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Ruin Probability for Discrete and Continuous Gaussian Risk Models

Jasnovidov Grigori

UNIL | Université de Lausanne

FACULTÉ DES HAUTES ÉTUDES COMMERCIALES

DÉPARTEMENT DE SCIENCES ACTUARIELLES

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THÈSE DE DOCTORAT

présentée à la

Faculté des Hautes Études Commerciales
de l'Université de Lausanne

pour l'obtention du grade de
Docteur en sciences actuarielles

par

Grigori JASNOVIDOV

Directeur de thèse
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Jury

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UNIL | Université de Lausanne
HEC Lausanne
Le Décanat
Bâtiment Internef
CH-1015 Lausanne

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RUIN PROBABILITY FOR DISCRETE AND CONTINUOUS GAUSSIAN RISK MODELS

Lausanne, le 18 mai 2021

Le doyen

Jean-Philippe Bonardi

HEC Lausanne
Le Décanat
Tél. +41 21 692 33 40 | Fax +41 21 692 33 05
www.hec.unil.ch | hecdoyen@unil.ch

Members of the Jury

Prof. **Felicitas Morhart**

President of the jury, University of Lausanne, Department of Marketing.

Prof. **Enkelejd Hashorva**

Thesis director, University of Lausanne, Department of Actuarial Science.

Prof. **François Dufresne**

Internal Expert, University of Lausanne, Department of Actuarial Science.

Prof. **Krzysztof Dębicki**

External Expert, University of Wroclaw.

Prof. **Mikhail Lifshits**

External Expert, St. Petersburg Department of Steklov Mathematical Institute.

Prof.**Dmitry Zaporozhets**

External Expert, St.Petersburg Department of Steklov Mathematical Institute.

University of Lausanne
Faculty of Business and Economics

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have been addressed to my entire satisfaction.

Signature: ______________________ Date: _____09/05/2021___________

Prof. Enkelejd HASHORVA
Thesis supervisor

University of Lausanne
Faculty of Business and Economics

PhD in Actuarial Science

I hereby certify that I have examined the doctoral thesis of

Grigori JASNOVIDOV

and have found it to meet the requirements for a doctoral thesis.
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made during the doctoral colloquium
have been addressed to my entire satisfaction.

Signature: F. Dufresne Date: 8 May 2021

Prof. François DUFRESNE
Internal member of the doctoral committee

University of Lausanne

University of Lausanne
Faculty of Business and Economics

PhD in Actuarial Science

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have been addressed to my entire satisfaction.

Signature: Krzysztof Dębicki Date: 8.05.2021

Prof. Krzysztof DEBICKI
External member of the doctoral committee

PhD in Actuarial Science

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Grigori JASNOVIDOV

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have been addressed to my entire satisfaction.

Signature: ________________________ Date: 9.5.2021

Prof. Mikhail LIFSHITS
External member of the doctoral committee

University of Lausanne
Faculty of Business and Economics

PhD in Actuarial Science

University of Lausanne
Faculty of Business and Economics

PhD in Actuarial Science

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Signature: ______________ Date: 9.05.2021

Prof. Dmitry ZAPOROZHETS
External member of the doctoral committee

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Contents

Chapter 1

Introduction & Notation

In this dissertation our main aim is the study of the asymptotics of the classical continuous and discrete ruin probabilities of Gaussian processes and their generalizations: Parisian, sojourn, γ-reflected and storage ruins. The classical ruin problem for Gaussian process is the computation of the asymptotics of

$$\mathbb{P}\left\{\exists t \in T : X(t) > u\right\}, \qquad u \to \infty, \tag{1.1}$$

where $X(t),\ t \in T$ is a Gaussian process with almost surely continuous sample paths and T is some measurable subset of the real line. Typically, in this work T is a closed finite interval, uniform grid or positive ray of the real line, while X is a centered process with stationary increments (often fractional Brownian motion or Brownian motion) and negative linear drift.

The asymptotic analysis of the probability in (1.1) is an important part of the extreme value theory that applies in many fields: insurance, reinsurance, finance and physics. We refer to a list of various applications of the extreme value theory to monograph [31], while for the particular role of the study of the ruin probabilities for Gaussian models to [57]. Currently a lot of Gaussian ruin problems have been investigated under continuous-time setup, we refer to, e.g., [27, 49, 56, 57, 59, 60] and references therein. In this work we mainly focus on the discrete-time settings of the ruin problem. Here we usually observe different scenarios of the asymptotic behavior of the discrete ruin probabilities if the variance of X in (1.1) is regularly varying at infinity with index not exceeding 1.

The second aim of the thesis is the study of the Pickands constants that commonly appear in the asymptotics of the ruin probabilities of Gaussian processes, see, e.g., [12, 30]. We devote Chapter 7 to study various properties of the discrete and continuous classical Pickands constants. Also, throughout the dissertation we introduce numerous Pickands and Piterbarg type constants and study their basic properties.

In Chapter 2 we explain how to apply the double-sum method, i.e., the main technique for deriving approximations of the ruin probabilities of Gaussian processes. By this approach we solve the classical, Parisian, sojourn and γ-reflected problems for Brownian motion discrete-time setting. To prove our results we rely on the self-similarity and independence of the increments of Brownian motion, these properties and the discrete time setup allow us to give relatively simple rigorous proofs.

In Chapter 3 we study the aymptotics of the ruin probabilities in discrete fractional Brownian motion risk models. We observe, that discretization of time leads to different asymptotical behavior even in some most simple and natural models. Moreover, for some cases discretization does not allow to derive the asymptotics, there we present optimal bounds.

In Chapter 4 we study the simultaneous Parisian ruin problem for fractional Brownian motion. Here the scenario of behavior of the ruin probability is determined by the length of the interval needed to clarify that a Parisian ruin occurs. Also, we suggest an approach for approximation of the numerical values of the Pickands and Piterbarg type constants appearing in the asymptotics via Monte-Carlo simulations.

In Chapter 5 we study the sojourn ruin problem for the model introduced in the previous chapter. The main difficulty here comparing with the problems of Chapters 3 and 4 is the less developed instruments for approximation of the sojourn ruin.

In Chapter 6 we solve the classical discrete ruin problem for the class of Gaussian process with stationary increments, almost surely continuous sample paths and regularly varying at infinity variance satisfying some smoothness conditions. We generalize this problem to the ruin problem of the suprema and infima of the corresponding storage process. In some special cases we observe that the discrete asymptotics are exponentially smaller than their continuous counterparts; in other case we detect the strong Piterbarg property for the storage process.

In Chapter 7 we investigate some properties of the classical discrete and continuous time Pickands constants. First of all, we give a relatively precise upper bound for the difference between the continuous and discrete Pickands constants with the same Hurst index. This bound is useful for estimation of the discretization-error appearing in the approach of approximation of the continuous Pickands constants introduced in [30]. Secondly, we present an explicit representation of the classical discrete Pickands constant for Brownian motion in terms of converging series. This representation allows us to show:

1) that the discrete Brownian motion Pickands constant is strictly decreasing with respect to the size of the grid;

2) the exact speed of convergence of the discrete Brownian motion Pickands constant to the corresponding continuous one. It is interesting, that the Riemann zeta function effects the answer.

In Chapter 8 we give alternative proofs of some results of the previous chapters. These proofs are based on the useful properties of Brownian motion: the independence and stationarity of the increments and self-similarity. We also avoid using special tools and rely only on well-known results.

Next, we present the basic notation throughout the thesis. The notation above is applicable for all following chapters.

Let Φ and $\overline{\Phi}$ be the distribution and survival functions of a standard Gaussian random variable, respectively.

Standard fractional Brownian motion is a centered Gaussian process with a.s. continuous sample paths, $B_H(0) = 0$ and covariance function

$$\mathrm{cov}(B_H(t), B_H(s)) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \qquad t, s \in \mathbb{R}.$$

If $H = 1/2$, then $B_{1/2} := B$ is a standard Brownian motion. Unless otherwise stated explicitly, we suppose that $B(t)$ is defined for $t \geq 0$, while $B_H(t)$ for $t \in \mathbb{R}$. We use abbreviations fBM and BM for fractional Brownian motion and Brownian motion, respectively.

For any $a < b \in \mathbb{R} \cup \{-\infty, \infty\}$ and $\tau \geq 0$ we set

$$[a,b]_\tau = \begin{cases} [a,b] \cap \tau\mathbb{Z}, & \tau > 0 \\ [a,b], & \tau = 0. \end{cases}$$

For any $\eta \geq 0$ we denote a discrete uniform grid by

$$G(\eta) = \begin{cases} \{0, \eta, 2\eta, ...\}, & \eta > 0 \\ [0, \infty), & \eta = 0. \end{cases}$$

Define the classical Pickands constant for $\delta \geq 0$ and $H \in (0,1)$ by

$$\mathcal{H}_{2H}^{\delta} = \lim_{S \to \infty} \frac{1}{S} \mathbb{E}\left\{ \sup_{t \in [0,S]_\delta} e^{\sqrt{2}B_H(t) - t^{2H}} \right\}.$$

It is known (see, e.g., [30]) that the constant above is finite and positive for any $\delta \geq 0$ and $H \in (0,1)$. Let $\mathcal{H}_{2H} := \mathcal{H}_{2H}^0$ and $\mathcal{H}^\delta := \mathcal{H}_1^\delta$.

Let $\mathbb{I}(\cdot)$ be the indicator function.

Let $C, C_1, C_2, \mathbb{C}, \mathbb{C}_1, \mathbb{C}_2, \bar{C}, \overline{C}, \widetilde{C}$, etc. be positive constants, that do not depend on any non-fixed considered parameter. Their numerical values are out of importance and they can be different in different places.

We use abbreviations "a.s.", "df(s)", "rv(s)" and "i.i.d." to mean almost surely, distribution function(s), random variable(s) and independent identically distributed, respectively.

We suppose that all stochastic processes and random variables are defined on the complete general probability space Ω with the probability measure $\mathbb{P}$. By default we assume that Ω consists of outcomes ω.

Chapter 2

Brownian Motion Discrete-Time Models

This chapter is based on G. Jasnovidov: Approximation of Ruin Probability and Ruin Time in Discrete Brownian Risk Models, Scandinavian Actuarial Journal, 718-735, 8, 2020.

2.1 Introduction

The classical Brownian risk model of an insurance portfolio

$$R_u(t) = u + ct - B(t), \qquad t \geq 0,$$

with the initial capital $u > 0$ and the premium rate $c > 0$, is a key benchmark model in risk theory; see e.g., [39]. For any $u > 0$ define the ruin time

$$\tau(u) = \inf\{t \geq 0 : B(t) - ct > u\}$$

and thus the corresponding ruin probability is given by the well-known formula (see e.g., [26])

$$\psi_\infty(u) := \mathbb{P}\left\{\tau(u) < \infty\right\} = \mathbb{P}\left\{\inf_{t\geq 0} R_u(t) < 0\right\} = e^{-2cu}. \tag{2.1}$$

In insurance practice however the ruin probability is relevant not on a continuous time scale, but on a discrete one, due to the operational time (which is discrete). For a given discrete uniform grid $G(\delta)$ we define the corresponding ruin probability by

$$\psi_{\delta,\infty}(u) := \mathbb{P}\left\{\inf_{t\in G(\delta)} R_u(t) < 0\right\} = \mathbb{P}\left\{\sup_{t\in G(\delta)} (B(t) - ct) > u\right\}. \tag{2.2}$$

For any $u > 0$ it is not possible to calculate $\psi_{\delta,\infty}(u)$ explicitly and no formulas are available for the distributional characteristics of the corresponding ruin time which we shall denote by $\tau_\delta(u)$. A natural question when explicit formulas are lacking is how can we approximate $\psi_{\delta,\infty}(u)$ and $\tau_\delta(u)$ for large u? Also of interest is to know what the role of δ is: does it influence the ruin probability in this classical risk model? The first question has been considered recently in [47] for fBM risk process.

When dealing with the Brownian risk model, both the independence of increments and the self-similarity property are crucial. In particular, those properties are the key to a rigorous and (relatively) simple proof.

Our first result presented next shows that the grid plays a role only with respect to the pre-factor specified by some constant. Specifically, that constant is well-known in the extremes of Gaussian processes being the Pickands constant $\mathcal{H}^{2c^2\delta}$, where

$$\mathcal{H}^{\eta} = \frac{1}{\eta}\mathbb{E}\left\{\frac{\sup_{t\in\eta\mathbb{Z}} e^{W(t)}}{\sum_{t\in\eta\mathbb{Z}} e^{W(t)}}\right\} = \frac{1}{\eta}\mathbb{E}\left\{\max_{t\ge 0, t\in\eta\mathbb{Z}} e^{W(t)} - \max_{t\ge\eta, t\in\eta\mathbb{Z}} e^{W(t)}\right\} \in (0,\infty) \tag{2.3}$$

for any $\eta > 0$, with $W(t) = \sqrt{2}B(t) - |t|$. The first formula in (2.3) is derived in [30], whereas the second in [13].

Theorem 2.1.1 *For any $\delta > 0$ we have*

$$\psi_{\delta,\infty}(u) \sim \mathcal{H}^{2c^2\delta}\psi_\infty(u), \quad u \to \infty \tag{2.4}$$

and further for any $s \in \mathbb{R}$

$$\lim_{u\to\infty} \mathbb{P}\left\{c^{3/2}(\tau_\delta(u) - u/c)/\sqrt{u} \le s \middle| \tau_\delta(u) < \infty\right\} = \Phi(s). \tag{2.5}$$

We note that the above results hold for the continuous case too, where the grid $G(\delta)$ is substituted by $[0,\infty)$. For that case (2.5) follows from [38]. The approximation in (2.5) shows that the ruin time is not affected by the density of the grid (i.e., it is independent of δ) and thus we conclude that the grid influences only the ruin probability. This is not the case for the ruin probability approximated in (2.4). For the Pickands constants we have, see e.g., [13, 30, 54]

$$\mathcal{H}^{2c^2\delta} \le 1 = \lim_{\delta\downarrow 0} \mathcal{H}^{2c^2\delta}.$$

In particular we see that via self-similarity in the Brownian risk model the role of the grid is coupled with the premium rate $c > 0$.

The objective of Section 2 is to explain in detail the main ideas and techniques adequate for the classical Brownian risk model. Section 3 discusses the ruin probability for the γ-reflected Brownian risk model, see also [18, 34, 35, 52]. The approximation of Parisian ruin (see [15, 16, 53]) and sojourn ruin (see [17, 21, 22]) is the topic of Section 4. Our findings show that also for those ruin probabilities, the influence of the grid, i.e., the choice of δ concerns only the leading constant in the asymptotic expansion being further coupled with the premium rate. Given the technical nature of several proofs, we shall relegate them to Section 5, which is followed by an Appendix containing auxiliary calculations.

2.2 Approximation Techniques for Brownian Risk Model

Both the independence of increments and the self-similarity property of BM render the Brownian risk model very tractable. In order to approximate $\psi_{\delta,\infty}(u)$ for given $\delta > 0$ we start with the following lower bound

$$\psi_{\delta,\infty}(u) = \mathbb{P}\left\{\exists t \in G(\delta) : B(t) > u + ct\right\} \geq \mathbb{P}\left\{B(ut_u) > u(1 + ct_u)\right\}$$

valid for t_u such that $ut_u \in G(\delta)$ for all u large. It is clear that such t_u exists and moreover

$$t_u = \frac{1}{c} + \frac{\theta_u}{u} \in G(\delta) \tag{2.6}$$

holds for some $\theta_u \in [0, \delta)$ and all large u. Consequently, by the well-known inequality (see, e.g., Lemma 2.1 in [56])

$$(1 - \frac{1}{u^2})\frac{1}{\sqrt{2\pi}u}e^{-u^2/2} \leq \overline{\Phi}(u) \leq \frac{1}{\sqrt{2\pi}u}e^{-u^2/2}, \quad u > 0 \tag{2.7}$$

we obtain for all large u and some positive constant C

$$\psi_{\delta,\infty}(u) \geq \overline{\Phi}(\sqrt{u/t_u}(1 + ct_u)) \geq \frac{C}{\sqrt{u}}e^{-2cu}, \tag{2.8}$$

where $\varphi = \Phi'$. Although the lower bound above is not precise enough, it is useful to localize a short interval around

$$t_0 := 1/c$$

that will lead eventually to the exact approximation of the ruin probability. Indeed, we have with

$$T_u^{\pm} = u(t_0 \pm u^{-1/2}\ln u), \qquad Z(t) = B(t) - ct$$

for all large u and any $C > 0, \ p < 0$ (the proof is given in the Appendix)

$$\mathbb{P}\left\{\sup_{t \notin [T_u^-, T_u^+]} Z(t) > u\right\} \leq Cu^p e^{-2cu}. \tag{2.9}$$

Since for any $u > 0$

$$\mathbb{P}\left\{\sup_{t \in [T_u^-, T_u^+]_\delta} Z(t) > u\right\} \leq \psi_{\delta,\infty}(u) \leq \mathbb{P}\left\{\sup_{t \in [T_u^-, T_u^+]_\delta} Z(t) > u\right\} + \mathbb{P}\left\{\sup_{t \notin [T_u^-, T_u^+]} Z(t) > u\right\} \tag{2.10}$$

by (2.8) and (2.9) we obtain that (set $\Delta_\delta(u) = [t_u - u^{-1/2}\ln u, t_u + u^{-1/2}\ln u]_{\frac{\delta}{u}}$)

$$\psi_{\delta,\infty}(u) \sim \mathbb{P}\left\{\sup_{t \in [T_u^-, T_u^+]_\delta,} Z(t) > u\right\} = \mathbb{P}\left\{\exists t \in \Delta_\delta(u) : \frac{B(t)}{1 + ct} > \sqrt{u}\right\} =: P_\delta(u), \quad u \to \infty,$$

where for the last equality we used the self-similarity property of BM.

