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EXTREMES AND LIMIT THEOREMS FOR DIFFERENCE OF CHI-TYPE PROCESSES *

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Abstract. Let $\{\zeta_{m,k}^{(\kappa)}(t), t \geq 0\}, \kappa > 0$ be random processes defined as the differences of two independent stationary chi-type processes with m and k degrees of freedom. In applications such as physical sciences and engineering dealing with structure reliability, of interest is the approximation of the probability that the random process $\zeta_{m,k}^{(\kappa)}$ stays in some safety region up to a fixed time T. In this paper we derive the asymptotics of $\mathbb{P}\left\{\sup_{t\in[0,T]} \zeta_{m,k}^{(\kappa)}(t) > u\right\}, u \to \infty$ under some assumptions on the covariance structures of the underlying Gaussian processes. Further, we establish a Berman sojourn limit theorem and a Gumbel limit result.

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1. Introduction

Let $\boldsymbol{X}(t) = (X_1(t), \ldots, X_{m+k}(t)), t \geq 0, m \geq 1, k \geq 0$ be a vector process with independent components which are centered stationary Gaussian processes with almost surely (a.s.) continuous sample paths and covariance functions satisfying

$$r_i(t) = 1 - C_i\,|t|^\alpha + o(|t|^\alpha), \quad t \to 0 \quad \text{and} \quad r_i(t) < 1, \quad \forall t\neq 0,\ 1 \leq i \leq m+k, \tag{1}$$

where $\alpha \in (0, 2]$ and $\boldsymbol{C} := (C_1, \ldots, C_{m+k}) \in (0, \infty)^{m+k}$. Define in the following $\left\{\zeta_{m,k}^{(\kappa)}(t), t \geq 0\right\}$, $\kappa > 0$ by

$$\zeta_{m,k}^{(\kappa)}(t) := \left(\sum_{i=1}^{m} X_i^2(t)\right)^{\kappa/2} - \left(\sum_{i=m+1}^{m+k} X_i^2(t)\right)^{\kappa/2} =: |\boldsymbol{X}^{(1)}(t)|^\kappa - |\boldsymbol{X}^{(2)}(t)|^\kappa, \quad t \geq 0. \tag{2}$$

In this paper we shall investigate the asymptotics of

$$\mathbb{P}\left\{\sup_{t\in[0,T]} \zeta_{m,k}^{(\kappa)}(t) > u\right\}, \quad u \to \infty,$$

with some constant $T > 0$.

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Our study of the above problem of considerable interest in engineering sciences dealing with the safety of structures; see, e.g., [18–20] and the references therein. More specifically, of interest is the probability that the Gaussian vector process $\boldsymbol{X}$ exits a predefined safety region $\boldsymbol{S}_u \subset \mathbb{R}^{m+k}$ up to the time T, namely

$$\mathbb{P}\left\{\boldsymbol{X}(t) \notin \boldsymbol{S}_u, \quad \text{for some } t \in [0,T]\right\}.$$

Various types of safety regions $\boldsymbol{S}_u$ have been considered for smooth Gaussian vector processes in the aforementioned papers. Particularly, a safety region given by a ball centered at 0 with radius $u > 0$

$$\boldsymbol{B}_u = \left\{(x_1, \ldots, x_{m+k}) \in \mathbb{R}^{m+k} : \left(\sum_{i=1}^{m+k} x_i^2\right)^{1/2} \le u\right\}$$

has been extensively studied; see, e.g., [2,6,15,22]. Referring to [1,2], we know that for $k = 0$

$$\begin{aligned}\mathbb{P}\left\{\boldsymbol{X}(t) \notin \boldsymbol{B}_u, \quad \text{for some } t \in [0,T]\right\} &= \mathbb{P}\left\{\sup_{t\in[0,T]} |\boldsymbol{X}(t)| > u\right\} \\ &= T H_{\alpha,1}^{m,0}(\boldsymbol{C}) u^{\frac{2}{\alpha}} \mathbb{P}\left\{|\boldsymbol{X}(0)| > u\right\}(1+o(1)), \quad u \to \infty,\end{aligned}$$

where $H_{\alpha,1}^{m,0}(\boldsymbol{C})$ is a positive constant (see (5) below for a precise definition). Very recently [23] obtained the tail asymptotics of the product of two Gaussian processes which has the same tail asymptotic behavior as $\sup_{t\in[0,T]} \zeta_{1,1}^{(2)}(t)$. Our first result extends the findings of [2,23] and suggests an asymptotic approximation for the exit probability of the Gaussian vector process $\boldsymbol{X}$ from the safety regions $\boldsymbol{S}_u^{(\kappa)}$ defined by

$$\boldsymbol{S}_u^{(\kappa)} = \left\{(x_1, \ldots, x_{m+k}) \in \mathbb{R}^{m+k} : |\boldsymbol{x}^{(1)}|^\kappa - |\boldsymbol{x}^{(2)}|^\kappa \le u\right\}.$$

Since chi-type processes appear naturally as limiting processes (see, e.g., [4,5,21]), when one considers two independent asymptotic models, the study of the supremum of the difference of the two chi-type processes is of some interest in mathematical statistics and its applications. Another motivation for considering the tail asymptotics of the supremum of the difference of chi-type processes is from ruin theory, where the tail asymptotics can be considered as the expansion of the ruin probability since the net loss of an insurance company is ususally modeled by the difference of two positive random processes; see, e.g., [10].
Although for $k \ge 1$ the random process $\zeta_{m,k}^{(\kappa)}$ is not Gaussian and the analysis of the supremum can not be transformed into the study of the supremum of a related Gaussian random field (which is the case for chi-type processes; see, e.g., [11,18–20,22,25]), it turns out that it is possible to apply the techniques for dealing with extremes of stationary processes developed mainly in [2,6,7]. In the second part of Section 2 we derive a sojourn limit theorem for $\zeta_{m,k}^{(\kappa)}$. Further, we show a Gumbel limit theorem for the supremum of $\zeta_{m,k}^{(\kappa)}$ over an increasing infinite interval. We refer to [2,4,5,16,22,26] for results on the Gumbel limit theorem for Gaussian processes and chi-type processes.
Brief outline of the paper: our main results are stated in Section 2. In Section 3 we present proofs of Theorem 2.1, Theorem 2.2 and Theorem 2.3 followed then by an appendix containing the somewhat complicated proofs of three lemmas utilized in Section 3.

2. Main Results

We start by introducing some notation. Let $\{Z(t), t \ge 0\}$ be a standard fractional Brownian motion (fBm) with Hurst index $\alpha/2 \in (0,1]$, i.e., it is a centered Gaussian process with a.s. continuous sample paths and covariance function

$$\mathrm{Cov}(Z(s), Z(t)) = \frac{1}{2}\Big(s^\alpha + t^\alpha - |s-t|^\alpha\Big), \quad s, t \ge 0.$$

In the following, let $\{Z_i(t), t \ge 0\}, 1 \le i \le m+k$ be independent copies of Z and define $\mathcal{W}_\kappa$ to be a Gamma distributed random variable with parameter $(k/\kappa, 1)$. Further let $\boldsymbol{O}_1 = (O_1, \ldots, O_m), \boldsymbol{O}_2 = (O_{m+1}, \ldots, O_{m+k})$ denote two random vectors uniformly distributed on the unit sphere of $\mathbb{R}^m$ and $\mathbb{R}^k$, respectively. Hereafter we shall suppose that $\boldsymbol{O}_1, \boldsymbol{O}_2, \mathcal{W}_\kappa$ and Z_i's are mutually independent. Define for $m \ge 1,\ k \ge 0,\ \kappa > 0$

$$\eta_{m,k}^{(\kappa)}(t) = \widetilde{Z}_{m,k}^{(\kappa)}(t) + E, \quad t \ge 0, \tag{3}$$

where E is a unit mean exponential random variable being independent of all the other random elements involved, and (recall $\boldsymbol{C}=(C_1,\ldots,C_{m+k})$ given in (1))

$$\widetilde{Z}^{(\kappa)}_{m,k}(t)=\begin{cases} L_1(t), & \kappa>1,\\ L_1(t)+L_2(t), & \kappa=1,\\ L_2(t), & \kappa<1,\end{cases}\quad\text{with}\quad \begin{cases} L_1(t)=\sum_{i=1}^m\sqrt{2C_i}O_iZ_i(t)-\left(\sum_{i=1}^m C_iO_i^2\right)t^\alpha,\\ L_2(t)=\mathcal{W}_\kappa-\Big(\mathcal{W}_\kappa^{2/\kappa}+2(\mathcal{W}_\kappa/\kappa)^{1/\kappa}\sum_{i=m+1}^{m+k}\sqrt{2C_i}O_iZ_i(t)\\ \qquad +2\kappa^{-2/\kappa}\sum_{i=m+1}^{m+k}C_iZ_i^2(t)\Big)^{\kappa/2},\end{cases}\tag{4}$$

and the convention that $\sum_{i=m+1}^m c_i=0$. In addition, denote by $\Gamma(\cdot)$ the Euler Gamma function. We state next our main result.

Theorem 2.1. *If $\{\zeta^{(\kappa)}_{m,k}(t),t\ge0\}$ is given by (2) with the involved Gaussian processes X_i's satisfying (1), then, for any $T>0$*

$$\mathbb{P}\left\{\sup_{t\in[0,T]}\zeta^{(\kappa)}_{m,k}(t)>u\right\}=TH^{m,k}_{\alpha,\kappa}(\boldsymbol{C})\mathbb{P}\left\{\zeta^{(\kappa)}_{m,k}(0)>u\right\}(1+o(1))\begin{cases} u^{\frac{2}{\alpha\kappa}}, & \kappa\ge1,\\ u^{\frac{2(2/\kappa-1)}{\alpha\kappa}}, & \kappa<1\end{cases}$$

holds as $u\to\infty$, where, with $\eta^{(\kappa)}_{m,k}$ given by (3),

$$H^{m,k}_{\alpha,\kappa}(\boldsymbol{C})=\lim_{a\downarrow0}\frac{1}{a}\mathbb{P}\left\{\sup_{j\ge1}\eta^{(\kappa)}_{m,k}(aj)\le0\right\}\in(0,\infty).\tag{5}$$

Remarks: a) The tail asymptotics of the Gaussian chaos $\zeta^{(\kappa)}_{m,k}(0)$ is discussed in Lemma 3.1 below.
b) The most obvious choice of κ is 1, which corresponds to the difference of L_2-norm of two independent multivariate Gaussian processes. For the case $\kappa=2$ and $m=k=1$ the problem was (implicitly) investigated by considering the product of two independent Gaussian processes in the recent contribution [23].
c) Since $\boldsymbol{O}_1$ is uniformly distributed on the unit sphere of $\mathbb{R}^m$, we have, for $\kappa>1$ and $\boldsymbol{C}=\boldsymbol{1}$, that $\eta^{(\kappa)}_{m,k}(t)\overset{d}{=}\sqrt{2}Z(t)-t^\alpha+E$. In such a case, the constant $H^{m,k}_{\alpha,\kappa}(\boldsymbol{1})$ coincides with the classical Pickands constant H_α (see, e.g., [3]). Approximation of Pickands constant H_α has been considered by a number of authors; see the recent contribution [9] which gives some simulation algorithms. Precise estimation of the general Pickands constant $H^{m,k}_{\alpha,\kappa}(\boldsymbol{C})$ seems to be hard to find, due to the complexity of the process η. However, some bounds for it could be relatively easier to derive; this will be addressed in a forthcoming project.
d) We see from Theorem 2.1 and Lemma 3.1 that, if $\kappa>2$, then, for any $m,k\ge1$

$$\mathbb{P}\left\{\sup_{t\in[0,T]}\zeta^{(\kappa)}_{m,k}(t)>u\right\}=\mathbb{P}\left\{\sup_{t\in[0,T]}\zeta^{(\kappa)}_{m,0}(t)>u\right\}(1+o(1))$$

holds as $u\to\infty$, which means that $X_{m+1},\ldots,X_{m+k}$ do not influence the tail asymptotic of $\sup_{t\in[0,T]}\zeta^{(\kappa)}_{m,k}(t)$. This is not so surprising as the tail asymptotic behavior of $\zeta^{(\kappa)}_{m,k}(0)$ is subexponential.
Next, we consider the sojourn time of $\zeta^{(\kappa)}_{m,k}$ above a threshold $u>0$ in the time interval $[0,t]$ defined by

$$L^{(\kappa)}_{m,k,t}(u)=\int_0^t\mathbb{I}\{\zeta^{(\kappa)}_{m,k}(s)>u\}\,ds,\quad t>0.$$

Our second result below establishes a Berman sojourn limit theorem for $\zeta^{(\kappa)}_{m,k}$. See [6] for related discussions on sojourn times of Gaussian processes and related processes.

