APPROXIMATION OF SUPREMUM OF MAX-STABLE STATIONARY PROCESSES & PICKANDS CONSTANTS

KRZYSZTOF DĘBICKI AND ENKELEJD HASHORVA

Abstract: Let $X(t), t \in \mathbb{R}$ be a stochastically continuous stationary max-stable process with Fréchet marginals $\Phi_\alpha, \alpha > 0$ and set $M_X(T) = \sup_{t \in [0,T]} X(t), T > 0$. In the light of the seminal articles [1, 2], it follows that $A_T = M_X(T)/T^{1/\alpha}$ converges in distribution as $T \to \infty$ to $\mathcal{H}^{1/\alpha} X(1)$, where $\mathcal{H}$ is the Pickands constant corresponding to the spectral process Z of X. In this contribution we derive explicit formulas for $\mathcal{H}$ in terms of Z and show necessary and sufficient conditions for its positivity. From our analysis it follows that $A_T^\beta, T > 0$ is uniformly integrable for any $\beta \in (0, \alpha)$. For Brown-Resnick X we show the validity of the celebrated Slepian inequality and discuss the finiteness of Piterbarg constants.

Key Words: Max-stable process; spectral tail process; Gaussian processes with stationary increments; Lévy processes; Pickands constants; Piterbarg constants; Slepian inequality; growth of supremum.

AMS Classification: Primary 60G15; secondary 60G70

1. Introduction

Let $X(t), t \in \mathbb{R}$ be a stochastically continuous stationary max-stable process with Fréchet marginals $\Phi_\alpha(x) = e^{-x^{-\alpha}}, x > 0, \alpha > 0$. Here max-stable means that the finite dimensional distributions (fidi's) of X are max-stable multivariate distributions, see e.g., [3, 4]. By a key theorem of de Haan [3], it is well-known that X can be represented (in distribution) as

$$X(t) = \max_{i \geq 1} P_i Z^{(i)}(t), \quad t \in \mathbb{R}, \tag{1.1}$$

where $\Pi = \sum_{i=1}^{\infty} \varepsilon_{P_i}$ is a Poisson point process (PPP) on $[0, \infty)$ with intensity $\alpha x^{-\alpha-1} dx$ independent of $Z^{(i)}$'s which are independent copies of a random process $Z(t), t \in \mathbb{R}$ which shall be referred to as the spectral process. The assumption that $X(t)$ has distribution Φ_α implies that

Date: December 10, 2018.

$\mathbb{E}\{Z^\alpha(t)\} = 1, \forall t \in \mathbb{R}$ Since we consider here only stationary max-stable processes, adapting the terminology of [4] we shall call the spectral process Z a *Brown-Resnick* stationary process.

The assumption that X is stochastically continuous implies that it has a separable and measurable version, see e.g., [5]; the same holds for Z, see [6]. Therefore in the following we suppose that both X and Z are jointly measurable and separable. Hereafter we shall assume further that X has locally bounded sample paths, and thus by (1.1) Z also has locally bounded sample paths. According to [6], this assumption is important for conditions that guarantee the existence of a dissipative Rosiński (or also called a mixed moving maxima) representation of X.

By separability and the local boundedness of the sample paths of X and Z both $M_X(T) = \sup_{t\in[0,T]} X(t)$ and $M_{Z^\alpha}(T)$ are well-defined and finite random variables for any $T > 0$. By (1.1), given $t_i \in \mathbb{R}, x_i \in (0,\infty), i \le k$ we have (see e.g., [7])

$$-\ln \mathbb{P}\{\forall_{1\le i\le k} X(t_i) \le x_i\} = \mathbb{E}\Big\{\max_{1\le i\le k} Z^\alpha(t_i)/x_i^\alpha\Big\}. \tag{1.2}$$

Since by the measurability of Z, for any $T > 0$ using Fubini theorem

$$\mathbb{E}\Big\{\int_0^T Z^\alpha(t)\lambda(dt)\Big\} = \int_0^T \mathbb{E}\{Z^\alpha(t)\}\lambda(dt) = T,$$

with $\lambda(\cdot)$ the Lebesgue measure on $\mathbb{R}$, we have

$$\mathbb{P}\{M_X(T) < \infty\} = \mathbb{P}\{M_{Z^\alpha}(T) < \infty\} = 1$$

and

$$-\ln \mathbb{P}\big\{M_X(T) \le (Tx)^{1/\alpha}\big\} = \frac{\mathbb{E}\{M_{Z^\alpha}(T)\}}{Tx}, \quad \forall T, x > 0. \tag{1.3}$$

The above shows that $\mathbb{E}\{M_{Z^\alpha}(T)\}, T > 0$ does not depend on the particular choice of the spectral process Z but only on X. By the stationarity of X it follows that

$$\mathbb{E}\{M_{Z^\alpha}(T)\} = \mathbb{E}\Big\{\sup_{S\le t\le S+T} Z^\alpha(t)\Big\} \in (0,\infty)$$

for any $S \in \mathbb{R}, T > 0$. Hence $\mathbb{E}\{M_{Z^\alpha}(T)\}, T > 0$ is sub-additive and consequently, Fekete lemma yields

$$\mathcal{H} := \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\big\{M_{Z^\alpha}(T)\big\} = \inf_{T>0} \frac{1}{T}\mathbb{E}\big\{M_{Z^\alpha}(T)\big\} \le \mathbb{E}\big\{M_{Z^\alpha}(1)\big\} \in (0,\infty). \tag{1.4}$$

Moreover, from the above we conculde that $\mathcal{H}$ does not depend on the particular choice of the spectral tail process Z but only on the stationary max-stable process X. Referring to [8], $\mathcal{H}$ is the so-called *generalised Pickands* constant defined with respect to some Brown-Resnick stationary process Z. In [9] $\mathcal{H}$ is introduced for the log-normal process

$$Z(t) \;=\; e^{B(t)-\sigma^2(t)/2}, \quad t \in \mathbb{R}, \tag{1.5}$$

where $B(t), t \in \mathbb{R}$ is a centered Gaussian process with stationary increments, continuous sample paths and variance function σ^2 which does not vanish in any interval of $\mathbb{R}$.

Taking B to be a fractional Brownian motion (fBm) with self-similarity Hurst index $\alpha/2 \in (0,1]$, we get that $\mathcal{H}$ is the *classical Pickands* constant, see e.g., [8, 10, 11]. The only known values of $\mathcal{H}$ are 1 and $1/\sqrt{\pi}$ corresponding to $\alpha = 1$ and $\alpha = 2$, respectively. In the case of Lévy processes the Pickands constant $\mathcal{H}$ appears explicitly in many contributions, see e.g., [8, 12] and references therein. Moreover, for the discrete-time case $X(t), t \in \mathbb{Z}$ we have that $\mathcal{H}$ (introduced similarly as for the continuous-time, see [13]) is the extremal index of the stationary time series $X(t), t \in \mathbb{Z}$. In that context, it has been also studied in [14] using the spectral representation of X. A considerable amount of research is dedicated to calculation and estimation of the extremal index of regularly varying time series, see e.g., [15] and references therein.

The main question that arises for Pickands constants $\mathcal{H}$ is:

Q1: Under what conditions are these constants positive or equal to 0?