In order to approximate $P_\delta(u)$ as $u \to \infty$ a common approach is to partition $\Delta_\delta(u)$ in small intervals and use Bonferroni inequality in order to determine the main contribution to the asymptotics.

This idea coupled with the continuous mapping theorem is essentially due to Piterbarg, see e.g., [57]. In this paper we use a modified approach in order to tackle some uniformity issues which arise in the approximations. In particular, we do not use continuous mapping theorem but rely instead on the independence of increments and self-similarity property of BM. We illustrate below briefly our approach.

We choose a partition $\Delta_{j,S,u}, -N_u \leq j \leq N_u$ of $\Delta_\delta(u)$ depending on some constant $S > 0$ as follows

$$\Delta_{j,S,u} = [t_u + jSu^{-1}, t_u + (j+1)Su^{-1}]_{\frac{\delta}{u}}, \quad N_u = \lfloor S^{-1}\ln(u)\sqrt{u}\rfloor. \tag{2.11}$$

Here $\lfloor\cdot\rfloor$ stands for the ceiling function. The Bonferroni inequality yields

$$p_1(S,u) \geq P_\delta(u) \geq p_1'(S,u) - p_2(S,u), \tag{2.12}$$

where

$$p_1(S,u) \;=\; \sum_{j=-N_u-1}^{N_u} p_{j,S,u}, \quad p_1'(S,u) \;=\; \sum_{j=-N_u}^{N_u-1} p_{j,S,u}, \quad p_2(S,u) = \sum_{-N_u-1\leq j<i\leq N_u} p_{i,j;S,u},$$

with

$$p_{j,S,u} = \mathbb{P}\left\{\exists_{t\in\Delta_{j,S,u}} \frac{B(t)}{1+ct} > \sqrt{u}\right\} \text{ and } p_{i,j;S,u} = \mathbb{P}\left\{\exists_{t\in\Delta_{i,S,u}} \frac{B(t)}{1+ct} > \sqrt{u}, \exists_{t\in\Delta_{j,S,u}} \frac{B(t)}{1+ct} > \sqrt{u}\right\}.$$

As shown in [17] [Eq. (43)] the term $p_2(S,u)$, also referred to as the double-sum term, is negligible compared with $p_1'(S,u)$ if we let $u\to\infty$ and then $S\to\infty$.

Moreover, $p_1(S,u)$ and $p_1'(S,u)$ are asymptotically equivalent with $P_\delta(u)$, i.e.,

$$\lim_{S\to\infty}\lim_{u\to\infty} p_1(S,u)/p_1'(S,u) = \lim_{S\to\infty}\lim_{u\to\infty} p_1(S,u)/P_\delta(u) = 1.$$

The main question is therefore how to approximate $p_1(S,u)$?

In order to answer the above question, we need to approximate each term $p_{j,S,u}$ as $u\to\infty$. Moreover, such approximation has to be uniform for all j satisfying $-N_u \leq j \leq N_u$, which is a subtle issue solved in this paper by utilizing the independence of increments of BM and the self-similarity property; see the proof of Theorem 2.1.1 in Section 5 and [17] for similar ideas in the continuous time setting.

2.3 γ-Reflected Risk Model

An interesting extension of the classical Brownian risk model is that of γ-reflected Brownian risk model introduced in [1]. The γ-reflected fBM risk model and its extensions are discussed in

[18, 34, 35, 52]. In this section we consider the approximation of the ruin probability over a discrete grid $G(\delta), \delta > 0$ for the γ-reflected BM model. Specifically, for $\gamma \in (0,1)$ we define the risk model

$$R^{\gamma}_{u,\delta}(t) = u + ct - B(t) + \gamma \inf_{s\in[0,t]_\delta}(B(s) - cs), \quad t \in G(\delta)$$

and the corresponding ruin time

$$\widetilde{\tau}_\gamma(u) = \inf\{t \geq 0 : R^{\gamma}_{u,\delta}(t) < 0\}, \qquad \gamma \in (0,1).$$

For given $\delta > 0$ we are interested in the ruin probability in discrete time, namely

$$\begin{aligned}
\Upsilon_{\gamma,\delta}(u) &:= \mathbb{P}\{\widetilde{\tau}_\gamma(u) < \infty\} \\
&= \mathbb{P}\left\{\exists t \in G(\delta) : R^{\gamma}_{u,\delta}(t) < 0\right\} \\
&= \mathbb{P}\{\exists t \in G(\delta), s \in [0,t]_\delta : u + ct - B(t) + \gamma B(s) - c\gamma s < 0\} \\
&= \mathbb{P}\{\exists t \in G(\delta), s \in [0,t]_\delta : B(t) - ct - \gamma B(s) + c\gamma s > u\},
\end{aligned}$$

which cannot be calculated explicitly. The risk process $R^{\gamma}_{u,\delta}(t)$ is not Gaussian anymore, however using the independence of the increments of BM and the self-similarity property, for any $u > 0$ we have with $t = k - l, \ s = l$

$$\begin{aligned}
\Upsilon_{\gamma,\delta}(u) &= \mathbb{P}\{\exists l \leq k \in G(\delta) : B(k) - kc - \gamma(B(l) - cl) > u\} \\
&= \mathbb{P}\{\exists l \leq k \in G(\delta) : (B(k) - B(l)) + (1-\gamma)B(l) - c(k - \gamma l) > u\} \\
&= \mathbb{P}\{\exists l \leq k \in G(\delta) : B(k-l) + (1-\gamma)B^*(l) - c(k - \gamma l) > u\} \\
&= \mathbb{P}\{\exists t, s \in G(\delta) : (B(t) - ct) + (1-\gamma)(B^*(s) - cs) > u\} \\
&= \mathbb{P}\left\{\exists t, s \in G(\delta/u) : \frac{B(t) + (1-\gamma)B^*(s)}{ct + (1-\gamma)cs + 1} > \sqrt{u}\right\},
\end{aligned}$$

where B^* is an independent copy of B. The above re-formulation shows that the ruin probability concerns the supremum of the random field Z given by

$$Z(t,s) = \frac{B(t) + (1-\gamma)B^*(s)}{ct + (1-\gamma)cs + 1}, \quad s,t \geq 0. \tag{2.13}$$

From [35] it follows, that for any $\eta, a > 0$

$$\mathcal{P}^a_\eta := \mathbb{E}\left\{\sup_{t\in[0,\infty)_\eta} e^{\sqrt{2}B(t) - t(1+a)}\right\} \in (0,\infty).$$

Our next result gives the approximation of the above ruin probability as $u \to \infty$.

Theorem 2.3.1 *For any $\delta > 0$ and any $\gamma \in (0,1)$*

$$\Upsilon_{\gamma,\delta}(u) \sim \mathcal{P}^{\frac{\gamma}{1-\gamma}}_{2c^2(1-\gamma)^2\delta}\mathcal{H}^{2c^2\delta}\psi_\infty(u), \quad u \to \infty. \tag{2.14}$$

We note that the basic properties of discrete Piterbarg constants are discussed in [3, 13].

2.4 Parisian & Sojourn Ruin

2.4.1 Parisian Ruin

In this section we expand our results to the Parisian ruin. For the continuous time [53] gives an exact formula for the Parisian ruin probability. Both finite and infinite Parisian ruin times for continuous setup of the problem are dealt with in [15, 16].
Next, for given δ, T positive (suppose for convenience that $T/\delta \in G(\delta)$) define the Parisian ruin time and probability for the discrete grid $G(\delta)$ by

$$\tau_\delta(u,T) = \{\inf t \in G(\delta) : \sup_{s\in[t,t+T]_\delta} R_u(s) < 0\}$$

and

$$\mathcal{P}_\delta(u,T) = \mathbb{P}\left\{\tau_\delta(u,T) < \infty\right\},$$

respectively. Our next result shows again that the grid determines the asymptotic approximation via the constant $\mathcal{H}_{\eta,T}$ defined for η, T positive by

$$\mathcal{H}_{\eta,T} = \mathbb{E}\left\{\frac{\sup\limits_{t\in\eta\mathbb{Z}} \inf\limits_{s\in[t,t+T]_\eta} e^{\sqrt{2}B(s)-|s|}}{\eta\sum_{t\in\eta\mathbb{Z}} e^{\sqrt{2}B(t)-|t|}}\right\} \in (0,\infty). \tag{2.15}$$

Note that if $T = 0$, then $\mathcal{H}_{\eta,0}$ equals the Pickands constant $\mathcal{H}^\eta$ defined in (2.3). The corresponding constant for the continuous case is introduced in [15].

Theorem 2.4.1 *For any $\delta, T > 0$*

$$\mathcal{P}_\delta(u,T) \sim \mathcal{H}_{2c^2\delta,2c^2T}\psi_\infty(u), \quad u \to \infty. \tag{2.16}$$

We see from the approximation above that the premium rate c influences also the leading constant in the asymptotics.

2.4.2 Sojourn Ruin

Sojourn ruin for fBM risk model has been discussed recently in [21]. As therein, adjusted for the discrete setup, we define the sojourn ruin time and probability by

$$\tau_\delta^k(u) = \{\inf t \in G(\delta) : \#\{s \in [0,t]_\delta : B(t) - ct > u\} > k\}$$

and

$$\mathcal{C}_\delta(u,k) = \mathbb{P}\left\{\tau_\delta^k(u) < \infty\right\},$$

where k is some non-negative integer and the symbol $\#$ stands for the number of the elements of a given set. Note in passing that $\mathcal{C}_\delta(u,0)=\psi_{\delta,\infty}(u)$. Next, for $\eta>0$ define the constant

$$\mathcal{B}_\eta(k)=\lim_{S\to\infty}\frac{\mathcal{B}_\eta(S,k)}{S},$$

where for any $S>0$

$$\mathcal{B}_\eta(S,k)=\int_{\mathbb{R}}\mathbb{P}\left\{\eta\sum_{s\in[0,S]_\eta}\mathbb{I}(\sqrt{2}B(s)-|s|+z>0)>k\right\}e^{-z}dz.$$

In view of [22] $\mathcal{B}_\eta(k)$ is positive and finite.

Theorem 2.4.2 *For any non-negative integer k we have as $u\to\infty$*

$$\mathcal{C}_\delta(u,k)\sim\mathcal{B}_{2c^2\delta}(k)\psi_\infty(u). \tag{2.17}$$

Remark 2.4.3 *i) Defining the ruin times corresponding to Parisian and sojourn ruin, it follows with similar arguments as in the proof of Theorem 2.1.1 that those can be approximated in the same way as* (2.5).
ii) If $k=0$, then the claim in (2.17) *reduces to* (2.4).

2.5 Proofs

Proof of Theorem 2.1.1: As mentioned in Section 2, the negligibility of the double-sum term follows by [17], hence the claim in (2.4) follows thus by approximating $p_1(S,u)$ as $u\to\infty$. We show first the approximation of $p_{j,S,u}$ as $u\to\infty$ uniformly for $-N_u\le j\le N_u$. Note that with $u=v^2$ and $\mathcal{N}$ being a standard Gaussian rv we have the distributional representation based on the independence of increments of BM

$$B(c_{j,S,u}+t/u)=\sqrt{c_{j,S,v}}\mathcal{N}+B(t)/v,\quad t\in[0,S],\ u>0,\ c_{j,S,v}=t_u+jSv^{-2}.$$

Recall that $t_u\in G(\delta)$ is given by $t_u=1/c+\theta_u/u$ for some $\theta_u\in[0,\delta)$. We have with $\varphi_{j,v}$ the probability density function of $\sqrt{c_{j,S,v}}\mathcal{N}$

$$\begin{aligned}
p_{j,S,u} &= \mathbb{P}\left\{\exists_{t\in\Delta_{j,S,u}}:(B(t)-\sqrt{u}ct)>\sqrt{u}\right\}\\
&= \int_{\mathbb{R}}\mathbb{P}\left\{\exists_{t\in[0,S]_\delta}:(B(t)/v-vc(c_{j,S,v}+t/v^2)>v-x|\sqrt{c_{j,S,v}}\mathcal{N}=x\right\}\varphi_{j,v}(x)dx\\
&= \frac{1}{v}\int_{\mathbb{R}}\mathbb{P}\left\{\exists_{t\in[0,S]_\delta}:(B(t)/v-vc(c_{j,S,v}+t/v^2)>v-(v-x/v)\right\}\varphi_{j,v}(v-x/v)dx\\
&= \frac{1}{v}\int_{\mathbb{R}}\mathbb{P}\left\{\exists_{t\in[0,S]_\delta}:Z(t)>x+cc_{j,S,v}v^2\right\}\varphi_{j,v}(v-x/v)dx\\
&= \frac{1}{v}\int_{\mathbb{R}}\mathbb{P}\left\{\exists_{t\in[0,S]_\delta}:Z(t)>x\right\}\varphi_{j,v}(v(1+cc_{j,S,v})-x/v)dx
\end{aligned}$$

$$= \frac{e^{-v^2(1+cc_{j,S,v})^2/(2c_{j,S,v})}}{v\sqrt{2\pi c_{j,S,v}}} \int_{\mathbb{R}} w(x)\omega(j,S,x)dx,$$

where (recall $Z(t) = B(t) - ct, t \ge 0$)

$$w(x) = \mathbb{P}\left\{\exists_{t\in[0,S]_\delta} : Z(t) > x\right\}, \quad \omega(j,S,x) = e^{x(1+cc_{j,S,v})/c_{j,S,v} - x^2/(2c_{j,S,v}v^2)}. \tag{2.18}$$

Using Borell-TIS inequality (see, e.g., [50]) we have (proof is given in the Appendix)

$$\int_{\mathbb{R}} w(x)\omega(j,S,x)dx = \int_{-M}^{M} w(x)e^{2cx}dx + A_{M,v}, \tag{2.19}$$

where $A_{M,v} \to 0$ as $u \to \infty$ and then $M \to \infty$, uniformly for $-N_u \le j \le N_u$ and $S > 0$. By the monotone convergence theorem

$$\begin{aligned}\lim_{M\to\infty} \int_{-M}^{M} w(x)e^{2cx}dx &= \frac{1}{2c}\mathbb{E}\left\{\sup_{t\in[0,S]_\delta} e^{2cB(t)-2c^2t}\right\} \\ &= \frac{1}{2c}\mathbb{E}\left\{\sup_{t\in[0,S]_\delta} e^{\sqrt{2}B(2c^2t)-2c^2t}\right\} = \frac{1}{2c}\mathbb{E}\left\{\sup_{t\in[0,2c^2S]_{2c^2\delta}} e^{\sqrt{2}B(t)-t}\right\}.\end{aligned}$$

In a view of the definition of discrete Pickands constants, see e.g., [12, 30]

$$\lim_{S\to\infty} \frac{1}{2c^2S}\mathbb{E}\left\{\sup_{t\in[0,2c^2S]_{2c^2\delta}} e^{\sqrt{2}B(t)-t}\right\} = \mathcal{H}^{2c^2\delta},$$

with $\mathcal{H}^\eta$ defined in (2.3). Consequently, the asymptotics of $p_1(S,u)$ as $u \to \infty$ and therefore also (2.5) follow by calculating the limit as $u \to \infty, S \to \infty$ of

$$K_{v,S} = e^{2v^2c}cS \sum_{j=-N_u-1}^{N_u} \frac{e^{-v^2(1+cc_{j,S,v})^2/(2c_{j,S,v})}}{v\sqrt{2\pi c_{j,S,v}}}.$$

Setting

$$f(t) = (1+ct)^2/2t = 1/(2t) + c + c^2t/2, \quad f'(t) = (-1/t^2 + c^2)/2, \quad f''(t) = 1/t^3$$

we have that $f'(t_0) = 0$ implying

$$f(t_0 + x) - f(t_0) = \frac{f''(t_0)}{2}x^2 + O(x^3)$$

as $x \to 0$ with $f''(t_0) = c^3$. Consequently, as $u \to \infty$

$$\begin{aligned}K_{v,S} &\sim \frac{c}{\sqrt{2\pi t_0}}\frac{S}{v} \sum_{j=-N_u-1}^{N_u} e^{-\left(v^2 f(t_0+(jS+\theta_u)/v^2) - v^2 f(t_0)\right)} \\ &\sim \frac{c}{\sqrt{2\pi t_0}}\frac{S}{v} \sum_{j=-N_u-1}^{N_u} e^{-f''(t_0)((jS+\theta_u)^2/v^2)/2}\end{aligned}$$

$$\sim \frac{c^{3/2}}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-f''(t_0)x^2/2}dx = 1, \tag{2.20}$$

where the last two steps follow with the same arguments as in the proof of (39) in [17]. Finally, we have that as $u \to \infty$ and then $S \to \infty$

$$P_\delta(u) \sim K_{v,S}\mathcal{H}^{2c^2\delta}e^{-2uc} \sim \mathcal{H}^{2c^2\delta}e^{-2uc}.$$

We show next (2.5). For any $u > 0, s \in \mathbb{R}$ we have

$$\mathbb{P}\left\{\tau_\delta(u) - ut_u \le s\sqrt{u}|\tau_\delta(u) < \infty\right\} = \frac{1}{\psi_{\delta,\infty}(u)}\mathbb{P}\left\{\exists_{t\in[0,ut_u+s\sqrt{u}]_\delta} : Z(t) > u\right\}.$$

