment stream whenever the surplus process is at level b. In practice, it is more reasonable for the board of the company to check the balance on a periodic basis and then decide whether to pay dividends to the shareholders, resulting in lump sum dividend payments at such discrete time points rather than continuous payment streams. This line of reasoning leads to the study of the horizontal dividend strategy in discrete-time risk models (cf. e.g. Dickson & Waters [7]). But the latter models have the drawback of leading to a (often large) system of linear equations for the quantities of interest. Consequently, this approach usually does not lead to explicit solutions and it is then difficult to gain structural insight in the influence of parameters and to identify optimal choices, such as the optimal barrier level.

In this paper, we want to pursue the idea of only acting at discrete points in time, but at the same time maintaining some of the transparency and elegance of the continuous-time approach. For that purpose we consider the continuous-time compound Poisson risk model (1), but 'look' at the process only at random times $\{Z_k\}_{k=0}^{\infty}$ (called observation times) with $Z_0 = 0$, at which a lump sum dividend payment of size $x - b$ will take place if the current surplus level x exceeds the barrier level b, and the process will be declared ruined if $x < 0$. Note in particular that ruin can now only be observed at these random observation times and so a surplus level below 0 between observation points will only result in actual ruin if it is also negative at the next observation time. The randomness of observation times will allow to carry over some of the properties of the classical continuous-time observation to this discretized version; in particular, $\{C(Z_k)\}_{k\geq 1}$ can be interpreted as a 'new' random walk.

Let $T_k = Z_k - Z_{k-1}$ ($k = 1, 2, \ldots$) be the k-th time interval between observations, and assume that $\{T_k\}_{k=1}^{\infty}$ is an i.i.d. sequence distributed as a generic r.v. T and independent of $\{N(t)\}_{t\geq 0}$ and $\{Y_i\}_{i=1}^{\infty}$. With the above-defined dividend rule with barrier b, denote the sequences of surplus levels at the time points $\{Z_k^-\}_{k=1}^{\infty}$ and $\{Z_k\}_{k=1}^{\infty}$ by $\{U_b(k)\}_{k=1}^{\infty}$ and $\{W_b(k)\}_{k=1}^{\infty}$ respectively, i.e., $\{U_b(k)\}_{k=1}^{\infty}$ and $\{W_b(k)\}_{k=1}^{\infty}$ are the surplus levels at the k-th observation *before* (*after*, respectively) potential dividends are paid. With initial surplus level $W_b(0) = x$ ($0 \leq x \leq b$), we then have the recursive relationship

$$U_b(k) = W_b(k-1) + c\,T_k - [S(Z_k) - S(Z_{k-1})], \quad W_b(k) = \min\{U_b(k), b\}, \qquad k = 1, 2, \ldots.$$

The time of ruin is defined by $\tau_b = Z_{k_b}$, where $k_b = \inf\{k \geq 1 : W_b(k) < 0\}$ is the number of observation intervals before ruin. A sample path under the present model is depicted in Figure 1.

For mathematical tractability, we will assume that the r.v. T is Erlang(n) distributed with density

$$f_T(t) := \frac{\gamma^n\, t^{n-1}\, e^{-\gamma t}}{(n-1)!}, \qquad t > 0$$

and corresponding Laplace transform $\widetilde{f}_T(s) = \int_0^\infty e^{-st} f_T(t)dt = \left(\frac{\gamma}{\gamma+s}\right)^n$, where $\gamma > 0$ is the rate parameter. Note that $n = 1$ refers to exponentially distributed observation intervals (which due to the lack-of-memory property of the exponential distribution reflects the case where the time until the next observation (dividend/ruin decision) does not depend on the time elapsed since the last decision).

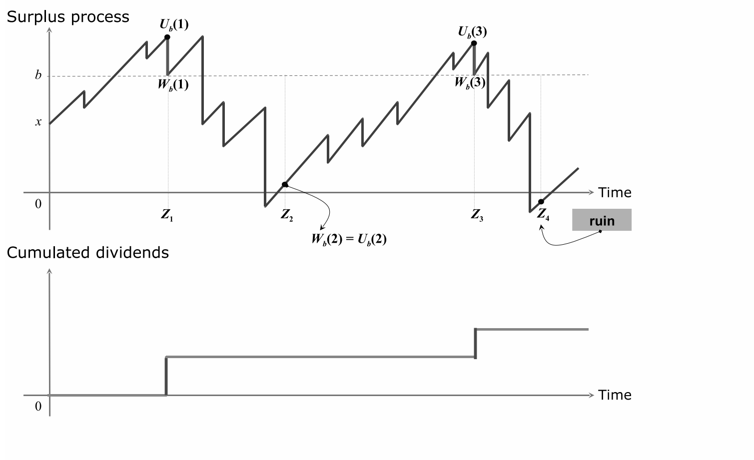

Figure 1: Sample path of a compound Poisson risk model under randomized observations

For any fixed n, the r.v. T converges in distribution to a point mass at 0 for $\gamma \to \infty$, so this limit corresponds to the classical continuous-time risk model with horizontal barrier strategy with barrier at b (i.e. continuous observation of the process and hence continuous decisions on dividends and ruin).

On the other hand, if one fixes $\mathbb{E}[T] = h$ and chooses n sufficiently large, this approximates the discrete-time risk model with time step h (i.e. deterministic observation intervals h), since the Erlang distribution for $n \to \infty$ and fixed expected value $\mathbb{E}[T] = h$ converges in distribution to a point mass in h. This so-called Erlangization technique and its computational advantages were exploited for other purposes (in particular for randomizing a finite time horizon for ruin problems) by Asmussen et al. [5] (see also Ramaswami et al. [13] and Stanford et al. [16, 17]). For statistical inference for continuous-time risk processes with deterministic discrete observation times, see Shimizu [15].

In the companion paper Albrecher et al. [1], we will investigate the expected discounted penalty function (Gerber & Shiu [9]) under random observation times. In the present paper we study the effect of the randomized observation times on the moments of the total discounted dividend payments until ruin for a discount rate $\delta \geq 0$. Let

$$\Delta_{M,\delta}(x;b) := \sum_{k=1}^{k_b} e^{-\delta Z_k}[U_b(k) - b]_+ \Big| W_b(0) = x, \qquad x \in \mathbb{R}. \tag{2}$$

With time 0 an intervention time, the total discounted dividend payments until ruin are represented by the r.v.

$$\Delta_\delta(x;b) := \begin{cases} 0, & x < 0, \\ \Delta_{M,\delta}(x;b), & 0 \leq x \leq b, \\ x - b + \Delta_{M,\delta}(b;b), & x > b. \end{cases}$$

In particular, the distribution of $\Delta_{M,\delta}(x;b)$ for $0 \leq x \leq b$ already determines $\Delta_\delta(x;b)$ for arbitrary x.

Denote the m-th moment of $\Delta_\delta(x;b)$ by

$$V_{m,\delta}(x;b) := \mathbb{E}\left[(\Delta_\delta(x;b))^m\right], \qquad m = 0, 1, 2, \ldots, \tag{3}$$

which is the main quantity of interest in this paper. We adopt the usual convention that $V_{0,\delta}(x;b) \equiv 1$ and shall use the abbreviation $V(x;b) := V_{1,\delta}(x;b)$. The quantities (3) have been studied for the classical compound Poisson model with continuous observation in Dickson & Waters [7].

We present three different approaches to study $V_{m,\delta}(x;b)$ for randomized observation intervals. In Section 2 we start with adapting the generator approach to the present model. If T is exponentially distributed, this leads to a system of integro-differential equations (IDEs) defined on different surplus layers that are connected by certain contact conditions (the resulting analysis has similarities with equations that appear in multi-layer dividend policies of the classical model, see Albrecher & Hartinger [3] and Lin & Sendova [12]). This approach is particularly instructive when analyzing conditions for the optimality of the dividend barrier in this model (see Section 5). In Section 3 the so-called discounted density of increment will be used to derive integral equations for $V_{m,\delta}(x;b)$ which are more tractable for a large class of claim and inter-observation time distributions. This is important in the Erlangization procedure because we would like to increase n gradually in the approximation. As a third alternative, in Section 4 the discounted density of overshoot is used for the analysis. This will lead to a factorization formula which is of independent interest and plays an important role in Section 4.1 when certain classical formulas are generalized. Section 6 gives numerical illustrations that underline the computational advantages of the method for approximating the discrete-time model. Moreover, the effect of random observation times on the quantity $V_{m,\delta}(x;b)$ is discussed.

2 Method 1: Integro-differential equations

Whenever the risk process has a Markovian structure, the classical approach of conditioning on events in a small time interval can be used to derive equations for the quantities of interest. In our context, exponential observation times (i.e. $n = 1$) lead to such a Markovian structure. For Erlang observation times the process can also be made Markovian by increasing the dimension of the state space (see e.g. Albrecher et al. [2] for details), so the method will still work, but in those situations the approaches of Sections 3 and 4 will be simpler to use, as the complexity of the equations increases substantially. For this reason, we will restrict the following derivations to the case of exponential observation times and to the first moment $V(x;b)$ (higher moments $V_{m,\delta}(x;b)$ can be handled analogously, see also Remark 2.2).

Since the conditioning technique exploits the removal of the time stamp, we will need to consider the definition (2) $\Delta_{M,\delta}(x;b)$ for all $x \in \mathbb{R}$, where now time 0 is a priori not an observation time. Note that for $0 \le x \le b$, $\mathbb{E}[\Delta_{M,\delta}(x;b)]$ and $\mathbb{E}[\Delta_\delta(x;b)]$ coincide, because no action needs to be taken at time 0.

In this approach, one has to distinguish between the 'usual' dynamics of the Markovian uncontrolled risk process $\{C(t)\}_{t\ge 0}$ and the occurrence of an observation time at which dividends may be paid out or ruin may be observed. We will see below that this results in an interacting system of IDEs with certain contact conditions. Both the observation time process and the claim number process are now homogeneous Poisson processes, independent of each other.