For a stationary max-stable process X this question is partially answered in [8] when Z is such that $Z(0) = 1$ almost surely. The case that $Z(0)$ is a non-negative random variable is treated in [16]. Specifically, the positivity of $\mathcal{H}$ has been shown under the assumption that

$$\mathbb{P}\{\mathcal{S}(Z) < \infty\} = 1, \quad \mathcal{S}(Z) := \int_{\mathbb{R}} Z^\alpha(t)\lambda(dt). \tag{1.6}$$

In view of [6], since X has locally bounded sample paths, then under (1.6) X has a dissipative Rosiński representation which is equivalent with X being generated by a non-singular dissipative flow, see [6, 17–19] for more details. As shown in [8, 16], if the spectral process Z has càdlàg sample paths and (1.6) holds, then

$$\mathcal{H} \ge \mathbb{E}\left\{ \frac{\sup_{t\in\mathbb{R}} Z^\alpha(t)}{\mathcal{S}(Z)} \right\}, \tag{1.7}$$

with equality shown under some technical assumptions for both Gaussian and Lévy spectral processes Z. The investigation therein was motivated by [7, 20]. The former contribution showed that (1.7) holds with equality for B in (1.5) being an fBm. Since $\mathcal{H}$ in (1.4) is defined as a limit, it turns out that the explicit calculation of $\mathcal{H}$ is for general Z too difficult. However, if (1.7) holds with equality, then $\mathcal{H}$ being an expectation, can be efficiently simulated, see e.g., [20].

An interesting question that arises here is:

Q2: Does (1.7) hold with equality for general Brown-Resnick stationary Z?

Clearly, if $\mathcal{H}=0$, then (1.4) means the convergence in probability

$$A_T := \frac{M_X(T)}{T^{1/\alpha}} \xrightarrow{p} 0, \quad T \to \infty, \tag{1.8}$$

whereas when $\mathcal{H}>0$ we have the convergence in distribution

$$A_T \xrightarrow{d} \mathcal{H}^{1/\alpha} X(1), \quad T \to \infty. \tag{1.9}$$

For X being a symmetric α-stable ($S\alpha S$) stationary process with $\alpha \in (0,2)$ the above convergence has been shown in the seminal articles [1, 2], see the recent contributions [21–24] for related results and new developments.

The findings of [1, 2] are important for the max-stable processes too, which is already pointed out in [14] for discrete max-stable processes. Indeed, using the link between max-stable and $S\alpha S$ processes established in [17] and [25] independently, it follows that when X is generated by a non-singular conservative flow, which by [6] (under the assumption of locally boundedness of sample paths of X) is equivalent with $\mathbb{P}\{\mathcal{S}(Z)=\infty\}=1$, then we have

$$\mathcal{H}=0. \tag{1.10}$$

Note that (1.10) holds also when we consider the discrete case $X(t), t \in \mathbb{Z}$, which can be shown for instance by utilising the expression of Pickands constant (which in this case coincides with the extremal index, [13, 16]) derived in [14].

We conclude that $\mathcal{H}$ is positive if and only if

$$\mathbb{P}\{\mathcal{S}(Z)=\infty\}<1.$$

Hence according to our argumentation above question **Q1** has a simple answer. Namely, if X is a stationary max-stable process with locally bounded sample paths, then (by [6]), $\mathcal{H}>0$ if and only if

$$\mathbb{P}\{\mathcal{S}(Z)<\infty\}>0. \tag{1.11}$$

Clearly, the convergence in probability in (1.8) implies that $A_T^\beta \xrightarrow{p} 0$ as $T \to \infty$ for any $\beta \in (0, \infty)$ and a similar implication holds for A_T^β when (1.9) is satisfied. For X being an $S\alpha S$ random field the recent contribution [26] strengthened those convergences to that of $\mathbb{E}\{A_T^\beta\}$ for $\beta \in (0, \alpha)$, i.e., showing the uniform integrability of A_T^β whenever $\beta \in (0, \alpha)$. The case that X is a stationary max-stable random process is easier to deal with, see Proposition 3.4 in Section 4.

Our main interest in this contribution is the derivation of expressions for $\mathcal{H}$ in terms of the spectral process Z that appears in the de Haan representation (1.1). In particular, motivated by **Q2**, we show that (1.7) (or a modification of it) holds with equality under (1.11) without further assumptions. Recall that so far it is only known that the inequality in (1.7) holds for X having a dissipative Rosiński representation.

As already shown in [8, 16], different representations for $\mathcal{H}$ relate to different dissipative Rosiński representations of X. Therefore, our analysis is also concerned with such representations for X.

Our study of Pickands constants (together with the criteria for its positivity) allows us to investigate the growth of the expectations of $M_X(T)$ and $M_{Z^\alpha}(T)$ as $T \to \infty$. The latter can be investigated under the further assumption of the Brown-Resnick model, i.e., when Z is a log-normal process. Moreover, for the Brown-Resnick model an extension of the celebrated Slepian inequality is possible, see Theorem 3.1 below.

Organisation of the paper: Our main results are displayed in Section 2 followed by discussions and some extensions presented in Section 3. Proofs are postponed to Section 4; an Appendix concludes this contribution.

2. Main Results

Let X, Z be (as in the Introduction) jointly measurable, separable and with locally bounded sample paths. By the measurability of Z we have that $\mathcal{S}(Z) = \int_{\mathbb{R}} Z^\alpha(t)\lambda(dt)$ is a random variable in $\mathbb{R} \cup \{+\infty\}$, see [5]. Write next $\mathbb{E}\{A; B\}$ instead of $\mathbb{E}\{A\mathbb{I}(B)\}$ for an event B with $\mathbb{P}\{B\} > 0$. Fixing $T > 0$ we have the following splitting formula

$$\mathbb{E}\{M_{Z^\alpha}(T)\} = \mathbb{E}\{M_{Z^\alpha}(T); \mathcal{S}(Z) < \infty\} + \mathbb{E}\{M_{Z^\alpha}(T); \mathcal{S}(Z) = \infty\}. \tag{2.1}$$

If $\mathbb{P}\{\mathcal{S}(Z) = \infty\} > 0$, then by [27][Lem 16] the random process Z_D defined by

$$Z_D(t) := Z(t)\mathbb{I}(\mathcal{S}(Z) = \infty), \quad t \in \mathbb{R}$$

is Brown-Resnick stationary. Since Z_D has also locally bounded sample paths and $\mathcal{S}(Z_D) = \infty$ almost surely, the corresponding max-stable process X_D is generated by a non-singular conservative flow. Moreover, under (1.11)

$$Z_C(t) = Z(t)\mathbb{I}(\mathcal{S}(Z) < \infty)$$

is also a Brown-Resnick stationary process which is generated by a non-singular dissipative flow. In order to omit technical details, we refer the reader to [6, 19, 28] for precise formulations are results on conservative and dissipative parts of max-stable processes.

By the discussions in the Introduction, condition (1.11) implies that

$$\mathcal{H} = \mathcal{H}_{Z_C} > 0, \tag{2.2}$$

where $\mathcal{H}_{Z_C}$ is the Pickands constant with respect to spectral process Z_C. We remark that (2.2) is new and not available in the literature so far.