Theorem 2.2. *Under the assumptions and notation of Theorem 2.1, we have, for any $t>0$*

$$\int_x^\infty\mathbb{P}\left\{u^{\frac{2\tau}{\alpha\kappa}}L^{(\kappa)}_{m,k,t}(u)>y\right\}dy=\mathbb{E}\left\{L^{(\kappa)}_{m,k,t}(u)\right\}\Upsilon_\kappa(x)(1+o(1))\begin{cases} u^{\frac{2}{\alpha\kappa}}, & \kappa\ge1,\\ u^{\frac{2(2/\kappa-1)}{\alpha\kappa}}, & \kappa<1\end{cases}$$

holds as $u\to\infty$ for all continuity point $x>0$ of $\Upsilon_\kappa(x):=\mathbb{P}\left\{\int_0^\infty\mathbb{I}\{\eta^{(\kappa)}_{m,k}(s)>0\}\,ds>x\right\}$, and $\tau=2/\kappa-1$ for $\kappa\in(0,1)$, and 1 otherwise.

In the following, we derive a Gumbel limit theorem for $\sup_{t\in[0,T]} \zeta_{m,k}^{(\kappa)}(t)$ under a linear normalization, which is also of interest in extreme value analysis and statistical tests. We refer to [5, 7, 14, 16] for its applications in deriving approximations of the critical values of the proposed test statistics.

Theorem 2.3. *Under the assumptions and notation of Theorem 2.1, if further the following Berman-type condition*

$$\lim_{t\to\infty} \max_{1\le l\le m+k} |r_l(t)|(\ln t)^c = 0, \quad \text{with } c := \begin{cases} 2/\kappa - 1, & 0<\kappa<1, \\ 1, & 1\le \kappa \le 2, \\ k+1-2k/\kappa, & \kappa > 2 \end{cases} \tag{6}$$

holds, then

$$\lim_{T\to\infty} \sup_{x\in\mathbb{R}} \left| \mathbb{P}\left\{ a_T^{(\kappa)} \Big(\sup_{t\in[0,T]} \zeta_{m,k}^{(\kappa)}(t) - b_T^{(\kappa)} \Big) \le x \right\} - \exp\left(-e^{-x}\right) \right| = 0,$$

where, for $T > e$

$$a_T^{(\kappa)} = \frac{(2\ln T)^{1-\kappa/2}}{\kappa}, \quad b_T^{(\kappa)} = (2\ln T)^{\kappa/2} + \frac{\kappa}{2(2\ln T)^{1-\kappa/2}} \left(K_0 \ln\ln T + \ln D_0\right), \tag{7}$$

with

$$D_0 = \left(\frac{H_{\alpha,\kappa}^{m,k}(\boldsymbol{C})}{\Gamma(m/2)\Gamma(k/2)} \right)^2 \times \begin{cases} 2^{\frac{2}{\alpha}\left(\frac{2}{\kappa}-1\right)+2\left(1-\frac{k}{\kappa}\right)} \left(\Gamma\left(\frac{k}{\kappa}\right)\kappa^{(k/\kappa-1)}\right)^2, & 0<\kappa\le 1, \\ 2^{\frac{2}{\alpha}+2\left(1-\frac{k}{\kappa}\right)} \left(\Gamma\left(\frac{k}{\kappa}\right)\kappa^{(k/\kappa-1)}\right)^2, & 1<\kappa<2, \\ 2^{\frac{2}{\alpha}-2} \left(\Gamma\left(\frac{k}{2}\right)\right)^2, & \kappa = 2, \\ 2^{\frac{2}{\alpha}} \left(\Gamma\left(\frac{k}{2}\right)\right)^2, & \kappa > 2 \end{cases}$$

$$K_0 = \begin{cases} m-2+(2/\alpha)(2/\kappa-1)+k(1-2/\kappa), & 0<\kappa\le 1, \\ m-2+2/\alpha+k(1-2/\kappa), & 1<\kappa<2, \\ m-2+2/\alpha, & \kappa\ge 2. \end{cases}$$

Under the assumptions of Theorem 2.3, we have the following convergence in probability (denoted by $\overset{p}{\to}$)

$$\frac{\sup_{t\in[0,T]} \zeta_{m,k}^{(\kappa)}(t)}{(2\ln T)^{\kappa/2}} \overset{p}{\to} 1, \quad T\to\infty,$$

which follows from the fact that $\lim_{T\to\infty} b_T^{(\kappa)}/(2\ln T)^{\kappa/2} = 1$ and that $a_T^{(\kappa)}$ is bounded away from zero, together with elementary considerations. In several cases such a convergence in probability can be strengthened to the pth mean convergence which is referred to as the Seleznjev pth mean convergence since the idea was first suggested by Seleznjev in [24], see also [13]. In order to show the Seleznjev pth mean convergence of crucial importance is the Piterbarg inequality (see [22], Theorem 8.1). Since the Piterbarg inequality holds also for chi-square processes (see [25], Proposition 3.2), using further the fact that

$$\zeta_{m,k}^{(\kappa)}(t) \le |\boldsymbol{X}^{(1)}(t)|^\kappa, \quad t\ge 0,$$

we immediately get the Piterbarg inequality for the difference of chi-type processes by simply applying the aforementioned proposition. Specifically, under the assumptions of Theorem 2.3 for any $T>0$ and all large u

$$\mathbb{P}\left\{ \sup_{t\in[0,T]} \zeta_{m,k}^{(\kappa)}(t) > u \right\} \le KTu^\beta \exp\left(-\frac{1}{2}u^{2/\kappa}\right),$$

where K and β are two positive constants not depending on T and u. Note that the above result also follows immediately from Theorem 2.1 combined with Lemma 3.1 below. Hence utilizing Lemma 4.5 in [25] we arrive at our last result.

Corollary 2.4. *(Seleznjev pth mean theorem) Under the assumptions of Theorem 2.3, we have, for any $p > 0$*

$$\lim_{T\to\infty} \mathbb{E}\left\{\left(\frac{\sup_{t\in[0,T]} \zeta^{(\kappa)}_{m,k}(t)}{(2\ln T)^{\kappa/2}}\right)^p\right\} = 1.$$

3. Further Results and Proofs

Before presenting the proof of Theorem 2.1 we first give some preliminary lemmas. Hereafter we use the same notation and assumptions as in Section 1. By $\overset{d}{\to}$ and $\overset{d}{=}$ we shall denote the convergence in distribution (or the convergence of finite dimensional distributions if both sides of it are random processes) and equality in distribution function, respectively. Further, we write $f_\xi(\cdot)$ for the pdf of a random variable ξ and write $h_1 \sim h_2$ if two functions $h_i(\cdot), i = 1,2$ are such that h_1/h_2 goes to 1 as the argument tends to some limit. For simplicity we shall denote, with $\kappa > 0$ and $\tau = 2\max(1/\kappa - 1, 0) + 1$,

$$q_\kappa = q_\kappa(u) = u^{-2\tau/(\alpha\kappa)}, \quad w_\kappa(u) = \frac{1}{\kappa}u^{2/\kappa-1}, \quad u > 0.$$

In the proofs of Lemmas 3.1–3.3, we denote $u_{\kappa,x} = u + x/w_\kappa(u)$ for all $u, x > 0$.

Lemma 3.1. *Let $\{\zeta^{(\kappa)}_{m,k}(t),\ t \ge 0\}$ be given by (2). For all integers $m \ge 1, k \ge 0$ we have as $u \to \infty$*

$$\mathbb{P}\left\{\zeta^{(\kappa)}_{m,k}(0) > u\right\} \sim \frac{f_{\zeta^{(\kappa)}_{m,k}(0)}(u)}{w_\kappa(u)} \sim \frac{2^{2-(m+k)/2}}{\kappa^2\Gamma(k/2)\Gamma(m/2)} \frac{u^{m/\kappa-1}}{w_\kappa(u)} \exp\left(-\frac{1}{2}u^{2/\kappa}\right) \begin{cases} \frac{\Gamma(k/\kappa)}{(w_\kappa(u))^{k/\kappa}}, & \kappa < 2, \\ \Gamma(k/2), & \kappa = 2, \\ \kappa 2^{k/2-1}\Gamma(k/2), & \kappa > 2, \end{cases}$$

where $\Gamma(k/\kappa)/\Gamma(k/2) := 1$ for $k = 0$ and all $\kappa > 0$.

Proof. For $k = 0$ the claim of the lemma is elementary (see, e.g., [2], p.117). Note that for any $k \ge 1$

$$f_{|\mathbf{X}^{(2)}(0)|^\kappa}(y) = \frac{2^{1-k/2}}{\kappa\Gamma(k/2)} y^{k/\kappa-1} \exp\left(-\frac{1}{2}y^{2/\kappa}\right), \quad y \ge 0. \tag{8}$$

We have, by the total probability law together with elementary considerations

$$\begin{aligned} f_{\zeta^{(\kappa)}_{m,k}(0)}(u) &= \frac{1}{w_\kappa(u)} \int_0^\infty f_{|\mathbf{X}^{(1)}(0)|^\kappa}(u_{\kappa,y})\, f_{|\mathbf{X}^{(2)}(0)|^\kappa}\left(\frac{y}{w_\kappa(u)}\right) dy \\ &= \frac{f_{|\mathbf{X}^{(1)}(0)|^\kappa}(u)}{w_\kappa(u)} \int_0^\infty \frac{f_{|\mathbf{X}^{(1)}(0)|^\kappa}(u_{\kappa,y})}{f_{|\mathbf{X}^{(1)}(0)|^\kappa}(u)} \frac{2^{1-k/2}}{\kappa\Gamma(k/2)} \left(\frac{y}{w_\kappa(u)}\right)^{k/\kappa-1} \exp\left(-\frac{1}{2}\left(\frac{y}{w_\kappa(u)}\right)^{2/\kappa}\right) dy \\ &\sim \frac{2^{1-k/2}}{\kappa\Gamma(k/2)} \frac{f_{|\mathbf{X}^{(1)}(0)|^\kappa}(u)}{w_\kappa(u)} \int_0^\infty \left(\frac{y}{w_\kappa(u)}\right)^{k/\kappa-1} \exp\left(-\frac{1}{2}\left(\frac{y}{w_\kappa(u)}\right)^{2/\kappa} - y\right) dy, \quad u \to \infty. \end{aligned} \tag{9}$$