Consider a time interval $(0,h)$ and distinguish the three cases that either an observation time occurs in this interval before a claim occurs, or a claim occurs before an observation time occurs, or neither a claim nor an observation time occurs until time h. By the Markovian structure we then have

$$
\begin{aligned}
V(x;b) &= e^{-(\lambda+\delta+\gamma)h} V(x+ch;b) \\
&\quad + \int_0^h \gamma e^{-\gamma t} e^{-\lambda t} e^{-\delta t} \Big([x+ct-b+V(b;b)] I_{\{x+ct>b\}} + V(x+ct;b) I_{\{0\le x+ct\le b\}} + 0\, I_{\{x+ct<0\}} \Big)\, dt \\
&\quad + \int_0^h \lambda e^{-\lambda t} e^{-\gamma t} e^{-\delta t} \int_0^\infty V(x+ct-y;b) f_Y(y)\, dy\, dt. \qquad (4)
\end{aligned}
$$

Here $I_{\{A\}}$ stands for the indicator function of the event A. Note again that before the first observation time Z_1 the process can become negative without leading to ruin, because ruin can only be observed at observation times. In addition, it is clear that $V(\cdot;b)$ is bounded by a linear function, hence (by letting $h \to 0$) one sees that $V(x;b)$ is continuous in x. One can now differentiate (4) with respect to h, and by taking the limit $h \to 0$ we arrive at the following system of IDEs:

$$0 = c\,\frac{d}{dx} V(x;b) - (\lambda+\gamma+\delta) V(x;b) + \lambda \int_0^\infty V(x-y;b) f_Y(y)\, dy, \qquad x<0, \qquad (5)$$

$$0 = c\,\frac{d}{dx} V(x;b) - (\lambda+\delta) V(x;b) + \lambda \int_0^\infty V(x-y;b) f_Y(y)\, dy, \qquad 0 \le x < b, \qquad (6)$$

$$0 = c\,\frac{d}{dx} V(x;b) - (\lambda+\gamma+\delta) V(x;b) + \lambda \int_0^\infty V(x-y;b) f_Y(y)\, dy + \gamma[x-b+V(b;b)], \qquad x \ge b. \qquad (7)$$

Within each of these three layers, $V(x;b)$ is indeed differentiable with respect to x, and upon comparison of (6) and (7), the continuity of $V(x;b)$ at $x=b$ also implies differentiability of $V(x;b)$ at $x=b$, i.e.

$$\frac{d}{dx} V(x;b)\Big|_{x=b-} = \frac{d}{dx} V(x;b)\Big|_{x=b+}. \qquad (8)$$

Analogously, one observes that $V(x;b)$ is not differentiable at $x=0$, as

$$c\,\frac{d}{dx} V(x;b)\Big|_{x=0-} = c\,\frac{d}{dx} V(x;b)\Big|_{x=0+} + \gamma V(0;b). \qquad (9)$$

For clarity of exposition, write now $V(x;b)$ as

$$V(x;b) := \begin{cases} V_L(x;b), & x<0, \\ V_M(x;b), & 0 \le x \le b, \\ V_U(x;b), & x>b, \end{cases}$$

where the subscripts 'L', 'M' and 'U' stand for 'lower', 'middle' and 'upper' layer respectively. Then

$$0 = c\,\frac{d}{dx} V_L(x;b) - (\lambda+\gamma+\delta) V_L(x;b) + \lambda \int_0^\infty V_L(x-y;b) f_Y(y)\, dy, \qquad x<0, \qquad (10)$$

$$0 = c\,\frac{d}{dx} V_M(x;b) - (\lambda+\delta) V_M(x;b) + \lambda \int_0^x V_M(x-y;b) f_Y(y)\, dy + \lambda \int_x^\infty V_L(x-y;b) f_Y(y)\, dy, \qquad 0 \le x < b, \qquad (11)$$

$$
\begin{aligned}
0 &= c\,\frac{d}{dx} V_U(x;b) - (\lambda+\gamma+\delta) V_U(x;b) + \lambda \int_0^{x-b} V_U(x-y;b) f_Y(y)\, dy \\
&\quad + \lambda \int_{x-b}^x V_M(x-y;b) f_Y(y)\, dy + \lambda \int_x^\infty V_L(x-y;b) f_Y(y)\, dy + \gamma[x-b+V_U(b;b)], \qquad x \ge b. \qquad (12)
\end{aligned}
$$

For a complete characterization of the solution of the above system of IDEs, one can use the continuity of $V(x;b)$ at $x=0$ and $x=b$. Furthermore, the linear boundedness and positivity of $V(x;b)$ for $x\in\mathbb{R}$ as well as the natural boundary condition $\lim_{x\to-\infty} V_L(x;b)=0$ can be employed (note that the derivative conditions (8) and (9) are consequences of the continuity in $x=0$ and $x=b$ and hence do not give extra information).

A crucial equation (in α) for this risk model turns out to be

$$\kappa(\alpha)=\delta+\gamma, \tag{13}$$

where $\kappa(\alpha):=\lambda[\widetilde{f}_Y(-\alpha)-1]-c\,\alpha$. It has a unique negative solution $-\rho_\gamma<0$. In addition, under a light-tailed assumption on the claim size distribution, it also has a positive solution $R_\gamma>0$ in the domain of convergence of $\widetilde{f}_Y(\cdot)$ (cf. Figure 2). Note that for $\gamma=0$, (13) reduces to the well-known Lundberg fundamental equation of the compound Poisson risk process.

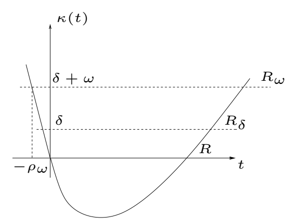

Figure 2: Roots of Equation (13)

2.1 Constructing a solution - the exponential claim case

We now illustrate how the above system of IDEs can be solved for exponentially distributed claim amounts with density $f_Y(y)=\nu e^{-\nu y}$ for $y>0$. We proceed by applying the operator $(d/dx+\nu)$ to (10), (11) and (12) respectively. First, for the lower layer $x<0$, the procedure reveals that $V_L(x;b)$ satisfies a second order homogeneous differential equation in x with constant coefficients and characteristic equation (in ξ)

$$\xi^2+\left(\nu-\frac{\lambda+\gamma+\delta}{c}\right)\xi-\frac{(\gamma+\delta)\nu}{c}=0\ . \tag{14}$$

The roots of the above equation are the negative of those of (13). Hence, the solution of (10) is of the form

$$V_L(x;b)=C_1e^{\rho_\gamma x}+C_2e^{-R_\gamma x},\qquad x\le 0 \tag{15}$$

for some constants C_1, C_2. Due to $\lim_{x\to-\infty} V(x;b)=0$, one immediately deduces $C_2=0$.

For the middle layer $0\le x<b$, one accordingly obtains the same homogeneous differential equation for $V_M(x;b)$, but with $\gamma=0$. Hence

$$V_M(x;b)=A_1e^{\rho_0 x}+A_2e^{-R_0 x},\qquad 0\le x\le b, \tag{16}$$

where the constants A_1, A_2 are still to be determined.

For the upper layer $x\ge b$, the same procedure results in a second-order differential equation in x for

$V_U(x;b)$ with constant coefficients and characteristic equation (14), but with a non-homogeneous term that is linear in x. Hence

$$V_U(x;b) = D_1 e^{\rho_\gamma x} + D_2 e^{-R_\gamma x} + D_3 x + D_4, \qquad x \geq b \tag{17}$$

for constants $D_1, \ldots, D_4$. From the linear boundedness of $V(x;b)$ it immediately follows that $D_1 = 0$.

For the determination of the remaining constants, the solutions (15), (16) and (17) are substituted into the IDE's (10), (11) and (12). First, (10) does not yield any information. For (11), equating coefficients of $e^{-\nu x}$ leads to

$$A_1 \frac{1}{\nu + \rho_0} + A_2 \frac{1}{\nu - R_0} - C_1 \frac{1}{\nu + \rho_\gamma} = 0. \tag{18}$$

As for (12), equating the coefficients of x yields

$$D_3 = \frac{\gamma}{\gamma + \delta}. \tag{19}$$

With D_3 determined, by equating the coefficients of $e^{-\nu x}$ along with the use of (18), we arrive at

$$A_1 \frac{\nu}{\nu + \rho_0} e^{\rho_0 b} + A_2 \frac{\nu}{\nu - R_0} e^{-R_0 b} - D_2 \frac{\nu}{\nu - R_\gamma} e^{-R_\gamma b} - D_4 = \frac{\gamma}{\gamma + \delta} \left(b - \frac{1}{\nu} \right), \tag{20}$$

while equating the constant term results in

$$\gamma e^{-R_\gamma b} D_2 - \delta D_4 = \frac{\gamma}{\gamma + \delta} \left(b\,\delta - c + \frac{\lambda}{\nu} \right). \tag{21}$$

In addition, the continuity of $V(x;b)$ at $x = 0$ and $x = b$ leads to the two further equations

$$A_1 + A_2 - C_1 = 0\,, \tag{22}$$

$$A_1 e^{\rho_0 b} + A_2 e^{-R_0 b} - D_2 e^{-R_\gamma b} - D_4 = \frac{\gamma\, b}{\gamma + \delta}\,. \tag{23}$$

Therefore, we now have a system of the five linear equations (18), (20), (21), (22) and (23) for the five remaining constants A_1, A_2, C_1, D_2 and D_4. This finally gives, after some elementary algebra and using equation (14),

$$V_L(x;b) = \frac{(\rho_0 + R_0)\, e^{\rho_\gamma x}}{\frac{R_\gamma + \rho_0}{1 - \rho_0/\rho_\gamma} \rho_0 e^{\rho_0 b} + \frac{R_\gamma - R_0}{1 + R_0/\rho_\gamma} R_0 e^{-R_0 b}}, \qquad x \leq 0,$$

$$V_M(x;b) = \frac{(R_\gamma + \rho_0) e^{\rho_0 x} - (R_\gamma - R_0) e^{-R_0 x}}{\frac{R_\gamma + \rho_0}{1 - \rho_0/\rho_\gamma} \rho_0 e^{\rho_0 b} + \frac{R_\gamma - R_0}{1 + R_0/\rho_\gamma} R_0 e^{-R_0 b}}, \qquad 0 \leq x \leq b. \tag{24}$$

The result for $V_U(x;b)$ is also explicit:

$$V_U(x;b) = \frac{\gamma (x - b)}{\gamma + \delta} + \frac{1}{R_\gamma} (e^{-R_\gamma (x-b)} - 1) \left(\frac{\gamma}{\gamma + \delta} - \frac{d}{dx} V_M(x;b) \Big|_{x=b} \right) + V_M(b;b), \qquad x \geq b,$$

where $V_M(b;b)$ and $\frac{d}{dx} V_M(x;b)\big|_{x=b}$ can be determined from (24).