In view of (2.2), in the following we can reduce our analysis by considering only the case that X is generated by a non-singular dissipative flow. In view of [6] this is equivalent with X having a dissipative Rosiński representation i.e., for some non-negative random process (called also random shape function) $L(t), t \in \mathbb{R}$ which is continuous in probability (we can consider here therefore L to be jointly measurable and separable) and for some $c > 0$ we have the representation (in distribution)

$$X(t) = \max_{i \ge 1} P_i L^{(i)}(t - T_i), \quad t \in \mathbb{R}, \tag{2.3}$$

where $\sum_{i=1}^{\infty} \epsilon_{(P_i, T_i)}$ is a PPP on $[0, \infty) \times \mathbb{R}$ with intensity $c \cdot \lambda(dt) \cdot \alpha x^{-\alpha-1} dx$, independent of $L^{(i)}$'s which are independent copies of L.

By (2.3) for any random variable $\mathcal{N}$ with density $p(t) > 0, t \in \mathbb{R}$

$$Z = (c/p(\mathcal{N}))^{1/\alpha} B^{\mathcal{N}} L \tag{2.4}$$

is a valid spectral process for X, where $\mathcal{N}$ is independent of L and $B^t L(\cdot) = L(\cdot - t), t \in \mathbb{R}$.

Theorem 2.1. *Let $X(t), t \in \mathbb{R}$ be a stochastically continuous max-stable stationary process with de Haan representation (1.1). If X has locally bounded sample paths with and condition (1.6) holds, then there exists some jointly measurable and separable non-negative random shape function L such that (2.4) holds and moreover we have*

$$\mathcal{H} = \frac{\mathbb{E}\{\sup_{t \in \mathbb{R}} L^\alpha(t)\}}{\mathbb{E}\{\mathcal{S}(L)\}} \in (0, \infty). \tag{2.5}$$

Remark 2.2. *When X has a dissipative Rosiński representation, it is possible to construct L such that $\sup_{t\in\mathbb{R}} L^\alpha(t) = 1$ almost surely, see [6]. Hence by (2.5) for such random shape functions L we have*

$$\mathcal{H} = \frac{1}{\mathbb{E}\{\mathcal{S}(L)\}}. \tag{2.6}$$

We shall show below that it is also possible to construct L such that $\mathcal{S}(L) = 1$ almost surely, and thus by (2.5) we obtain an alternative formula, namely

$$\mathcal{H} = \mathbb{E}\Big\{\sup_{t\in\mathbb{R}} L^\alpha(t)\Big\}. \tag{2.7}$$

For simplicity we shall assume in the following that both X and Z have càdlàg sample paths. Let D be the space of càdlàg functions $f : [0,\infty) \to \mathbb{R}$ equipped with a metric d which makes it complete and separable, see e.g., [29] for details. Let $\mathcal{D}$ be the Borel σ-algebra on D defined by this metric and let μ be a probability measure given by (interpret $0 : 0$ as 1 below)

$$\mu(A) = \mathbb{E}\{Z^\alpha(0)\mathbb{I}(Z/Z(0) \in A)\}, \quad A \in \mathcal{D}.$$

Since (D, d) is a Polish metric space (complete and separable), we can determine a stochastic process $\Theta(t), t \in \mathbb{R}$ with càdlàg sample paths and probability law μ; refer to Θ as the spectral tail process. By [16][Thm 4.1] all the fidi's of the max-stable process X are determined by Θ. Namely, we have the following inf-argmax formula valid for x_i's positive constants and t_i's in $\mathbb{R}$

$$-\ln \mathbb{P}\{X(t_i) \le x_i, 1 \le i \le n\} \;=\; \sum_{k=1}^n \frac{1}{x_k^\alpha}\mathbb{P}\Big\{\inf \operatorname{argmax}_{1\le i\le n}\Big(\frac{\Theta^\alpha(t_i - t_k)}{x_i^\alpha}\Big) = k\Big\}. \tag{2.8}$$

Consequently, Θ defines X and vice-versa, from X we can calculate the fidi's of Θ by the generalised Pareto distributions of X, see [16][Remark 6.4]. The next result gives an explicit construction for the random shape function L and confirms (1.7).

Theorem 2.3. *Under the setup of Theorem 2.1, if further X has càdlàg sample paths and (1.11) holds, then*

$$\mathcal{H} = \mathbb{E}\Big\{\frac{\sup_{t\in\mathbb{R}} \Theta^\alpha(t)}{\mathcal{S}(\Theta)}; \mathcal{S}(\Theta) \in (0,\infty)\Big\} \in (0,\infty). \tag{2.9}$$

Moreover, if (1.6) is valid, then X has dissipative Rosiński representation (2.4) with random shape function

$$L(t) = \frac{\Theta(t)}{(\mathcal{S}(\Theta))^{1/\alpha}}, \quad t \in \mathbb{R}. \tag{2.10}$$

Remark 2.4. *If X is as in Theorem 2.3 with càdlàg sample paths, then it follows straightforwardly that $\mathcal{H} = 0$ is equivalent with $\mathbb{P}\{\mathcal{S}(Z) = \infty\} = \mathbb{P}\{\mathcal{S}(\Theta) = \infty\} = 1$. Note further that since $\Theta(0) = 1$ almost surely, then $\mathcal{S}(\Theta) > 0$ almost surely since Θ has càdlàg sample paths.*

Example 1. Consider the Gaussian case with Z as in (1.5), where $B(t), t \in \mathbb{R}$ is a centered Gaussian process with stationary increments, continuous sample paths and variance function σ^2 that does not vanish in compact intervals of $\mathbb{R}$.

We can assume without loss of generality (see [4]) that $\sigma(0) = 0$. Hence $Z(0) = 1$ almost surely and for the corresponding spectral tail process Θ we simply have $\Theta = Z$. Since for this case $\alpha = 1$, then under (1.11)

$$\mathcal{H} = \mathbb{E}\left\{ \frac{\sup_{t\in\mathbb{R}} Z(t)}{\mathcal{S}(Z)}; \mathcal{S}(Z) \in (0,\infty) \right\} \in (0,\infty).$$

In view of [4], the following condition

$$\liminf_{t\to\infty} \frac{\sigma^2(t)}{\ln t} > 8 \tag{2.11}$$

implies (1.1) and thus X has a dissipative Rosiński representation with random shape function $L(t) = Z(t)/\mathcal{S}(Z), t \in \mathbb{R}$. Moreover

$$\mathcal{H} = \mathbb{E}\left\{ \frac{\sup_{t\in\mathbb{R}} Z(t)}{\mathcal{S}(Z)} \right\} \in (0,\infty), \tag{2.12}$$

which has been proved in [8][Thm 2] under some additional assumptions on the variance function of B.

Example 2. Stationary max-stable Lévy–Brown–Resnick processes X have spectral processes $Z(t) = e^{W(t)}, t \in \mathbb{R}$ constructed from two independent Lévy processes. Specifically, let $\{B^+(t), t \geq 0\}$ be a Lévy process with Laplace exponent $\Psi(\theta) = \ln \mathbb{E}\{\exp\left(\theta B^+(1)\right)\}$ being finite for $\theta = 1$. Write $-W^-$ for another independent Lévy process with Laplace exponent

$$\ln \mathbb{E}\{e^{\theta W^-(1)}\} = \Psi(1-\theta) - (1-\theta)\Psi(1).$$

Then we set $W(t) = W^+(t) := B^+(t) - \Psi(1)t, t \geq 0$, and $W(t) = W^-(-t)$ if $t < 0$. In view of [30] the max-stable process X with unit Fréchet marginals $\Phi_1(x) = e^{-1/x}, x > 0$ corresponding to the spectral process Z is stationary. Note that this fact is proved in a completely different context in [31][Lem 1].