Recalling that $\lim_{u\to\infty} w_\kappa(u) = \infty, 1/2, 0$ correspond to $\kappa <,=,> 2$, respectively, we conclude the second claimed asymptotic relation of the lemma. The first claimed asymptotic relation then follows similarly as

$$\mathbb{P}\left\{\zeta^{(\kappa)}_{m,k}(0) > u\right\} = \frac{f_{\zeta^{(\kappa)}_{m,k}(0)}(u)}{w_\kappa(u)} \int_0^\infty \frac{f_{\zeta^{(\kappa)}_{m,k}(0)}(u_{\kappa,x})}{f_{\zeta^{(\kappa)}_{m,k}(0)}(u)}\, dx \sim \frac{f_{\zeta^{(\kappa)}_{m,k}(0)}(u)}{w_\kappa(u)} \int_0^\infty e^{-x}\, dx, \quad u \to \infty.$$

□

Lemma 3.2. *If $\{\zeta^{(\kappa)}_{m,k}(t), t \ge 0\}$ is as in Theorem 2.1, then*

$$\left\{w_\kappa(u)(\zeta^{(\kappa)}_{m,k}(q_\kappa t) - u) | \{\zeta^{(\kappa)}_{m,k}(0) > u\},\ t \ge 0\right\} \overset{d}{\to} \left\{\eta^{(\kappa)}_{m,k}(t),\ t \ge 0\right\}, \quad u \to \infty,$$

with $\eta^{(\kappa)}_{m,k}$ given by (3). Recall that $\overset{d}{\to}$ stands for the convergence of finite dimensional distributions.

Proof. We henceforth adopt the notation introduced in Section 2. By Lemma 3.1, we have

$$w_\kappa(u)(\zeta_{m,k}^{(\kappa)}(0)-u)\Big|\{\zeta_{m,k}^{(\kappa)}(0)>u\}\overset{d}{\to}E,\quad u\to\infty.$$

Thus, in view of Theorem 5.1 in [6], it suffices to show that, for any $0<t_1<\cdots<t_n<\infty, n\in\mathbb{N}$

$$\begin{aligned}p_k(u)&:=\mathbb{P}\left\{\cap_{j=1}^n\{\zeta_{m,k}^{(\kappa)}(q_\kappa t_j)\le u_{\kappa,z_j}\}\Big|\zeta_{m,k}^{(\kappa)}(0)=u_{\kappa,x}\right\}\\&\to\mathbb{P}\left\{\cap_{j=1}^n\{\widetilde{Z}_{m,k}^{(\kappa)}(t_j)+x\le z_j\}\right\},\quad u\to\infty\end{aligned}\tag{10}$$

holds for all $x>0$ and $z_j\in\mathbb{R}, 1\le j\le n$. Define below

$$\Delta_{iu}(t_j)=X_i(q_\kappa t_j)-r_i(q_\kappa t_j)X_i(0),\quad 1\le i\le m+k,\ 1\le j\le n.$$

By (1) we have

$$\begin{aligned}&u^{2\tau/\kappa}\mathrm{Cov}(\Delta_{iu}(s),\Delta_{iu}(t))\to C_i(s^\alpha+t^\alpha-|s-t|^\alpha)\\&\quad=2C_i\mathrm{Cov}(Z_i(s),Z_i(t)),\quad u\to\infty,\ s,t>0,\ 1\le i\le m+k.\end{aligned}$$

Therefore,

$$\{u^{\tau/\kappa}\Delta_{iu}(t),t\ge 0\}\overset{d}{\to}\{\sqrt{2C_i}Z_i(t),t\ge 0\},\quad u\to\infty,\quad 1\le i\le m+k.$$

Furthermore, by the independence of $\Delta_{iu}(t)$'s and $X_i(0)$'s, the random processes Z_i's can be chosen such that they are independent of $\zeta_{m,k}^{(\kappa)}(0)$. Note that $\boldsymbol{X}^{(1)}(0)\overset{d}{=}R_1\boldsymbol{O}_1$ holds for some $R_1>0$ which is independent of $\boldsymbol{O}_1$. Then, using the Taylor's expansion of $(1+x)^{\kappa/2}=1+\kappa x/2+o(x), x\to 0$, we have, for any $z_j\in\mathbb{R}$, $1\le j\le n$

$$\begin{aligned}p_0(u)&=\mathbb{P}\left\{\bigcap_{j=1}^n\left\{|\boldsymbol{X}^{(1)}(q_\kappa t_j)|^\kappa\le u_{\kappa,z_j}\right\}\Big||\boldsymbol{X}^{(1)}(0)|^\kappa=u_{\kappa,x}\right\}\\&=\mathbb{P}\left\{\bigcap_{j=1}^n\left\{w_\kappa(u)\left(R_1^\kappa\left(1+\frac{1}{R_1^2}V_u(t_j)\right)^{\kappa/2}-R_1^\kappa\right)\le z_j-x\right\}\Big|R_1^\kappa=u_{\kappa,x}\right\}\\&=\mathbb{P}\left\{\bigcap_{j=1}^n\left\{\frac{\kappa}{2}w_\kappa(u)R_1^{\kappa-2}V_u(t_j)(1+o_p(1))\le z_j-x\right\}\Big|R_1^\kappa=u_{\kappa,x}\right\}\\&=\mathbb{P}\left\{\bigcap_{j=1}^n\left\{\sum_{i=1}^m\frac{\sqrt{2C_i}O_iZ_i(t_j)}{u^{(\tau-1)/\kappa}}(1+o_p(1))-\left(\sum_{i=1}^m\frac{C_iO_i^2}{u^{2(\tau-1)/\kappa}}\right)t_j^\alpha(1+o_p(1))+x\le z_j\right\}\right\},\quad u\to\infty,\end{aligned}\tag{11}$$

where $V_u(t_j):=\sum_{i=1}^m\Delta_{iu}^2(t_j)+2\sum_{i=1}^m\Delta_{iu}(t_j)r_i(q_\kappa t_j)X_i(0)-\sum_{i=1}^m(1-r_i^2(q_\kappa t_j))X_i^2(0)$. Consequently, the claim for $k=0$ follows. Next, for $k\ge 1$, we rewrite $p_k(u)$ as

$$\begin{aligned}p_k(u)&=\int_0^\infty\mathbb{P}\left\{\bigcap_{j=1}^n\left\{\zeta_{m,k}^{(\kappa)}(q_\kappa t_j)\le u_{\kappa,z_j}\right\}\Big||\boldsymbol{X}^{(1)}(0)|^\kappa=u_{\kappa,x+y},|\boldsymbol{X}^{(2)}(0)|^\kappa=\frac{y}{w_\kappa(u)}\right\}\\&\quad\times\frac{f_{|\boldsymbol{X}^{(1)}(0)|^\kappa}(u_{\kappa,x+y})f_{|\boldsymbol{X}^{(2)}(0)|^\kappa}(y/w_\kappa(u))}{w_\kappa(u)f_{\zeta_{m,k}^{(\kappa)}(0)}(u_{\kappa,x})}\,dy\\&=\int_0^\infty\mathbb{P}\left\{\bigcap_{j=1}^n\left\{|\boldsymbol{X}^{(1)}(q_\kappa t_j)|^\kappa\le u_{\kappa,z_j+w_\kappa(u)\cdot|\boldsymbol{X}^{(2)}(q_\kappa t_j)|^\kappa}\right\}\Big||\boldsymbol{X}^{(1)}(0)|^\kappa=u_{\kappa,x+y},|\boldsymbol{X}^{(2)}(0)|^\kappa=\frac{y}{w_\kappa(u)}\right\}\times h_{\kappa,u}(y)\,dy,\end{aligned}\tag{12}$$

where

$$\begin{aligned} h_{\kappa,u}(y) &:= \frac{f_{|\boldsymbol{X}^{(1)}(0)|^\kappa}(u_{\kappa,x+y}) f_{|\boldsymbol{X}^{(2)}(0)|^\kappa}(y/w_\kappa(u))}{w_\kappa(u) f_{\zeta^{(\kappa)}_{m,k}(0)}(u_{\kappa,x})} \\ &= \frac{f_{|\boldsymbol{X}^{(1)}(0)|^\kappa}(u_{\kappa,x+y})}{f_{|\boldsymbol{X}^{(1)}(0)|^\kappa}(u_{\kappa,y})} \frac{f_{\zeta^{(\kappa)}_{m,k}(0)}(u)}{f_{\zeta^{(\kappa)}_{m,k}(0)}(u_{\kappa,x})} \frac{f_{|\boldsymbol{X}^{(1)}(0)|^\kappa}(u_{\kappa,y}) f_{|\boldsymbol{X}^{(2)}(0)|^\kappa}(y/w_\kappa(u))}{w_\kappa(u) f_{\zeta^{(\kappa)}_{m,k}(0)}(u)} \\ &\sim \frac{f_{|\boldsymbol{X}^{(1)}(0)|^\kappa}(u_{\kappa,y}) f_{|\boldsymbol{X}^{(2)}(0)|^\kappa}(y/w_\kappa(u))}{\int_0^\infty f_{|\boldsymbol{X}^{(1)}(0)|^\kappa}(u_{\kappa,y}) f_{|\boldsymbol{X}^{(2)}(0)|^\kappa}(y/w_\kappa(u))\, dy}, \quad u\to\infty. \end{aligned} \tag{13}$$

Here the last step follows by Lemma 3.1 and (9).
Next, we derive the limit distribution of $w_\kappa(u)|\boldsymbol{X}^{(2)}(q_\kappa t)|^\kappa \Big| \{w_\kappa(u)|\boldsymbol{X}^{(2)}(0)|^\kappa = y\}$. Noting that $\boldsymbol{X}^{(2)}(0) \overset{d}{=} R_2\boldsymbol{O}_2$ holds for some $R_2>0$ which is independent of $\boldsymbol{O}_2$, we have by similar arguments as in (11) that, for any $t\ge 0$

$$\begin{aligned} &\left(w_\kappa(u)|\boldsymbol{X}^{(2)}(q_\kappa t)|^\kappa\right)^{2/\kappa} \Big| \{w_\kappa(u)|\boldsymbol{X}^{(2)}(0)|^\kappa = y\} \\ &= (w_\kappa(u))^{2/\kappa}\Bigg(\sum_{i=m+1}^{m+k} X_i^2(0) + 2\sum_{i=m+1}^{m+k} r_i(q_\kappa t)X_i(0)\Delta_{iu}(t) + \sum_{i=m+1}^{m+k}\Delta_{iu}^2(t) \\ &\quad - \sum_{i=m+1}^{m+k}(1-r_i(q_\kappa t)^2)X_i^2(0)\Bigg)\Bigg| \left\{R_2^\kappa = \frac{y}{(w_\kappa(u))^{1/\kappa}}\right\} \\ &= (w_\kappa(u))^{2/\kappa}\Bigg(R_2^2 + 2\frac{R_2}{u^{\tau/\kappa}}\sum_{i=m+1}^{m+k}\sqrt{2C_i}O_iZ_i(t)(1+o_p(1)) + \frac{2}{u^{2\tau/\kappa}}\sum_{i=m+1}^{m+k} C_iZ_i^2(t)(1+o_p(1)) \\ &\quad -2\left(\frac{R_2}{u^{\tau/\kappa}}\right)^2\sum_{i=m+1}^{m+k} C_iO_i^2t^\alpha(1+o_p(1))\Bigg)\Bigg| \left\{R_2^\kappa = \frac{y}{(w_\kappa(u))^{1/\kappa}}\right\} \\ &= y^{2/\kappa} + 2y^{1/\kappa}\left(\frac{w_\kappa(u)}{u^\tau}\right)^{1/\kappa}\sum_{i=m+1}^{m+k}\sqrt{2C_i}O_iZ_i(t)(1+o_p(1)) + 2\left(\frac{w_\kappa(u)}{u^\tau}\right)^{2/\kappa}\sum_{i=m+1}^{m+k} C_iZ_i^2(t)(1+o_p(1)) \\ &=: \theta_{\kappa,u}(y,t). \end{aligned} \tag{14}$$