Remark 2.1 The crucial result above is formula (24), which gives $V(x;b) = \mathbb{E}[\Delta_{M,\delta}(x;b)] = \mathbb{E}[\Delta_\delta(x;b)]$, since for $0 \le x \le b$ there is no action at time 0 regardless of whether or not it is an observation time. However, even if one would eventually only be interested in this middle layer, the consideration of all three interacting layers was necessary to determine the involved coefficients in the present approach. Note that (24) is expressed solely through the roots of equation (14) for different values of γ. Furthermore, it is 'almost' of the form $h(u)/h'(b)$ for some function $h(\cdot)$, which is the known form of $V(x;b)$ for a general class of Markov processes that are skip-free upwards (see for instance Gerber et al. [8]). Formula (24) can hence also be seen as an adaptation of such a form for a model with certain types of upward jumps, in view of the random walk $C(Z_k)$ ($k \in \mathbb{N}$) with state space $\mathbb{R}$. See also Remark 4.4.
From Figure 2 it is easily seen that for $\gamma \to \infty$ we have $\rho_\gamma \to +\infty$ and $R_\gamma \to \nu$ so that (24) tends to

$$V(x;b) = \frac{(\nu+\rho_0)e^{\rho_0 x} - (\nu - R_0)e^{-R_0 x}}{(\nu+\rho_0)\rho_0 e^{\rho_0 b} + (\nu - R_0)R_0 e^{-R_0 b}}, \qquad 0 \le x \le b,$$

which is indeed the corresponding formula for the classical continuous-time risk model (see for instance Gerber & Shiu [9, Eqn.(7.8)]).

Remark 2.2 In principle, the method presented in this section extends to the case of Erlang(n) observation intervals, to more general claim size distributions as well as to the determination of higher moments $V_{m,\delta}(x,b)$. However, this will typically lead to considerable computational effort, as one has to keep track of all three layers for each of the n exponential stages. In particular, $3n$ IDEs will have to be solved simultaneously and the complexity of these equations will further increase with the order m of the dividend moments as well as the claim size distribution. In Section 3 we will investigate an alternative approach that allows to avoid these difficulties.

3 Method 2: Discounted density of increment $g_\delta(y)$

We now follow another approach based on the increment of the uncontrolled process $\{C(t)\}_{t\ge 0}$ between successive observation intervals, exploiting the random walk structure of $\{C(Z_k)\}_{k\ge 1}$. This will simplify the analysis to some extent. In this setting, time 0 is a first observation point, so we can now directly work with definition (3).

Suppose we want to keep track of both the length of the interval $T_k = Z_k - Z_{k-1}$ and the change in the surplus between time Z_{k-1} and Z_k^- ($k = 1, 2, \ldots$). Due to the Markovian structure of $\{C(t)\}_{t\ge 0}$, this sequence of pairs is i.i.d. with generic distribution $(T, \sum_{i=1}^{N(T)} Y_i - cT)$ and joint Laplace transform

$$\mathbb{E}\left[e^{-\delta T - s\left(\sum_{i=1}^{N(T)} Y_i - cT\right)}\right] = \mathbb{E}\left[e^{-(\delta - cs)T}\mathbb{E}\left[e^{-s\sum_{i=1}^{N(T)} Y_i}\middle| T\right]\right] = \mathbb{E}\left[e^{-[\lambda+\delta-cs-\lambda\widetilde{f}_Y(s)]T}\right]. \tag{25}$$

On the other hand, one can also write

$$\mathbb{E}\left[e^{-\delta T - s\left(\sum_{i=1}^{N(T)} Y_i - cT\right)}\right] = \int_{-\infty}^{\infty} e^{-sy} g_\delta(y)\, dy, \tag{26}$$

where $g_\delta(y)$ ($-\infty < y < \infty$) represents the discounted density of the increment $\sum_{i=1}^{N(T)} Y_i - cT$ between successive observation times, discounted at rate δ with respect to time T. This quantity will be particularly useful in the sequel.

3.1 Exponential claim sizes and exponential observation times

Let us first again return to the case that Y and T are both exponentially distributed with mean $1/\nu$ and $1/\gamma$, respectively. Then (25) becomes

$$\mathbb{E}\left[e^{-\delta T - s\left(\sum_{i=1}^{N(T)} Y_i - cT\right)}\right] = \frac{\gamma}{\gamma + \lambda + \delta - cs - \lambda\frac{\nu}{\nu+s}},$$

which by the use of partial fractions can be written as

$$\mathbb{E}\left[e^{-\delta T - s\left(\sum_{i=1}^{N(T)} Y_i - cT\right)}\right] = \frac{\gamma(\nu+\rho_\gamma)}{c\rho_\gamma(\rho_\gamma+R_\gamma)}\left(\frac{\rho_\gamma}{\rho_\gamma - s}\right) + \frac{\gamma(\nu-R_\gamma)}{cR_\gamma(\rho_\gamma+R_\gamma)}\left(\frac{R_\gamma}{R_\gamma+s}\right). \tag{27}$$

Comparing (26) and (27), it is then clear that

$$g_\delta(y) = \frac{\gamma(\nu+\rho_\gamma)}{c\rho_\gamma(\rho_\gamma+R_\gamma)}\rho_\gamma e^{\rho_\gamma y} I_{\{y<0\}} + \frac{\gamma(\nu-R_\gamma)}{cR_\gamma(\rho_\gamma+R_\gamma)} R_\gamma e^{-R_\gamma y} I_{\{y>0\}}, \qquad -\infty < y < \infty$$

is a two-sided exponential density which is defective when $\delta > 0$. We can now condition on the pair $(T_1, \sum_{i=1}^{N(T_1)} Y_i - cT_1)$ to arrive at

$$\begin{aligned} V(x;b) = &\int_{b-x}^{\infty} \frac{\gamma(\nu+\rho_\gamma)}{c\rho_\gamma(\rho_\gamma+R_\gamma)}\rho_\gamma e^{-\rho_\gamma y}[y-(b-x)+V(b;b)]\,dy + \int_0^{b-x} \frac{\gamma(\nu+\rho_\gamma)}{c\rho_\gamma(\rho_\gamma+R_\gamma)}\rho_\gamma e^{-\rho_\gamma y} V(x+y;b)\,dy \\ &+ \int_0^x \frac{\gamma(\nu-R_\gamma)}{cR_\gamma(\rho_\gamma+R_\gamma)} R_\gamma e^{-R_\gamma y} V(x-y;b)\,dy, \qquad 0 \le x \le b. \end{aligned} \tag{28}$$

While the third integral term in (28) is a standard convolution, the first two integrals resemble those arising in derivations for the dual risk model under a dividend barrier (see e.g. Avanzi et al. [6, Eqns. (2.3) and (3.1)]).

Applying the operator $(d/dx - \rho_\gamma)(d/dx + R_\gamma)$ on both sides, one can transform (28) into a second-order homogeneous differential equation in x for $V(x;b)$ with constant coefficients which has a solution of the form

$$V(x;b) = A_1 e^{\alpha_1 x} + A_2 e^{\alpha_2 x}, \qquad 0 \le x \le b. \tag{29}$$

The constants A_1, A_2 and α_1, α_2 still have to be determined. Substituting (29) into (28) and matching the coefficients of the various exponential terms, one obtains the equations

$$\frac{\gamma(\nu+\rho_\gamma)}{c(\rho_\gamma+R_\gamma)}\frac{1}{\rho_\gamma-\alpha_i} + \frac{\gamma(\nu-R_\gamma)}{c(\rho_\gamma+R_\gamma)}\frac{1}{R_\gamma+\alpha_i} = 1, \qquad i = 1,2, \tag{30}$$

$$A_1\frac{\alpha_1 e^{\alpha_1 b}}{\rho_\gamma-\alpha_1} + A_2\frac{\alpha_2 e^{\alpha_2 b}}{\rho_\gamma-\alpha_2} = \frac{1}{\rho_\gamma}, \tag{31}$$

$$A_1\frac{1}{R_\gamma+\alpha_1} + A_2\frac{1}{R_\gamma+\alpha_2} = 0. \tag{32}$$

Equation (30) implies that α_1, α_2 are the roots (in ξ) of the quadratic equation

$$\rho_\gamma R_\gamma + (\rho_\gamma - R_\gamma)\xi - \xi^2 = \frac{\gamma\nu}{c} + \frac{\gamma}{c}\xi. \tag{33}$$

Since ρ_γ and $-R_\gamma$ are roots to the quadratic equation (14), by Vieta's rule they satisfy

$$\rho_\gamma R_\gamma = \frac{(\gamma+\delta)\nu}{c} \qquad \text{and} \qquad \rho_\gamma - R_\gamma = \frac{\lambda+\gamma+\delta}{c} - \nu.$$

Applying these relationships to (33), one verifies that indeed $\alpha_1 = \rho_0$ and $\alpha_2 = -R_0$. In particular, α_1 and α_2 are independent of γ. The constants A_1 and A_2 now follow from the system of the two linear equations (31) and (32) and one finally again obtains (24).

The use of the discounted density $g_\delta(y)$ turned out to simplify the analysis as compared to the IDE approach of Section 2, as only the middle layer needs to be considered. In the next subsection we extend this method to higher dividend moments for Erlang(n) observation intervals and claim sizes with rational Laplace transform.