By [30] the Lévy-Brown–Resnick process X admits a dissipative Rosiński representation and thus Theorem 2.3 and [8][Thm 3.2] imply that (note that since $Z(0) = 1$ almost surely, then $\Theta = Z$)

$$\mathcal{H} = \mathbb{E}\left\{ \frac{\sup_{t\in\mathbb{R}} Z(t)}{\mathcal{S}(Z)} \right\} = \frac{\underline{k}(0,1)}{\underline{k}'(0,0)} > 0, \tag{2.13}$$

where $\underline{k}$ is the bivariate Laplace exponent of the descending ladder process corresponding to W^+. If B^+ is a spectrally negative Lévy process, we have the alternative formula $\mathcal{H} = \Psi'(1) > 0$, which is already derived in [32].

3. Discussions & Extensions

3.1. **Slepian inequality for Brown-Resnick max-stable processes.** Slepian inequality is essential in the theory of extremes and sample path properties of Gaussian and related processes. A commonly used version of Slepian inequality given for instance in [33][Thm 1.1] is as follows:
If $B_1(t), B_2(t), t \in \mathbb{R}$ are two centered Gaussian processes, then for any $t_1, \ldots, t_n \in \mathbb{R}, n \geq 1$ we have

$$\mathbb{E}\left\{ \max_{1\le i\le n} B_1(t_i) \right\} \geq \mathbb{E}\left\{ \max_{1\le i\le n} B_2(t_i) \right\},$$

provided that for all $1 \le i \neq j \le n$

$$\mathbb{E}\{(B_1(t_i) - B_1(t_j))^2\} \geq \mathbb{E}\{(B_2(t_i) - B_2(t_j))^2\}. \tag{3.1}$$

Moreover, in view of [34][Eq. 6] (applied to $g(x) = e^x$) for any real-valued function f

$$\mathbb{E}\left\{ \max_{1\le i,j\le n} e^{[B_1(t_i)-B_1(t_j)]-f(t_i)} \right\} \geq \mathbb{E}\left\{ \max_{1\le i,j\le n} e^{[B_2(t_i)-B_2(t_j)]-f(t_i)} \right\}. \tag{3.2}$$

Let $X_i(t), t \in \mathbb{R}, i = 1,2$ be max-stable processes with spectral processes $Z_i(t) = e^{B_i(t)-f(t)}, i = 1,2$. Max-stable processes that are constructed from log-normal Gaussian spectral processes are commonly referred to in the literature as Brown-Resnick max-stable processes.
By (1.2) if $B_i, i = 1,2$ are separable with locally bounded sample paths such that (3.1) holds, then using (3.2) we obtain

$$\mathbb{P}\left\{ \sup_{t\in K} X_1(t) > x \right\} \geq \mathbb{P}\left\{ \sup_{t\in K} X_2(t) > x \right\}, \quad \forall x > 0. \tag{3.3}$$

Both processes Z_1 and Z_2 are Brown-Resnick stationary if defined by

$$Z_i(t) = e^{B_i(t)-\sigma_i^2(t)/2}, \quad i = 1,2, \quad t \in \mathbb{R},$$

where $B_i, i = 1,2$ are two centered Gaussian processes with stationary increments and variance functions σ_1^2 and σ_2^2, respectively. Since in general σ_1 is different from σ_2 we cannot use the refinement of Vitale [34] to Slepian inequality stated in (3.2) to arrive at (3.3).

Our next result states the Slepian inequality for Brown-Resnick max-stable processes X_1 and X_2 which are stationary. Moreover, it implies a comparison criterium for the corresponding Pickands constants denoted by $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Since the law of X_i's depends only on their variograms $\gamma_i(t) = Var(B_i(t) - B_i(0)),\ i = 1,2,$ then we suppose without loss of generality that $\sigma_i(0) = 0$ for $i = 1,2$.

Theorem 3.1. *Let X_1, X_2 be two stationary max-stable Brown-Resnick processes with spectral processes Z_1 and Z_2, respectively. Suppose that $\sigma_1(0) = \sigma_2(0) = 0$ and $Z_i, i = 1,2$ are separable with locally bounded sample paths. If further for any $t \in \mathbb{R}$*

$$\sigma_1(t) \quad \geq \quad \sigma_2(t), \tag{3.4}$$

then for any compact set $K \subset \mathbb{R}$

$$\mathbb{P}\left\{\sup_{t\in K} X_1(t) > x\right\} \geq \mathbb{P}\left\{\sup_{t\in K} X_2(t) > x\right\}, \quad \forall x > 0 \tag{3.5}$$

and $\mathcal{H}_1 \geq \mathcal{H}_2$.

3.2. **Piterbarg constants.** For a given Brown-Resnick stationary process Z as in Section 2 we can define the so-called Piterbarg constants by

$$\mathcal{P}_f(K) = \mathbb{E}\left\{\sup_{t\in K} e^{B(t)-\sigma^2(t)/2-f(t)}\right\}$$

for a given positive measurable function f, $K = [0,\infty)$ or $K = \mathbb{R}$ and B a centered Gaussian processes with stationary increments as in the previous section. In the literature, Piterbarg constants appear naturally in the tail asymptotics of supremum of non-stationary Gaussian processes, see e.g. [11].

Clearly, the main question that arises is if $\mathcal{P}_f(K)$ is finite. So far for $f(t) = a\sigma^2(t), t \in K$ the finiteness of Piterbarg constants is shown using ideas from the double-sum technique of Piterbarg, see e.g., [11]. Utilising the properties of max-stable processes, we are able to show the finiteness of Piterbarg constants for more general class of functions f. In the following proposition we consider only the case $K = [0, \infty)$; scenario $K = \mathbb{R}$ follows by analogous line of reasoning. We denote next by $\mathcal{W}$ the Will's functional (see e.g., [35]) defined by

$$\mathcal{W}(s) = \mathbb{E}\left\{ \sup_{t \in [0,s]} e^{B(t) - \sigma^2(t)/2} \right\}, \quad s > 0.$$

Theorem 3.2. *If B is a centered Gaussian process with stationary increments, bounded sample paths and variance function σ^2, then for any locally bounded measurable $f : [0, \infty] \to \mathbb{R}$ such that $f(t) > a \ln t, a > 1, t > 0$ we have that $\mathcal{P}_f([0, \infty)) < \infty$ and moreover*

$$\mathcal{P}_f[(0, \infty)) \le \inf_{\delta > 0} \mathcal{W}(\delta) \sum_{i=0}^{\infty} \sup_{i \in [0,1]} e^{-f(i\delta)} < \infty. \tag{3.6}$$

Remark 3.3. *With B specified in Theorem 3.2, for any $T > 0$, [35][Thm 1] implies*

$$\ln \mathcal{W}(T) \le \mathbb{E}\left\{ \sup_{t \in [0,T]} B(t) \right\} < \infty. \tag{3.7}$$

Consequently, by Example 1

$$\liminf_{T \to \infty} \left[\mathbb{E}\left\{ \sup_{t \in [0,T]} B(t) \right\} - \ln T \right] \ \ge \ \ln \mathcal{H} > -\infty, \tag{3.8}$$

provided that (2.11) holds.