This together with (11) and (12) implies that

$$\begin{aligned} p_k(u) &= \int_0^\infty \mathbb{P}\Bigg\{\bigcap_{j=1}^n\Bigg\{\sum_{i=1}^m \frac{\sqrt{2C_i}O_iZ_i(t_j)}{u^{(\tau-1)/\kappa}}(1+o_p(1)) - \left(\sum_{i=1}^m\frac{C_iO_i^2}{u^{2(\tau-1)/\kappa}}\right)t_j^\alpha(1+o_p(1)) + x + y \\ &\qquad \le z_j + w_\kappa(u)|\boldsymbol{X}^{(2)}(q_\kappa t_j)|^\kappa\Bigg\}\Bigg| |\boldsymbol{X}^{(2)}(0)|^\kappa = \frac{y}{w_\kappa(u)}\Bigg\} h_{\kappa,u}(y)\, dy \\ &= \int_0^\infty \mathbb{P}\Bigg\{\bigcap_{j=1}^n\Bigg\{\sum_{i=1}^m \frac{\sqrt{2C_i}O_iZ_i(t_j)}{u^{(\tau-1)/\kappa}}(1+o_p(1)) - \left(\sum_{i=1}^m\frac{C_iO_i^2}{u^{2(\tau-1)/\kappa}}\right)t_j^\alpha(1+o_p(1)) + x + y \\ &\qquad \le z_j + (\theta_{\kappa,u}(y,t_j))^{\frac{\kappa}{2}}\Bigg\}\Bigg\} h_{\kappa,u}(y)\, dy. \end{aligned} \tag{15}$$

Recalling that $\tau = 1+\max(0, 2(1/\kappa-1))$ and $w_\kappa(u) = (1/\kappa)u^{2/\kappa-1}$, we have by (14) that $(\theta_{\kappa,u}(y,t_j))^{\kappa/2} = y + o_p(1)$ for $\kappa>1$. While for $\kappa\in(0,1]$, it follows by (13) and Lemma 3.1 that,

$$h_{\kappa,\infty}(y) := \lim_{u\to\infty} h_{\kappa,u}(y) = \frac{1}{\Gamma(k/\kappa)} y^{k/\kappa-1}e^{-y}, \quad y>0, \tag{16}$$

which is the pdf of a Gamma distributed random variable with parameter $(k/\kappa, 1)$. Hence, combining (13)–(16) and (4) for the definition of $\widetilde{Z}^{(\kappa)}_{m,k}(t)$, the claim in (10) follows. Consequently, the proof of Lemma 3.2 is complete. □
The next lemma corresponds to Condition B in [2]; see also [1,4]. We note in passing that this condition, motivated by [6], is often referred to as the "short-lasting-exceedance" condition, which is crucial in ensuring that the double sum

part is asymptotically negligible with respect to the principle sum of the discrete approximations to the continuous maximum; see, e.g., Chapter 5 in [1]. Denote in the following by $[x]$ the integer part of $x \in \mathbb{R}$.

Lemma 3.3. *If $\{\zeta_{m,k}^{(\kappa)}(t), t \geq 0\}$ is as in Theorem 2.1, then for any $T, a > 0$*

$$\limsup_{u\to\infty} \sum_{j=N}^{[T/(aq_\kappa)]} \mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(aq_\kappa j) > u \middle| \zeta_{m,k}^{(\kappa)}(0) > u\right\} \to 0, \quad N \to \infty.$$

Proof. Note first that the case $k = 0$ is treated in [2], p.119. Using the fact that the standard bivariate Gaussian distribution is exchangeable, we have, for $u > 0$

$$\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(q_\kappa t) > u \middle| \zeta_{m,k}^{(\kappa)}(0) > u\right\} = 2\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(q_\kappa t) > u, |\boldsymbol{X}^{(1)}(q_\kappa t)| > |\boldsymbol{X}^{(1)}(0)| \middle| \zeta_{m,k}^{(\kappa)}(0) > u\right\} =: 2\Theta(u).$$

Further, it follows from Lemma 3.1 that, for any $k \geq 1$

$$\begin{aligned}
\Theta(u) &= \int_0^\infty \int_0^\infty \mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(q_\kappa t) > u, |\boldsymbol{X}^{(1)}(q_\kappa t)| > |\boldsymbol{X}^{(1)}(0)| \middle| |\boldsymbol{X}^{(1)}(0)|^\kappa = u_{\kappa,x+y}, |\boldsymbol{X}^{(2)}(0)|^\kappa = \frac{y}{w_\kappa(u)}\right\} \\
&\quad \times \frac{f_{|\boldsymbol{X}^{(1)}(0)|^\kappa}(u_{\kappa,x+y}) f_{|\boldsymbol{X}^{(2)}(0)|^\kappa}\left(\frac{y}{w_\kappa(u)}\right)}{w_\kappa^2(u)\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0) > u\right\}} dxdy \\
&\leq \int_0^\infty \int_0^\infty \mathbb{P}\left\{|\boldsymbol{X}^{(1)}(q_\kappa t)|^\kappa > u_{\kappa,y} \middle| |\boldsymbol{X}^{(1)}(0)|^\kappa = u_{\kappa,x+y}\right\} \frac{f_{|\boldsymbol{X}^{(1)}(0)|^\kappa}(u_{\kappa,x+y}) f_{|\boldsymbol{X}^{(2)}(0)|^\kappa}\left(\frac{y}{w_\kappa(u)}\right)}{w_\kappa^2(u)\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0) > u\right\}} dxdy \\
&= \int_0^\infty \mathbb{P}\left\{|\boldsymbol{X}^{(1)}(q_\kappa t)|^\kappa > u_{\kappa,y} \middle| |\boldsymbol{X}^{(1)}(0)|^\kappa > u_{\kappa,y}\right\} \frac{\mathbb{P}\left\{|\boldsymbol{X}^{(1)}(0)|^\kappa > u_{\kappa,y}\right\}}{w_\kappa(u)\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0) > u\right\}} f_{|\boldsymbol{X}^{(2)}(0)|^\kappa}\left(\frac{y}{w_\kappa(u)}\right) dy.
\end{aligned}$$

Moreover, in view of the treatment of the case $k = 0$ in [2], p.119 we readily see that, for any $p \geq 1$, with $R(t) := \max_{1\leq i\leq m} r_i(t), r(t) := \min_{1\leq i \leq m} r_i(t)$ and $\Phi(\cdot)$ denoting the $N(0,1)$ distribution function,

$$\begin{aligned}
\mathbb{P}\left\{|\boldsymbol{X}^{(1)}(q_\kappa t)|^\kappa > u_{\kappa,y} \middle| |\boldsymbol{X}^{(1)}(0)|^\kappa > u_{\kappa,y}\right\} &\leq 4m\left(1 - \Phi\left(\frac{(1 - R(q_\kappa t))u^{1/\kappa}}{\sqrt{m(1 - r^2(q_\kappa t))}}\right)\right) \\
&\leq K_p t^{-\alpha p/2}, \quad \forall q_\kappa t \in (0, T]
\end{aligned}$$

holds for some $K_p > 0$ not depending on u, t and y. Consequently,

$$\begin{aligned}
\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(q_\kappa t) > u \middle| \zeta_{m,k}^{(\kappa)}(0) > u\right\} &\leq 2K_p t^{-\alpha p/2} \int_0^\infty \frac{\mathbb{P}\left\{|\boldsymbol{X}^{(1)}(0)|^\kappa > u_{\kappa,y}\right\}}{w_\kappa(u)\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0) > u\right\}} f_{|\boldsymbol{X}^{(2)}(0)|^\kappa}\left(\frac{y}{w_\kappa(u)}\right) dy \\
&= 2K_p t^{-\alpha p/2}, \quad \forall q_\kappa t \in (0, T].
\end{aligned} \tag{17}$$

Therefore, with $p = 4/\alpha$,

$$\begin{aligned}
&\limsup_{u\to\infty} \sum_{j=N}^{[T/(aq_\kappa)]} \mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(aq_\kappa j) > u \middle| \zeta_{m,k}^{(\kappa)}(0) > u\right\} \\
&\leq 2K_p \int_{aN}^\infty x^{-2}\, dx = \frac{2K_p}{aN} \to 0, \quad N \to \infty
\end{aligned}$$

establishing the proof. □

The lemma below concerns the accuracy of the discrete approximation to the continuous process, which is related to Condition C in [2]. As shown in [4] (see Eq. (7) therein), in order to verify Condition C the following lemma is sufficient. Its proof is relegated to the appendix.

Lemma 3.4. *If $\{\zeta_{m,k}^{(\kappa)}(t), t \ge 0\}$ is as in Theorem 2.1, then there exist some constants $C, p > 0, d > 1$ and $\lambda_0, u_0 > 0$ such that*

$$\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(q_\kappa t) > u + \frac{\lambda}{w_\kappa(u)}, \zeta_{m,k}^{(\kappa)}(0) \le u\right\} \le C t^d \lambda^{-p} \mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0) > u\right\}$$

for $0 < t^{\varpi} < \lambda < \lambda_0$ and $u > u_0$. Here ϖ is $\alpha/2$ for $\kappa \ge 1$, and $(\alpha/2)\min(\kappa/(4(1-\kappa)), 1)$ otherwise.

Proof of Theorem 2.1: It follows from Lemmas 3.1–3.4 that all the assumptions of Theorem 1 in [2] are satisfied by the process $\zeta_{m,k}^{(\kappa)}$, which immediately establishes the proof. □

Proof of Theorem 2.2: In view of (17) with $p = 4/\alpha$ and letting $v_\kappa = v_\kappa(u) = 1/q_\kappa(u) = u^{2\tau/(\alpha\kappa)}$, we obtain

$$\begin{aligned} v_\kappa \int_{N/v_\kappa}^{T} \mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(s) > u \middle| \zeta_{m,k}^{(\kappa)}(0) > u\right\} ds &= \int_{N}^{v_\kappa T} \mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(s/v_\kappa) > u \middle| \zeta_{m,k}^{(\kappa)}(0) > u\right\} ds \\ &\le K_{4/\alpha} \int_{N}^{v_\kappa T} s^{-2}\, ds \le \frac{K_{4/\alpha}}{N}, \quad u \to \infty. \end{aligned}$$

Hence

$$\lim_{N\to\infty} \limsup_{u\to\infty} v_\kappa \int_{N/v_\kappa}^{T} \mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(s) > u \middle| \zeta_{m,k}^{(\kappa)}(0) > u\right\} ds = 0.$$

Since further Lemma 3.2 holds, the claim follows by Theorem 3.1 in [6]. □

As shown by Theorem 10 in [2], in order to derive the Gumbel limit theorem for the random process $\zeta_{m,k}^{(\kappa)}$ two additional conditions, which were first addressed by the seminal contributions [16, 17], need to be checked, namely the mixing Condition D and the Condition D' therein. These two conditions will follow from Lemma 3.5 and Lemma 3.6 below; their proofs are displayed in the appendix.