Considering the approximations of $p_{j,S,u}$ uniformly for all $-N_u \le j \le N'_u$ with $N'_u = \lfloor s\sqrt{u}/S\rfloor$ we obtain as above

$$\lim_{u\to\infty}\frac{1}{\psi_{\delta,\infty}(u)}\mathbb{P}\left\{\exists_{t\in[0,ut_u+s\sqrt{u}]_\delta} : Z(t) > u\right\} = \frac{\int_{-\infty}^s e^{-f''(t_0)x^2/2}dx}{\int_{\mathbb{R}} e^{-f''(t_0)x^2/2}dx} = \frac{c^{3/2}}{\sqrt{2\pi}}\int_{-\infty}^s e^{-c^3x^2/2}dx = \Phi(sc^{3/2}).$$

Hence

$$\lim_{u\to\infty}\mathbb{P}\left\{(\tau_\delta(u) - ut_u)/\sqrt{u} \le s\middle|\tau_\delta(u) < \infty\right\} = \Phi(sc^{3/2}), \quad s \in \mathbb{R}.$$

Since Φ is continuous, by Dini's theorem, the above convergence holds also substituting s by s_u such that $\lim_{u\to\infty} s_u = s \in \mathbb{R}$. Consequently, since $\theta_u \in [0,\delta)$ we have also

$$\lim_{u\to\infty}\mathbb{P}\left\{c^{3/2}(\tau_\delta(u) - ut_0)/\sqrt{u} \le s\middle|\tau_\delta(u) < \infty\right\} = \Phi(s), \quad s \in \mathbb{R}.$$

Proof of Theorem 2.3.1. Recall that $t_u = t_0 + \theta_u/u = 1/c + \theta_u/u$ and denote $\beta = 1-\gamma$. We analyze the variance function σ_Z^2 of the process $Z(t,s)$. For any non-negative s,t we have

$$\sigma_Z^2(t,s) = \frac{t+\beta^2 s}{(ct+\beta cs+1)^2} = \frac{t+\beta s}{(ct+\beta cs+1)^2} - \frac{\beta(1-\beta)s}{(ct+\beta cs+1)^2} =: A(t,s) - A^*(t,s).$$

Note that $A(t,s)$ depends only on $t+\beta s$ and achieves its global maxima on the line $t+\beta s = t_0 = 1/c$, while $A^*(t,s)$ is negative for all $s>0$ and equals zero for $s=0$. Hence $(t,s) = (1/c,0)$ is the unique global maxima of $\sigma_Z^2(t,s)$ and $\sigma_Z^2(1/c,0) = \frac{1}{4c}$. We define next

$$\mathbb{D}_\delta(u) = \left\{s,t \in G(\delta/u) : (t,s) \in (-\frac{\ln u}{\sqrt{u}} + t_u, \frac{\ln u}{\sqrt{u}} + t_u) \times (0, \frac{\ln u}{\sqrt{u}})\right\}.$$

We have (proof see in the Appendix)

$$\Upsilon_{\gamma,\delta}(u) \sim \mathbb{P}\left\{\exists t,s \in \mathbb{D}_\delta(u) : Z(t,s) > \sqrt{u}\right\} =: \zeta(u), \quad u \to \infty. \tag{2.21}$$

Let $\Delta_{i,S,u}$ be as in (2.11) and set

$$p(i,j) = \mathbb{P}\left\{\exists t,s \in \Delta_{i,S,u} \times \Delta^*_{j,S,u} : (B(t) - \sqrt{u}ct) + \beta(B^*(s) - \sqrt{u}cs) > \sqrt{u}\right\},$$
$$p(i,j;i',j') = \mathbb{P}\{\exists t,s \in \Delta_{i,S,u} \times \Delta^*_{j,S,u} : (B(t) - \sqrt{u}ct) + \beta(B^*(s) - \sqrt{u}cs) > \sqrt{u},$$
$$\exists t,s \in \Delta_{i',S,u} \times \Delta^*_{j',S,u} : (B(t) - \sqrt{u}ct) + \beta(B^*(s) - \sqrt{u}cs) > \sqrt{u}\}$$

for $-N_u \le i \le N_u$, $0 \le j \le N_u$, fixed $S > 0$ and

$$\Delta^*_{j,S,u} = [\frac{jS}{u}, \frac{j(S+1)}{u}].$$

By Bonferroni inequality

$$\sum_{0\le j\le N_u-1, -N_u\le i\le N_u-1} p(i,j) - \sum_{0\le j,j'\le N_u, -N_u-1\le i,i'\le N_u, (i,i')\neq(j,j')} p(i,j;i',j') \le \zeta(u) \le \sum_{0\le j\le N_u, -N_u-1\le i\le N_u} p(i,j).$$

The term

$$\sum_{0\le j,j'\le N_u, -N_u-1\le i,i'\le N_u, (i,i')\neq(j,j')} p(i,j;i',j')$$

is negligible by the proof of Theorem 2.1, Eq. [14] in [35] and consequently

$$\zeta(u) \sim \sum_{0\le j\le N_u, -N_u\le i\le N_u} p(i,j), \quad u \to \infty.$$

Next, we approximate $p(i,j)$ uniformly. Recall, that $v^2 = u$, $c_{i,S,v} = t_u + \frac{iS}{v^2}$, $\varphi_{i,v}$ is the density function of $\sqrt{c_{i,S,v}}\mathcal{N}$ and set $G_j = [jS, (j+1)S]_\delta$. We have

$$\begin{aligned}
& p(i,j) \\
= \; & \mathbb{P}\{\exists (t,s) \in \Delta_{i,S,u} \times \Delta^*_{j,S,u} : B(t) - B(c_{i,S,v}) - c\sqrt{u}(t - c_{i,S,v}) + B(c_{i,S,v}) - \sqrt{u}cc_{i,S,v} \\
& + \beta(B^*(s) - \sqrt{u}cs) > \sqrt{u}\} \\
= \; & \int_{\mathbb{R}} \mathbb{P}\left\{\exists (t,s) \in [0, \frac{S}{u}]_{\delta/u} \times \Delta^*_{j,S,u} : B(t) - \sqrt{u}ct - \sqrt{u}cc_{i,S,v} + \beta(B^*(s) - \sqrt{u}cs) > \sqrt{u} - x\right\} \varphi_{i,v}(x)dx \\
= \; & \int_{\mathbb{R}} \mathbb{P}\left\{\exists (t,s) \in [0,S]_\delta \times G_j : \frac{B(t)}{v} - vc(c_{i,S,v} + \frac{t}{v^2}) + \beta(\frac{B^*(s)}{v} - \frac{cs}{v}) > v - x\right\} \varphi_{i,v}(x)dx \\
= \; & \frac{1}{v} \int_{\mathbb{R}} \mathbb{P}\left\{\exists (t,s) \in [0,S]_\delta \times G_j : \frac{B(t)}{v} - vc(c_{i,S,v} + \frac{t}{v^2}) + \beta(\frac{B^*(s)}{v} - \frac{cs}{v}) > v - (v - \frac{x}{v})\right\} \varphi_{i,v}(v - \frac{x}{v})dx \\
= \; & \frac{1}{v} \int_{\mathbb{R}} \mathbb{P}\left\{\exists (t,s) \in [0,S]_\delta \times G_j : B(t) - ct + \beta(B^*(s) - cs) > x + v^2 cc_{i,S,v}\right\} \varphi_{i,v}(v - \frac{x}{v})dx \\
= \; & \frac{1}{v} \int_{\mathbb{R}} \mathbb{P}\left\{\exists (t,s) \in [0,S]_\delta \times G_j : B(t) - ct + \beta(B^*(s) - cs) > x\right\} \varphi_{i,v}(v(1 + cc_{i,S,v}) - \frac{x}{v})dx \\
= \; & \frac{e^{-\frac{v^2(1+cc_{i,S,v})^2}{2c_{i,S,v}}}}{v\sqrt{2\pi c_{i,S,v}}} \int_{\mathbb{R}} W_j(x)\omega(i,S,x)dx,
\end{aligned}$$

where

$$W_j(x) = \mathbb{P}\{\exists (t,s) \in [0,S]_\delta \times G_j : B(t) - ct + \beta(B^*(s) - cs) > x\}$$

and $\omega(i,S,x)$ is defined in (2.18). By Borell-TIS inequality for all $|i|,|j| \le N_u$ (proof is in the Appendix)

$$\int_{\mathbb{R}} W_j(x)\omega(i,S,x)dx \sim \int_{\mathbb{R}} W_j(x)e^{2cx}dx, \quad u \to \infty. \tag{2.22}$$

Next we have with $G_j^* = [2jc^2S, 2(j+1)c^2S]_{2c^2\delta}$

$$\begin{aligned}
\int_{\mathbb{R}} W_j(x)e^{2cx}dx &= \frac{1}{2c}\int_{\mathbb{R}} \mathbb{P}\left\{\sup_{(t,s)\in[0,S]_\delta\times G_j} \left(2cB(t) - 2c^2t + \beta(2cB^*(s) - 2c^2s)\right) > 2cx\right\} e^{2cx}d(2cx)\\
&= \frac{1}{2c}\int_{\mathbb{R}} \mathbb{P}\left\{\sup_{(t,s)\in[0,S]_\delta\times G_j} \left(\sqrt{2}B(2c^2t) - 2c^2t + \beta(\sqrt{2}B^*(2c^2s) - 2c^2s)\right) > x\right\} e^{x}dx\\
&= \frac{1}{2c}\mathbb{E}\left\{\sup_{(t,s)\in[0,2c^2S]_{2c^2\delta}\times G_j^*} \exp\left(\sqrt{2}B(t) - t + \beta(\sqrt{2}B^*(s) - s)\right)\right\}\\
&= \frac{1}{2c}\mathbb{E}\left\{\sup_{t\in[0,2c^2S]_{2c^2\delta}} e^{\sqrt{2}B(t)-t}\right\}\mathbb{E}\left\{\sup_{s\in G_j^*} e^{\beta(\sqrt{2}B^*(s)-s)}\right\}.
\end{aligned} \tag{2.23}$$

By (2.22) combined with the line above we write

$$\begin{aligned}
\zeta(u) &\sim \frac{1}{2c}\mathbb{E}\left\{\sup_{t\in[0,2c^2S]_{2c^2\delta}} e^{\sqrt{2}B(t)-t}\right\}\\
&\times \sum_{0\le j\le N_u} \mathbb{E}\left\{\sup_{s\in G_j^*} e^{\beta(\sqrt{2}B^*(s)-s)}\right\} \sum_{-N_u\le i\le N_u} \frac{e^{-\frac{v^2(1+cc_{i,S,v})^2}{2c_{i,S,v}}}}{v\sqrt{2\pi c_{i,S,v}}}, \quad u \to \infty.
\end{aligned} \tag{2.24}$$

As was shown in the proof of Theorem 2.1.1 as $u \to \infty$ and then $S \to \infty$

$$\frac{1}{2c}\mathbb{E}\left\{\sup_{t\in[0,2c^2S]_{2c^2\delta}} e^{\sqrt{2}B(t)-t}\right\} \sum_{-N_u\le i\le N_u} \frac{e^{-\frac{v^2(1+cc_{i,S,v})^2}{2c_{i,S,v}}}}{v\sqrt{2\pi c_{i,S,v}}} \sim \mathcal{H}^{2c^2\delta}e^{-2cu}. \tag{2.25}$$

We have as $S \to \infty$ (proof of the first line below is in the Appendix)

$$\begin{aligned}
\sum_{0\le j\le N_u} \mathbb{E}\left\{\sup_{s\in G_j^*} e^{\beta(\sqrt{2}B^*(s)-s)}\right\} &\sim \mathbb{E}\left\{\sup_{s\in[0,2c^2S]_{2c^2\delta}} e^{\beta(\sqrt{2}B(s)-s)}\right\} \\
&= \int_{\mathbb{R}} \mathbb{P}\left\{\sup_{s\in[0,2c^2S]_{2c^2\delta}} \left(\sqrt{2}B(s\beta^2) - \frac{s\beta^2}{\beta} > x\right)\right\} e^{x}dx\\
&= \int_{\mathbb{R}} \mathbb{P}\left\{\sup_{s\in[0,2c^2\beta^2S]_{2c^2\beta^2\delta}} \left(\sqrt{2}B(s) - s(1 + \frac{1-\beta}{\beta})\right) > x\right\} e^{x}dx
\end{aligned} \tag{2.26}$$

$$\to \mathcal{P}_{2c^2\beta^2\delta}^{\frac{1-\beta}{\beta}} = \mathcal{P}_{2c^2(1-\gamma)^2\delta}^{\frac{\gamma}{1-\gamma}} \in (0,\infty).$$

Combining the statement above with (2.24) and (2.25) we conclude

$$\zeta(u) \sim \mathcal{P}_{2c^2(1-\gamma)^2\delta}^{\frac{\gamma}{1-\gamma}} \mathcal{H}^{2c^2\delta} e^{-2cu}, \quad u \to \infty$$

and hence by (2.21) the claim follows. □

Proof of Theorem 4.2.1. The proof is similar to that of Theorem 2.1.1 and we use similar notation as therein. We have by (2.9)

$$\mathcal{P}_\delta(u,T) \sim \mathbb{P}\left\{ \sup_{t\in[T_u^-,T_u^+]_\delta} \inf_{s\in[t,t+T]_\delta} Z(s) > u \right\} =: \widetilde{P}_\delta(u), \quad u \to \infty \tag{2.27}$$

if we show that $\widetilde{P}_\delta(u) \geq Ce^{-2cu}$. By the self-similarity of BM

$$\begin{aligned} \widetilde{P}_\delta(u) &= \mathbb{P}\left\{\exists t \in [T_u^-, T_u^+]_\delta : \forall s \in [t, t+T]_\delta \; Z(s) > u)\right\} \\ &= \mathbb{P}\left\{\exists t \in [\frac{T_u^-}{u}, \frac{T_u^+}{u}]_\delta : \forall s \in [t, t+\frac{T}{u}]_{\frac{\delta}{u}} \; \frac{B(s)}{1+cs} > \sqrt{u}\right\}. \end{aligned}$$

We choose the same partition $\Delta_{j,S,u}$, $-N_u \leq j \leq N_u$ of the interval $\Delta_u = [\frac{T_u^-}{u}, \frac{T_u^+}{u}]$ as in the proof of Theorem 2.1.1. The Bonferroni inequality yields

$$\widetilde{p}_1(S,u) \geq \widetilde{P}_\delta(u) \geq \widetilde{p}_1'(S,u) - \widetilde{p}_2(S,u), \tag{2.28}$$

where

$$\widetilde{p}_1(S,u) = \sum_{j=-N_u-1}^{N_u} \widetilde{p}_{j,S,u}, \quad \widetilde{p}_1'(S,u) = \sum_{j=-N_u}^{N_u-1} \widetilde{p}_{j,S,u}, \quad \widetilde{p}_2(S,u) = \sum_{-N_u-1\leq j<i\leq N_u} \widetilde{p}_{i,j;S,u},$$

with

$$\widetilde{p}_{j,S,u} = \mathbb{P}\left\{ \sup_{t\in\Delta_{j,S,u}} \inf_{s\in[t,t+\frac{T}{u}]_{\frac{\delta}{u}}} \frac{B(s)}{1+cs} > \sqrt{u} \right\}$$

and

$$\widetilde{p}_{i,j;S,u} = \mathbb{P}\left\{ \sup_{t\in\Delta_{i,S,u}} \inf_{s\in[t,t+\frac{T}{u}]_{\frac{\delta}{u}}} \frac{B(s)}{1+cs} > \sqrt{u}, \sup_{t\in\Delta_{j,S,u}} \inf_{s\in[t,t+\frac{T}{u}]_{\frac{\delta}{u}}} \frac{B(s)}{1+cs} > \sqrt{u} \right\}.$$

Clearly, $\widetilde{p}_{i,j;S,u} \leq p_{i,j;S,u}$ and hence

$$\widetilde{p}_2(S,u) \;\leq\; p_2(S,u).$$

Thus, if we show that $\widetilde{p}_1(S,u) \sim C_1 e^{-2cu}$ we conclude that $\widetilde{p}_2(S,u)$ is negligible. We approximate each summand in $\widetilde{p}_1(S,u)$ uniformly. As in the proof of Theorem 2.1.1 we obtain

$$\widetilde{p}_{j,S,u} \;=\; \frac{e^{-v^2(1+cc_{j,S,v})^2/(2c_{j,S,v})}}{v\sqrt{2\pi c_{j,S,v}}} \int_{\mathbb{R}} w(T,x)\omega(j,S,x)dx,$$

where

$$w(T,x)=\mathbb{P}\left\{\sup_{t\in[0,S]_\delta}\inf_{s\in[t,t+T]_\delta} Z(s)>x\right\}$$

and $\omega(j,S,x)$ is defined in (2.18). By Borell-TIS inequality (similarly the proof of (2.19)) it follows that

$$\int_{\mathbb{R}} w(T,x)\omega(j,S,x)dx \to \int_{\mathbb{R}} w(T,x)e^{2cx}dx, \qquad u\to\infty.$$