3.3. **Growth of supremum.** Let X be a separable max-stable stationary process with locally bounded sample paths and spectral process Z such that (1.1) holds. For any $T > 0$, let $A_T := M_X(T) T^{-1/\alpha}, T > 0$. Since A_T is non-negative, then for any $\beta \in (0, \alpha)$ we have

$$\begin{aligned} \mathbb{E}\{A_T^\beta\} &= \int_0^\infty \mathbb{P}\{A_T > x^{1/\beta}\} dx \\ &= \int_0^\infty \left(1 - e^{-\mathbb{E}\{\sup_{t \in [0,T]} Z^\alpha(t)\}/(T x^{\alpha/\beta})}\right) dx \\ &= \left(\mathbb{E}\left\{ \sup_{t \in [0,T]} Z^\alpha(t) \right\} / T\right)^{\beta/\alpha} \Gamma(1 - \beta/\alpha), \end{aligned} \tag{3.9}$$

where $\Gamma(\cdot)$ stands for the Euler Gamma function and in (3.9) we used (1.3). Consequently, as $T \to \infty$

$$\mathbb{E}\{A_T^\beta\} \to \mathcal{H}^{\beta/\alpha} \Gamma(1 - \beta/\alpha) < \infty. \tag{3.10}$$

A direct implication of (1.8), (1.9) and (3.10) is the following result.

Proposition 3.4. *If $X(t), t \in \mathcal{T}$ where $\mathcal{T} = \mathbb{R}$ or $\mathbb{Z}$ is a max-stable stationary process as above, then (3.10) holds and moreover $A_T^\beta, T > 0$ is uniformly integrable for any $\beta \in (0, \alpha)$.*

Remark 3.5. *For X a $S\alpha S$ stationary random field (3.10) has been shown in [26][Thm 3.1]. Extension of (3.10) to stationary max-stable random fields is straightforward and omitted here.*

4. Proofs

Proof of Theorem 2.1 We adapt the arguments of the proof of [2][Thm 2.1] for our max-stable process. Note first that by (1.11), in view of [6] X has a dissipative Rosiński representation with some process L, which by the construction in the aforementioned paper is stochastically continuous and locally bounded (these properties are inherited from X). Therefore there exists jointly measurable and separable version of L which is locally bounded; we shall consider this version below.

Step 1: Since Z is given by (2.4), and moreover Z is locally bounded, then by (1.4), for any $T > 0$, we have that $\mathcal{H} \le \mathbb{E}\{M_{Z^\alpha}(T)\}/T$. Moreover, by stationarity of X together with the local boundedness of the sample paths

$$H(T) := \mathbb{E}\left\{\sup_{0 \le t \le T} Z^\alpha(t)\right\} = \mathbb{E}\left\{\sup_{0 \le t \le T} Z^\alpha(-t)\right\} \in (0, \infty).$$

Consequently, as in [2] we obtain the following lower bound

$$
\begin{aligned}
\infty > H(2) &= \mathbb{E}\left\{\sup_{0 \le t \le 2} Z^\alpha(-t)\right\} \\
&= c\mathbb{E}\left\{\sup_{0 \le t \le 2} \frac{B^{\mathcal{N}} L^\alpha(-t)}{p(\mathcal{N})}\right\} \\
&= c\mathbb{E}\left\{\int_{\mathbb{R}} \sup_{0 \le t \le 2} L^\alpha(x-t)\lambda(dx)\right\} \qquad (4.1)\\
&= c\sum_{i \in \mathbb{Z}} \mathbb{E}\left\{\int_i^{i+1} \sup_{0 \le t \le 2} L^\alpha(x-t)\lambda(dx)\right\} \\
&\ge c\sum_{i \in \mathbb{Z}} \mathbb{E}\left\{\sup_{i-1 \le t \le i} L^\alpha(t)\right\} \int_i^{i+1} \lambda(dx) \\
&\ge c\mathbb{E}\left\{\sup_{t \in \mathbb{R}} L^\alpha(t)\right\}.
\end{aligned}
$$

Further, since we assume that $X(0)$ has Fréchet Φ_α distribution, by (1.2)

$$-\ln \mathbb{P}\{X(0) \le 1\} = \mathbb{E}\{Z^\alpha(0)\} = c\mathbb{E}\left\{\int_{\mathbb{R}} (B^t L^\alpha)(0)\lambda(dt)\right\} = c\mathbb{E}\left\{\int_{\mathbb{R}} L^\alpha(t)\lambda(dt)\right\} > 0.$$

Hence we conclude that

$$\mathbb{E}\left\{\sup_{t\in\mathbb{R}} L^\alpha(t)\right\} \in (0,\infty). \tag{4.2}$$

Step 2: If for some positive M we have $\mathbb{P}\{\sup_{|t|\geq M} L(t)=0\}=1$, then for $T>2M$

$$\begin{aligned}
\frac{H(T)}{T} &= \frac{c}{T}\int_{\mathbb{R}} \mathbb{E}\left\{\sup_{0\le s\le T} L^\alpha(x+s)\right\}\lambda(dx)\\
&= \frac{c}{T}\int_{\mathbb{R}} \mathbb{E}\left\{\sup_{t\le v\le T+t} L^\alpha(v)\right\}\lambda(dt)\\
&= \frac{c}{T}\int_{-M-T}^{M} \mathbb{E}\left\{\sup_{t\le v\le T+t} L^\alpha(v)\right\}\lambda(dt)\\
&= c\int_{-M/T}^{1+M/T} \mathbb{E}\left\{\sup_{-Tx\le v\le T(1-x)} L^\alpha(v)\right\}\lambda_T(dx)/T\\
&= c(1-2M/T)\mathbb{E}\left\{\sup_{-M\le v\le M} L^\alpha(v)\right\}\int_0^1 \lambda_T(dx)/T\\
&\quad +O(M/T)\\
&\to c\mathbb{E}\left\{\sup_{v\in\mathbb{R}} L^\alpha(v)\right\}, \quad T\to\infty,
\end{aligned}$$

where $\lambda_T(dx)=\lambda(Tdx)$.

Step 3: For $M>0$ set $L_M(t)=L(t)\mathbb{I}(|t|\le M)$. Using (4.2) and applying the bounded convergence theorem we obtain

$$\lim_{M\to\infty}\Delta_M := \lim_{M\to\infty}\left[\mathbb{E}\left\{\sup_{t\in\mathbb{R}} L_M^\alpha(t)\right\} - \mathbb{E}\left\{\sup_{t\in\mathbb{R}} L^\alpha(t)\right\}\right]=0. \tag{4.3}$$

Further by (4.1), again the bounded convergence theorem yields

$$\lim_{M\to\infty}\mathbb{E}\left\{\int_{\mathbb{R}}\sup_{t\in[0,1]}[L^\alpha(t-x)-L_M^\alpha(t-x)]\lambda(dx)\right\}=0. \tag{4.4}$$