Lemma 3.5. *Let T, a be any given positive constants and $M \in (0, T)$. If $\{\zeta_{m,k}^{(\kappa)}(t), t \ge 0\}$ is as in Theorem 2.1, then for any $0 \le s_1 < \cdots < s_p < t_1 < \cdots < t_{p'}$ in $\{a q_\kappa j : j \in \mathbb{Z}, 0 \le a q_\kappa j \le T\}$ such that $t_1 - s_p \ge M$*

$$\begin{aligned} &\left|\mathbb{P}\left\{\cap_{i=1}^{p}\{\zeta_{m,k}^{(\kappa)}(s_i) \le u\}, \cap_{j=1}^{p'}\{\zeta_{m,k}^{(\kappa)}(t_j) \le u\}\right\} - \mathbb{P}\left\{\cap_{i=1}^{p}\{\zeta_{m,k}^{(\kappa)}(s_i) \le u\}\right\}\mathbb{P}\left\{\cap_{j=1}^{p'}\{\zeta_{m,k}^{(\kappa)}(t_j) \le u\}\right\}\right| \\ &\quad \le K u^{\varsigma} \sum_{1\le i\le p, 1\le j\le p'} \widetilde{r}(t_j - s_i) \exp\left(-\frac{u^{2/\kappa}}{1 + \widetilde{r}(t_j - s_i)}\right) \end{aligned} \tag{18}$$

and

$$\begin{aligned} &\left|\mathbb{P}\left\{\cap_{i=1}^{p}\{\zeta_{m,k}^{(\kappa)}(s_i) > u\}, \cap_{j=1}^{p'}\{\zeta_{m,k}^{(\kappa)}(t_j) > u\}\right\} - \mathbb{P}\left\{\cap_{i=1}^{p}\{\zeta_{m,k}^{(\kappa)}(s_i) > u\}\right\}\mathbb{P}\left\{\cap_{j=1}^{p'}\{\zeta_{m,k}^{(\kappa)}(t_j) > u\}\right\}\right| \\ &\quad \le K u^{\varsigma} \sum_{1\le i\le p, 1\le j\le p'} \widetilde{r}(t_j - s_i) \exp\left(-\frac{u^{2/\kappa}}{1 + \widetilde{r}(t_j - s_i)}\right) \end{aligned} \tag{19}$$

hold for all $u > 0$ and some $K > 0$ not depending on u. Here $\varsigma = 2/\kappa\big(m - k(2/\kappa - 1) - 1 + \max(0, 2(1/\kappa - 1)\big)$ and $\widetilde{r}(t) := \max_{1\le l\le m+k} |r_l(t)|, t > 0$.

Remark: In contrast to the Normal Comparison Lemma in [16] for the Gaussian processes, we obtain the crucial comparison inequality of the chi-type processes in Lemma 3.5 to ensure the mixing D condition, which was first addressed by the seminal paper [17] (see Lemma 3.5 therein and also Theorem 10 in [2]). Lemma 3.5 is also expected to be useful in extreme value analysis when concerned with chi-type processes; see, e.g., [26] avoiding the technical verification of mixed-Gumbel limit theorems for the strongly dependent cyclo-stationary χ-processes.

Lemma 3.6. *Under the assumptions of Theorem 2.3, for ς, $\widetilde{r}(\cdot)$ as in Lemma 3.5 and T_κ given by*

$$T_\kappa = T_\kappa(u) = \frac{1}{H_{\alpha,\kappa}^{m,k}(\boldsymbol{C})} \frac{q_\kappa(u)}{\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0) > u\right\}}, \tag{20}$$

we have, for any given constant $\varepsilon \in (0, T_\kappa)$

$$u^{\varsigma} \frac{T_\kappa}{q_\kappa} \sum_{\varepsilon \le aq_\kappa j \le T_\kappa} \widetilde{r}(aq_\kappa j) \exp\left(-\frac{u^{2/\kappa}}{1+\widetilde{r}(aq_\kappa j)}\right) \to 0, \qquad u \to \infty. \tag{21}$$

Proof of Theorem 2.3: To establish Conditions D and D' in [2], we shall make use of Lemma 3.5 with $T = T_\kappa$ given by (20) and $M = \varepsilon \in (0, T_\kappa)$, and Lemma 3.6. First note that the right-hand side of (18) is bounded from above by

$$K u^{\varsigma} \frac{T_\kappa}{aq_\kappa} \sum_{\varepsilon \le aq_\kappa j \le T_\kappa} \widetilde{r}(aq_\kappa j) \exp\left(-\frac{u^{2/\kappa}}{1+\widetilde{r}(aq_\kappa j)}\right),$$

which by an application of (21) implies that the mixing Condition D in [2] holds for the random process $\zeta_{m,k}^{(\kappa)}$. Next, we prove Condition D' in [2], i.e., for any given positive constants a and $\widetilde{T}$

$$\limsup_{u\to\infty} \sum_{j=[\widetilde{T}/(aq_\kappa)]}^{\left[\varepsilon/\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0)>u\right\}\right]} \mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(aq_\kappa j) > u \middle| \zeta_{m,k}^{(\kappa)}(0) > u\right\} \to 0, \quad \varepsilon \downarrow 0. \tag{22}$$

Indeed, by (19) for some $\widetilde{M} > \widetilde{T}$ and a positive constant K

$$\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(aq_\kappa j) > u \middle| \zeta_{m,k}^{(\kappa)}(0) > u\right\} \le \mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0) > u\right\} + K u^{\varsigma} \frac{T_\kappa}{q_\kappa} \widetilde{r}(aq_\kappa j) \exp\left(-\frac{u^{2/\kappa}}{1+\widetilde{r}(aq_\kappa j)}\right)$$

holds for $u > 0$ and $aq_\kappa j > \widetilde{M}$. Consequently,

$$\begin{aligned}
&\limsup_{u\to\infty} \sum_{j=[\widetilde{T}/(aq_\kappa)]}^{\left[\varepsilon/\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0)>u\right\}\right]} \mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(aq_\kappa j) > u \middle| \zeta_{m,k}^{(\kappa)}(0) > u\right\} \\
&\le \limsup_{u\to\infty} \sum_{j=[\widetilde{T}/(aq_\kappa)]}^{[\widetilde{M}/(aq_\kappa)]} \mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(aq_\kappa j) > u \middle| \zeta_{m,k}^{(\kappa)}(0) > u\right\} + \varepsilon \\
&+ \limsup_{u\to\infty} K u^{\varsigma} \frac{T_\kappa}{q_\kappa} \sum_{j=[\widetilde{M}/(aq_\kappa)]}^{\left[\varepsilon/\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0)>u\right\}\right]} \widetilde{r}(aq_\kappa j) \exp\left(-\frac{u^{2/\kappa}}{1+\widetilde{r}(aq_\kappa j)}\right),
\end{aligned}$$

which equals ϵ by an application of Lemma 3.3 and (21), respectively. It follows then that (22) holds. Consequently, in view of Theorem 10 in [2] we have, for T_κ given by (20)

$$\lim_{u\to\infty} \mathbb{P}\left\{\sup_{t\in[0,T_\kappa]} \zeta_{m,k}^{(\kappa)}(t) \le u + \frac{x}{w_\kappa(u)}\right\} = \exp\left(-e^{-x}\right), \quad x \in \mathbb{R}.$$

Expressing u in terms of T_κ using (20) (see also (33)) we obtain the required claim with $a_T^{(\kappa)}, b_T^{(\kappa)}$ given by (7) for any $x \in \mathbb{R}$; the uniform convergence in x follows since all functions (with respect to x) are continuous, bounded and increasing. □

4. Appendix

Proof of Lemma 3.4: By (1), for any small $\epsilon \in (0,1)$ there exists some positive constant B such that

$$r_i(t) \ge \frac{1}{2} \quad \text{and} \quad 1 - r_i(t) \le B t^{\alpha}, \quad \forall t \in (0, \epsilon],\ 1 \le i \le m+k.$$

Furthermore, for any positive t satisfying (recall $\varpi = \alpha/2\mathbb{I}\{\kappa \ge 1\} + \alpha/2 \min(\kappa/(4(1-\kappa)), 1)\mathbb{I}\{0 < \kappa < 1\}$)

$$0 < t^{\varpi} < \lambda < \lambda_0 := \min\left(\frac{1}{2^{\kappa+4}B}, \frac{\kappa}{2^{\kappa+2}}, \epsilon^{\varpi}\right)$$

and any $u > 2$

$$u^{2\tau/\kappa}\theta_\kappa(t) \le 2^\kappa \kappa B t^\alpha \le \frac{\kappa t^{\alpha/2}}{16} \quad \text{with} \quad \theta_\kappa(t) := \frac{1}{(r(q_\kappa t))^\kappa} - 1,\ r(t) := \min_{1\le i\le m+k} r_i(t). \tag{23}$$

Let $(\boldsymbol{X}^{(1)}_{1/r}(t), \boldsymbol{X}^{(2)}_{1/r}(t)) := \big(X_1(t) - r_1^{-1}(t)X_1(0), \ldots, X_{m+k}(t) - r_{m+k}^{-1}(t)X_{m+k}(0)\big)$ which by definition is independent of $\{\zeta^{(\kappa)}_{m,k}(t), t \ge 0\}$. For $j = 1, 2$

$$\mathbb{P}\left\{|\boldsymbol{X}^{(j)}_{1/r}(q_\kappa t)| > x\right\} \le \mathbb{P}\left\{|\boldsymbol{X}^{(j)}(0)| > \frac{x}{2\sqrt{2Bu^{-2\tau/\kappa}t^\alpha}}\right\}, \quad u\theta_\kappa(t) \le \frac{\lambda}{2w_\kappa(u)}. \tag{24}$$

In the following, the cases $\kappa = 1, \kappa \in (1, \infty)$ and $\kappa \in (0, 1)$ will be considered in turn.
Case $\kappa = 1$: Note by the triangular inequality that

$$\zeta^{(1)}_{m,k}(q_1 t) \le |\boldsymbol{X}^{(1)}_{1/r}(q_1 t)| + |\boldsymbol{X}^{(2)}_{1/r}(q_1 t)| + \frac{1}{r(q_1 t)}\zeta^{(1)}_{m,k}(0) + \theta_1(t)|\boldsymbol{X}^{(2)}(0)|.$$

Consequently, from (24) we get

$$\begin{aligned}
&\mathbb{P}\left\{\zeta^{(1)}_{m,k}(q_1 t) > u + \frac{\lambda}{u}, \zeta^{(1)}_{m,k}(0) \le u\right\} \\
&\le \mathbb{P}\left\{|\boldsymbol{X}^{(1)}_{1/r}(q_1 t)| + |\boldsymbol{X}^{(2)}_{1/r}(q_1 t)| + \theta_1(t)|\boldsymbol{X}^{(2)}(0)| > \frac{\lambda}{2u}, \zeta^{(1)}_{m,k}(q_1 t) > u\right\} \\
&\le \mathbb{P}\left\{|\boldsymbol{X}^{(1)}_{1/r}(q_1 t)| + |\boldsymbol{X}^{(2)}_{1/r}(q_1 t)| > \frac{\lambda}{3u}\right\}\mathbb{P}\left\{\zeta^{(1)}_{m,k}(q_1 t) > u\right\} + \mathbb{P}\left\{\theta_1(t)|\boldsymbol{X}^{(2)}(0)| > \frac{\lambda}{6u}\right\} \\
&=: I_{1u} + I_{2u}.
\end{aligned}$$