3.2 Higher moments of discounted dividends and Erlang observation times

When the observation interval T is Erlang(n) distributed and the claim size Y has an arbitrary distribution, the joint Laplace transform (25) has the representation

$$\mathbb{E}\left[e^{-\delta T - s\left(\sum_{i=1}^{N(T)} Y_i - cT\right)}\right] = \left(\frac{\gamma}{\gamma + \lambda[1-\widetilde{f}_Y(s)] + (\delta - cs)}\right)^n. \tag{34}$$

The zeros of the denominator inside the bracket on the right-hand side, namely the roots of the equation (in ξ)

$$c\xi - (\lambda+\gamma+\delta) + \lambda\widetilde{f}_Y(\xi) = 0, \tag{35}$$

are the negative of those of equation (13). In particular, there is a unique positive root $\rho_\gamma > 0$.

Recall (26) in connection with (34). Using the notation

$$g_\delta(y) = g_{\delta,-}(-y)I_{\{y<0\}} + g_{\delta,+}(y)I_{\{y>0\}}, \qquad -\infty < y < \infty, \tag{36}$$

along the same line of arguments as in Section 3.1, an integral equation for the m-th moment of discounted dividend payments can be obtained as

$$\begin{aligned} V_{m,\delta}(x;b) = &\sum_{k=0}^{m}\binom{m}{k}\int_{b-x}^{\infty}[y-(b-x)]^{m-k}V_{k,\delta}(b;b)g_{m\delta,-}(y)\,dy + \int_0^{b-x}V_{m,\delta}(x+y;b)g_{m\delta,-}(y)\,dy \\ &+ \int_0^x V_{m,\delta}(x-y;b)g_{m\delta,+}(y)\,dy, \qquad 0 \le x \le b \end{aligned} \tag{37}$$

for $m = 1, 2, \ldots$. Here the quantities $g_{m\delta,-}(\cdot)$ and $g_{m\delta,+}(\cdot)$ refer to a discount rate $m\delta$ instead of δ.

Remark 3.1 Also in this approach one can obtain the moments of $\Delta_{M,\delta}(x;b)$ (for which time 0 is not an observation time). The corresponding adaptations lead to

$$V_{m,\delta}(x;b) = \sum_{k=0}^{m}\binom{m}{k}\int_{b-x}^{\infty}[y-(b-x)]^{m-k}V_{k,\delta}(b;b)g_{m\delta,-}(y)\,dy + \int_{-x}^{b-x}V_{m,\delta}(x+y;b)g_{m\delta,-}(y)\,dy, \qquad x<0,$$

$$V_{m,\delta}(x;b) = \sum_{k=0}^{m}\binom{m}{k}\left(\int_0^\infty [y-(b-x)]^{m-k}V_{k,\delta}(b;b)g_{m\delta,-}(y)\,dy \right.$$
$$\left. + \int_0^{x-b}[(x-b)-y]^{m-k}V_{k,\delta}(b;b)g_{m\delta,+}(y)\,dy\right) + \int_{x-b}^{x} V_{m,\delta}(x-y;b)g_{m\delta,+}(y)\,dy, \quad x > b$$

for $m = 1, 2, \ldots$. For $0 \leq x \leq b$, the expression obviously coincides with the one where time 0 is an observation time. Hence the solution of (37) also leads to the formulas for $x < 0$ and $x > b$ for the moments of $\Delta_{M,\delta}(x;b)$.

The quantities $g_{\delta,-}(\cdot)$ and $g_{\delta,+}(\cdot)$ will not always have a tractable form, but if $f_Y(\cdot)$ has a rational Laplace transform, i.e.

$$\widetilde{f}_Y(s) = \frac{Q_{2,r-1}(s)}{Q_{1,r}(s)}, \tag{38}$$

where $Q_{1,r}(s)$ is a polynomial in s of degree exactly r with leading coefficient of 1 and $Q_{2,r-1}(s)$ is a polynomial in s of degree at most $r-1$ (and the two polynomials have distinct zeros), then it is shown in [1] that

$$g_{\delta,-}(y) = \sum_{j=1}^{n} B_j^* \frac{y^{j-1}e^{-\rho_\gamma y}}{(j-1)!} \qquad \text{and} \qquad g_{\delta,+}(y) = \sum_{i=1}^{r}\sum_{j=1}^{n} B_{ij}\frac{y^{j-1}e^{-R_{\gamma,i}y}}{(j-1)!}, \tag{39}$$

where $-R_{\gamma,1}, \ldots, -R_{\gamma,r}$ are the r roots of equation (35) with negative real parts (with $\widetilde{f}_Y(\cdot)$ analytically extended beyond the abscissa of convergence), and the constants B_j^* and B_{ij} are given by

$$B_j^* = (-1)^{n-j}\left(\frac{\gamma}{c}\right)^n \frac{1}{(n-j)!}\frac{d^{n-j}}{ds^{n-j}}\frac{[Q_{1,r}(s)]^n}{\prod_{l=1}^{r}(s+R_{\gamma,l})^n}\Bigg|_{s=\rho_\gamma}, \qquad j = 1, 2, \ldots, n, \tag{40}$$

and

$$B_{ij} = \left(\frac{\gamma}{c}\right)^n \frac{1}{(n-j)!}\frac{d^{n-j}}{ds^{n-j}}\frac{[Q_{1,r}(s)]^n}{(\rho_\gamma - s)^n\prod_{l=1,l\neq i}^{r}(s+R_{\gamma,l})^n}\Bigg|_{s=-R_{\gamma,i}}, \qquad i = 1, 2, \ldots, r; \;\; j = 1, 2, \ldots, n. \tag{41}$$

Since the above quantities depend on δ, we write $\rho_{\gamma,m}$, $R_{\gamma,i,m}$, $B^*_{j,m}$ and $B_{ij,m}$ if δ is replaced by $m\delta$.

If one now applies the operator $(d/dx - \rho_{\gamma,m})^n \prod_{i=1}^{r}(d/dx + R_{\gamma,i,m})^n$ to both sides of (37), it can be seen that $V_{m,\delta}(x;b)$ satisfies a homogeneous differential equation of order $n(r+1)$ in x with constant coefficients and a solution of the form

$$V_{m,\delta}(x;b) = \sum_{i=1}^{n(r+1)} A_{i,m}e^{\alpha_{i,m}x}\;, \quad 0 \leq x \leq b\;, \tag{42}$$

for constants $\{A_{i,m}\}_{i=1}^{n(r+1)}$ and $\{\alpha_{i,m}\}_{i=1}^{n(r+1)}$. We directly substitute (42) and the densities (39) into the integral equation (37) and perform some straightforward but tedious calculations. Omitting the details, the first integral on the right-hand side of (37) is evaluated as

$$\sum_{k=0}^{m}\binom{m}{k}\int_{b-x}^{\infty}[y-(b-x)]^{m-k}V_{k,\delta}(b;b)g_{m\delta,-}(y)\,dy$$
$$= \sum_{i=1}^{n}\left[\sum_{k=0}^{m}\binom{m}{k}V_{k,\delta}(b;b)\sum_{j=i}^{n}B^*_{j,m}\sum_{l=i}^{j}\frac{1}{\rho_{\gamma,m}^{m-k+j-l+1}}\frac{(m-k+j-l)!}{(j-l)!(l-i)!}b^l\right]\frac{(-1)^{i-1}}{(i-1)!}b^{-i}e^{-\rho_{\gamma,m}b}x^{i-1}e^{\rho_{\gamma,m}x}. \tag{43}$$

Similarly, the second integral in (37) is

$$\begin{aligned}
&\int_0^{b-x} V_{m,\delta}(x+y;b) g_{m\delta,-}(y)\,dy \\
&= \sum_{i=1}^{n(r+1)} A_{i,m} \left(\sum_{j=1}^{n} \frac{B^*_{j,m}}{(\rho_{\gamma,m}-\alpha_{i,m})^j} \right) e^{\alpha_{i,m}x} \\
&\quad - \sum_{i=1}^{n} \left(\sum_{p=1}^{n(r+1)} A_{p,m} \sum_{j=i}^{n} B^*_{j,m} \sum_{l=i}^{j} \frac{1}{(\rho_{\gamma,m}-\alpha_{p,m})^{j-l+1}} \frac{1}{(l-i)!} b^l e^{\alpha_{p,m}b} \right) \frac{(-1)^{i-1}}{(i-1)!} b^{-i} e^{-\rho_{\gamma,m}b} x^{i-1} e^{\rho_{\gamma,m}x},
\end{aligned} \tag{44}$$

whereas the third integral is given by

$$\begin{aligned}
&\int_0^{x} V_{m,\delta}(x-y;b) g_{m\delta,+}(y)\,dy \\
&= \sum_{i=1}^{n(r+1)} A_{i,m} \left(\sum_{k=1}^{r} \sum_{j=1}^{n} \frac{B_{kj,m}}{(R_{\gamma,k,m}+\alpha_{i,m})^j} \right) e^{\alpha_{i,m}x} \\
&\quad - \sum_{k=1}^{r} \sum_{i=1}^{n} \left(\sum_{p=1}^{n(r+1)} A_{p,m} \sum_{j=i}^{n} \frac{B_{kj,m}}{(R_{\gamma,k,m}+\alpha_{p,m})^j} \right) \frac{(R_{\gamma,k,m}+\alpha_{p,m})^{i-1}}{(i-1)!} x^{i-1} e^{-R_{\gamma,k,m}x}.
\end{aligned} \tag{45}$$

Incorporating (43), (44) and (45) into (37) and equating the coefficients of $x^{i-1}e^{\rho_{\gamma,m}x}$ yields

$$\begin{aligned}
&\sum_{p=1}^{n(r+1)} A_{p,m} \sum_{j=i}^{n} B^*_{j,m} \sum_{l=i}^{j} \left(\frac{1}{(\rho_{\gamma,m}-\alpha_{p,m})^{j-l+1}} - \frac{1}{\rho_{\gamma,m}^{j-l+1}} \right) \frac{1}{(l-i)!} b^l e^{\alpha_{p,m}b} \\
&\qquad = \sum_{k=0}^{m-1} \binom{m}{k} V_{k,\delta}(b;b) \sum_{j=i}^{n} B^*_{j,m} \sum_{l=i}^{j} \frac{1}{\rho_{\gamma,m}^{m-k+j-l+1}} \frac{(m-k+j-l)!}{(j-l)!(l-i)!} b^l, \qquad i=1,2,\ldots,n.
\end{aligned} \tag{46}$$