Next we have

$$\begin{aligned}\int_{\mathbb{R}} w(T,x)e^{2cx}dx &= \frac{1}{2c}\mathbb{E}\left\{\sup_{t\in[0,S]_\delta}\inf_{s\in[t,t+T]_\delta} e^{2cB(s)-2c^2s}\right\}\\ &= \frac{1}{2c}\mathbb{E}\left\{\sup_{t\in[0,S]_\delta}\inf_{s\in[t,t+T]_\delta} e^{\sqrt{2}B(2c^2s)-2c^2s}\right\}\\ &= \frac{1}{2c}\mathbb{E}\left\{\sup_{t\in[0,2c^2S]_{2c^2\delta}}\inf_{s\in[t,t+2c^2T]_{2c^2\delta}} e^{\sqrt{2}B(s)-s}\right\}.\end{aligned}$$

It follows with similar arguments as in [12] that as $S\to\infty$

$$\lim_{S\to\infty}\frac{1}{2c^2S}\mathbb{E}\left\{\sup_{t\in[0,2c^2S]_{2c^2\delta}}\inf_{s\in[t,t+2c^2T]_{2c^2\delta}} e^{\sqrt{2}B(t)-t}\right\}=\mathcal{H}_{2c^2\delta,2Tc^2}\in(0,\infty), \tag{2.29}$$

where the constant $\mathcal{H}_{2c^2\delta,2Tc^2}$ is given by (2.15). Hence by (2.20) we have

$$\widetilde{P}_\delta(u)\sim\mathcal{H}_{2c^2\delta,2Tc^2}e^{-2cu}, \quad u\to\infty$$

and (2.27) holds, establishing the claim. □

Proof of Theorem 2.4.2. We use below the same notation as in the previous proofs. By (2.9) we have

$$\mathcal{C}_\delta(u,k)\sim\mathbb{P}\left\{\#\{t(T_u^-,T_u^+)_\delta : Z(t)>u\}>k\right\}=:\hat{\psi}_k^\delta(u), \quad u\to\infty \tag{2.30}$$

if we show that $\hat{\psi}_k^\delta(u)\geq Ce^{-2cu}$. Using the self-similarity of BM for any $u>0$

$$\hat{\psi}_k^\delta(u)=\mathbb{P}\left\{\#\{t\in(-\frac{\ln u}{\sqrt{u}}+t_0,t_0+\frac{\ln u}{\sqrt{u}})\cap G(\frac{\delta}{u}) : \frac{B(t)}{ct+1}>\sqrt{u}\}>k\right\}.$$

Letting

$$A_{j,u}:=\#\{t\in\Delta_{j,S,u} : \frac{B(t)}{ct+1}>\sqrt{u}\}$$

we have using the idea from [17]

$$\hat{\psi}_\delta^k(u) \quad\leq\quad \mathbb{P}\left\{\sum_{j=-N_u-1}^{N_u} A_{j,u}>k\right\}$$

$$
\begin{aligned}
&= \mathbb{P}\left\{ \sum_{j=-N_u-1}^{N_u} A_{j,u} > k, \{\text{there exists only one } j \text{ such that } A_{j,u} > 0\} \right\} \\
&\quad + \mathbb{P}\left\{ \sum_{j=-N_u-1}^{N_u} A_{j,u} > k, \{\text{there exists } i \neq j \text{ such that } A_{i,u} > 0 \text{ and } A_{j,u} > 0 \ \} \right\} \\
&=: p_{1,k}(u) + \Pi_0(u).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\hat{\psi}_k^\delta(u) &\geq \mathbb{P}\left\{ \sum_{j=-N_u}^{N_u-1} A_{j,u} > k \right\} \\
&= \mathbb{P}\left\{ \sum_{j=-N_u}^{N_u-1} A_{j,u} > k, \{\text{there exists only one } j \text{ such that } A_{j,u} > 0 \ \} \right\} \\
&\quad + \mathbb{P}\left\{ \sum_{j=-N_u}^{N_u-1} A_{j,u} > k, \{\text{there exists } i \neq j \text{ such that } A_{i,u} > 0 \text{ and } A_{j,u} > 0 \ \} \right\} \\
&=: p_{2,k}(u) + \Pi_0'(u).
\end{aligned}
$$

Notice, that $\Pi_0(u)$ and $\Pi_0'(u)$ are less than the double-sum term in Theorem 2.1.1. They are negligible if we prove that $p_{2,k}(u) \sim p_{1,k}(u) \geq Ce^{-2cu}$ as $u \to \infty$ for some $C > 0$. We have

$$
\begin{aligned}
p_{1,k}(u) &= \sum_{j=-N_u-1}^{N_u} \Big(\mathbb{P}\{A_{j,u} > k\} - \mathbb{P}\{A_{j,u} > k, \exists i \neq j : A_{i,u} > 0\} \Big) \\
&= \sum_{j=-N_u-1}^{N_u} \mathbb{P}\{A_{j,u} > k\} - \sum_{j=-N_u-1}^{N_u} \mathbb{P}\{A_{j,u} > k, \exists i \neq j : A_{i,u} > 0\}.
\end{aligned}
$$

The last summand is less than the double-sum term in Theorem 2.1.1 and is negligible. Thus, we need to compute the asymptotics of

$$
Q_{\delta,k}(u) := \sum_{j=-N_u}^{N_u} \mathbb{P}\{A_{j,u} > k\}. \tag{2.31}
$$

With similar arguments as in the proof of Theorem 2.1.1

$$
\mathbb{P}\{A_{j,u} > k\} = \mathbb{P}\left\{ \#\{t \in \Delta_{j,S,u} : \frac{B(t)}{ct+1} > \sqrt{u}\} > k \right\} = \frac{e^{-v^2(1+cc_{j,S,v})^2/(2c_{j,S,v})}}{v\sqrt{2\pi c_{j,S,v}}} \int_{\mathbb{R}} w_k(x)\omega(j,S,x)dx,
$$

where $\omega(j,S,x)$ is defined in (2.18) and

$$
w_k(x) = \mathbb{P}\{\#\{t \in [0,S]_\delta : Z(t) > x\} > k\}.
$$

Similarly to the proof of (2.19) we have

$$
\int_{\mathbb{R}} w_k(x)\omega(j,S,x)dx \to \int_{\mathbb{R}} w_k(x)e^{2cx}dx, \qquad u \to \infty.
$$

Next

$$\int_{\mathbb{R}} w_k(x)e^{2cx}dx = \frac{1}{2c}\int_{\mathbb{R}} \mathbb{P}\left\{\#\{t\in[0,2c^2S]_\delta : \sqrt{2}B(t)-t>x\}>k\right\}e^x dx.$$

As shown in [22]

$$\lim_{S\to\infty}\frac{1}{2c^2S}\int_{\mathbb{R}} \mathbb{P}\left\{\#\{t\in[0,2c^2S]_{2c^2\delta} : \sqrt{2}B(t)-t>x\}>k\right\}e^x dx = \mathcal{B}_{2c^2\delta}(k)\in(0,\infty).$$

Consequently, by (2.20) as $u\to\infty$ and then $S\to\infty$

$$Q_{\delta,k}(u)\sim cS\mathcal{B}_{2c^2\delta}(k)\sum_{j=-N_u}^{N_u}\frac{e^{-v^2(1+cc_{j,S,v})^2/(2c_{j,S,v})}}{v\sqrt{2\pi c_{j,S,v}}} \quad = \quad \mathcal{B}_{2c^2\delta}(k)e^{-2cu}K_{v,S}\sim \mathcal{B}_{2c^2\delta}(k)e^{-2cu}.$$

Since $\mathcal{B}_{2c^2\delta}(k)\in(0,\infty)$, then $Q_{\delta,k}(u)\sim\hat{\psi}_k^\delta(u)$ as $u\to\infty$ implying

$$\hat{\psi}_k^\delta(u)\sim e^{-2cu}\mathcal{B}_{2c^2\delta}(k),\quad u\to\infty$$

and the claim follows. □

2.6 Appendix

Proof of (2.21)**.** Recall that

$$Z(t,s)=\frac{B(t)+(1-\gamma)B^*(s)}{ct+(1-\gamma)cs+1},\quad s,t\ge 0,$$

where B and B^* are independent BM. For some positive ε and large u denote

$$\begin{aligned} A(\varepsilon) &= \Big([0,\infty)\times[0,\infty)\Big)\setminus\Big([-\varepsilon+t_u,\varepsilon+t_u]\times[0,\varepsilon]\Big),\\ R(\varepsilon,u) &= \Big([-\varepsilon+t_u,\varepsilon+t_u]\times[0,\varepsilon]\Big)\setminus\Big([-\tfrac{\ln u}{\sqrt{u}}+t_u,\tfrac{\ln u}{\sqrt{u}}+t_u]\times[0,\tfrac{\ln u}{\sqrt{u}}]\Big). \end{aligned} \tag{2.32}$$

We have

$$\begin{aligned} &\mathbb{P}\left\{\exists(t,s)\in\mathbb{D}_\delta(u): Z(t,s)>\sqrt{u}\right\}\\ \le\ &\mathbb{P}\left\{\exists(t,s)\in G(\delta/u): Z(t,s)>\sqrt{u}\right\}\\ \le\ &\mathbb{P}\left\{\exists(t,s)\in\mathbb{D}_\delta(u): Z(t,s)>\sqrt{u}\right\}+\mathbb{P}\left\{\exists(t,s)\in A(\varepsilon): Z(t,s)>\sqrt{u}\right\}\\ &+\mathbb{P}\left\{\exists(t,s): R(\varepsilon,u): Z(t,s)>\sqrt{u}\right\}. \end{aligned} \tag{2.33}$$

We show next that $Z(t,s)$ is a.s. bounded for $t,s\ge0$. According to Chapter 4, p. 31 in [59] it is equivalent that $Z(t,s)$ is bounded with positive probability. We have

$$\mathbb{P}\left\{\sup_{t,s\ge0}Z(t,s)\le 1\right\} \quad = \quad \mathbb{P}\left\{\text{for all } t,s\ge0\ B(t)-ct+(1-\gamma)(B^*(s)-cs)\le 1\right\}$$

$$\begin{aligned}
&\geq \mathbb{P}\{\text{for all } t,s\geq 0\ B(t)-ct\leq 1/2, (1-\gamma)(B^*(s)-cs)\leq 1/2\}\\
&= \left(1-\mathbb{P}\{\exists t\geq 0: B(t)-ct>1/2\}\right)\left(1-\mathbb{P}\left\{\exists s\geq 0: B^*(s)-cs>1/2(1-\gamma)^{-1}\right\}\right)\\
&= (1-e^{-c})(1-e^{-c(1-\gamma)^{-1}})>0,
\end{aligned}$$

where we used (2.1) for the last equation. Hence by Borell-TIS inequality (see [50])

$$\mathbb{P}\left\{\exists (t,s)\in A(\varepsilon): Z(t,s)>\sqrt{u}\right\}=o(\mathbb{P}\left\{Z(1/c,0)>\sqrt{u}\right\}),\qquad u\to\infty. \tag{2.34}$$

Next we shall prove that

$$\mathbb{P}\left\{\exists (t,s)\in R(\varepsilon,u): Z(t,s)>\sqrt{u}\right\}=o(\mathbb{P}\left\{Z(1/c,0)>\sqrt{u}\right\}),\qquad u\to\infty. \tag{2.35}$$

If we show that for any $(t,s)\in R(\varepsilon,u)$ and for some positive constant C holds, that

$$\sigma_Z^2(1/c,0)-\sigma_Z^2(t,s)\geq C\frac{\ln^2 u}{u}$$

we can immediately claim (2.35) by Piterbarg's inequality (Proposition 9.2.5 in [59]). Notice that if

i) $s\notin[0,\frac{\ln u}{\sqrt{u}}]$, then

$$\begin{aligned}
\sigma_Z^2(1/c,0)-\sigma_Z^2(t,s) &= (A(1/c,0)-A(t,s))+A^*(t,s)\\
&\geq A^*(t,s)=\frac{s\beta(1-\beta)}{(ct+\beta cs+1)^2}\geq C\frac{\ln u}{\sqrt{u}}\geq C\frac{\ln^2 u}{u},
\end{aligned}$$

hence the claim follows.

ii) assume that $s\in[0,\frac{\ln u}{\sqrt{u}}]$. Setting

$$L(x)=\frac{x}{(cx+1)^2},$$

we have that $L(x)$ attains its unique maxima at point $x=1/c$, $L'(1/c)=0$ and $L''(1/c)<0$. We have

$$\sigma_Z^2(1/c,0)-\sigma_Z^2(t,s)=\sigma_Z^2(1/c,0)-A(t,s)+A^*(t,s)\geq\sigma_Z^2(1/c,0)-A(t,s)=L(1/c)-L(t+\beta s).$$

For all (t,s) such that $(t,s)\in R(\varepsilon,u),\ s\in[0,\frac{\ln u}{\sqrt{u}}]$ we have that $|1/c-(t+\beta s)|\geq C\frac{\ln u}{\sqrt{u}}$. Hence

$$L(1/c)-L(t+\beta s)\geq C|L''(1/c)|(1/c-(t+\beta s))^2\geq C\frac{\ln^2 u}{u}$$

and (2.35) holds.

Notice that for some positive constant C

$$\mathbb{P}\left\{\exists t,s\in\mathbb{D}_\delta(u): Z(t,s)>\sqrt{u}\right\}\geq\mathbb{P}\left\{Z(t_u,0)>\sqrt{u}\right\}\geq C\mathbb{P}\left\{Z(1/c,0)>\sqrt{u}\right\}.$$

Combining the statement above with (2.33),(2.34) and (2.35) we establish (2.21). □

Proof of (2.9)**.** Notice that

$$\mathbb{P}\left\{\sup_{t\notin[T_u^-,T_u^+]} Z(t)>u\right\}\le \mathbb{P}\left\{\exists t,s\in A(\varepsilon):Z(t,s)>\sqrt{u}\right\}+\mathbb{P}\left\{\exists t,s\in R(\varepsilon,u):Z(t,s)>\sqrt{u}\right\},$$

where $A(\varepsilon)$ and $R(\varepsilon,u)$ are defined in (2.32). Hence the claim follows by (2.34) and (2.35). □

Proof of (2.22)**.** We shall prove that

$$\int_{\mathbb{R}} W_j(x)e^{-\frac{x^2}{2v^2c_i}+x\frac{1+cc_i}{c_i}}dx=\int_{-M}^{M} W_j(x)e^{2cx}dx+\overline{A}_{M,v}, \tag{2.36}$$

where $\overline{A}_{M,v}\to 0$ as $u\to\infty$ and then $M\to\infty$ uniformly for all $|i|,|j|\le N_u$. We have

$$\begin{aligned}
&\int_{\mathbb{R}} W_j(x)e^{-\frac{x^2}{2v^2c_i}+x\frac{1+cc_i}{c_i}}dx-\int_{-M}^{M} W_j(x)e^{2cx}dx\\
\le\;&\int_{|x|>M} W_j(x)e^{-\frac{x^2}{2uc_i}+x\frac{1+cc_i}{c_i}}dx+\left|\int_{-M}^{M} W_j(x)e^{2cx}(e^{-\frac{x^2}{2uc_i}-\frac{x}{u}\frac{(\theta_u+iS)c}{c_i}}-1)dx\right|\\
=:\;&\overline{I}_1+|\overline{I}_2|.
\end{aligned} \tag{2.37}$$

Let $u\ge M^6$. For any integer $|i|,|j|\le N_u$, $x\in[-M,M]$ and u large we have

$$|\frac{(\theta_u+iS)cx}{c_i})|\le CM\sqrt{u}\ln u,\qquad |\frac{x^2}{2c_i}|\le CM^2,$$

hence

$$|\frac{1}{u}(-\frac{x^2}{2c_i}-\frac{(\theta_u+iS)cx}{c_i})|\le\frac{C}{u}(M^2+M\sqrt{u}\ln u)\le\frac{C}{M^4}+\frac{CM}{u^{2/5}}\le\frac{1}{M}.$$

We have by (2.23) and (2.40) that for all $|j|\le N_u$

$$\int_{\mathbb{R}} W_j(x)e^{2cx}dx\le C$$

and hence

$$|\overline{I}_2|\le\frac{1}{M}\int_{-M}^{M} W_j(x)e^{2cx}dx\le\frac{1}{M}\int_{\mathbb{R}} W_j(x)e^{2cx}dx=\frac{C}{M}\to 0,\quad M\to\infty. \tag{2.38}$$

Next we have for large u

$$\overline{I}_1\;\le\;\int_{|x|>M}\mathbb{P}\left\{\exists t\in[0,S],s\ge 0:B(t)-ct+\beta(B^*(s)-cs)>x\right\}e^{x\frac{1+cc_i}{c_i}}dx$$

$$\leq \int\limits_{x>M} \mathbb{P}\left\{\exists t \in [0, \frac{S}{x}], s \geq 0 : Z(t,s) > \sqrt{x}\right\} e^{x\frac{1+cc_i}{c_i}} dx + \int\limits_{x<-M} e^{cx} dx.$$

We analyze behavior of $\sigma_Z^2(t,s)$ on the set $\{(t,s) \in [0, \frac{S}{x}] \times [0,\infty)\}$. Since

$$\sigma_Z^2(t,s) = \frac{t + \beta^2 s}{(ct + c\beta s + 1)^2} \leq \frac{S}{x} + \frac{\beta^2 s}{(c\beta s + 1)^2} \leq \frac{S}{x} + \frac{\beta}{4c}$$

taking large enough x we can write for any fixed $\varepsilon > 0$ that

$$\sigma_Z^2(t,s) \leq \frac{\beta(1+\varepsilon)}{4c}.$$

Hence by Borell-TIS inequality for large x

$$\mathbb{P}\left\{\exists t \in [0, \frac{S}{x}], s \geq 0 : Z(t,s) > \sqrt{x}\right\} \leq e^{-x\frac{2c}{\beta(1+2\varepsilon)}}.$$