By (23) and (24), we have, for any $p > 1$

$$\mathbb{P}\left\{|\boldsymbol{X}^{(1)}_{1/r}(q_1 t)| > \frac{\lambda}{6u}\right\} \le \mathbb{P}\left\{|\boldsymbol{X}^{(1)}(0)| > \frac{\lambda}{12\sqrt{2B}t^{\alpha/2}}\right\} \le K\left(\frac{\lambda}{t^{\alpha/2}}\right)^{-p}$$

holds with some $K > 0$ (the values of p and K might change from line to line below). Similarly,

$$\mathbb{P}\left\{|\boldsymbol{X}^{(2)}_{1/r}(q_1 t)| > \frac{\lambda}{6u}\right\} \le K\left(\frac{\lambda}{t^{\alpha/2}}\right)^{-p}$$

and hence

$$I_{1u} \le K\left(\frac{\lambda}{t^{\alpha/2}}\right)^{-p}\mathbb{P}\left\{\zeta^{(1)}_{m,k}(0) > u\right\}. \tag{25}$$

Moreover, in view of Lemma 3.1 and (23) we have for sufficiently large u that

$$I_{2u} \le \frac{\mathbb{P}\left\{|\boldsymbol{X}^{(2)}(0)| > \frac{2\lambda u}{t^{\alpha/2}}\right\}}{\mathbb{P}\left\{\zeta^{(1)}_{m,k}(0) > u\right\}}\mathbb{P}\left\{\zeta^{(1)}_{m,k}(0) > u\right\} \le K\left(\frac{\lambda}{t^{\alpha/2}}\right)^{-(p-k+2)} u^{-(p+m-2k)}\mathbb{P}\left\{\zeta^{(1)}_{m,k}(0) > u\right\}. \tag{26}$$

Hence, the claim for $\kappa = 1$ follows from (25) and (26) by choosing $p > \max(4/\alpha + k, 2k)$.
Case $\kappa \in (1, \infty)$: Denote below by $(\boldsymbol{Y}^{(1)}(t), \boldsymbol{Y}^{(2)}(t)) := \big(r_1^{-1}(t)X_1(0), \ldots, r_{m+k}^{-1}(t)X_{m+k}(0)\big)$. Note that $|\boldsymbol{Y}^{(1)}(t)| \le |\boldsymbol{X}^{(1)}(0)|/r(t)$ and $|\boldsymbol{X}^{(2)}(0)| \le |\boldsymbol{Y}^{(2)}(t)| \le |\boldsymbol{X}^{(2)}(0)|/r(t)$ for all $t < \varepsilon$, and for some constants $K_1, K_2 > 0$ whose values might change from line to line below

$$|1 + x|^\kappa \ge 1 + \kappa x, \quad x \in \mathbb{R} \quad \text{and} \quad (1 + x)^\kappa \le 1 + K_1 x + K_2 x^\kappa, \quad x \ge 0.$$

We have further by the triangle inequality

$$\begin{aligned}
\zeta_{m,k}^{(\kappa)}(q_\kappa t) &\le \left(|\boldsymbol{Y}^{(1)}(q_\kappa t)| + |\boldsymbol{X}_{1/r}^{(1)}(q_\kappa t)|\right)^\kappa - \left||\boldsymbol{Y}^{(2)}(q_\kappa t)| - |\boldsymbol{X}_{1/r}^{(2)}(q_\kappa t)|\right|^\kappa \\
&\le |\boldsymbol{Y}^{(1)}(q_\kappa t)|^\kappa + K_1 |\boldsymbol{X}_{1/r}^{(1)}(q_\kappa t)||\boldsymbol{Y}^{(1)}(q_\kappa t)|^{\kappa-1} + K_2 |\boldsymbol{X}_{1/r}^{(1)}(q_\kappa t)|^\kappa \\
&\quad -|\boldsymbol{Y}^{(2)}(q_\kappa t)|^\kappa + \kappa |\boldsymbol{X}_{1/r}^{(2)}(q_\kappa t)||\boldsymbol{Y}^{(2)}(q_\kappa t)|^{\kappa-1} \\
&\le K_1 |\boldsymbol{X}_{1/r}^{(1)}(q_\kappa t)||\boldsymbol{X}^{(1)}(0)|^{\kappa-1} + K_2 |\boldsymbol{X}_{1/r}^{(1)}(q_\kappa t)|^\kappa \\
&\quad + K_3 |\boldsymbol{X}_{1/r}^{(2)}(q_\kappa t)||\boldsymbol{X}^{(2)}(0)|^{\kappa-1} + \frac{\zeta_{m,k}^{(\kappa)}(0)}{(r(q_\kappa t))^\kappa} + \theta_\kappa(t)|\boldsymbol{X}^{(2)}(0)|^\kappa
\end{aligned}$$

holds for $q_\kappa t \le \epsilon$ and some constant $K_3 > 0$. Therefore, with $\mu = 1/(2(\kappa-1))$ and $\varphi = \alpha/(4(\kappa-1))$,

$$\begin{aligned}
&\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(q_\kappa t) > u + \frac{\lambda}{w_\kappa(u)}, \zeta_{m,k}^{(\kappa)}(0) \le u\right\} \\
&\le \mathbb{P}\left\{|\boldsymbol{X}^{(1)}(0)| > \frac{\lambda^\mu u^{1/\kappa}}{t^\varphi}\right\} + \mathbb{P}\left\{|\boldsymbol{X}^{(2)}(0)| > \frac{\lambda^\mu u^{1/\kappa}}{t^\varphi}\right\} \\
&\quad + \mathrm{P}\left\{K_1 |\boldsymbol{X}_{1/r}^{(1)}(q_\kappa t)| \left(\frac{\lambda^\mu u^{1/\kappa}}{t^\varphi}\right)^{\kappa-1} + K_2 |\boldsymbol{X}_{1/r}^{(1)}(q_\kappa t)|^\kappa + K_3 |\boldsymbol{X}_{1/r}^{(2)}(q_\kappa t)| \left(\frac{\lambda^\mu u^{1/\kappa}}{t^\varphi}\right)^{\kappa-1}\right. \\
&\quad \left. + \theta_\kappa(t)|\boldsymbol{X}^{(2)}(0)|^\kappa \ge \frac{\lambda}{2 w_\kappa(u)}, \zeta_{m,k}^{(\kappa)}(q_\kappa t) > u\right\} \\
&=: \tilde{I}_{1u} + \tilde{I}_{2u} + \tilde{I}_{3u}.
\end{aligned} \tag{27}$$

Note by (23) that $\lambda^\mu / t^\varphi > 1$. Similar arguments as in (26) yield that

$$\begin{aligned}
\tilde{I}_{1u} &\le K \left(\tfrac{\lambda^\mu}{t^\varphi}\right)^{-(p-m+2)} u^{-(p-k(2/\kappa-1)\mathbb{I}\{\kappa\le 2\})/\kappa} \mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0) > u\right\} \\
\tilde{I}_{2u} &\le K \left(\tfrac{\lambda^\mu}{t^\varphi}\right)^{-(p-k+2)} u^{-(p-k+m-k(2/\kappa-1)\mathbb{I}\{\kappa\le 2\})/\kappa} \mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0) > u\right\}
\end{aligned}$$

and

$$\begin{aligned}
\tilde{I}_{3u} &\le \left(\mathbb{P}\left\{K_1 |\boldsymbol{X}_{1/r}^{(1)}(q_\kappa t)| \left(\frac{\lambda^\mu u^{1/\kappa}}{t^\varphi}\right)^{\kappa-1} > \frac{\lambda}{8 w_\kappa(u)}\right\} + \mathbb{P}\left\{K_2 |\boldsymbol{X}_{1/r}^{(1)}(q_\kappa t)|^\kappa > \frac{\lambda}{8 w_\kappa(u)}\right\}\right. \\
&\quad \left. + \mathbb{P}\left\{K_3 |\boldsymbol{X}_{1/r}^{(2)}(q_\kappa t)| \left(\frac{\lambda^\mu u^{1/\kappa}}{t^\varphi}\right)^{\kappa-1} > \frac{\lambda}{8 w_\kappa(u)}\right\}\right) \mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(q_\kappa t) > u\right\} \\
&\quad + \mathbb{P}\left\{\theta_\kappa(t)|\boldsymbol{X}^{(2)}(0)|^\kappa > \frac{\lambda}{8 w_\kappa(u)}\right\} \\
&=: (\mathit{II}_{1u} + \mathit{II}_{2u} + \mathit{II}_{3u}) \mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0) > u\right\} + \mathit{II}_{4u}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\mathit{II}_{1u} &\le \mathbb{P}\left\{|\boldsymbol{X}^{(1)}(0)| > K_1 \frac{\lambda^{1/2} u^{-1/\kappa}}{t^{-\alpha/4}(r^{-2}(q_\kappa t) - 1)^{1/2}}\right\} \\
&\le \mathbb{P}\left\{|\boldsymbol{X}^{(1)}(0)| > K_1 \frac{\lambda^{1/2}}{t^{\alpha/4}}\right\} \le K \left(\frac{\lambda}{t^{\alpha/2}}\right)^{-p/2}.
\end{aligned}$$

Similarly,

$$\mathit{II}_{2u} \le K \left(\frac{\lambda u^{2(1-1/\kappa)}}{t^{\alpha\kappa/2}}\right)^{-p/\kappa}, \quad \mathit{II}_{3u} \le K \left(\frac{\lambda}{t^{\alpha/2}}\right)^{-p/2}.$$

Next, we deal with II_{4u}. We have by (23) that $2^{\kappa+4}Bt^{\alpha/2}\le 1$. Therefore, similar arguments as for (26) yield that

$$\begin{aligned} II_{4u} &\le \mathbb{P}\left\{|\boldsymbol{X}^{(2)}(0)|^{\kappa} > \frac{2\lambda u}{t^{\alpha/2}}\frac{1}{2^{\kappa+4}Bt^{\alpha/2}}\right\} \\ &\le K\left(\frac{\lambda}{t^{\alpha/2}}\right)^{-(p-k+2)} u^{-(p-k+m-k(2/\kappa-1)\mathbb{I}\{\kappa\le 2\})/\kappa}\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0)>u\right\}. \end{aligned} \tag{28}$$

Therefore, the claim for $\kappa\in(1,\infty)$ follows from (27) and the inequalities for $\tilde{I}_{1u},\tilde{I}_{2u}$ and II_{1u}– II_{4u} by choosing $p>\max(8(\kappa-1)/\alpha+k+m,2k)$.
Case $\kappa\in(0,1)$: Note that

$$(1+x)^{\kappa}\le 1+x,\quad x\ge 0 \quad\text{and}\quad -|1-x|^{\kappa}\le -(1-x),\quad x\in[0,\infty).$$