Note that we have separated the term $V_{m,\delta}(b;b)$ and used (42) at $x=b$ in obtaining the above equation. Similarly, by equating the coefficients of $e^{\alpha_{i,m}x}$ we arrive at

$$\sum_{j=1}^{n} \frac{B^*_{j,m}}{(\rho_{\gamma,m}-\alpha_{i,m})^j} + \sum_{k=1}^{r} \sum_{j=1}^{n} \frac{B_{kj,m}}{(R_{\gamma,k,m}+\alpha_{i,m})^j} = 1, \qquad i=1,2,\ldots,n(r+1).$$

Due to the representation (26) and the form of the densities (39), the above equation implies that for each fixed $m=1,2,\ldots$, $\{\alpha_{i,m}\}_{i=1}^{n(r+1)}$ are roots of the equation (in ξ)

$$\mathbb{E}\left[e^{-m\delta T - \xi\left(\sum_{i=1}^{N(T)} Y_i - cT\right)} \right] = 1, \tag{47}$$

which is the Lundberg fundamental equation of the present compound Poisson risk model under Erlang(n) observation intervals (this is natural in view of the embedded random walk structure of the uncontrolled process $\{C(t)\}_{t\geq 0}$ observed at discrete time points).

Finally, equating the coefficients of $x^{i-1}e^{-R_{\gamma,k,m}x}$ gives

$$\sum_{p=1}^{n(r+1)} A_{p,m} \sum_{j=i}^{n} \frac{B_{kj,m}}{(R_{\gamma,k,m}+\alpha_{p,m})^j} = 0, \qquad k = 1,2,\ldots,r; \;\; i = 1,2,\ldots,n. \tag{48}$$

Hence, for each fixed $m \in \mathbb{N}$, $\{\alpha_{i,m}\}_{i=1}^{n(r+1)}$ are obtained as the roots of (47), and $\{A_{i,m}\}_{i=1}^{n(r+1)}$ are the solutions of the system of $n(r+1)$ linear equations (46) and (48). Then a complete characterization of $V_{m,\delta}(x;b)$ is given by (42). In view of (46) this procedure is recursive in m.

Remark 3.2 From the Lundberg equation (47) along with the representation (34), one observes that the roots $\{\alpha_{i,m}\}_{i=1}^{n(r+1)}$ are independent of γ when claims have rational Laplace transform, as long as the observation intervals remain exponential (i.e. $n = 1$). Indeed, they are the roots of (35) with $\gamma = 0$ (and $m\delta$ instead of δ) and are hence the negative of the roots of (13) with $\gamma = 0$ (also with $m\delta$ in place of δ). However, for arbitrary Erlang(n) observation intervals, $\{\alpha_{i,m}\}_{i=1}^{n(r+1)}$ will in general not be independent of the value of γ.

4 Method 3: Discounted density of overshoot $h_\delta(y|x;b)$

We now present yet another, although related approach to analyze this model. This method is based on the fact that, from any present surplus level, further dividends can only be collected if the uncontrolled process overshoots level b before it becomes negative at an observation time. As we shall see towards the end of the section, this method leads to expressions from which certain classical results can be retrieved as special cases. These include the Laplace transform of a two-sided upper exit time and the expected discounted dividends paid until ruin.

Assume that time 0 is the first observation time and let $k_b^* = \min\{k \geq 1 : U_b(k) > b\}$ be the number of observation intervals before the first overshoot of the process $\{U_b(k)\}_{k=1}^{\infty}$ over level b. Clearly $\tau_b^* = Z_{k_b^*}$ is the first time a dividend payment is made as long as $\tau_b^* < \tau_b$ (i.e. ruin has not occurred yet). In the spirit of Gerber & Shiu [9], suppose a 'penalty function' $w^*(\cdot)$ is applied to the first overshoot of $\{U_b(k)\}_{k=1}^{\infty}$ over level b avoiding ruin until then and define the quantity

$$\chi_\delta(x;b) = \mathbb{E}\left[e^{-\delta\tau_b^*} w^*\big(U_b(k_b^*) - b\big) I_{\{\tau_b^* < \tau_b\}} \big| W_b(0) = x\right], \qquad 0 \leq x \leq b. \tag{49}$$

Recall the discounted density $g_\delta(y)$ from (26) in its decomposed form (36). Akin to the derivation of (28), we have by conditioning

$$\chi_\delta(x;b) = \int_{b-x}^{\infty} w^*(y-(b-x)) g_{\delta,-}(y)\,dy + \int_0^{b-x} \chi_\delta(x+y;b) g_{\delta,-}(y)\,dy + \int_0^{x} \chi_\delta(x-y;b) g_{\delta,+}(y)\,dy,$$
$$0 \leq x \leq b. \tag{50}$$

In Section 4.1 we will solve this integral equation for claim sizes with rational Laplace transform along the same lines as the one for $V_{m,\delta}(x;b)$ in Section 3.2 was dealt with. We shall now first show how the quantity $\chi_\delta(x;b)$ can be used to study the dividend moment function $V_{m,\delta}(x;b)$.

Suppose in general that $\chi_\delta(x;b)$ can be expressed in the form

$$\chi_\delta(x;b) = \int_0^\infty w^*(y) h_\delta(y|x;b)\, dy, \qquad 0 \le x \le b, \tag{51}$$

where $h_\delta(y|x;b)$ is the discounted density of the overshoot over level b avoiding ruin. Then by conditioning on such a first overshoot, one arrives at

$$V_{m,\delta}(x;b) = \sum_{k=0}^{m} \binom{m}{k} V_{k,\delta}(b;b) \int_0^\infty y^{m-k} h_{m\delta}(y|x;b)\, dy, \qquad 0 \le x \le b. \tag{52}$$

Moreover, for $m = 1$ this simplifies to

$$V(x;b) = \int_0^\infty [y + V(b;b)]\, h_\delta(y|x;b)\, dy, \qquad 0 \le x \le b. \tag{53}$$

Putting $x = b$ in (52) and solving for $V_{m,\delta}(b;b)$ yields

$$V_{m,\delta}(b;b) = \frac{1}{1 - \int_0^\infty h_{m\delta}(y|b;b)\, dy} \sum_{k=0}^{m-1} \binom{m}{k} V_{k,\delta}(b;b) \int_0^\infty y^{m-k} h_{m\delta}(y|b;b)\, dy. \tag{54}$$

Thus, (54) is a recursive formula to evaluate $V_{m,\delta}(b;b)$ for all m, and then $V_{m,\delta}(x;b)$ for $0 \le x < b$ can be obtained via (52).

Remark 4.1 The representations (52) and (54) are valid for arbitrary distributions for the claim size and the observation intervals.

Remark 4.2 Assume again that both the claim sizes and the observation intervals are exponentially distributed with mean $1/\nu$ and $1/\gamma$, respectively. Then, skipping the details, the solution of (50) leads to

$$\chi_\delta(x;b) = \left(\int_0^\infty e^{-\rho_\gamma y} w^*(y)\, dy\right) \frac{(\rho_\gamma - \rho_0)(\rho_\gamma + R_0)[(R_\gamma + \rho_0)e^{\rho_0 x} - (R_\gamma - R_0)e^{-R_0 x}]}{(R_\gamma + \rho_0)(\rho_\gamma + R_0)e^{\rho_0 b} - (R_\gamma - R_0)(\rho_\gamma - \rho_0)e^{-R_0 b}}, \qquad 0 \le x \le b,$$

and therefore

$$h_\delta(y|x;b) = e^{-\rho_\gamma y} \frac{(\rho_\gamma - \rho_0)(\rho_\gamma + R_0)[(R_\gamma + \rho_0)e^{\rho_0 x} - (R_\gamma - R_0)e^{-R_0 x}]}{(R_\gamma + \rho_0)(\rho_\gamma + R_0)e^{\rho_0 b} - (R_\gamma - R_0)(\rho_\gamma - \rho_0)e^{-R_0 b}}, \qquad y > 0;\; 0 \le x \le b.$$

This factorization form makes it particularly easy to compute the integral terms in (52) and (54).

4.1 Solution of $\chi_\delta(x;b)$ for claims with rational Laplace transform

If (as in Section 3.2) the claim sizes have rational Laplace transform and the observation intervals are Erlang(n) distributed, then (50) can be solved to give

$$\chi_\delta(x;b) = \sum_{i=1}^{n(r+1)} \eta_i e^{\alpha_i x}, \qquad 0 \le x \le b, \tag{55}$$

where $\{\alpha_i\}_{i=1}^{n(r+1)} \equiv \{\alpha_{i,1}\}_{i=1}^{n(r+1)}$, and $\{\eta_i\}_{i=1}^{n(r+1)}$ are the solution of the system of $n(r+1)$ linear equations consisting of

$$\sum_{p=1}^{n(r+1)} \eta_p \sum_{j=i}^{n} B_j^* \sum_{l=i}^{j} \frac{1}{(\rho_\gamma - \alpha_p)^{j-l+1}} \frac{1}{(l-i)!} b^l e^{\alpha_p b} = \sum_{j=i}^{n} B_j^* \sum_{l=i}^{j} \frac{b^l}{(j-l)!(l-i)!} \left(\int_0^\infty y^{j-l} e^{-\rho_\gamma y} w^*(y)\, dy \right), \qquad i = 1, 2, \ldots, n, \tag{56}$$

$$\sum_{p=1}^{n(r+1)} \eta_p \sum_{j=i}^{n} \frac{B_{kj}}{(R_{\gamma,k} + \alpha_p)^j} = 0, \qquad k = 1, 2, \ldots, r; \quad i = 1, 2, \ldots, n. \tag{57}$$