Choosing ε such that $\beta(1+2\varepsilon) < 1$, uniformly for all $|i|, |j| \leq N_u$ we have with $a = 2c - \frac{2c}{\beta(1+2\varepsilon)} < 0$

$$\overline{I}_1 \leq o(1) + \int\limits_{|x|>M} e^{x\frac{1+cc_i}{c_i} - x\frac{2c}{\beta(1+2\varepsilon)}} dx = o(1) + \int\limits_{|x|>M} e^{x(2c + o(1) - \frac{2c}{\beta(1+2\varepsilon)})} dx$$
$$\leq o(1) + 2\int\limits_{|x|>M} e^{ax} \to 0, \qquad M \to \infty. \tag{2.39}$$

Combination of (2.38) and (2.39) establishes (2.36). By the monotone convergence theorem (2.36) implies (2.22). □

Proof of (2.19)**.** We have

$$\int_{\mathbb{R}} w(x)\omega(j,S,x)dx - \int\limits_{-M}^{M} w(x)e^{2cx}dx \leq \int\limits_{|x|>M} w(x)\omega(j,S,x)dx + |\int\limits_{|x|<M} w(x)(\omega(j,S,x) - e^{2cx})|dx$$
$$=: I_1 + |I_2|.$$

Since $W_j(x) \geq w(x)$ we have that ($\overline{I}_1$ and $\overline{I}_2$ are defined in (2.37))

$$|I_2| \leq |\overline{I}_2|, \qquad I_1 \leq \overline{I}_1$$

implying

$$I_1 + I_2 \leq \overline{I}_1 + |\overline{I}_2| \to 0$$

as $u \to \infty$ and then $M \to \infty$ by (2.38) and (2.39). Thus, (2.19) is established. □

Proof of (2.26)**.** For any $j \geq 1$ we have (set $b_j = 2jc^2 S$)

$$\mathbb{E}\left\{\sup_{s \in G_j^*} e^{\beta(\sqrt{2}B^*(s) - s)}\right\}$$

$$
\begin{aligned}
&\le \int_{\mathbb{R}} \mathbb{P}\left\{\sup_{s\in[2jc^2S,2(j+1)c^2S]} \beta(\sqrt{2}B(s)-s-\sqrt{2}B(b_j)+b_j) > x+\beta(b_j-\sqrt{2}B(b_j))\right\} e^x dx \\
&= \frac{1}{\sqrt{2\pi b_j}} \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{P}\left\{\sup_{s\in[0,2c^2S]} \beta(\sqrt{2}B(s)-s) > x+\beta(b_j-\sqrt{2}y)\right\} e^x dx e^{-\frac{y^2}{2b_j}} dy \\
&= \frac{1}{\sqrt{2\pi b_j}} \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{P}\left\{\sup_{s\in[0,2c^2S]} \beta(\sqrt{2}B(s)-s) > x\right\} e^{x-\beta(b_j-\sqrt{2}y)} e^{-\frac{y^2}{2b_j}} dx dy \\
&= \frac{e^{-\beta b_j}}{\sqrt{2\pi b_j}} \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{P}\left\{\sup_{s\in[0,2c^2S]} \beta(\sqrt{2}B(s)-s) > x\right\} e^x e^{-\frac{y^2}{2b_j}+\sqrt{2}\beta y} dx dy \\
&= \frac{e^{-\beta b_j}}{\sqrt{2\pi b_j}} \int_{\mathbb{R}} \mathbb{P}\left\{\sup_{s\in[0,2c^2S]} \beta(\sqrt{2}B(s)-s) > x\right\} e^x dx \int_{\mathbb{R}} e^{-\frac{y^2}{2b_j}+\sqrt{2}\beta y} dy.
\end{aligned}
$$

Next we have by (2.1)

$$
\begin{aligned}
\int_{\mathbb{R}} \mathbb{P}\left\{\sup_{s\in[0,2c^2S]} \beta(\sqrt{2}B(s)-s) > x\right\} e^x dx &\le \int_{\mathbb{R}} \mathbb{P}\left\{\sup_{s\in[0,\infty)} (B(s)-\frac{s}{\beta\sqrt{2}}) > \frac{x}{\beta\sqrt{2}}\right\} e^x dx \\
&= 1+\int_0^\infty e^{-x/\beta+x} \le C.
\end{aligned}
$$

Since

$$
\int_{\mathbb{R}} e^{-\frac{y^2}{2b_j}+\sqrt{2}\beta y} dy = \sqrt{2\pi b_j} e^{\beta^2 b_j},
$$

we have for some fixed small enough ε and large S

$$
\mathbb{E}\left\{\sup_{s\in G_j^*} e^{\beta(\sqrt{2}B(s)-s)}\right\} \le Ce^{-\beta(1-\beta)b_j} \le e^{-jS\varepsilon}. \tag{2.40}
$$

Thus, as $S\to\infty$

$$
\sum_{1\le j\le N_u} \mathbb{E}\left\{\sup_{s\in G_j^*} e^{\beta(\sqrt{2}B(s)-s)}\right\} \le e^{-\varepsilon S}(1+o(1))
$$

establishing the claim. □

Chapter 3

Simultaneous Fractional Brownian Motion Discrete-Time Ruin

This chapter is based on G. Jasnovidov: Simultaneous Ruin Probability for Two-Dimensional Fractional Brownian motion risk Process over Discrete Grid, in press, Lithuanian Mathematical Journal, 2021.

3.1 Introduction

Define two risk processes

$$R_{i,u}^{(H)}(t) = q_i u + c_i t - B_H(t), \quad i = 1, 2,$$

where $c_i, q_i > 0$. The discrete simultaneous ruin time and ruin probability over the infinite time horizon are defined by

$$\bar{\tau}_{\delta,H}(u) = \inf\{t \in G(\delta) : R_{1,u}^{(H)}(t) < 0, R_{2,u}^{(H)}(t) < 0\}$$

and

$$\bar{\psi}_{\delta,H}(u) = \mathbb{P}\left\{\bar{\tau}_{\delta,H}(u) < \infty\right\}, \tag{3.1}$$

respectively. For positive δ the simultaneous ruin probability is of interest both for theory-oriented studies and for applications in reinsurance (see, e.g., [45] and references therein). In this paper we investigate only the discrete setup; the continuous problem has been already solved in [45]. For any possible choices of positive δ and $H \in (0, 1)$ it is not possible to calculate $\bar{\psi}_{\delta,H}(u)$ explicitly. A natural question when lack of explicit formulas is the case, is how can we approximate $\bar{\psi}_{\delta,H}(u)$ for large u. Also of interest is to know what is the role of δ, does it affects the ruin probability in the considered risk model. Theorem 3.2.1 gives detailed answers for these questions. Our results show that the discrete time ruin probabilities behave differently from continuous if $H \leq 1/2$. We refer to [41] for some alternative proofs of the results.

Also of certain interest is the finite time horizon setup of the problem. For fixed $T > 0$ the discrete simultaneous ruin probability over a finite time horizon is

$$\bar{\zeta}_{H,T}(u) = \mathbb{P}\left\{\exists t \in [0,T] : R_{1,u}^{(H)}(t) < 0, R_{2,u}^{(H)}(t) < 0\right\}. \tag{3.2}$$

The corresponding discrete ruin problem over a finite time horizon is trivial, since set $[0,T]\cap G(\delta)$ consists of finite number of elements and hence asymptotics of the large deviation is determined by the unique maximizer of the variance of the process (this, e.g., follows immediately from Lemma 2.3 in [56] or Proposition 2.4.2 in [59]). Thus, we shall be concerned only with the continuous ruin problem over a finite horizon. Asymptotics of $\bar{\zeta}_{H,T}(u)$ is discussed in Remark 3.2.5.

3.2 Main Results

First we eliminate the trends via self-similarity of fBM. For any $u > 0$ we have

$$\begin{aligned}\bar{\psi}_{\delta,H}(u) &= \mathbb{P}\left\{\exists t \in G(\delta) : B_H(t) > q_1 u + c_1 t, B_H(t) > q_2 u + c_2 t\right\} \\ &= \mathbb{P}\left\{\exists t \in G(\delta/u) : \frac{B_H(t)}{\max(c_1 t + q_1, c_2 t + q_2)} > u^{1-H}\right\}.\end{aligned}$$

If the two lines $q_1 + c_1 t$ and $q_2 + c_2 t$ do not intersect over $(0, \infty)$, then the problem degenerates to the one-dimensional case, which is discussed in Theorem 3.2.3. In consideration of that dealing with $\bar{\psi}_{\delta,H}(u)$ we always suppose that

$$c_1 > c_2, \quad q_2 > q_1. \tag{3.3}$$

It turns out that the variance of $\frac{B_H(t)}{\max(c_1 t+q_1, c_2 t+q_2)}$ can achieve its unique maxima only at one of the following points:

$$t_1 = \frac{Hq_1}{c_1(1-H)}, \qquad t_2 = \frac{Hq_2}{c_2(1-H)}, \qquad t^* = \frac{q_2 - q_1}{c_1 - c_2}. \tag{3.4}$$

It follows from (4.3) that $t_1 < t_2$. As we show later, the order between t_1, t_2 and t^* determines the asymptotics of $\bar{\psi}_{\delta,H}(u)$ as $u \to \infty$.

For notational simplicity we write $\bar{\psi}_\delta(u)$ instead of $\bar{\psi}_{\delta,1/2}(u)$. Define for some function $k(t)$ constant

$$\widetilde{\mathcal{P}}_\eta^k = \lim_{T\to\infty} \mathbb{E}\left\{\sup_{t\in[-T,T]_\eta} e^{\sqrt{2}B(t)-|t|+k(t)}\right\}$$

when the expectation above is finite and set for $\delta \geq 0$

$$d_\delta(t) = \mathbb{I}(t<0)\frac{(q_2 c_1 + c_2 q_1 - 2q_2 c_2)t}{c_1 q_2 - q_1 c_2} + \mathbb{I}(t\geq 0)\frac{(2c_1 q_1 - c_1 q_2 - q_1 c_2)t}{c_1 q_2 - q_1 c_2} - \delta\mathbb{I}(t\geq 0)\frac{(c_1 q_2 - q_1 c_2)(c_1 - c_2)}{q_2 - q_1}.$$

Define constants

$$C_H^{(i)} = \frac{c_i^H q_i^{1-H}}{H^H(1-H)^{1-H}}, \quad i = 1, 2.$$

The theorem below establishes the asymptotics of $\bar{\psi}_{\delta,H}(u)$.

Theorem 3.2.1 *For $\delta > 0$ as $u \to \infty$*
1) if $t^* \notin (t_1, t_2)$,

$$\bar{\psi}_{\delta,H}(u) \sim (\frac{1}{2})^{\mathbb{I}(t^*=t_i)} \times \begin{cases} \mathcal{H}_{2H} \frac{2^{\frac{1}{2}-\frac{1}{2H}}\sqrt{\pi}}{H^{1/2}(1-H)^{1/2}} (C_H^{(i)} u^{1-H})^{\frac{1}{H}-1} \overline{\Phi}(C_H^{(i)} u^{1-H}), & H > 1/2 \\ \mathcal{H}^{2c_i^2\delta} e^{-2c_i q_i u}, & H = 1/2 \\ \frac{\sqrt{2\pi} H^{H+1/2} q_i^H u^H}{\delta c_i^{H+1}(1-H)^{H+1/2}} \overline{\Phi}(C_H^{(i)} u^{1-H}), & H < 1/2, \end{cases}$$

where $i = 1$ if $t^* \le t_1$ and $i = 2$ if $t^* \ge t_2$,
2) if $t^* \in (t_1, t_2)$, then with $\mathbb{D}_H = \frac{c_1 t^* + q_1}{(t^*)^H}$ when $H > 1/2$

$$\bar{\psi}_{\delta,H}(u) \sim \overline{\Phi}(\mathbb{D}_H u^{1-H}),$$

when $H = 1/2$

$$\widetilde{\mathcal{P}}_\gamma^{d_\delta} \overline{\Phi}(\mathbb{D}_{1/2}\sqrt{u})(1 + o(1)) \le \bar{\psi}_\delta(u) \le A \widetilde{\mathcal{P}}_\gamma^{d_0} \overline{\Phi}(\mathbb{D}_{1/2}\sqrt{u})(1 + o(1)), \tag{3.5}$$

where $\widetilde{\mathcal{P}}_\gamma^{d}, \widetilde{\mathcal{P}}_\gamma^{d_\delta} \in (0, \infty)$ and

$$A = e^{\delta \frac{(c_1 q_2 - c_2 q_1)(c_1 q_2 + q_1 c_2 - 2c_2 q_2)}{2(q_2 - q_1)^2}} > 1, \qquad \gamma = \frac{\delta(c_1 q_2 - q_1 c_2)^2}{2(q_2 - q_1)^2}, \tag{3.6}$$

when $H < 1/2$

$$2e^{-Bu^{1-H}} \overline{\Phi}(\mathbb{D}_H u^{1-H})(1 + o(1)) \le \bar{\psi}_{\delta,H}(u) \le \overline{\Phi}(\mathbb{D}_H u^{1-H})(1 + o(1)), \tag{3.7}$$

where

$$B = -\frac{\delta w_1'(t^*) w_2'(t^*)}{2(w_1'(t^*) - w_2'(t^*))} > 0, \qquad w_i(t) = \frac{(q_i + c_i t)^2}{t^{2H}}, \ i = 1, 2. \tag{3.8}$$

Remark 3.2.2 *The bounds in (3.7) are exact. Namely, there exist two tending to infinity sequences $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ such that as $n \to \infty$*

$$\bar{\psi}_{\delta,H}(u_n) \sim \overline{\Phi}(\mathbb{D}_H u_n^{1-H}), \quad \bar{\psi}_{\delta,H}(v_n) \sim 2e^{-Bv_n^{1-H}} \overline{\Phi}(\mathbb{D}_H v_n^{1-H}).$$

To study the asymptotics of the two-dimensional ruin probability over the infinite time horizon crucial is the asymptotic approximation of the one-dimensional one. The asymptotics of this ruin probability has been already studied in [47]. Since there are some inaccuracies we give the following corrected result.

Theorem 3.2.3 *For any $\delta > 0$ with $C_H = \frac{c^H}{H^H(1-H)^{1-H}}$ as $u \to \infty$*

$$\mathbb{P}\left\{\exists t \in G(\delta) : B_H(t) - ct > u\right\} \sim \begin{cases} \mathcal{H}_{2H} \frac{2^{\frac{1}{2}-\frac{1}{2H}}\sqrt{\pi}}{H^{1/2}(1-H)^{1/2}} (C_H u^{1-H})^{1/H-1} \overline{\Phi}(C_H u^{1-H}), & H > 1/2, \\ \mathcal{H}^{2c^2\delta} e^{-2cu}, & H = 1/2 \\ \frac{\sqrt{2\pi} H^{H+1/2} u^H}{\delta c^{H+1}(1-H)^{H+1/2}} \overline{\Phi}(C_H u^{1-H}), & H < 1/2. \end{cases} \quad (3.9)$$

Remark 3.2.4 *If $H > 1/2$ the asymptotics of the discrete probabilities in Theorems 3.2.1 and 3.2.3 are the same as in the continuous case and do not depend on δ. If $H = 1/2$ the asymptotics differ only in the constants. If $H < 1/2$ the discrete asymptotics are infinitely smaller than the corresponding continuous. All these statements directly follow from Theorems 3.2.1, 3.2.3 and Corollary 2 in [37] and Theorem 3.1 in [45].*

Next we discuss the finite time-horizon case. Here for large u the two-dimensional ruin probability always reduces to the one-dimensional one, that has been already studied in [10],[25]. More precisely, we have

Remark 3.2.5 *For any $T > 0$ with $\lambda(u) = \frac{\max(q_1 u + c_1 T, q_2 u + c_2 T)}{T^H}$ as $u \to \infty$*

$$\bar{\zeta}_{H,T}(u) \sim \begin{cases} \mathcal{H}_{2H} (\lambda(u))^{\frac{1-2H}{H}} \frac{(1/2)^{(1/2H)}}{H} \overline{\Phi}(\lambda(u)), & H < 1/2 \\ \overline{\Phi}(\lambda(u)), & H > 1/2 \end{cases}$$

and

$$\bar{\zeta}_{\frac{1}{2},T}(u) = \overline{\Phi}\left(\frac{uq_i}{\sqrt{T}} + c_i\sqrt{T}\right) + e^{-2c_i q_i u}\overline{\Phi}\left(\frac{uq_i}{\sqrt{T}} - c_i\sqrt{T}\right), \quad i = 1, 2,$$

where $i = 1$ if $(q_1, c_1) \geq (q_2, c_2)$ in the alphabetical order and $i = 2$, otherwise.