Write next for $T>0$

$$\begin{aligned}
\left|\frac{H(T)}{T}-\mathcal{H}\right| &\le \frac{1}{T}\left|H(T)-c\mathbb{E}\left\{\int_{\mathbb{R}}\sup_{0\le s\le T} L_M^\alpha(x+t)\lambda(dx)\right\}\right|\\
&\quad +c\left|\frac{1}{T}\mathbb{E}\left\{\int_{\mathbb{R}}\sup_{0\le t\le T} L_M^\alpha(x+t)\lambda(dx)\right\}-\mathbb{E}\left\{\sup_{t\in\mathbb{R}} L_M^\alpha(t)\right\}\right|+c\left|\Delta_M\right|.
\end{aligned}$$

The second and third terms converge as $T\to\infty$ and $M\to\infty$, respectively by Step 2 and (4.3). Further by (4.4) (write $[T]$ for the smallest integer larger than T)

$$\frac{1}{T}\left|H(T)-c\mathbb{E}\left\{\int_{\mathbb{R}}\sup_{0\le t\le T} L_M^\alpha(t-x)\lambda(dx)\right\}\right|$$

$$
\begin{aligned}
&\le \frac{c}{T}\mathbb{E}\Big\{\int_{\mathbb{R}} \sup_{0\le t\le T} [L^\alpha(t-x) - L_M^\alpha(t-x)]\lambda(dx)\Big\}\\
&\le \frac{c}{T}\mathbb{E}\left\{\int_{\mathbb{R}} \sum_{1\le j\le [T]} \sup_{j-1\le t\le j} [L^\alpha(t-x) - L_M^\alpha(t-x)]\lambda(dx)\right\}\\
&\le \frac{c[T]}{T}\mathbb{E}\Big\{\int_{\mathbb{R}} \sup_{0\le t\le 1} [L^\alpha(t-x) - L_M^\alpha(t-x)]\lambda(dx)\Big\}\\
&\to 0, \quad M\to\infty,
\end{aligned}
$$

hence the proof is complete. □

Proof of Theorem 2.3 We consider first the case that $\mathbb{P}\{\mathcal{S}(Z)<\infty\}=1$. The assumption that $\mathbb{E}\{Z^\alpha(t)\}=1, t\in\mathbb{R}$ implies that $\mathcal{S}(Z)>0$ almost surely, since by Fubini theorem

$$
\mathbb{E}\{\mathcal{S}(Z)\} \ge \mathbb{E}\Big\{\int_a^b Z^\alpha(t)\lambda(dt)\Big\} = \int_a^b \mathbb{E}\{Z^\alpha(t)\}\lambda(dt) = b-a.
$$

Hence almost surely $\mathcal{S}(Z)\in(0,\infty)$.

The map $H:D\to[0,\infty]$ defined by $H(f)\to\int_{\mathbb{R}}|f(t)|^\alpha\lambda(dt)=:\mathcal{S}(f)$ is $\mathcal{D}/\mathcal{B}(\mathbb{R})$ measurable since $\mathcal{S}(f)$ for f càdlàg is determined by $f(t), t\in\mathcal{Q}$ with $\mathcal{Q}$ the set of all rational numbers, and $\mathcal{D}$ coincides with the σ-algebra $\sigma(\pi_t, t\in\mathcal{Q})$, see [29][Prop 7.1]. Hence we have

$$
\begin{aligned}
1 &= \mathbb{E}\{Z^\alpha(0)\} = \mathbb{E}\{Z^\alpha(0)\mathbb{I}(\mathcal{S}(Z)\in(0,\infty))\}\\
&= \mathbb{E}\{Z^\alpha(0)\mathbb{I}(0<Z(0)<\infty, \mathcal{S}(Z)\in(0,\infty))\}\\
&= \mathbb{E}\{Z^\alpha(0)\mathbb{I}(\mathcal{S}(Z/Z(0))\in(0,\infty))\}\\
&= \mathbb{E}\{\mathbb{I}(\mathcal{S}(\Theta)\in(0,\infty))\} \qquad (4.5)
\end{aligned}
$$

implying

$$
\mathbb{P}\{\mathcal{S}(Z)\in(0,\infty)\}=1 \Leftrightarrow \mathbb{P}\{\mathcal{S}(\Theta)\in(0,\infty)\}=1. \qquad (4.6)
$$

Consequently, the random shape function L given by

$$
L(t) = \frac{\Theta(t)}{(\mathcal{S}(\Theta))^{1/\alpha}}, \quad t\in\mathbb{R}
$$

is well-defined with càdlàg sample paths and $\mathcal{S}(L)=1$ almost surely.

We continue by showing that $\widetilde{Z} := (p(\mathcal{N}))^{-1/\alpha}B^{\mathcal{N}}L$ is a spectral process such that its corresponding max-stable process $\widetilde{X}$ with de Haan representation (1.1) (taking $\widetilde{Z}$ instead of Z) is stationary. Since for any $h \in \mathbb{R}$

$$B^h \widetilde{Z} = (p(\mathcal{N}))^{-1/\alpha} B^{\mathcal{N}+h} L = (p_h(\mathcal{N}_h))^{-1/\alpha} B^{\mathcal{N}_h} L,$$

with $\mathcal{N}_h = \mathcal{N} + h$ which has density function $p_h(t) = p(t-h)$ it follows that $\widetilde{X}$ is stationary. Next, we prove that $\widetilde{X}$ has the same fidi's as X. By the stationarity, in view of (2.8) this follows if we show that the spectral tail process $\widetilde{\Theta}$ of $\widetilde{X}$ has the same fidi's as Θ. For any $A \in \mathcal{A}$

$$\begin{aligned}
\mathbb{P}\{\widetilde{\Theta} \in A\} &= \mathbb{E}\Big\{ \widetilde{Z}^\alpha(0) \mathbb{I}(\widetilde{Z}/\widetilde{Z}(0) \in A) \Big\} \\
&= \mathbb{E}\Big\{ \frac{(B^{\mathcal{N}}\Theta)^\alpha(0)}{\mathcal{S}(\Theta)} \mathbb{I}((B^{\mathcal{N}}\Theta)/(B^{\mathcal{N}}\Theta)(0) \in A, \mathcal{S}(Z) \in (0,\infty)) \Big\} \\
&= \int_{\mathbb{R}} \mathbb{E}\Big\{ \frac{(B^t\Theta)^\alpha(0)}{\mathcal{S}(B^t\Theta)} \mathbb{I}((B^t\Theta)/(B^t\Theta)(0) \in A, \mathcal{S}(B^t\Theta) \in (0,\infty)) \Big\} \lambda(dt) \\
&= \int_{\mathbb{R}} \mathbb{E}\big\{ H(B^t\Theta) \big\} \lambda(dt).
\end{aligned}$$

The functional $H(f) = F(f)$ is 0-homogeneous non-negative and $\mathcal{D}/\mathcal{B}(\mathbb{R})$ measurable (we use the convention $0\infty =: 0$). Since $\mathcal{S}(B^t f) = \mathcal{S}(f), t \in \mathbb{R}$, then applying Lemma 5.1 in Appendix we obtain

$$\begin{aligned}
\int_{\mathbb{R}} \mathbb{E}\big\{ H(B^t\Theta) \big\} \lambda(dt) &= \int_{\mathbb{R}} \mathbb{E}\{ Z^\alpha(t) H(Z) \} \lambda(dt) \\
&= \int_{\mathbb{R}} \mathbb{E}\Big\{ Z^\alpha(t) \frac{Z^\alpha(0)}{\mathcal{S}(Z)} \mathbb{I}(Z/Z(0) \in A, \mathcal{S}(Z) \in (0,\infty) \Big\} \lambda(dt) \\
&= \mathbb{E}\Big\{ \frac{\int_{\mathbb{R}} Z^\alpha(t)\lambda(dt)}{\mathcal{S}(Z)} Z^\alpha(0) \mathbb{I}(Z/Z(0) \in A, \mathcal{S}(Z) \in (0,\infty) \Big\} \\
&= \mathbb{E}\{ Z^\alpha(0) \mathbb{I}(Z/Z(0) \in A) \} \\
&= \mathbb{P}\{\Theta \in A\},
\end{aligned}$$

establishing that X has the same fidi's as $\widetilde{X}$ and the dissipative Rosiński representation of X with L constructed above.