For exponential observation times (i.e. $n = 1$) one can get more explicit results and we shall restrict ourselves to this case for the rest of this subsection. Equations (56) and (57) then reduce to the set of linear equations

$$\sum_{p=1}^{r+1} \eta_p(b) \frac{1}{\rho_\gamma - \alpha_p} e^{\alpha_p b} = \int_0^\infty e^{-\rho_\gamma y} w^*(y)\, dy, \tag{58}$$

$$\sum_{p=1}^{r+1} \eta_p(b) \frac{1}{R_{\gamma,k} + \alpha_p} = 0, \qquad k = 1, 2, \ldots, r, \tag{59}$$

where we emphasized the dependence of $\{\eta_i\}_{i=1}^{r+1}$ on the barrier b. For $i = 1, 2, \ldots, r+1$, we define ζ_i to be the cofactor of the $(1, i)$-th element of the coefficient matrix of the above linear system (with (58) listed in the first row). It is instructive to note that $\{\zeta_i\}_{i=1}^{r+1}$ do not depend on b, since b only appears in the first row of the above-mentioned coefficient matrix. Moreover, each ζ_i can be computed via the determinant of a Cauchy matrix with the appropriate sign (see the Appendix of Gerber & Shiu [10]). Then, solving the system by Cramer's rule followed by cofactor expansion (along the i-th column for the numerator and along the first row for the denominator) in the evaluation of determinants, we arrive at

$$\eta_i(b) = \left(\int_0^\infty e^{-\rho_\gamma y} w^*(y)\, dy \right) \frac{\zeta_i}{\sum_{p=1}^{r+1} \zeta_p \frac{1}{\rho_\gamma - \alpha_p} e^{\alpha_z b}}, \qquad i = 1, 2, \ldots, r+1. \tag{60}$$

Incorporating (60) into (55) (for $n = 1$) gives

$$\chi_\delta(x; b) = \left(\int_0^\infty e^{-\rho_\gamma y} w^*(y)\, dy \right) \frac{\sum_{i=1}^{r+1} \zeta_i e^{\alpha_i x}}{\sum_{i=1}^{r+1} \zeta_i \frac{1}{\rho_\gamma - \alpha_i} e^{\alpha_i b}}, \qquad 0 \le x \le b.$$

Due to relationship (51), one concludes that $h_\delta(y|x; b)$ admits the factorization as a product of a function of y, a function of x and a function of b as

$$h_\delta(y|x; b) = \rho_\gamma e^{-\rho_\gamma y} \frac{\varpi_1(x)}{\varpi_2(b)}, \qquad y > 0; \quad 0 \le x \le b, \tag{61}$$

where

$$\varpi_1(x) = \sum_{i=1}^{r+1} \zeta_i e^{\alpha_i x} \qquad \text{and} \qquad \varpi_2(b) = \sum_{i=1}^{r+1} \zeta_i \frac{\rho_\gamma}{\rho_\gamma - \alpha_i} e^{\alpha_i b}.$$

Remark 4.3 The definition (49) together with the representation (51) with $w^*(\cdot) \equiv 1$ means that

$$\int_0^\infty h_\delta(y|x;b)\,dy = \frac{\varpi_1(x)}{\varpi_2(b)} = \frac{\sum_{i=1}^{r+1} \zeta_i e^{\alpha_i x}}{\sum_{i=1}^{r+1} \zeta_i \frac{\rho_\gamma}{\rho_\gamma - \alpha_i} e^{\alpha_i b}}, \qquad 0 \le x \le b$$

is the Laplace transform of the time of the first overshoot above level b avoiding ruin, when the initial capital is $W_b(0) = x$. If, for example, the claim size follows a mixture of exponentials with distinct means $1/\nu_1, 1/\nu_2, \ldots, 1/\nu_r$, then a simple graphical plot reveals that, as $\gamma \to \infty$, one can write $\rho_\gamma \to \infty$ and $R_{\gamma,k} \to \nu_k$ for $k = 1, 2, \ldots, r$, so that

$$\lim_{\gamma\to\infty} \int_0^\infty h_\delta(y|x;b)\,dy = \frac{\sum_{i=1}^{r+1} \zeta_i e^{\alpha_i x}}{\sum_{i=1}^{r+1} \zeta_i e^{\alpha_i b}}, \qquad 0 \le x \le b, \tag{62}$$

where $\{\zeta_i\}_{i=1}^{r+1}$ are now calculated from the coefficients of the system (59) with $R_{\gamma,k}$ replaced by ν_k. Note that with $\gamma \to \infty$, the time of first overshoot over level b avoiding ruin in the present model is essentially the time of first upcrossing level b avoiding ruin in the classical continuous-time barrier model. Indeed, (62) coincides with Gerber & Shiu [11, Eqn.(A.9)].

Remark 4.4 It is also worthwhile to note that (61) implies the normalized discounted density of the amount of overshoot to be exponential with mean $1/\rho_\gamma$, regardless of the initial capital x and the barrier level b.

Finally, substitution of the factorization (61) into (53) leads to

$$V(x;b) = \frac{\varpi_1(x)}{\rho_\gamma[\varpi_2(b) - \varpi_1(b)]}, \qquad 0 \le x \le b.$$

As expected, the optimal barrier level $b = b^*$ which maximizes $V(x;b)$ with respect to b is independent of the initial surplus $0 \le x \le b^*$. Note also that in the limit $\gamma \to \infty$, (due to $\rho_\gamma \to \infty$) the denominator in the above expression is

$$\lim_{\gamma\to\infty} \rho_\gamma[\varpi_2(b) - \varpi_1(b)] = \lim_{\gamma\to\infty} \sum_{i=1}^{r+1} \zeta_i \frac{\rho_\gamma \alpha_i}{\rho_\gamma - \alpha_i} e^{\alpha_i b} = \sum_{i=1}^{r+1} \zeta_i \alpha_i e^{\alpha_i b} = \varpi_1'(b),$$

with the understanding that $\{\zeta_i\}_{i=1}^{r+1}$ are calculated at the limit as $\gamma \to \infty$ (see Remark 4.3). Hence, in the limit one obtains the well-known form $V(x;b) = \varpi_1(x)/\varpi_1'(b)$ of the continuous-time model.

5 On the optimal barrier choice for exponential inter-observation times

In this section we will discuss the issue of the optimal dividend barrier further according to the definition (2), i.e. time 0 is not an intervention time. For the entire section we assume that the inter-observation time T is exponentially distributed with mean $1/\gamma$. Let us start with the case of exponential claims.

Example 5.1 Assume the claim size Y is exponentially distributed with mean $1/\nu$. Since in equation (24) only the denominator depends on the barrier level b, one can identify the optimal barrier b^* which maximizes $V(x;b)$ for a given initial capital x by minimizing the denominator with respect to b. This immediately leads to

$$b^* = \max\left\{0, \frac{1}{\rho_0 + R_0} \ln \frac{(R_\gamma - R_0)(\rho_\gamma - \rho_0)R_0^2}{(R_\gamma + \rho_0)(\rho_\gamma + R_0)\rho_0^2}\right\}, \tag{63}$$

which generalizes Gerber & Shiu [9, Eqn.(7.10)]. Also, one readily checks that

$$\frac{d}{dx}V(x;b^*)\bigg|_{x=b^*} = 1. \tag{64}$$

At the same time, b^* is the only value b for which (64) holds. □

Recall that in the classical continuous-time model $\frac{d}{dx}V(x;b)|_{x=b} = 1$ for all b, whereas in the above example with exponential observation times and exponential claims, this derivative was equal to 1 at the barrier only if the barrier is optimal. In the sequel we will show that this property holds more generally.

With a general claim size density $f_Y(\cdot)$, differentiating (11) and (12) and evaluating in $x = b$, together with the continuity conditions $V_L(0;b) = V_M(0;b)$ and $V_M(b;b) = V_U(b;b)$ that were established in Section 2 we obtain

$$\begin{aligned}
0 = c\,\frac{d^2}{dx^2}V_M(x;b)\bigg|_{x=b} &- (\lambda+\delta)\frac{d}{dx}V_M(x;b)\bigg|_{x=b} + \lambda\int_0^b \left(\frac{d}{dx}V_M(x-y;b)\bigg|_{x=b}\right) f_Y(y)\,dy \\
&+ \lambda\int_b^\infty \left(\frac{d}{dx}V_L(x-y;b)\bigg|_{x=b}\right) f_Y(y)\,dy, \\
0 = c\,\frac{d^2}{dx^2}V_U(x;b)\bigg|_{x=b} &- (\lambda+\gamma+\delta)\frac{d}{dx}V_U(x;b)\bigg|_{x=b} + \gamma + \lambda\int_0^b \left(\frac{d}{dx}V_M(x-y;b)\bigg|_{x=b}\right) f_Y(y)\,dy \\
&+ \lambda\int_b^\infty \left(\frac{d}{dx}V_L(x-y;b)\bigg|_{x=b}\right) f_Y(y)\,dy.
\end{aligned}$$

Since we know from (8) that the derivative of $V(x,b)$ in $x = b$ exists, from the above expressions it is clear that the second-order derivatives of $V_M(x,b)$ and $V_U(x,b)$ in $x = b$ match (i.e. the second derivative of $V(x,b)$ in $x = b$ exists) if and only if $\frac{d}{dx}V_M(x;b)\big|_{x=b} = \frac{d}{dx}V_U(x;b)\big|_{x=b} = 1$.

If $V(x,b)$ is differentiable in its second component b, then an obvious necessary condition for optimality is $\frac{\partial}{\partial b}V(x;b) = 0$ at $b = b^*$ for $x \in \mathbb{R}$. But we will now show that this implies (64). For fixed b, Dynkin's formula (see e.g. Rolski et al. [14]) can be applied for $V(x;b)$ and states that

$$\begin{aligned}
&e^{-\delta t}V(C(t);b) - V(C(0);b) \\
&\quad - \int_0^t e^{-\delta s}\left(c\,\frac{\partial}{\partial x}V(x;b)\bigg|_{x=C(s)} - (\lambda+\delta)V(C(s);b) + \lambda\int_0^\infty V(C(s)-y;b)\,f_Y(y)\,dy\right) ds
\end{aligned}$$

defines a zero-mean martingale (note that the generator of the uncontrolled surplus $\{C(t)\}_{t\geq 0}$ applied to $e^{-\delta t}V(x;b)$, namely

$$c\,\frac{\partial}{\partial x}V(x;b) - (\lambda+\delta)V(x;b) + \lambda\int_0^\infty V(x-y;b)f_Y(y)\,dy, \tag{65}$$

is part of the IDEs (5), (6) and (7) for $V(x;b)$ in Section 2).