3.3 Proofs

Proof of Theorem 3.2.1. Denote

$$V_i(t) = \frac{B_H(t)}{c_i t + q_i}, \quad i = 1, 2. \quad (3.10)$$

Case (1). Assume that $t^* < t_1$. We have by the self-similarity of fBM

$$\bar{\psi}_{\delta,H}(u) \leq \mathbb{P}\left\{\exists t \in G(\delta/u) : V_1(t) > u^{1-H}\right\} =: \psi^{(1)}_{\delta,H}(u). \quad (3.11)$$

Since $t^* < t_1$ for any $0 < \varepsilon < t_1 - t^*$ we have

$$\bar{\psi}_{\delta,H}(u) \;\; \geq \;\; \mathbb{P}\left\{\exists t \in [t_1 - \varepsilon, t_1 + \varepsilon]_{\frac{\delta}{u}} : V_1(t) > u^{1-H}, V_2(t) > u^{1-H}\right\}$$

$$
\begin{aligned}
&= \mathbb{P}\left\{\exists t \in [t_1-\varepsilon, t_1+\varepsilon]_{\frac{\delta}{u}} : V_1(t) > u^{1-H}\right\} \\
&\sim \psi^{(1)}_{\delta,H}(u), \qquad u \to \infty. \qquad (3.12)
\end{aligned}
$$

For a detailed proof of the last line above see the Appendix. Thus, by (3.11)

$$
\bar{\psi}_{\delta,H}(u) \sim \psi^{(1)}_{\delta,H}(u), \qquad u \to \infty
$$

and by Theorem 3.2.3 the claim is established.

Let $t^* = t_1$. We have

$$
\begin{aligned}
\mathbb{P}\left\{\sup_{t\in[t^*,\infty)_{\frac{\delta}{u}}} V_1(t) > u^{1-H}\right\} &\leq \bar{\psi}_{\delta,H}(u) \\
&\leq \mathbb{P}\left\{\sup_{t\in[t^*,\infty)_{\frac{\delta}{u}}} V_1(t) > u^{1-H}\right\} + \mathbb{P}\left\{\sup_{t\in[0,t^*)_{\frac{\delta}{u}}} V_2(t) > u^{1-H}\right\}. \qquad (3.13)
\end{aligned}
$$

Since t^* is the unique maximizer of $\mathrm{Var}\{V_1(t)\}$ (details are given in the Appendix)

$$
\mathbb{P}\left\{\sup_{t\in[t^*,\infty)_{\frac{\delta}{u}}} V_1(t) > u^{1-H}\right\} \sim \frac{1}{2}\psi^{(1)}_{\delta,H}(u), \quad H \in (0,1), \qquad u \to \infty. \qquad (3.14)
$$

Next we prove that

$$
\mathbb{P}\left\{\sup_{t\in[0,t^*)_{\frac{\delta}{u}}} V_2(t) > u^{1-H}\right\} = o(\psi^{(1)}_{\delta,H}(u)), \quad u \to \infty. \qquad (3.15)
$$

Case $H \geq 1/2$. As follows from Corollary 2 in [37] and Theorem 3.2.3 for $H \geq 1/2$, all large u

$$
C\psi^{(1)}_{0,H}(u) \leq \psi^{(1)}_{\delta,H}(u) \leq \psi^{(1)}_{0,H}(u).
$$

Hence with the same constant C as in the line above it holds that

$$
\mathbb{P}\left\{\sup_{t\in[0,t^*)_{\frac{\delta}{u}}} V_2(t) > u^{1-H}\right\} (\psi^{(1)}_{\delta,H}(u))^{-1} \leq C^{-1}\mathbb{P}\left\{\sup_{t\in[0,t^*)} V_2(t) > u^{1-H}\right\} (\psi^{(1)}_{0,H}(u))^{-1} \to 0, \quad u \to \infty,
$$

where the last convergence follows from the proof of Theorem 3.1, case (4), $H \geq 1/2$ in [45].
Case $H < 1/2$. Let $\theta_u \in [0,\delta)$ be such that $t^* + \frac{\theta_u}{u} \in G(\frac{\delta}{u})$. Denote

$$
t_u = t^* + \frac{\theta_u}{u}. \qquad (3.16)
$$

Notice that with $t_u^- = t_u - \delta/u$ by (2.7)

$$
\mathbb{P}\left\{\sup_{t\in[0,t^*)_{\frac{\delta}{u}}} V_2(t) > u^{1-H}\right\} \leq \mathbb{P}\left\{V_2(t_u^-) > u^{1-H}\right\} + Cu \sup_{t\in[0,t_u^- - \delta/u]_{\frac{\delta}{u}}} \mathbb{P}\left\{V_2(t) > u^{1-H}\right\}
$$

$$= \mathbb{P}\left\{V_2(t_u^-) > u^{1-H}\right\} + Cu\mathbb{P}\left\{V_2(t_u^- - \frac{\delta}{u}) > u^{1-H}\right\}$$
$$\leq \left(1 + Cu\exp(u^{1-2H}\frac{\delta w_2'(t^*)}{2})\right)\mathbb{P}\left\{V_2(t_u^-) > u^{1-H}\right\},$$

where $w_2(t)$ is defined in (3.8). Since $H < 1/2$ and $w_2'(t^*) < 0$ it follows from (3.9) that the expression above equals $o(\psi_{\delta,H}^{(1)}(u))$ as $u \to \infty$ and (3.15) holds. Thus, from (3.13), (3.14) and (3.15) it follows that

$$\bar{\psi}_{\delta,H}(u) \sim \frac{1}{2}\psi_{\delta,H}^{(1)}(u), \quad u \to \infty$$

establishing the claim by (3.9). Case $t^* \geq t_2$ follows by the same arguments.

Case (2). Denote

$$Z_H(t) = \frac{B_H(t)}{\max(c_1t + q_1, c_2t + q_2)} \qquad \text{and} \qquad \sigma_H^2(t) = \mathrm{Var}\{Z_H(t)\}. \tag{3.17}$$

Notice that if $t^* \in [t_1, t_2]$, then t^* is the unique maximizer of $\sigma_H(t)$. Moreover, $\sigma_H(t)$ increases over $[0, t^*]$ and decreases over $[t^*, \infty)$.

Case $H > 1/2$. From Theorem 3.1 case (3), $H > \frac{1}{2}$ in [45] it follows that

$$\bar{\psi}_{\delta,H}(u) \leq \overline{\Phi}(\mathbb{D}_H u^{1-H})(1 + o(1)), \qquad u \to \infty.$$

We have (recall, t_u is defined in (3.16))

$$\bar{\psi}_{\delta,H}(u) = \mathbb{P}\left\{\sup_{t\in G(\frac{\delta}{u})} Z_H(t) > u^{1-H}\right\} \geq \mathbb{P}\left\{Z_H(t_u) > u^{1-H}\right\} \sim \overline{\Phi}(\mathbb{D}_H u^{1-H}), \quad u \to \infty.$$

Combining two statements above we establish the claim.

Case $H = 1/2$. For notational simplicity we write $Z(t)$ instead of $Z_{1/2}(t)$. It follows from [45] and (3.22) that with $\Delta = [t_u - S/u, t_u + S/u]_{\frac{\delta}{u}}$ as $u \to \infty$ and then $S \to \infty$

$$\bar{\psi}_\delta(u) \sim \mathbb{P}\left\{\exists t \in \Delta : Z(t) > \sqrt{u}\right\}. \tag{3.18}$$

Let $B^*(t)$ be an independent copy of BM, $\overline{B}^*(t) = B^*(t) - c_1 t$, $\phi_u(x)$ be the probability density function of $B(ut_u)$ and define

$$\eta = q_1 + c_1t^* = q_2 + c_2t^* = \frac{c_1q_2 - q_1c_2}{c_1 - c_2}. \tag{3.19}$$

By the self-similarity and independence of the increments of BM we have as $u \to \infty$

$$\mathbb{P}\left\{\sup_{t\in\Delta} Z(t) > \sqrt{u}\right\}$$

$$
\begin{aligned}
&= \mathbb{P}\{\exists \hat{t} \in [ut_u - S, ut_u)_\delta : B(\hat{t}) > q_2 u + c_2 \hat{t} \\
&\quad \text{or } \exists t \in [ut_u, ut_u + S]_\delta : (B(t) - B(ut_u)) + B(ut_u) > q_1 u + c_1 t\} \\
&= \mathbb{P}\{\exists \hat{t} \in [ut_u - S, ut_u)_\delta : B(\hat{t}) > q_2 u + c_2 ut_u + c_2(\hat{t} - ut_u) \\
&\quad \text{or } \exists t \in [ut_u, ut_u + S]_\delta : B^*(t - ut_u) + B(ut_u) > q_1 u + c_1 ut_u + c_1(t - ut_u)\} \\
&= \int_{\mathbb{R}} \phi_u(\eta u - x) \times \mathbb{P}\{\exists \hat{t} \in [ut_u - S, ut_u)_\delta : B(\hat{t}) > q_2 u + c_2 ut_u + c_2(\hat{t} - ut_u) \\
&\quad \text{or } \exists t \in [ut_u, ut_u + S]_\delta : B^*(t - ut_u) + \eta u - x > q_1 u + c_1 ut_u + c_1(t - ut_u) | B(ut_u) = \eta u - x\} dx \\
&= \int_{\mathbb{R}} \mathbb{P}\{\exists \hat{t} \in [-S, 0)_\delta : Z_u(\hat{t}) > x + c_2\theta_u \text{ or } \exists t \in [0, S]_\delta : B^*(t) - c_1 t > x + c_1\theta_u\} \phi_u(\eta u - x) dx \\
&= \frac{e^{\frac{-\eta^2 u}{2t_u}}}{\sqrt{2\pi u t_u}} \int_{\mathbb{R}} \mathbb{P}\left\{\exists \hat{t} \in [-S, 0)_\delta : Z_u(\hat{t}) > x + c_2\theta_u \text{ or } \exists t \in [0, S]_\delta : \overline{B}^*(t) > x + c_1\theta_u\right\} e^{\frac{\eta x}{t_u} - \frac{x^2}{2ut_u}} dx \\
&\sim \frac{e^{\frac{-\eta^2 u}{2t^*}}}{\sqrt{2\pi u t^*}} e^{\frac{\eta^2\theta_u}{2(t^*)^2} - \frac{\eta c_2 \theta_u}{t^*}} \int_{\mathbb{R}} \mathbb{P}\left\{\exists \hat{t} \in [-S, 0)_\delta : Z_u(\hat{t}) > x \text{ or } \exists t \in [0, S]_\delta : \overline{B}^*(t) > x + (c_1 - c_2)\theta_u\right\} e^{\frac{\eta x}{t_u} - \frac{(x - c_2\theta_u)^2}{2ut_u}} dx,
\end{aligned}
$$

where $Z_u(\hat{t})$ is an independent of $\overline{B}^*(t)$ Gaussian process with expectation and covariance defined below:

$$
\mathbb{E}\left\{Z_u(\hat{t})\right\} = \frac{uq_2 - x - c_2\theta_u}{ut_u}\hat{t}, \qquad \mathrm{cov}(Z_u(\hat{s}), Z_u(\hat{t})) = \frac{-\hat{s}\hat{t}}{ut_u} - \hat{t}, \qquad -S \le \hat{s} \le \hat{t} \le 0.
$$

Since $\eta - 2t^* c_2 > 0$ we have

$$
\begin{aligned}
&\int_{\mathbb{R}} \mathbb{P}\left\{\exists \hat{t} \in [-S, 0)_\delta : Z_u(\hat{t}) > x \text{ or } \exists t \in [0, S]_\delta : \overline{B}^*(t) > x + (c_1 - c_2)\delta\right\} e^{\frac{\eta x}{t_u} - \frac{(x - c_2\theta_u)^2}{2ut_u}} dx \qquad (3.20) \\
&\le e^{\frac{\theta_u \eta(\eta - 2t^* c_2)}{2(t^*)^2}} \int_{\mathbb{R}} \mathbb{P}\left\{\exists \hat{t} \in [-S, 0)_\delta : Z_u(\hat{t}) > x \text{ or } \exists t \in [0, S]_\delta : \overline{B}^*(t) > x + (c_1 - c_2)\theta_u\right\} e^{\frac{\eta x}{t_u} - \frac{(x - c_2\theta_u)^2}{2ut_u}} dx \\
&\le e^{\frac{\delta \eta(\eta - 2t^* c_2)}{2(t^*)^2}} \int_{\mathbb{R}} \mathbb{P}\left\{\exists \hat{t} \in [-S, 0)_\delta : Z_u(\hat{t}) > x \text{ or } \exists t \in [0, S]_\delta : \overline{B}^*(t) > x\right\} e^{\frac{\eta x}{t_u} - \frac{(x - c_2\theta_u)^2}{2ut_u}} dx.
\end{aligned}
$$

We estimate the integral in the lower bound. Assume that BM is defined on $\mathbb{R}$ (centered Gaussian process with $\mathrm{cov}(B(t), B(s)) = \frac{|t| + |s| - |s - t|}{2}$). When $u \to \infty$ covariance and expectation of $Z_u(t) - \frac{q_2 t}{t^*}$ converge to those of BM, hence $Z_u(t) - \frac{q_2 t}{t^*}$ converges to $B(t)$ for $t < 0$ in the sense of convergence of finite-dimensional distributions. Thus, with $\zeta = \frac{q_2}{t^*}$ as $u \to \infty$ (proof is given in the Appendix)

$$
\begin{aligned}
&\int_{\mathbb{R}} \mathbb{P}\left\{\exists \hat{t} \in [-S, 0)_\delta : Z_u(\hat{t}) > x \text{ or } \exists t \in [0, S]_\delta : \overline{B}^*(t) > x + (c_1 - c_2)\delta\right\} e^{\frac{\eta x}{t_u} - \frac{(x - c_2\theta_u)^2}{2ut_u}} dx \\
&\sim \int_{\mathbb{R}} \mathbb{P}\left\{\exists \hat{t} \in [-S, 0)_\delta : B(\hat{t}) + \zeta\hat{t} > x \text{ or } \exists t \in [0, S]_\delta : B(t) - c_1 t > x + (c_1 - c_2)\delta\right\} e^{\frac{\eta x}{t^*}} dx \qquad (3.21) \\
&=: I(S).
\end{aligned}
$$

By the explicit formula $\mathbb{P}\left\{\sup_{t\ge 0}(B(t)-ct)>x\right\}=e^{-2cx},\ c,x>0$ (see [26]) we have

$$
\begin{aligned}
I(S) &\le \int_{-\infty}^{0} e^{\frac{\eta x}{t^*}}dx+\int_{0}^{\infty}\Big(\mathbb{P}\left\{\exists t\ge 0: B(t)-\zeta t>x\right\}+\mathbb{P}\left\{\exists t\ge 0: B(t)-c_1t>x\right\}\Big)e^{\frac{\eta x}{t^*}}dx\\
&= \frac{t^*}{\eta}+\int_0^\infty \big(e^{(-2\zeta+\frac{\eta}{t^*})x}+e^{(-2c_1+\frac{\eta}{t^*})x}\big)dx<\infty,
\end{aligned}
$$

provided by $\min(2\zeta t^*,2c_1t^*)>\eta$. Since $I(S)$ is non-decreasing it implies $\lim_{S\to\infty}I(S)\in(0,\infty)$. We have with $\xi=\frac{\eta^2}{(t^*)^2}$ and $\hat{d}_\delta(t)=\mathbb{I}(t<0)\frac{\zeta t^*t}{\eta}-\mathbb{I}(t\ge 0)(\frac{c_1t^*t}{\eta}+\frac{\eta(c_1-c_2)\delta}{t^*})$ that

$$
\begin{aligned}
&I(S)\\
&= \frac{t^*}{\eta}\int_{\mathbb{R}}\mathbb{P}\left\{\exists t\in[-S,S]_\delta:\frac{\eta}{t^*}B(t)+t\Big(\mathbb{I}(t\le 0)\frac{\eta\zeta}{t^*}-\mathbb{I}(t\ge 0)\frac{\eta c_1}{t^*}\Big)-\mathbb{I}(t\ge 0)\frac{\eta(c_1-c_2)\delta}{t^*}>\frac{\eta x}{t^*}\right\}e^{\frac{\eta x}{t^*}}d(\frac{x\eta}{t^*})\\
&= \frac{t^*}{\eta}\int_{\mathbb{R}}\mathbb{P}\left\{\exists \xi t\in[-\xi S,\xi S]_{\xi\delta}: B(\xi t)+\xi t\Big(\mathbb{I}(t\le 0)\frac{t^*\zeta}{\eta}-\mathbb{I}(t\ge 0)\frac{t^*c_1}{\eta}\Big)-\mathbb{I}(t\ge 0)\frac{\eta(c_1-c_2)\delta}{t^*}>x\right\}e^{x}dx\\
&= \int_{\mathbb{R}}\mathbb{P}\left\{\exists t\in[-S\xi,S\xi]_{\xi\delta}: B(t)+\hat{d}_\delta(t)>x\right\}e^{x}dx\\
&= \mathbb{E}\left\{\sup_{t\in[-\frac{S\xi}{2},\frac{S\xi}{2}]_{\frac{\delta\xi}{2}}} e^{\sqrt{2}B(t)-|t|+\hat{d}_\delta(2t)+|t|}\right\}.
\end{aligned}
$$

Since $\lim_{S\to\infty}I(S)\in(0,\infty)$, $\hat{d}_\delta(2t)+|t|=d_\delta(t)$ and $\frac{\delta\xi}{2}=\gamma$, we have that the expression above tends to $\widetilde{\mathcal{P}}_\gamma^{d_\delta}\in(0,\infty)$ as $S\to\infty$. Thus, summarizing all calculations above we conclude that as $u\to\infty$ and then $S\to\infty$

$$
\mathbb{P}\left\{\sup_{t\in\Delta}Z(t)>\sqrt{u}\right\}\ge \widetilde{\mathcal{P}}_\gamma^{d_\delta}\overline{\Phi}(\mathbb{D}_{1/2}\sqrt{u})(1+o(1)). \tag{3.22}
$$

For the same reasons estimating the upper bound in (3.20) we have that as $u\to\infty$ and then $S\to\infty$

$$
\mathbb{P}\left\{\sup_{t\in\Delta}Z(t)>\sqrt{u}\right\}\le \widetilde{\mathcal{P}}_\gamma^{d_0}e^{\frac{\delta\eta(\eta-2t^*c_2)}{2(t^*)^2}}\overline{\Phi}(\mathbb{D}_{1/2}\sqrt{u})(1+o(1))
$$

and the claim is established.