Next, for a given spectral process Y we denote by $\mathcal{H}_Y$ the corresponding Pickands constant. Next, if $\mathbb{P}\{\mathcal{S}(Z) < \infty\} \in (0,1)$ by (2.2) we have

$$\mathcal{H} = a \lim_{T\to\infty} \mathbb{E}\Big\{ \sup_{t\in[0,T]} Z_C^\alpha(t)/a \Big\} = a\mathcal{H}_Y,$$

where $Y = Z_C/a^{1/\alpha}$ with $Z_C(t) = Z(t)\mathbb{I}(0 < \mathcal{S}(Z) < \infty)$ and $a = \mathbb{P}\{0 < \mathcal{S}(\Theta) < \infty\} \in (0,1)$. Note in passing that

$$\mathbb{E}\{Y^\alpha(0)\} = \mathbb{E}\{Z_C^\alpha(0)/a\} = \mathbb{E}\{Z^\alpha(0)\mathbb{I}(0 < \mathcal{S}(Z) < \infty)\}/a = 1.$$

The spectral tail process Θ_C of Y is calculated for any $A \in \mathcal{A}$ by

$$\begin{aligned}
\mathbb{P}\{\Theta_C \in A\} &= \mathbb{E}\left\{\frac{Z_C^\alpha(0)}{a}\mathbb{I}(Z_C/Z_C(0) \in A)\right\} \\
&= \mathbb{E}\left\{\frac{Z^\alpha(0)}{a}\mathbb{I}(Z/Z(0) \in A)\mathbb{I}(0 < \mathcal{S}(Z) < \infty)\right\} \\
&= \mathbb{E}\left\{\frac{\mathbb{I}(\Theta \in A)}{a}\mathbb{I}(0 < \mathcal{S}(Z) < \infty)\right\} \\
&= \mathbb{P}\{\Theta \in A | 0 < \mathcal{S}(\Theta) < \infty\}.
\end{aligned}$$

As above for the stationary max-stable process X_C^* with spectral process Y we have that it has a dissipative Rosiński representation with random shape function L_C given by

$$L_C(t) = \frac{\Theta_C(t)}{\mathcal{S}(\Theta_C)} = \frac{\Theta(t)}{\mathcal{S}(\Theta)}\Big|(0 < \mathcal{S}(\Theta) < \infty), \quad t \in \mathbb{R}.$$

Consequently, using further Theorem 2.1

$$\begin{aligned}
\mathcal{H} &= a\mathcal{H}_Y \\
&= a\frac{\mathbb{E}\{\sup_{t\in\mathbb{R}} L_C^\alpha(t)\}}{\mathbb{E}\{\mathcal{S}(L_C)\}} \\
&= a\frac{\mathbb{E}\{\sup_{t\in\mathbb{R}} \Theta_C^\alpha(t)/\mathcal{S}(\Theta_C)\}}{\mathbb{E}\{\mathcal{S}(\Theta_C/\mathcal{S}(\Theta_C))\}} \\
&= a\mathbb{E}\left\{\sup_{t\in\mathbb{R}} \frac{\Theta_C^\alpha(t)}{\mathcal{S}(\Theta_C)}\right\} \\
&= \mathbb{E}\left\{\sup_{t\in\mathbb{R}} \frac{\Theta^\alpha(t)}{\mathcal{S}(\Theta)}; 0 < \mathcal{S}(\Theta) < \infty\right\}
\end{aligned}$$

establishing the proof. $\square$

Proof of Theorem 3.1 In view of [36] both X_1 and X_2 are max-stable statoinary processes. Hence in view of (1.2), in order to show (3.5) we need to prove that for any compact set $K \subset \mathbb{R}$

$$\mathbb{E}\left\{\sup_{t\in K} Z_2(t)\right\} \le \mathbb{E}\left\{\sup_{t\in K} Z_1(t)\right\} < \infty \tag{4.7}$$

is valid with $Z_i(t) = e^{B_i(t) - \sigma_i^2(t)/2}$, $i = 1, 2$. Note in passing that the finiteness of $\mathbb{E}\{\sup_{t\in K} Z_1(t)\}$ follows from [35][Thm 1].

By the stationarity of increments of B_i's the variance functions $\sigma_i^2(t), t \in \mathbb{R}, i = 1, 2$ are negative definite functions. Consequently, by Schoenberg theorem, for each $u > 0$, $i = 1, 2$ the function

$$R_u^{(i)}(s,t) := \exp\left(-\frac{1}{2u^2}\sigma_i^2(s-t)\right), \quad s,t \in \mathbb{R}$$

is positive definite and thus a valid covariance function.

Let $W_u^{(i)}(t), t \in \mathbb{R}$, $u > 0$ be a family of separable centered stationary Gaussian processes with covariance functions

$$\mathrm{Cov}(W_u^{(i)}(s), W_u^{(i)}(t)) = R_u^{(i)}(s,t), \quad s,t \in \mathbb{R}$$

for $i = 1, 2$. Since by assumption $\sigma_1(t) \geq \sigma_2(t)$ for any $t \in \mathbb{R}$, then for any $s, t \in \mathbb{R}$ we have

$$R_u^{(1)}(s,t) \leq R_u^{(2)}(s,t).$$

Hence, for a given compact set $K \subset \mathbb{R}$, applying Slepian inequality, see e.g., [37][Thm 3] for any $u > 0$

$$\mathbb{P}\left\{\sup_{t\in K} W_u^{(1)}(t) > u\right\} \geq \mathbb{P}\left\{\sup_{t\in K} W_u^{(2)}(t) > u\right\}. \tag{4.8}$$

The definition of the covariance functions above yields for $i = 1, 2$

$$\lim_{u\to\infty} \sup_{s,t\in K} \left| \frac{1 - \mathrm{Cov}(W_u^{(i)}(s), W_u^{(i)}(t))}{\frac{1}{2u^2}\sigma_i^2(s-t)} - 1 \right| = 0.$$

Applying [38][Lem 6.1] for any $\eta > 0$ and any given compact $K \subset \mathbb{R}$ we obtain

$$\lim_{u\to\infty} \frac{\mathbb{P}\{\sup_{t\in\eta\mathbb{Z}\cap K} W_u^{(i)}(t) > u\}}{\Psi(u)} = \mathbb{E}\left\{\sup_{t\in\eta\mathbb{Z}\cap K} Z_i(t)\right\}, \tag{4.9}$$

where $\Psi(\cdot)$ is the tail distribution of the standard Normal random variable. Hence, by the separability and local boundedness of the sample paths, (4.8) combined with (4.9) implies (4.7).