Let us condition on the observation time $T=t$ and replace (65) by the specific inhomogeneities, then for $0<b<b'$ we obtain

$$\begin{aligned} V(x;b)-V(x;b') = \mathbb{E}\Bigg[\mathbb{E}\Bigg[& e^{-\delta t}[V(C(t);b)-V(C(t);b')] \\ & -\int_0^t e^{-\delta s}\Bigg(\gamma[V(C(s);b)-V(C(s);b')]I_{\{C(s)<0\}} \\ & +\gamma\left\{V(C(s);b)-V(C(s);b')+b-b'-[V(b;b)-V(b';b')]\right\} I_{\{C(s)>b'\}} \\ & +\gamma\left\{V(C(s);b)-[C(s)-b+V(b;b)]\right\} I_{\{b<C(s)\le b'\}}\Bigg) ds \Bigg| \, T=t\Bigg]\Bigg]. \end{aligned} \tag{66}$$

Before dividing by $b-b'$ we look at $\gamma\{V(C(s);b)-[C(s)-b+V(b;b)]\}I_{\{b<C(s)\le b'\}}$ and notice that for $|b'-b|$ small we can apply a second-order Taylor expansion around b, for a fixed surplus path,

$$V(C(s);b)=V(b;b)+[C(s)-b]\frac{\partial}{\partial x}V(x;b)\Big|_{x=b}+\frac{[C(s)-b]^2}{2}\frac{\partial^2}{\partial x^2}V(x;b)\Big|_{x=\xi}$$

for some $\xi\in(b,b')$ such that the second derivative exists and is finite. Now let us divide equation (66) by $b-b'$,

$$\begin{aligned} \frac{V(x;b)-V(x;b')}{b-b'} = \mathbb{E}\Bigg[\mathbb{E}\Bigg[& e^{-\delta t}\frac{V(C(t);b)-V(C(t);b')}{b-b'} \\ & -\int_0^t e^{-\delta s}\Bigg\{\gamma\frac{V(C(s);b)-V(C(s);b')}{b-b'}I_{\{C(s)<0\}} \\ & +\gamma\left(\frac{V(C(s);b)-V(C(s);b')}{b-b'}+1-\frac{V(b;b)-V(b';b')}{b-b'}\right)I_{\{C(s)>b'\}} \\ & +\gamma\,\frac{[C(s)-b]\frac{\partial}{\partial x}V(x;b)\big|_{x=b}+\frac{[C(s)-b]^2}{2}\frac{\partial^2}{\partial x^2}V(x;b)\big|_{x=\xi}-[C(s)-b]}{b-b'}I_{\{b<C(s)\le b'\}}\Bigg\} ds\Bigg| \, T=t\Bigg]\Bigg]. \end{aligned} \tag{67}$$

Note that

$$\frac{[C(s)-b]\frac{\partial}{\partial x}V(x;b)\big|_{x=b}+\frac{[C(s)-b]^2}{2}\frac{\partial^2}{\partial x^2}V(x;b)\big|_{x=\xi}-[C(s)-b]}{b-b'}I_{\{b<C(s)\le b'\}}$$

tends to zero if $b\to b'$ exactly if $\lim_{b\to b'}\frac{\partial}{\partial x}V(x;b)\big|_{x=b}=1$, or equivalently $\frac{\partial}{\partial x}V(x;b)\big|_{x=b'}=1$. When writing $\frac{V(b;b)-V(b';b')}{b-b'}=\frac{V(b;b)-V(b;b')}{b-b'}+\frac{V(b;b')-V(b';b')}{b-b'}$ in (67) we get that $\frac{\partial}{\partial b}V(x;b)=0$ for some $b=b^*$ and arbitrary $x\in\mathbb{R}$ can only hold if (64) holds.

Because $V(x;b)$ is linearly bounded and monotone we are allowed to interchange expectations and the limit $b\to b'$ and can conclude that a positive maximizing barrier height b^* implies (64), which itself implies that $V(x;b^*)$ is twice differentiable in x at $x=b^*$. These arguments are also valid for $b>b'$ and $b\to b'$, therefore the fact that $V(x;b^*)$ is twice differentiable in x at the barrier turns out to be a necessary criterion for the optimality of the barrier in this model.

6 Numerical illustrations

Let us now look at some numerical illustrations. Consider first the case of exponential claims with mean $1/\nu$ and exponential inter-observation times with mean $1/\gamma$. In this situation, the optimal barrier level b^* can be calculated via formula (63). Figure 3 depicts b^* as a function of γ for a particular set of parameters. As can be expected, b^* increases with γ, as a larger value of γ leads to more frequent observations of the process, which implies a higher chance of observing early ruin, and as a result a higher b^* is required for safety (otherwise ruin may occur before dividends are paid). Let us now fix the value of $\gamma = 10$, for which

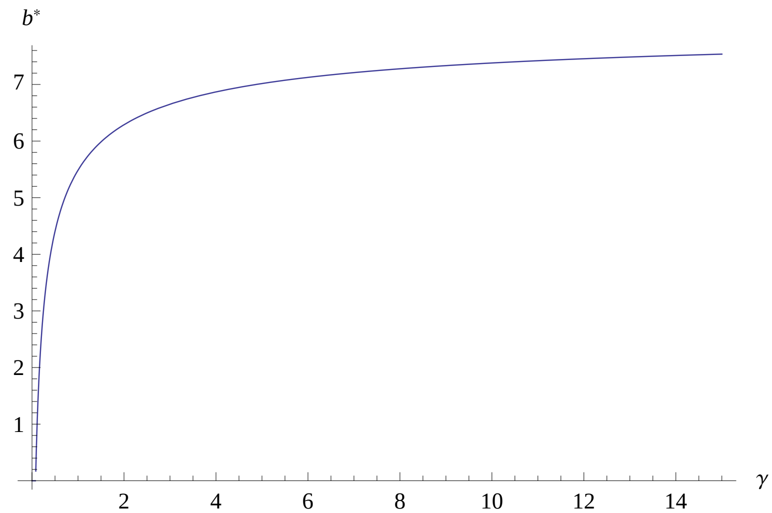

Figure 3: Optimal barrier b^* as a function of γ for exponential claims and exponential inter-observation times with $c = 6,\ \nu = 3,\ \lambda = 15,\ \delta = 0.05$.

the optimal barrier level is $b^* = 7.379$. Figures 4 and 5 give $V(x; b)$ for three different barrier levels b and illustrate the smooth-fit property of the maximizing barrier b^*. We see that the first order derivatives with respect to x fit together in $x = b$ for each of the three barrier levels. However, only for the barrier level b^* we have that $V(x; b)$ is twice differentiable in x and the necessary criterion (64) for the optimal barrier level b^* holds.

Next, we compare the values of the expected values $V(x; b)$ and standard deviations $SD(x; b)$ of the total discounted dividend payments until ruin for our model with random observation times to the ones of the classical continuous observation model for three different parameter sets. At the same time, we investigate how much the values of $V(x; b)$ and $SD(x; b)$ are affected by the 'randomness' of the observation times. This is done by using observation intervals with Erlang(n) distribution, for which we fix the expected time between observations ($\mathbb{E}[T] = 2.5$), but increase the value of n. Note again that for large n we approach the case of deterministic periodic observation intervals (i.e. the discrete-time risk model), yet utilizing the computational advantages of the random approach. We shall consider three different claim size distributions, each of which leads to an expected value of 1. Concretely, we consider a sum of two exponentials with mean 1/3 and 2/3 (Table 1), an exponential claim size distribution with mean 1 (Table 2) and a mixture of two exponentials (one exponential with mean 2 (mixing probability 1/3) and one exponential with mean 1/2 (mixing probability 2/3)) (Table 3). The variances of these

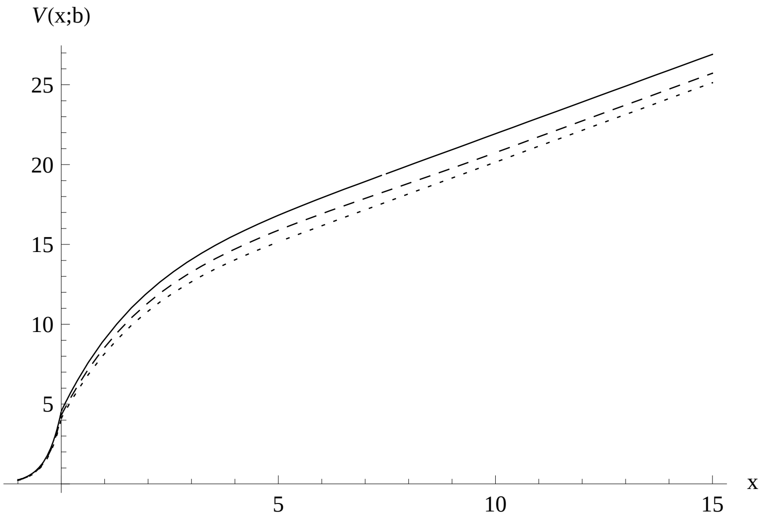

Figure 4: $V(x;b)$ for three barrier levels $b = 5$ (dotted line), $b = b^* = 7.379$ (solid line) and $b = 10$ (dashed line) and parameters $c = 6$, $\nu = 3$, $\lambda = 15$, $\delta = 0.05$, $\gamma = 10$.

claim distributions are 0.56, 1 and 2, respectively.

Note that all the above claim distributions have rational Laplace transforms $\widetilde{f}_Y(s)$ in the form of (38). Therefore, in producing the following tables, the algorithm in Section 3.2 for Erlang(n) observation intervals can be used. Our procedure is summarized below.