Case $H<1/2$. As shown in the Appendix (recall, $t_u^-=t_u-\frac{\delta}{u}$ and $V_1(t),V_2(t)$ are defined in (3.10))

$$
\bar{\psi}_{\delta,H}(u)\ \sim\ \mathbb{P}\left\{V_2(t_u^-)>u^{1-H}\right\}+\mathbb{P}\left\{V_1(t_u)>u^{1-H}\right\},\quad u\to\infty. \tag{3.23}
$$

We have (recall, $w_i(t) = \frac{(q_i+c_it)^2}{t^{2H}},\ i=1,2$)

$$
\begin{aligned}
\mathbb{P}\left\{V_1(t_u) > u^{1-H}\right\} &\sim \overline{\Phi}(\mathbb{D}_H u^{1-H}) \exp(-\frac{\theta_u w_1'(t^*) u^{1-2H}}{2}),\\
\mathbb{P}\left\{V_2(t_u^-) > u^{1-H}\right\} &\sim \overline{\Phi}(\mathbb{D}_H u^{1-H}) \exp(-\frac{-(\delta-\theta_u) w_2'(t^*) u^{1-2H}}{2}), \quad u\to\infty.
\end{aligned}
$$

Thus,

$$
\bar{\psi}_{\delta,H}(u) \sim \overline{\Phi}(\mathbb{D}_H u^{1-H}) \left(\exp(-\frac{\theta_u w_1'(t^*)}{2} u^{1-2H}) + \exp(-\frac{-(\delta-\theta_u) w_2'(t^*)}{2} u^{1-2H}) \right), \quad u\to\infty,
$$

hence the claim follows from the inequality (recall, $B = -\frac{\delta w_1'(t^*) w_2'(t^*)}{2(w_1'(t^*)-w_2'(t^*))} > 0$)

$$
\begin{aligned}
2e^{-Bu^{1-2H}}(1+o(1)) &\leq \exp(-\frac{\theta_u w_1'(t^*)}{2} u^{1-2H}) + \exp(-\frac{-(\delta-\theta_u) w_2'(t^*)}{2} u^{1-2H}) \qquad (3.24)\\
&\leq 1+o(1), \quad u\to\infty
\end{aligned}
$$

and the proof is established. □

Proof of Remark 3.2.2. Consider a sequence $\{u_n\}_{n\in\mathbb{N}}$ such $u_n\to\infty$ and for all n $t^*\in G(\delta/u_n)$. From the proof of Theorem 3.2.1 case (2), $H<1/2$ it follows, that

$$
\bar{\psi}_{\delta,H}(u_n) = \overline{\Phi}(\mathbb{D}_H u_n^{1-H})(1+o(1)), \quad n\to\infty.
$$

Next we choose a sequence $\{v_n\}_{n\in\mathbb{N}}$ such $v_n\to\infty$ and for all n $t^* - \frac{\delta w_2'(t^*)}{v_n(w_1'(t^*)-w_2'(t^*))} \in G(\delta/v_n)$. For such sequence inequality in (3.24) becomes equality, hence

$$
\bar{\psi}_{\delta,H}(v_n) \sim 2e^{-Bv_n^{1-H}} \overline{\Phi}(\mathbb{D}_H v_n^{1-H}), \qquad n\to\infty
$$

and the claim follows. □

Proof of Theorem 3.2.3. When $H=\frac{1}{2}$ the assertion of the theorem follows from [40] and [47].
Case $H>1/2$. For large u we have

$$
\mathbb{P}\left\{\exists t\geq 0: \inf_{s\in[t,t+u^{\frac{2H-1}{2H}}]} (B_H(s)-cs) > u\right\} \leq \mathbb{P}\left\{\sup_{t\in G(\delta)} (B_H(t)-ct) > u\right\} \leq \mathbb{P}\left\{\sup_{t\geq 0}(B_H(t)-ct) > u\right\}.
$$

In view of Remark 3.2 in [15] the lower and the upper bounds above are asymptotically equivalent, hence

$$
\mathbb{P}\left\{\sup_{t\in G(\delta)} (B_H(t)-ct) > u\right\} \sim \mathbb{P}\left\{\sup_{t\geq 0}(B_H(t)-ct) > u\right\}, \quad u\to\infty.
$$

The asymptotics of the last probability above is given, e.g., in Corollary 3.1 in [15], thus the claim follows.

Case $H < 1/2$. By the self-similarity of fBM we have

$$\begin{aligned}\psi_{\delta,H}(u) &:= \mathbb{P}\left\{\exists t \in G(\delta) : B_H(t) > u + ct\right\}\\ &= \mathbb{P}\left\{\exists t \in G(\frac{\delta}{u}) : \frac{B_H(t)}{1+ct} > u^{1-H}\right\}\\ &=: \mathbb{P}\left\{\exists t \in G(\frac{\delta}{u}) : V(t) > u^{1-H}\right\}.\end{aligned}$$

Note that the variance of $V(t)$ achieves its unique maxima at $t_0 = \frac{H}{c(1-H)}$. As shown in the Appendix

$$\psi_{\delta,H}(u) \sim \sum_{t\in I(t_0)} \mathbb{P}\left\{V(t) > u^{1-H}\right\}, \qquad u \to \infty, \tag{3.25}$$

where $I(t_0) = (-1/\sqrt{u} + t_0, 1/\sqrt{u} + t_0)_{\frac{\delta}{u}}$. We have with $\hat{u} = \frac{u^{1-H}c^H}{H^H(1-H)^{1-H}}$ as $u \to \infty$

$$\begin{aligned}\sum_{t\in I(t_0)} \mathbb{P}\left\{V(t) > u^{1-H}\right\} &= \sum_{t\in I(t_0)} \overline{\Phi}(u^{1-H}\frac{1+ct}{t^H})\\ &\sim \sum_{t\in I(t_0)} \frac{1}{\sqrt{2\pi}\hat{u}} e^{-\frac{1}{2}(u^{1-H}\frac{1+ct}{t^H})^2}.\end{aligned}$$

Setting $f_H(t) = \frac{(1+ct)^2}{t^{2H}}$ we have $f'_H(t_0) = 0$ and $f''_H(t_0) = \frac{2c^{2+2H}(1-H)^{2H+1}}{H^{2H+1}} > 0$. Since $f_H(t) \approx f_H(t_0) + \frac{(t-t_0)^2}{2} f''_H(t_0),\ t \in I(t_0)$ we write (a strict proof is given in the Appendix in [41])

$$\begin{aligned}\sum_{t\in I(t_0)} \frac{1}{\sqrt{2\pi}\hat{u}} e^{-\frac{1}{2}(u^{1-H}\frac{1+ct}{t^H})^2} &= \frac{1}{\sqrt{2\pi}\hat{u}} e^{-\hat{u}^2/2} \sum_{t\in I(t_0)} e^{-\frac{1}{2}u^{2-2H}\left(\frac{(1+ct)^2}{t^{2H}} - \frac{(1+ct_0)^2}{t_0^{2H}}\right)}\\ &\sim \overline{\Phi}(\hat{u}) \sum_{t\in I(t_0)} e^{-\frac{1}{2}u^{2-2H}\frac{f''_H(t_0)}{2}(t-t_0)^2}, \quad u \to \infty.\end{aligned} \tag{3.26}$$

Next (set $F = \frac{f''_H(t_0)}{4} = \frac{c^{2+2H}(1-H)^{2H+1}}{2H^{2H+1}}$)

$$\begin{aligned}\sum_{t\in I(t_0)} e^{-\frac{1}{2}u^{2-2H}\frac{f''_H(t_0)}{2}(t-t_0)^2} &\sim 2 \sum_{t\in(0,u^{-1/2})_{\delta/u}} e^{-Fu^{2-2H}t^2}\\ &= 2 \sum_{tu^{1-H}\in(0,u^{1/2-H})_{\delta u^{-H}}} e^{-F(tu^{1-H})^2}\\ &= \frac{2u^H}{\delta}\Big(\delta u^{-H} \sum_{t\in(0,u^{1/2-H})_{\delta u^{-H}}} e^{-Ft^2}\Big)\\ &\sim \frac{2u^H}{\delta\sqrt{F}} \int_0^\infty e^{-Ft^2} d(\sqrt{F}t)\\ &= \frac{\sqrt{\pi}u^H}{\delta\sqrt{F}}, \quad u \to \infty.\end{aligned}$$

Combining the line above with (3.26) and (3.25) we have

$$\psi_{\delta,H}(u) \sim \overline{\Phi}(\frac{u^{1-H}c^H}{H^H(1-H)^{1-H}})\frac{\sqrt{2\pi}H^{H+1/2}u^H}{\delta c^{H+1}(1-H)^{H+1/2}}, \qquad u \to \infty \tag{3.27}$$

and the claim follows. □

Proof of Remark 3.2.5. Assume, that $(q_1, c_1) \geq (q_2, c_2)$ in the alphabetical order, the other case follows by the same arguments. For large u we have that $q_1 u + c_1 t \geq q_2 u + c_2 t$ for all $t \in [0, T]$ implying

$$\bar{\zeta}_H(u) = \mathbb{P}\left\{\exists t \in [0, T] : B_H(t) > c_1 t + q_1 u\right\}.$$

Thus, for $H = 1/2$ the claim follows by [26]. For $H \neq 1/2$ Theorem 2.1 in [10] completes the proof. □

3.4 Appendix

Proof of (3.12)**.** To establish the claim, it is enough to show that

$$\mathbb{P}\left\{\exists t \notin [t_1 - \varepsilon, t_1 + \varepsilon] : V_1(t) > u^{1-H}\right\} = o(\psi^{(1)}_{\delta,H}(u)), \qquad u \to \infty.$$

We shall prove that $V_1(t)$ is a.s. bounded on $[0, \infty)$. By Chapter 4, p. 31 in [59] it is equivalent with $\mathbb{P}\left\{V_1(t) \text{ is bounded for } t \geq 0\right\} > 0$. We have by Corollary 2 in [37]

$$\mathbb{P}\left\{\sup_{t \geq 0} V_1(t) \leq u\right\} \quad = \quad 1 - \mathbb{P}\left\{\sup_{t \geq 0} V_1(t) > u\right\} \to 1, \quad u \to \infty.$$

Thus, $V_1(t)$ is bounded a.s. Note that the variance $v(t)$ of $V_1(t)$ achieves its unique maxima at t_1. Denote

$$m = \max_{t \in [0, t_1 - \varepsilon] \cup [t_1 + \varepsilon, \infty)} v(t), \quad M = \mathbb{E}\left\{\sup_{t \in [0, t_1 - \varepsilon] \cup [t_1 + \varepsilon, \infty)} V_1(t)\right\}.$$

By Borell-TIS inequality (see Lemma 5.3 in [45]) we have that $M < \infty$ and for all u large enough

$$\mathbb{P}\left\{\exists t \notin [t_1 - \varepsilon, t_1 + \varepsilon] : V_1(t) > u^{1-H}\right\} \leq e^{-\frac{(u^{1-H} - M)^2}{2m}}.$$

From Theorem 3.2.3 and inequality $m < v(t_1)$ it follows that

$$e^{-\frac{(u^{1-H} - M)^2}{2m}} = o(\psi^{(1)}_{\delta,H}(u)), \quad u \to \infty$$

and thus (3.12) holds. □

Proof of (3.14)**.** Assume that $H < 1/2$. Since $t^* = t_1$ is the unique maximizer of $\mathrm{Var}\{V_1(t)\}$, then repeating the proof of Theorem 3.2.3 we obtain

$$\mathbb{P}\left\{\sup_{t \in [t^*, \infty)_{\frac{\delta}{u}}} V_1(t) > u^{1-H}\right\} \sim \sum_{t \in [t_1, t_1 + 1/\sqrt{u})_{\frac{\delta}{u}}} \mathbb{P}\left\{V_1(t) > u^{1-H}\right\}, \qquad u \to \infty.$$

The method of computation of the asymptotics of the sum above is the same as in the proof of Theorem 3.2.3, see the calculation of the analogous sum in (3.25). The difference is only in the intervals of summation, in Theorem 3.2.3 $I(t_0)$ is symmetric about t_0, while $[t_1, t_1 + 1/\sqrt{u})_{\frac{\delta}{u}}$ has only the right part. Thus, multiplier 1/2 appears before the final asymptotics.

Assume that $H = 1/2$. Then the claim follows from the proof of Theorem 1.1 in [40]. The index of summation in (19) in [40] in our case will be $1 \leq j \leq N_u$, thus, multiplier 1/2 appears before the final asymptotics. The claim can also be established by Theorem 1, ii) in [47].

Assume that $H > 1/2$. As in the proof of Theorem 3.2.3, case $H > 1/2$ we have

$$\mathbb{P}\left\{\exists t \geq t^* : \inf_{s\in[t,t+u^{\frac{-1}{2H}}]} V_1(t) > u^{1-H}\right\} \leq \mathbb{P}\left\{\sup_{t\in[t^*,\infty)_{\frac{\delta}{u}}} V_1(t) > u^{1-H}\right\} \leq \mathbb{P}\left\{\sup_{t\in[t^*,\infty)} V_1(t) > u^{1-H}\right\}.$$

As follows from [59] the upper bound in the inequality above is equivalent with $\frac{1}{2}\psi^{(1)}_{0,H}(u),\ u \to \infty$ and by Theorem 2.1 in [15] the lower bound has the same asymptotics. Since for $H > 1/2$ (see Theorem 3.2.3) it holds that $\psi^{(1)}_{0,H}(u) \sim \psi^{(1)}_{\delta,H}(u),\ u \to \infty$ we obtain the claim. □

Proof of (3.21)**.** First we show that with $\bar{\delta} = (c_1 - c_2)\delta$

$$\begin{aligned}
&\int_{\mathbb{R}} \mathbb{P}\left\{\exists \hat{t} \in [-S, 0)_\delta : Z_u(\hat{t}) > x \text{ or } \exists t \in [0, S]_\delta : \overline{B}^*(t) > x + \bar{\delta}\right\} e^{\frac{\eta x}{t_u} - \frac{(x-c_2\theta_u)^2}{2ut_u}} dx \\
&= \int_{-M}^{M} \mathbb{P}\left\{\exists \hat{t} \in [-S, 0)_\delta : Z_u(\hat{t}) > x \text{ or } \exists t \in [0, S]_\delta : \overline{B}^*(t) > x + \bar{\delta}\right\} e^{\frac{\eta x}{t^*}} dx + B_{M,v},
\end{aligned} \tag{3.28}$$

where $B_{M,v} \to 0$ as $u \to \infty$ and then $M \to \infty$. We have

$$\begin{aligned}
|B_{M,v}| &\leq \left|\int_{-M}^{M} \mathbb{P}\left\{\exists \hat{t} \in [-S, 0)_\delta : Z_u(\hat{t}) > x \text{ or } \exists t \in [0, S]_\delta : \overline{B}^*(t) > x + \bar{\delta}\right\} (e^{\frac{\eta x}{t_u} - \frac{(x-c_2\theta_u)^2}{2ut_u}} - e^{\frac{\eta x}{t^*}}) dx\right| \\
&\quad + \int_{|x|>M} \mathbb{P}\left\{\exists \hat{t} \in [-S, 0)_\delta : Z_u(\hat{t}) > x \text{ or } \exists t \in [0, S]_\delta : \overline{B}^*(t) > x + \bar{\delta}\right\} e^{\frac{\eta x}{t_u} - \frac{(x-c_2\theta_u)^2}{2ut_u}} dx \\
&=: |I_1| + I_2.
\end{aligned}$$

Since $\mathrm{Var}\{Z_u(t)\}$ is bounded and $\mathbb{E}\{Z_u(t)\} < 0$ for large u and all $t \in [-S, 0]$ by Borell-TIS inequality for $x > 0$ and some $C > 0$ we have

$$\begin{aligned}
\mathbb{P}\left\{\sup_{\hat{t}\in[-S,0)_\delta} Z_u(\hat{t}) > x \text{ or } \sup_{t\in[0,S]_\delta} \overline{B}^*(t) > x + \bar{\delta}\right\} &\leq \mathbb{P}\left\{\sup_{t\in[-S,0]} (Z_u(t) - \mathbb{E}\{Z_u(t)\}) > x\right\} + \mathbb{P}\left\{\sup_{t\in[0,S]} B(t) > x\right\} \\
&\leq e^{-x^2/C}.
\end{aligned}$$

Thus, as $u \to \infty$

$$I_2 \le \int\limits_{x>M} e^{-\frac{x^2}{C}+\frac{\eta x}{t_u}} dx + \int\limits_{x<-M} e^{\frac{\eta x}{2t^*}} dx \to 0, \quad M \to \infty.$$