The assumption $\mathbb{E}\{Z_i(t)\} = 1$ for any $t \in \mathbb{R}$ implies that the max-stable processes corresponding to Z_1 and Z_2 have unit Fréchet marginals Φ_1. The Pickands constants corresponding to Z_1 and Z_2 denoted by $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively exist and are finite since X_1 and X_2 are stationary (see the argument given for the derivation of (1.4)). Hence a direct application of (4.7) implies that $\mathcal{H}_1 \geq \mathcal{H}_2$. This completes the proof. $\square$

Proof of Theorem 3.2 First note that by the fact that $Z(t) = e^{B(t)-\sigma^2(t)/2}$ is Brown-Resnick stationary, we have that

$$\mathbb{E}\left\{\sup_{t\in[\delta i,\delta(i+1)]} Z(t)\right\} = \mathbb{E}\left\{\sup_{t\in[0,\delta]} Z(t)\right\} =: \mathcal{W}(\delta) \in (0,\infty)$$

for any $\delta \in \mathbb{R}, i > 0$. Hence for any positive δ using the assumption that f is locally bounded and $f(t) > a \ln t$ for all t large with some $a > 0$, we obtain

$$\begin{aligned}\mathbb{E}\Big\{\sup_{t\ge 0} Z(t)e^{-f(t)}\Big\} &\le \sum_{i=0}^{\infty}\mathbb{E}\Big\{\sup_{t\in[\delta i,\delta(i+1)]} Z(t)e^{-f(t)}\Big\}\\ &\le \sum_{i=0}^{\infty}\mathbb{E}\Big\{\sup_{t\in[\delta i,\delta(i+1)]} Z(t)\Big\}\sup_{t\in[\delta i,\delta(i+1)]} e^{-f(t)}\\ &= \mathcal{W}(\delta)\sum_{i=0}^{\infty}\sup_{t\in[i\delta,(i+1)\delta]} e^{f(t)} < \infty,\end{aligned}$$

hence the proof follows. □

5. Appendix: Tilt-Shift and inf-argmax fomula

Next we present the tilt-shift formula which is initially shown for the special case of Brown-Resnick max-stable processes with log-normal Z in [7]. The inf-argmax formula mentioned above is shown in [16], we present below a shorter proof.

Lemma 5.1. *Let $X(t), t \in \mathbb{R}$ be a max-stable process with Fréchet marginals Φ_α, de Haan representation* (1.1) *and càdlàg sample paths.*

i) If X is stationary, then for any non-negative 0-homogeneous $\mathcal{D}/\mathcal{B}(\mathbb{R})$-measurable functional H we have

$$\mathbb{E}\{Z^\alpha(h)H(Z)\} = \mathbb{E}\{H(B^h\Theta)\} = \mathbb{E}\{Z^\alpha(0)H(B^h Z)\}. \tag{5.1}$$

ii) If (5.1) *holds for any $h \in \mathbb{R}$, then X with representation* (1.1) *is stationary.*

Proof of Lemma 5.1 i) As in [8] we have that the stationarity of X implies the shift invariance of the exponent measure, i.e., for any $h \in \mathbb{R}, A \subset \mathcal{D}$

$$\nu(A) = \int_0^\infty \mathbb{P}\{uZ \in A\}\alpha u^{-\alpha-1}\,du = \int_0^\infty \mathbb{P}\{uB^hZ \in A\}\alpha u^{-\alpha-1}\,du.$$

If A is 0-homogeneous (meaning $cA = A, c > 0$) set in $\mathcal{D}$, then since further $\mathbb{E}\{Z^\alpha(h)\} = 1$ implies that $\mathbb{P}\{Z^\alpha(h) \in [0,\infty)\} = 1$ for any $h \in \mathbb{R}$ using $z^\alpha = \int_0^\infty \mathbb{I}(rz > 1)\alpha r^{-\alpha-1}\,dr$ valid for any $z \in (0,\infty)$ we obtain

$$\begin{aligned}\mathbb{E}\{Z^\alpha(h)\mathbb{I}(Z \in A)\} &= \mathbb{E}\{Z^\alpha(h)\mathbb{I}(Z \in A, Z(h) \in (0,\infty)\}\\ &= \mathbb{E}\Big\{\int_0^\infty \mathbb{I}(rZ(h) > 1)\mathbb{I}(rZ \in A)\alpha r^{-\alpha-1}\,dr\Big\}\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\Big\{\int_0^\infty \mathbb{I}(rZ(0)>1, rB^hZ\in A)\alpha r^{-\alpha-1}\,dr\Big\}\\
&= \mathbb{E}\Big\{\int_0^\infty \mathbb{I}(rZ(0)>1)\alpha r^{-\alpha-1}\,dr\mathbb{I}(B^hZ\in A)\Big\}\\
&= \mathbb{E}\big\{Z^\alpha(0)\mathbb{I}(B^hZ\in A)\big\}\\
&= \mathbb{E}\big\{Z^\alpha(0)\mathbb{I}(B^hZ/Z(0)\in A)\big\}\\
&= \mathbb{E}\big\{\mathbb{I}(B^h\Theta\in A)\big\},
\end{aligned}
$$

hence the claim for any 0-homogeneous $\mathcal{D}/\mathcal{B}(\mathbb{R})$-measurable functional H follows easily.

ii) If (5.1) holds, then for any $n\ge 1, t_i\in\mathbb{R}, x_i>0, i\le n$ since $\inf\arg\max$ functional is 0-homogeneous and $\mathcal{D}/\mathcal{B}(\mathbb{R})$-measurable, by (1.2) we have

$$
\begin{aligned}
-\ln\mathbb{P}\{X(t_i)\le x_i, 1\le i\le n\} &= \mathbb{E}\Big\{\max_{1\le i\le n} x_i^{-\alpha}Z^\alpha(t_i)\Big\}\\
&= \sum_{k=1}^n x_k^{-\alpha}\mathbb{E}\Big\{Z^\alpha(t_k)\mathbb{I}\Big(\inf\arg\max_{1\le i\le n} Z^\alpha(t_i)/x_i^\alpha = k\Big)\Big\}\\
&=: \sum_{k=1}^n x_k^{-\alpha}\mathbb{E}\{Z^\alpha(t_k)F_k(Z)\}\\
&= \sum_{k=1}^n x_k^{-\alpha}\mathbb{E}\big\{F_k(B^{t_k}\Theta)\big\}\\
&= \sum_{k=1}^n x_k^{-\alpha}\mathbb{P}\Big\{\inf\arg\max_{1\le i\le n}\Theta^\alpha(t_i-t_k)/x_i^\alpha = k\Big\},
\end{aligned}
$$

where we used (5.1) in the second last line above. Consequently, X is stationary and thus the proof if complete. □

Acknowledgments

Many thanks to Parthanil Roy for discussions and suggestion of the key reference [2]. We thank the referees for numerous suggestions that improved the original manuscript. EH was supported by SNSF Grant 200021-175752/1. KD was partially supported by NCN Grant No 2015/17/B/ST1/01102 (2016-2019).

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Krzysztof Dębicki, Mathematical Institute, University of Wrocław, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

E-mail address: `Krzysztof.Debicki@math.uni.wroc.pl`

Enkelejd Hashorva, Department of Actuarial Science, University of Lausanne,, Chamberonne 1015 Lausanne, Switzerland

E-mail address: `Enkelejd.Hashorva@unil.ch`