We have further by the triangle inequality

$$\begin{aligned} \zeta_{m,k}^{(\kappa)}(q_\kappa t) &\le \left(|\boldsymbol{Y}^{(1)}(q_\kappa t)|+|\boldsymbol{X}^{(1)}_{1/r}(q_\kappa t)|\right)^{\kappa} - \left||\boldsymbol{Y}^{(2)}(q_\kappa t)|-|\boldsymbol{X}^{(2)}_{1/r}(q_\kappa t)|\right|^{\kappa} \\ &\le |\boldsymbol{Y}^{(1)}(q_\kappa t)|^{\kappa}+|\boldsymbol{X}^{(1)}_{1/r}(q_\kappa t)||\boldsymbol{X}^{(1)}(0)|^{\kappa-1}-|\boldsymbol{Y}^{(2)}(q_\kappa t)|^{\kappa}+|\boldsymbol{X}^{(2)}_{1/r}(q_\kappa t)||\boldsymbol{Y}^{(2)}(q_\kappa t)|^{\kappa-1} \\ &\le \frac{|\boldsymbol{X}^{(1)}(0)|^{\kappa}}{(r(q_\kappa t))^{\kappa}}+|\boldsymbol{X}^{(1)}_{1/r}(q_\kappa t)||\boldsymbol{X}^{(1)}(0)|^{\kappa-1}-|\boldsymbol{X}^{(2)}(0)|^{\kappa}+|\boldsymbol{X}^{(2)}_{1/r}(q_\kappa t)||\boldsymbol{X}^{(2)}(0)|^{\kappa-1} \\ &= \frac{\zeta_{m,k}^{(\kappa)}(0)}{(r(q_\kappa t))^{\kappa}}+\theta_\kappa(t)|\boldsymbol{X}^{(2)}(0)|^{\kappa}+|\boldsymbol{X}^{(1)}_{1/r}(q_\kappa t)||\boldsymbol{X}^{(1)}(0)|^{\kappa-1}+|\boldsymbol{X}^{(2)}_{1/r}(q_\kappa t)||\boldsymbol{X}^{(2)}(0)|^{\kappa-1}. \end{aligned}$$

Therefore, we have by (24), with $\psi=\alpha/(4(1-\kappa))$

$$\begin{aligned} &\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(q_\kappa t)>u+\frac{\lambda}{w_\kappa(u)},\zeta_{m,k}^{(\kappa)}(0)\le u\right\} \\ &\le \mathbb{P}\left\{\theta_\kappa(t)|\boldsymbol{X}^{(2)}(0)|^{\kappa}+\frac{|\boldsymbol{X}^{(1)}_{1/r}(q_\kappa t)|}{(u^{-\tau/\kappa}t^{\psi})^{1-\kappa}}+\frac{|\boldsymbol{X}^{(2)}_{1/r}(q_\kappa t)|}{(u^{-\tau/\kappa}t^{\psi})^{1-\kappa}}>\frac{\lambda}{2w_\kappa(u)},\zeta_{m,k}^{(\kappa)}(q_\kappa t)>u\right\} \\ &\quad +\mathbb{P}\left\{|\boldsymbol{X}^{(1)}(0)|\le u^{-\frac{\tau}{\kappa}}t^{\psi},\zeta_{m,k}^{(\kappa)}(q_\kappa t)>u\right\}+\mathbb{P}\left\{|\boldsymbol{X}^{(2)}(0)|\le u^{-\frac{\tau}{\kappa}}t^{\psi},\zeta_{m,k}^{(\kappa)}(0)\le u,\zeta_{m,k}^{(\kappa)}(q_\kappa t)>u+\frac{\lambda}{w_\kappa(u)}\right\} \\ &=: I_{1u}^*+I_{2u}^*+I_{3u}^*. \end{aligned}$$

Now we deal with the three terms one by one. Clearly, for any $u>2$

$$\begin{aligned} I_{1u}^* &\le \mathbb{P}\left\{\theta_\kappa(t)|\boldsymbol{X}^{(2)}(0)|^{\kappa}>\frac{\lambda}{6w_\kappa(u)}\right\}+\mathbb{P}\left\{|\boldsymbol{X}^{(1)}_{1/r}(q_\kappa t)|>\frac{\lambda\kappa t^{\alpha/4}}{6u^{\tau/\kappa}}\right\}\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(q_\kappa t)>u\right\} \\ &\quad +\mathbb{P}\left\{|\boldsymbol{X}^{(2)}_{1/r}(q_\kappa t)|>\frac{\lambda\kappa t^{\alpha/4}}{6u^{\tau/\kappa}}\right\}\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(q_\kappa t)>u\right\}, \end{aligned}$$

where the first term can be treated as for II_{4u}, see (28). For the rest two terms, we have, by using (24)

$$\mathbb{P}\left\{|\boldsymbol{X}^{(j)}_{1/r}(q_\kappa t)|>\frac{\lambda\kappa t^{\alpha/4}}{6u^{\tau/\kappa}}\right\}\le\mathbb{P}\left\{|\boldsymbol{X}^{(j)}(0)|>\frac{\kappa}{12\sqrt{2B}}\frac{\lambda}{t^{\alpha/4}}\right\}\le K\left(\frac{\lambda}{t^{\alpha/4}}\right)^{-p},\quad j=1,2. \tag{29}$$

In order to deal with I_{2u}^* and I_{3u}^*, set below $(\boldsymbol{X}_r^{(1)}(t),\boldsymbol{X}_r^{(2)}(t)):=\left(X_1(0)-r_1(t)X_1(t),\dots,X_{m+k}(0)-r_{m+k}(t)X_{m+k}(t)\right)$ which by definition is independent of $\{\zeta_{m,k}^{(\kappa)}(t),t\ge 0\}$. For $j=1,2$

$$\mathbb{P}\left\{|\boldsymbol{X}^{(j)}_{r}(q_\kappa t)|>x\right\}\le\mathbb{P}\left\{|\boldsymbol{X}^{(j)}(0)|>\frac{2\sqrt{\lambda}x}{\sqrt{u^{-2\tau/\kappa}t^{\alpha}}}\right\}. \tag{30}$$

Using further the triangle inequality $|\boldsymbol{X}_r^{(1)}(q_\kappa t)|^\kappa \ge (r(q_\kappa t))^\kappa |\boldsymbol{X}^{(1)}(q_\kappa t)|^\kappa - |\boldsymbol{X}^{(1)}(0)|^\kappa$ and (23) (recalling $|\boldsymbol{X}^{(1)}(q_\kappa t)|^\kappa \ge \zeta_{m,k}^{(\kappa)}(q_\kappa t) > u$), we have

$$
\begin{aligned}
I_{2u}^* &\le \mathbb{P}\left\{|\boldsymbol{X}_r^{(1)}(q_\kappa t)|^\kappa > u\left((r(q_\kappa t))^\kappa - \frac{t^{\psi\kappa}}{u^{1+\tau}}\right)\right\}\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(q_\kappa t) > u\right\}\\
&\le \mathbb{P}\left\{|\boldsymbol{X}_r^{(1)}(q_\kappa t)|^\kappa > \frac{(1-2^{-\kappa})u}{2^\kappa}\right\}\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0) > u\right\}\\
&\le \mathbb{P}\left\{|\boldsymbol{X}^{(1)}(0)| > (1-2^{-\kappa})^{1/\kappa}\frac{\sqrt{\lambda}}{t^{\alpha/2}}\right\}\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0) > u\right\}\\
&\le K\left(\frac{\lambda}{t^\alpha}\right)^{-p/2}\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0) > u\right\}. \qquad (31)
\end{aligned}
$$

For I_{3u}^*, using $|\boldsymbol{X}^{(1)}(q_\kappa t)|^\kappa > u + \lambda/w_\kappa(u)$ and

$$
|\boldsymbol{X}^{(1)}(0)|^\kappa = \zeta_{m,k}^{(\kappa)}(0) + |\boldsymbol{X}^{(2)}(0)|^\kappa \le u\left(1 + \frac{t^{\psi\kappa}}{u^{1+\tau}}\right)
$$

we have

$$
\begin{aligned}
I_{3u}^* &\le \mathbb{P}\left\{|\boldsymbol{X}_r^{(1)}(q_\kappa t)|^\kappa > u\left((r(q_\kappa t))^\kappa\left(1+\frac{\lambda}{uw_\kappa(u)}\right) - \left(1+\frac{t^{\psi\kappa}}{u^{1+\tau}}\right)\right)\right\}\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(q_\kappa t) > u\right\}\\
&= \mathbb{P}\left\{|\boldsymbol{X}_r^{(1)}(q_\kappa t)|^\kappa > u^{-\tau}\left(\lambda\kappa(r(q_\kappa t))^\kappa - u^{1+\tau}(1-(r(q_\kappa t))^\kappa) - t^{\psi\kappa}\right)\right\}\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0) > u\right\},
\end{aligned}
$$

where by (23)

$$
\lambda\kappa(r(q_\kappa t))^\kappa - u^{1+\tau}(1-(r(q_\kappa t))^\kappa) - t^{\psi\kappa} \ge \frac{\lambda\kappa}{2^{\kappa+1}} - t^{\psi\kappa} \ge \frac{\lambda\kappa}{2^{\kappa+2}}.
$$

Consequently, it follows further by (30) that

$$
\begin{aligned}
I_{3u}^* &\le \mathbb{P}\left\{|\boldsymbol{X}^{(1)}(0)| > 2^{-2/\kappa}\kappa^{1/\kappa}\frac{\lambda^{1/\kappa+1/2}}{t^{\alpha/2}}\right\}\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0) > u\right\}\\
&\le K\left(\frac{\lambda^{1/\kappa+1/2}}{t^{\alpha/2}}\right)^{-p}\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0) > u\right\},
\end{aligned}
$$

which together with (28), (29) and (31) completes the proof for $\kappa \in (0,1)$ by taking $p > 4/\alpha + k$. Consequently, the desired claim of Lemma 3.4 follows. This completes the proof. □

Proof of Lemma 3.5: We give only the proof for (18) since (19) follows by similar arguments. Since the claims for $k = 0$ are already shown in [1], we only consider that $k \ge 1$ below. Define, for $j = 1, 2$, independent random vectors $\left(|\boldsymbol{Y}^{(j)}(s_1)|, \ldots, |\boldsymbol{Y}^{(j)}(s_p)|\right)$ and $\left(|\widetilde{\boldsymbol{Y}}^{(j)}(t_1)|, \ldots, |\widetilde{\boldsymbol{Y}}^{(j)}(t_{p'})|\right)$, which are independent of the process $\zeta_{m,k}^{(\kappa)}$ and have the same distributions as those of $\left(|\boldsymbol{X}^{(j)}(s_1)|, \ldots, |\boldsymbol{X}^{(j)}(s_p)|\right)$ and $\left(|\boldsymbol{X}^{(j)}(t_1)|, \ldots, |\boldsymbol{X}^{(j)}(t_{p'})|\right)$, respectively. Note that, for any $u > 0$, the left-hand side of (18) is clearly bounded from above by

$$
\begin{aligned}
&\Bigg|\mathbb{P}\left\{\bigcap_{i=1}^{p}\left\{|\boldsymbol{X}^{(2)}(s_i)|^\kappa \ge |\boldsymbol{X}^{(1)}(s_i)|^\kappa - u\right\}, \bigcap_{j=1}^{p'}\left\{|\boldsymbol{X}^{(2)}(t_j)|^\kappa \ge |\boldsymbol{X}^{(1)}(t_j)|^\kappa - u\right\}\right\}\\
&\quad - \mathbb{P}\left\{\bigcap_{i=1}^{p}\left\{|\boldsymbol{Y}^{(2)}(s_i)|^\kappa \ge |\boldsymbol{X}^{(1)}(s_i)|^\kappa - u\right\}, \bigcap_{j=1}^{p'}\left\{|\widetilde{\boldsymbol{Y}}^{(2)}(t_j)|^\kappa \ge |\boldsymbol{X}^{(1)}(t_j)|^\kappa - u\right\}\right\}\Bigg|\\
&+\Bigg|\mathbb{P}\left\{\bigcap_{i=1}^{p}\left\{|\boldsymbol{X}^{(1)}(s_i)|^\kappa \le |\boldsymbol{Y}^{(2)}(s_i)|^\kappa + u\right\}, \bigcap_{j=1}^{p'}\left\{|\boldsymbol{X}^{(1)}(t_j)|^\kappa \le |\widetilde{\boldsymbol{Y}}^{(2)}(t_j)|^\kappa + u\right\}\right\}\\
&\quad - \mathbb{P}\left\{\bigcap_{i=1}^{p}\left\{|\boldsymbol{Y}^{(1)}(s_i)|^\kappa \le |\boldsymbol{Y}^{(2)}(s_i)|^\kappa + u\right\}, \bigcap_{j=1}^{p'}\left\{|\widetilde{\boldsymbol{Y}}^{(1)}(t_j)|^\kappa \le |\widetilde{\boldsymbol{Y}}^{(2)}(t_j)|^\kappa + u\right\}\right\}\Bigg|. \qquad (32)
\end{aligned}
$$