1. For various positive integers m (up to the order of dividend moments of interest), solve (35) with δ replaced by $m\delta$, which has a unique positive root $\rho_{\gamma,m}$ and r roots with negative real parts, namely $-R_{\gamma,1,m}, \dots, -R_{\gamma,r,m}$.

2. One may use (40) and (41) (with ρ_γ and $R_{\gamma,i}$ replaced by $\rho_{\gamma,m}$ and $R_{\gamma,i,m}$ respectively) to determine $B^*_{j,m}$ and $B_{ij,m}$. However, they also may be determined as the coefficients in the partial fractions expansion

$$\left(\frac{\gamma}{c}\right)^n \frac{[Q_{1,r}(s)]^n}{(\rho_{\gamma,m} - s)^n \prod_{i=1}^r (s + R_{\gamma,i,m})^n} = \sum_{j=1}^n \frac{B^*_{j,m}}{(\rho_{\gamma,m} - s)^j} + \sum_{i=1}^r \sum_{j=1}^n \frac{B_{ij,m}}{(s + R_{\gamma,i,m})^j}.$$

3. Solve the Lundberg fundamental equation (47) for $\{\alpha_{i,m}\}_{i=1}^{n(r+1)}$ with its left-hand side evaluated using the right-hand side of (34).

4. Solve the system of $n(r+1)$ linear equations (46) and (48) to obtain $\{A_{i,m}\}_{i=1}^{n(r+1)}$, where $V_{k,\delta}(b;b)$ appearing in (46) (for $k = 1, 2, \dots, m-1$) is given by (42) at $x = b$ with the trivial starting value $V_{0,\delta}(b;b) = 1$. This procedure is recursive in m.

5. The dividend moment $V_{m,\delta}(x;b)$ is finally given by (42).

In Tables 1–3, the optimal barrier b^* in the respective scenario is used as the barrier level for the calculations. Note that b^* does not depend on the initial surplus x (for $0 \le x \le b^*$), so that the value of

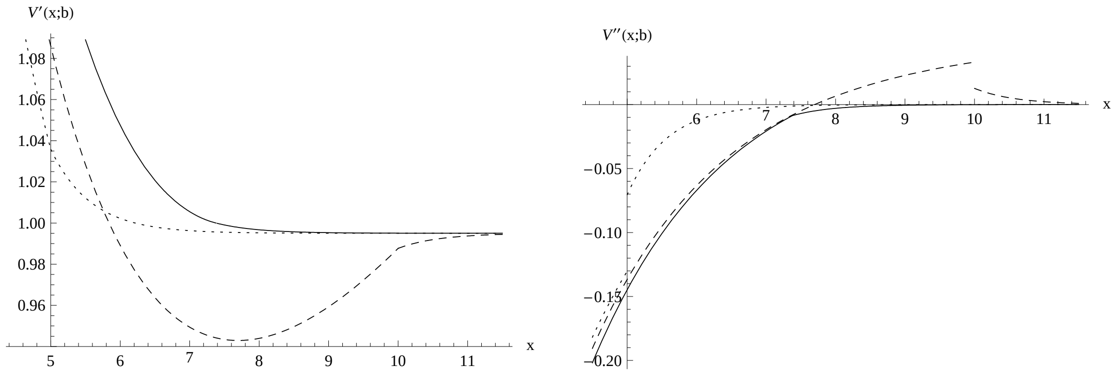

Figure 5: First and second derivative in x of $V(x;b)$ for three barrier levels $b = 5$ (dotted line), $b = b^* = 7.379$ (solid line) and $b = 10$ (dashed line) and parameters $c = 6$, $\nu = 3$, $\lambda = 15$, $\delta = 0.05$, $\gamma = 10$.

b^* is the same within each column, but usually will be different for different columns.

From Tables 1–3, one can observe that for initial surplus $x = 0$, the discrete random observation model produces much higher expected total discounted dividends than the classical continuous-time model in the first column. The reason is that with random observation ruin cannot occur very early (namely not before the first observation time). Another observation is that in all cases the maximizing barrier b^* in the classical continuous case is larger than the ones for discrete observations, while for x sufficiently larger than zero the expected discounted dividends are of a similar size. Hence, not observing instantly allows to lower the dividend barrier without lowering the dividend performance. This again can be explained by the fact that in the random observation model ruin between observations is not observed if the process is again positive at the next observation time and so on average one can expect dividend payments to occur for a longer time period than in the classical model (this seems to be realistic, since in practice the risk process will also be monitored at certain time points only). One also sees from the tables that the standard deviation of the total discounted dividend payments decreases for increasing initial capital.

Comparing the values of the same cells across Tables 1–3, the optimal barrier level b^* appears to increase with the variance of the claim size distribution (which can again be explained by the need to avoid early ruin so that later dividend payments can take place). Moreover, the expectation $V(x;b^*)$ appears to decrease as the variance of the claim size increases for any given initial capital x.

It is worthwhile to mention that moderate values of n (say, $n = 7$ or $n = 8$) already seem to be a good approximation of the discrete-time model, as the values do not change significantly any more when increasing n. One particular benefit of the present method hence also is in terms of a 'randomized approximation scheme' for the discrete-time model. Due to the compound Poisson aggregate claims distribution, it would be computationally very hard to obtain these numbers with the usual techniques for discrete-time risk models.

Sum Exp	Classical	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
b^*	15.81	12.98	13.27	13.37	13.42	13.45	13.47	13.49	13.50
$V(0;b^*)$	29.20	55.46	55.27	55.34	55.42	55.48	55.52	55.56	55.59
$SD(0;b^*)$	42.19	43.81	43.93	43.94	43.94	43.93	43.92	43.92	43.91
$V(5;b^*)$	83.17	86.83	86.67	86.61	86.57	86.55	86.53	86.52	86.52
$SD(5;b^*)$	28.04	23.39	23.80	23.94	24.02	24.07	24.10	24.13	24.14
$V(10;b^*)$	93.17	94.27	94.30	94.30	94.30	94.30	94.30	94.30	94.30
$SD(10;b^*)$	17.98	17.08	17.18	17.22	17.23	17.24	17.25	17.26	17.26
$V(b^*;b^*)$	99.29	97.34	97.68	97.79	97.84	97.88	97.90	97.92	97.93
$SD(b^*;b^*)$	16.57	16.50	16.53	16.54	16.54	16.54	16.54	16.54	16.54

Table 1: $c = 1.5$, $\lambda = 1$, $\delta = 0.005$, $f_Y(y) = 3\,e^{-1.5y} - 3e^{-3y}$, $T \sim \text{Erlang}(n)$ with $\mathbb{E}[T] = 2.5$

Exp	Classical	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
b^*	19.06	15.93	16.28	16.40	16.46	16.50	16.53	16.54	16.56
$V(0;b^*)$	28.45	51.66	51.18	51.10	51.09	51.10	51.11	51.11	51.12
$SD(0;b^*)$	41.38	43.71	43.84	43.87	43.88	43.89	43.89	43.89	43.89
$V(5;b^*)$	76.49	81.48	81.17	81.05	80.98	80.94	80.91	80.90	80.88
$SD(5;b^*)$	33.15	28.43	28.90	29.07	29.15	29.21	29.24	29.27	29.28
$V(10;b^*)$	88.86	90.56	90.50	90.48	90.46	90.45	90.45	90.44	90.44
$SD(10;b^*)$	22.29	20.67	20.85	20.91	20.94	20.96	20.98	20.99	20.99
$V(15;b^*)$	94.88	96.02	96.02	96.01	96.01	96.01	96.00	96.00	96.00
$SD(15;b^*)$	19.24	18.88	18.94	18.96	18.97	18.98	18.98	18.99	18.98
$V(b^*;b^*)$	99.00	96.95	97.30	97.42	97.48	97.52	97.54	97.56	97.57
$SD(b^*;b^*)$	18.88	18.81	18.83	18.84	18.85	18.85	18.85	18.85	18.85

Table 2: $c = 1.5$, $\lambda = 1$, $\delta = 0.005$, $f_Y(y) = e^{-y}$, $T \sim \text{Erlang}(n)$ with $\mathbb{E}[T] = 2.5$

Mixed Exp	Classical	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
b^*	25.49	21.87	22.35	22.51	22.60	22.65	22.69	22.72	22.73
$V(0;b^*)$	27.03	46.22	45.58	45.40	45.31	45.27	45.24	45.22	45.20
$SD(0;b^*)$	39.87	42.90	42.97	42.99	43.00	43.00	43.00	43.01	43.01
$V(5;b^*)$	64.91	71.03	70.47	70.26	70.15	70.09	70.04	70.01	69.98
$SD(5;b^*)$	38.58	35.18	35.60	35.75	35.83	35.88	35.91	35.93	35.95
$V(10;b^*)$	79.11	82.03	81.78	81.68	81.62	81.59	81.57	81.55	81.54
$SD(10;b^*)$	30.42	28.17	28.46	28.57	28.62	28.65	28.67	28.69	28.71
$V(15;b^*)$	86.88	88.72	88.59	88.53	88.50	88.48	88.47	88.46	88.45
$SD(15;b^*)$	25.71	24.66	24.81	24.86	24.89	24.91	24.92	24.92	24.93
$V(20;b^*)$	92.55	94.05	93.96	93.91	93.89	93.88	93.87	93.86	93.86
$SD(20;b^*)$	23.80	23.42	23.49	23.52	23.53	23.54	23.54	23.54	23.55
$V(b^*;b^*)$	98.11	95.94	96.32	96.45	96.52	96.56	96.58	96.60	96.61
$SD(b^*;b^*)$	23.34	23.28	23.31	23.31	23.32	23.32	23.32	23.32	23.33

Table 3: $c = 1.5$, $\lambda = 1$, $\delta = 0.005$, $f_Y(y) = (1/6)\,e^{-0.5y} + (4/3)e^{-2y}$, $T \sim \text{Erlang}(n)$ with $\mathbb{E}[T] = 2.5$

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