Next, note by Cauchy-Schwarz inequality that $u^2+v^2 \le (u^2-2\rho uv+v^2)/(1-|\rho|)$ for all $\rho \in (-1,1)$ and $u,v \in \mathbb{R}$. It follows that, $f_{ij}(\cdot,\cdot)$, the joint density function of $\Big(|\boldsymbol{X}^{(1)}(s_i)|, |\boldsymbol{X}^{(1)}(t_j)|\Big)$, satisfies

$$\begin{aligned}
f_{i,j}(x,y) &= \int_{|\boldsymbol{x}|=x,|\boldsymbol{y}|=y} \prod_{l=1}^{m} \frac{1}{2\pi\sqrt{1-r_l^2(t_j-s_i)}} \exp\left(-\frac{x_l^2-2r_l(t_j-s_i)x_l y_l+y_l^2}{2(1-r_l^2(t_j-s_i))}\right) d\boldsymbol{x}d\boldsymbol{y}\\
&\le \frac{1}{(2\pi)^m(1-(\widetilde{r}(t_j-s_i))^2)^{m/2}} \int_{|\boldsymbol{x}|=x,|\boldsymbol{y}|=y} \prod_{l=1}^{m} \exp\left(-\frac{x_l^2+y_l^2}{2(1+|r_l(t_j-s_i)|)}\right) d\boldsymbol{x}d\boldsymbol{y}\\
&\le \frac{1}{(2\pi)^m(1-(\widetilde{r}(t_j-s_i))^2)^{m/2}} \exp\left(-\frac{x^2+y^2}{2(1+\widetilde{r}(t_j-s_i))}\right) \int_{|\boldsymbol{x}|=x,|\boldsymbol{y}|=y} d\boldsymbol{x}d\boldsymbol{y}\\
&= \frac{(xy)^{m-1}}{2^{m-2}(\Gamma(m/2))^2(1-(\widetilde{r}(t_j-s_i))^2)^{m/2}} \exp\left(-\frac{x^2+y^2}{2(1+\widetilde{r}(t_j-s_i))}\right), \quad x,y>0.
\end{aligned}$$

Therefore, in view of Lemma 2 in [1], with K a constant whose value might change from line to line, the first absolute value in (32) is bounded from above by

$$\begin{aligned}
&K\sum_{i=1}^{p}\sum_{j=1}^{p'} \int_{x^\kappa>u}\int_{y^\kappa>u} \widetilde{r}(t_j-s_i)\Big((x^\kappa-u)(y^\kappa-u)\Big)^{(k-1)/\kappa} \exp\left(-\frac{(x^\kappa-u)^{2/\kappa}+(y^\kappa-u)^{2/\kappa}}{2(1+\widetilde{r}(t_j-s_i))}\right) f_{ij}(x,y)\,dxdy\\
&\le K\sum_{i=1}^{p}\sum_{j=1}^{p'} \widetilde{r}(t_j-s_i)\left(\int_u^\infty (x-u)^{(k-1)/\kappa}x^{m/\kappa-1}\exp\left(-\frac{x^{2/\kappa}}{2(1+\widetilde{r}(t_j-s_i))}\right)dx\right)^2\\
&\le K u^{(2/\kappa)(m-(k-1)(2/\kappa-1)-2)} \sum_{i=1}^{p}\sum_{j=1}^{p'} \widetilde{r}(t_j-s_i)\exp\left(-\frac{u^{2/\kappa}}{1+\widetilde{r}(t_j-s_i)}\right),
\end{aligned}$$

where in the first inequality, we use first the bound $e^{-x}\le 1, x\ge 0$ and then a change of variable $x'=x^\kappa$, while the second inequality follows by a change of variable $x'=u^{2/\kappa-1}(x-u)$ and Taylor's expansion of $(u+x'/u^{2/\kappa-1})^{2/\kappa} = u^{2/\kappa}+(2/\kappa)x'+O(u^{-2/\kappa})$ for large u and $x'\ge 0$. Similarly, denoting by $g(\cdot)$ the pdf of $|\boldsymbol{X}^{(2)}(0)|$ (see also (8)), we obtain that the second absolute value in (32) is bounded from above by

$$\begin{aligned}
&K\sum_{i=1}^{p}\sum_{j=1}^{p'} \widetilde{r}(t_j-s_i)\int_0^\infty\int_0^\infty \Big((x^\kappa+u)(y^\kappa+u)\Big)^{(m-1)/\kappa} \exp\left(-\frac{(x^\kappa+u)^{2/\kappa}+(y^\kappa+u)^{2/\kappa}}{2(1+\widetilde{r}(t_j-s_i))}\right) g(x)g(y)\,dxdy\\
&\le K\sum_{i=1}^{p}\sum_{j=1}^{p'} \widetilde{r}(t_j-s_i)\left(\int_0^\infty (x^\kappa+u)^{(m-1)/\kappa}x^{k-1}\exp\left(-\frac{(x^\kappa+u)^{2/\kappa}}{2(1+\widetilde{r}(t_j-s_i))}\right)dx\right)^2\\
&\le K u^{(2/\kappa)(m-k(2/\kappa-1)-1)} \sum_{i=1}^{p}\sum_{j=1}^{p'} \widetilde{r}(t_j-s_i)\exp\left(-\frac{u^{2/\kappa}}{1+\widetilde{r}(t_j-s_i)}\right),
\end{aligned}$$

where the last step follows by a change of variable $x'=u^{2/\kappa-1}x^\kappa$. Hence the proof of (18) is established since

$$\Big(m-k(2/\kappa-1)-1\Big)-\Big(m-(k-1)(2/\kappa-1)-2\Big) = -2(1/\kappa-1).$$

The desired result in Lemma 3.5 follows. □

Proof of Lemma 3.6: The proof follows by the same arguments as for Lemma 12.3.1 in [16], using alternatively the following asymptotic relation (recall (20) and Lemma 3.1)

$$u^{2/\kappa} = 2\ln T_\kappa + K_0\ln\ln T_\kappa + \ln D_0(1+o(1)), \quad T_\kappa\to\infty \tag{33}$$

with D_0, K_0 defined in Theorem 2.3. We split the sum in (21) at T_κ^β, where β is a constant such that $0<\beta<(1-\delta)/(1+\delta)$ and $\delta=\sup\{\widetilde{r}(t): t\ge\epsilon\}<1$ (see, e.g., Lemma 8.1.1 (i) in [16]). Below K is again a positive constant which value might change from line to line. From (33) we conclude that $\exp\left(-u^{2/\kappa}/2\right)\le K/T_\kappa$ and

$u^{2/\kappa} = 2\ln T_\kappa(1+o(1))$. Further,

$$u^\varsigma \frac{T_\kappa}{q_\kappa} \sum_{\varepsilon \le a q_\kappa j \le T_\kappa^\beta} \widetilde{r}(a q_\kappa j) \exp\left(-\frac{u^{2/\kappa}}{1+\widetilde{r}(a q_\kappa j)}\right)$$
$$\le u^{\varsigma + \frac{4\tau}{\alpha\kappa}} T_\kappa^{\beta+1} \exp\left(-\frac{u^{2/\kappa}}{1+\delta}\right) \le K (\ln T_\kappa)^{\frac{\kappa\varsigma}{2} + \frac{2\tau}{\alpha}} T_\kappa^{\beta+1-\frac{2}{1+\delta}},$$

which tends to 0 as $T_\kappa \to \infty$ since $\beta+1-2/(1+\delta) < 0$. For the remaining sum, denoting $\delta(t) = \sup\{|\widetilde{r}(s)\ln s| : s \ge t\}$, $t>0$, we have $\widetilde{r}(t) \le \delta(t)/\ln t$ as $t \to \infty$, and thus in view of (33) for $a q_\kappa j \ge T_\kappa^\beta$

$$\exp\left(-\frac{u^{2/\kappa}}{1+\widetilde{r}(a q_\kappa j)}\right) \le \exp\left(-u^{2/\kappa}\left(1 - \frac{\delta(T_\kappa^\beta)}{\ln T_\kappa^\beta}\right)\right)$$
$$\le K \exp(-u^{2/\kappa}) \le K T_\kappa^{-2} (\ln T_\kappa)^{-K_0}.$$

Consequently, with c given by Theorem 2.3 (recall $\tau = 2\max(1/\kappa - 1, 0) + 1$),

$$u^\varsigma \frac{T_\kappa}{q_\kappa} \sum_{T_\kappa^\beta \le a q_\kappa j \le T_\kappa} \widetilde{r}(a q_\kappa j) \exp\left(-\frac{u^{2/\kappa}}{1+\widetilde{r}(a q_\kappa j)}\right)$$
$$\le K u^\varsigma \left(\frac{T_\kappa}{q_\kappa}\right)^2 T_\kappa^{-2} (\ln T_\kappa)^{-K_0} \frac{1}{\left(\ln T_\kappa^\beta\right)^c} \frac{1}{T_\kappa/q_\kappa} \sum_{T_\kappa^\beta \le a q_\kappa j \le T_\kappa} \widetilde{r}(a q_\kappa j)\big(\ln(a q_\kappa j)\big)^c$$
$$\le K (\ln T_\kappa)^{\frac{\kappa\varsigma}{2} + \frac{2\tau}{\alpha} - K_0 - c} \frac{1}{T_\kappa/q_\kappa} \sum_{T_\kappa^\beta \le a q_\kappa j \le T_\kappa} \widetilde{r}(a q_\kappa j)\big(\ln(a q_\kappa j)\big)^c. \quad (34)$$

Since $K_0 = m - 2 + 2\tau/\alpha + k\min(1-2/\kappa, 0)$ and $\varsigma := 2/\kappa(m - k(2/\kappa - 1) - 1 + \max(0, 2(1/\kappa - 1)))$, we have $\kappa\varsigma/2 + 2\tau/\alpha - K_0 - c = 0$ for all $\kappa > 0$. Noting further that the Berman-type condition $\lim_{t\to\infty} \widetilde{r}(t)\big(\ln t\big)^c = 0$ holds and $\beta < 1$, the right-hand side of (34) tends to 0 as $u \to \infty$. Thus the proof is complete.

□

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