For $u \ge M^3$ we have

$$|I_1| \le \int\limits_{-M}^{M} e^{-\frac{x^2}{C}+\frac{\eta x}{t^*}} \Big| e^{-\frac{x\eta\theta_u}{ut^* t_u}-\frac{(x-c_2\theta_u)^2}{2ut_u}} - 1\Big| dx \le \int\limits_{\mathbb{R}} e^{-\frac{x^2}{C}+\frac{\eta x}{t^*}} dx \sup_{x\in[-M,M]} |e^{-\frac{x\eta\theta_u}{ut^* t_u}-\frac{(x-c_2\theta_u)^2}{2ut_u}} - 1| \le \frac{C}{M}.$$

Thus, $\lim\limits_{M\to\infty}\lim\limits_{u\to\infty}(|I_1|+I_2)=0$ and (3.28) holds. Since for $t \in [-S,0]$ $Z_u(t)$ converges to $B(t)+\zeta t$ as $u \to \infty$ in the sense of convergence of finite-dimensional distributions we have

$$\begin{aligned} &\int\limits_{-M}^{M} \mathbb{P}\left\{\exists \hat{t} \in [-S,0)_\delta : Z_u(\hat{t}) > x \text{ or } \exists t \in [0,S]_\delta : \overline{B}^*(t) > x+\bar{\delta}\right\} e^{\frac{\eta x}{t^*}} dx \\ \to &\int\limits_{-M}^{M} \mathbb{P}\left\{\exists \hat{t} \in [-S,0)_\delta : B(t)+\zeta\hat{t} > x \text{ or } \exists t \in [0,S]_\delta : \overline{B}^*(t) > x+\bar{\delta}\right\} e^{\frac{\eta x}{t^*}} dx, \qquad u \to \infty. \end{aligned}$$

By the monotone convergence theorem the expression above tends to

$$\int\limits_{\mathbb{R}} \mathbb{P}\left\{\exists \hat{t} \in [-S,0)_\delta : B(\hat{t})+\zeta\hat{t} > x \text{ or } \exists t \in [0,S]_\delta : B(t)-c_1 t > x+\bar{\delta}\right\} e^{\frac{\eta x}{t^*}} dx, \qquad M \to \infty$$

and the claim is established. □

Proof of (3.23)**.** We have by Lemma 2.3 in [56] for all large u with $w = u^{1-H}$ (recall, $t_u^- = t_u - \delta/u$)

$$\begin{aligned} \bar{\psi}_{\delta,H}(u) &\ge \mathbb{P}\left\{\sup_{t\in\{t_u^-,t_u\}} Z_H(t) > w\right\} \\ &= \mathbb{P}\left\{V_1(t_u) > w\right\} + \mathbb{P}\left\{V_2(t_u^-) > w\right\} - \mathbb{P}\left\{V_1(t_u) > w, V_2(t_u^-) > w\right\} \\ &\sim \mathbb{P}\left\{V_1(t_u) > w\right\} + \mathbb{P}\left\{V_2(t_u^-) > w\right\}, \quad u \to \infty. \end{aligned} \tag{3.29}$$

Next we prove that

$$\mathbb{P}\left\{\exists t \in G(\delta/u), t \ge t^* : V_1(t) > w\right\} \sim \mathbb{P}\left\{V_1(t_u) > w\right\}, \quad u \to \infty. \tag{3.30}$$

Fix some $\varepsilon > 0$. Since $\sigma_H^2(t)$ is decreasing over $[t^*,\infty)$ we have by Borell-TIS inequality as $u \to \infty$

$$\mathbb{P}\left\{\exists t \in G(\delta/u), t \ge t^*+\varepsilon : V_1(t) > w\right\} = o(\mathbb{P}\left\{V_1(t_u) > w\right\}). \tag{3.31}$$

We have with $t_u^+ = t_u + \delta/u$ and $w_1(t)$ defined in (3.8) as $u \to \infty$

$$\mathbb{P}\left\{\exists t \in G(\delta/u), t_u^+ \le t \le t^*+\varepsilon : V_1(t) > w\right\} \le Cu \sup_{t\in G(\delta/u), t_u^+\le t\le t^*+\varepsilon} \mathbb{P}\left\{V_1(t) > w\right\}$$

$$
\begin{aligned}
&\leq \; Cu\mathbb{P}\left\{V_1(t_u^+) > w\right\} \\
&\sim \; Cu\mathbb{P}\left\{V_1(t_u) > w\right\}\exp(-\frac{w_1'(t^*)\delta}{2}w) \\
&= \; o(\mathbb{P}\left\{V_1(t_u) > w\right\}).
\end{aligned}
$$

Combining the lines above with (3.31) we establish (3.30). By the same arguments we have

$$
\mathbb{P}\left\{\exists t \in G(\delta/u), t < t^* : V_2(t) > w\right\} \sim \mathbb{P}\left\{V_2(t_u^-) > w\right\}, \quad u \to \infty
$$

implying with (3.30)

$$
\begin{aligned}
\bar{\psi}_{\delta,H}(u) \; &\leq \; \mathbb{P}\left\{\exists t \in G(\delta/u), t < t^* : V_2(t) > w\right\} + \mathbb{P}\left\{\exists t \in G(\delta/u), t \geq t^* : V_1(t) > w\right\} \\
&= \; \left(\mathbb{P}\left\{V_1(t_u) > w\right\} + \mathbb{P}\left\{V_2(t_u^-) > w\right\}\right)(1 + o(1)), \quad u \to \infty.
\end{aligned}
$$

By (3.29) and the line above we obtain the claim. □

Proof of (3.25)**.** First we prove that with $\overline{I}(t_0) = (-\frac{1}{\sqrt{u}} + t_0, t_0 + \frac{1}{\sqrt{u}})$

$$
\mathbb{P}\left\{\sup_{t \in G(\delta/u)\setminus\overline{I}(t_0)} V(t) > u^{1-H}\right\} = o(\psi_{\delta,H}(u)), \quad u \to \infty. \tag{3.32}
$$

Denote $\varepsilon(t_0) = (-\varepsilon + t_0, \varepsilon + t_0)_{\delta/u}$ and $\overline{\varepsilon}(t_0) = (-\varepsilon + t_0, \varepsilon + t_0)$ for some $\varepsilon > 0$. We have

$$
\mathbb{P}\left\{\sup_{t \in G(\delta/u)\setminus\overline{I}(t_0)} V(t) > u^{1-H}\right\} \leq \mathbb{P}\left\{\sup_{t \in \varepsilon(t_0)\setminus\overline{I}(t_0)} V(t) > u^{1-H}\right\} + \mathbb{P}\left\{\sup_{t \in [0,\infty)\setminus\overline{\varepsilon}(t_0)} V(t) > u^{1-H}\right\}.
$$

The second summand in the line above is negligible by Borell-TIS inequality. Notice that

$$
\begin{aligned}
\mathbb{P}\left\{\sup_{t \in \varepsilon(t_0)\setminus\overline{I}(t_0)} V(t) > u^{1-H}\right\} \; &\leq \; Cu \sup_{t \in \varepsilon(t_0)\setminus\overline{I}(t_0)} \mathbb{P}\left\{V(t) > u^{1-H}\right\} \\
&\leq \; Cu\left(\mathbb{P}\left\{V(t_0 - 1/\sqrt{u}) > u^{1-H}\right\} + \mathbb{P}\left\{V(t_0 + 1/\sqrt{u}) > u^{1-H}\right\}\right) \\
&\leq \; 3Cu\overline{\Phi}(u^{1-H}\frac{1 + ct_0}{t_0^H})\exp(-\frac{1}{4}f_H''(t_0)u^{1-2H}), \quad u \to \infty,
\end{aligned}
$$

recall that $f_H(t) = \frac{(1+ct)^2}{t^{2H}}$ and $f_H''(t_0) > 0$. Hence we have

$$
\mathbb{P}\left\{\sup_{t \in \varepsilon(t_0)\setminus\overline{I}(t_0)} V(t) > u^{1-H}\right\} = o(\overline{\Phi}(u^{1-H}\frac{1 + ct_0}{t_0^H})), \quad u \to \infty
$$

and thus (3.32) follows from (3.27). Next by Bonferroni inequality

$$
\sum_{t \in I(t_0)} \mathbb{P}\left\{V(t) > u^{1-H}\right\} - \Pi(u) \leq \mathbb{P}\left\{\sup_{t \in I(t_0)} V(t) > u^{1-H}\right\} \leq \sum_{t \in I(t_0)} \mathbb{P}\left\{V(t) > u^{1-H}\right\}, \tag{3.33}
$$

where

$$
\Pi(u) = \sum_{t_1 < t_2 \in I(t_0)} \mathbb{P}\left\{V(t_1) > u^{1-H}, V(t_2) > u^{1-H}\right\}.
$$

Fix some numbers $t_1, t_2 \in I(t_0)$. We have (recall, $\hat{u} = \frac{u^{1-H}c^H}{H^H(1-H)^{1-H}}$)

$$\begin{aligned}\mathbb{P}\left\{V(t_1) > u^{1-H}, V(t_2) > u^{1-H}\right\} &\leq \mathbb{P}\left\{\frac{\varepsilon^{(1)}V(t_1)}{\sigma_H(t_0)} > \frac{u^{1-H}}{\sigma_H(t_0)}, \frac{\varepsilon^{(2)}V(t_2)}{\sigma_H(t_0)} > \frac{u^{1-H}}{\sigma_H(t_0)}\right\}\\ &=: \mathbb{P}\left\{W_1 > \hat{u}, W_2 > \hat{u}\right\},\end{aligned}$$

where numbers $\varepsilon^{(1)}, \varepsilon^{(2)} \geq 1$ are chosen such that $\mathrm{Var}\{\frac{\varepsilon^{(1)}V(t_1)}{\sigma_H(t_0)}\} = \mathrm{Var}\{\frac{\varepsilon^{(2)}V(t_2)}{\sigma_H(t_0)}\} = 1$. We have that correlation r_w of (W_1, W_2) has expansion

$$r_w(t_1, t_2) = 1 - C|t_1 - t_2|^{2H} + o(|t_1 - t_2|^{2H}), \qquad t_1, t_2 \to t_0$$

and hence for all $t_1, t_2 \in I(t_0)$ it holds that $\sqrt{|t_1 - t_2|^{2H}} \geq \delta^H u^{-H}$. Thus, by Lemma 2.3 in [56] we have

$$\mathbb{P}\left\{W_1 > \hat{u}, W_2 > \hat{u}\right\} \leq \overline{\Phi}(\hat{u})\overline{\Phi}(\hat{C}u^{1-2H}), \quad u \to \infty$$

implying that for all $t_1, t_2 \in I(t_0)$

$$\mathbb{P}\left\{V(t_1) > u^{1-H}, V(t_2) > u^{1-H}\right\} \leq \overline{\Phi}(\hat{u})\overline{\Phi}(\hat{C}u^{1-2H}), \quad u \to \infty.$$

There are less then Cu^2 summands in $\Pi(u)$, hence from the line above, (3.33) and (3.27) it follows that

$$\mathbb{P}\left\{\sup_{t\in I(t_0)} V(t) > u^{1-H}\right\} \sim \sum_{t\in I(t_0)} \mathbb{P}\left\{V(t) > u^{1-H}\right\}, \quad u \to \infty.$$

Thus, the claim follows by the line above combined with (3.32). □

Chapter 4

Parisian Ruin for Insurer and Reinsurer under Quota-Share Treaty

This chapter is based on G. Jasnovidov and A. Shemenduyk: Parisian Ruin for Insurer and Reinsurer under Quota-Share Treaty, submitted.

4.1 Introduction

Consider the risk model defined by

$$R(t) = u + \rho t - X(t), \qquad t \geq 0, \tag{4.1}$$

where $X(t)$ is a centered Gaussian risk process with a.s. continuous sample paths, $\rho > 0$ is the net profit rate and $u > 0$ is the initial capital. This model is relevant to insurance and financial applications, see, e.g., [31]. Some contributions (see, e.g., [2, 15, 16]), extend the classical ruin problem to the so-called Parisian ruin problem which allows the surplus process to spend a pre-specified time below zero before a ruin is recognized. Formally, the classical Parisian ruin time and ruin probability for $T \geq 0$ are defined by

$$\tau(u,T) = \{\inf t \geq 0 : \forall s \in [t, t+T] \ R(s) < 0\}$$

and

$$\mathbb{P}\left\{\tau(u,T) < \infty\right\}, \tag{4.2}$$

respectively. As in the classical case, only for X being a BM the probability above can be calculated explicitly (see [53]):

$$\mathbb{P}\left\{\exists t \geq 0 : \forall s \in [t, t+T] \ B(s) - cs > u\right\} = \frac{e^{-c^2T/2} - c\sqrt{2\pi T}\Phi(-c\sqrt{T})}{e^{-c^2T/2} + c\sqrt{2\pi T}\Phi(c\sqrt{T})} e^{-2cu}, \quad T \geq 0.$$

Note in passing, that the asymptotics of the Parisian ruin probability for X being a self-similar Gaussian processes is derived in [15]. We refer to [16, 40] for other investigations of some relevant problems.

Motivated by [45] (see also [42]), we study a model where two companies share the net losses in proportions $\delta_1, \delta_2 > 0$, with $\delta_1 + \delta_2 = 1$, and receive the premiums at rates $\rho_1, \rho_2 > 0$, respectively. Further, the risk process of the i-th company is defined by

$$R_i(t) = x_i + \rho_i t - \delta_i B(t), \quad t \geq 0, \ i = 1, 2,$$

where $x_i > 0$ is the initial capital of the i-th company. In this model both claims and net losses are distributed between the companies, which corresponds to the proportional reinsurance dependence of the companies. Define the simultaneous Parisian ruin time for $T \geq 0$ by

$$\tau(u, T) = \inf\{t \geq 0 : \forall s \in [t, t+T] \ R_1(s) < 0, R_2(s) < 0\}.$$

In this paper we study the asymptotics of the simultaneous Parisian ruin probability defined by

$$\mathbb{P}\left\{\tau(u, T) < \infty\right\}, \quad T \geq 0.$$

Since the probability above does not change under a scaling of (R_1, R_2), it equals to

$$\mathbb{P}\left\{\exists t \geq 0 : \forall s \in [t, t+T] \ u_1 + c_1 s - B(s) < 0, u_2 + c_2 s - B(s) < 0\right\}, \quad T \geq 0,$$

where $u_i = x_i/\delta_i$ and $c_i = \rho_i/\delta_i, \ i = 1, 2$. Later on, we derive the asymptotics of the probability above as u_1, u_2 tend to infinity at the constant speed (i.e., u_1/u_2 is constant). Therefore, we let $u_i = q_i u$ be fixed constants with $q_i > 0, \ i = 1, 2$ and deal with asymptotics of

$$\mathcal{P}_T(u) := \mathbb{P}\left\{\exists t \geq 0 : \forall s \in [t, t+T] \ B(s) > q_1 u + c_1 s, B(s) > q_2 u + c_1 s\right\}, \quad T \geq 0$$

as $u \to \infty$. Letting the initial capital tends to infinity is not just a mathematical assumption, but also an economic requirement stated by authorities in all developed countries, see [55]. In many countries a new insurance company is required to retain a sufficient initial capital for the first economic period. It aims to prevents the company from the bankruptcy because of excessive number of small claims and/or several major claims, before the premium income is able to balance the losses and profits.

Observe that $\mathcal{P}_T(u)$ can be rewritten as

$$\mathbb{P}\left\{\exists t \geq 0 : \forall s \in [t, t+T] \ B(s) - \max(c_1 s + q_1 u, c_2 s + q_2 u) > 0\right\}.$$

Thus, the two-dimensional problem may also be considered as a one-dimensional crossing problem over a piece-wise linear barrier. If the two lines $q_1 u + c_1 t$ and $q_2 u + c_2 t$ do not intersect over

$(0,\infty)$, then the problem reduces to the classical one-dimensional BM risk model, which has been discussed in [15, 16] and thus will not be the focus of this paper. In consideration of that, we shall assume that

$$c_1 > c_2, \quad q_2 > q_1. \tag{4.3}$$

Under the assumption above the lines $q_1 u + c_1 t$ and $q_2 u + c_2 t$ intersects at point ut_* with

$$t_* = \frac{q_2 - q_1}{c_1 - c_2} > 0 \tag{4.4}$$

that plays a crucial role in the following. The first usual step when dealing with asymptotics of a ruin probability of a Gaussian process is centralizing the process. In our case it can be achieved by the self-similarity of BM:

$$\begin{aligned}
\mathcal{P}_T(u) &= \mathbb{P}\left\{\exists tu \geq 0 : \inf_{su\in[tu,tu+T]}(B(su) - c_1 su) > q_1 u, \inf_{su\in[tu,tu+T]}(B(su) - c_2 su) > q_2 u\right\} \\
&= \mathbb{P}\left\{\exists t \geq 0 : \inf_{s\in[t,t+T/u]}(B(s) - (c_1 s + q_1)\sqrt{u}) > 0, \inf_{s\in[t,t+T/u]}(B(s) - (c_2 s + q_2)\sqrt{u}) > 0\right\} \\
&= \mathbb{P}\left\{\exists t \geq 0 : \inf_{s\in[t,t+T/u]} \frac{B(s)}{\max(c_1 s + q_1, c_2 s + q_2)} > \sqrt{u}\right\}.
\end{aligned}$$

The next step is analysis of the variance of the centered process. Note that the variance of $\frac{B(t)}{\max(c_1 t+q_1, c_2 t+q_2)}$ can achieve its unique maxima only at one of the following points:

$$t_*, \quad \bar{t}_1 := \frac{q_1}{c_1}, \quad \bar{t}_2 := \frac{q_2}{c_2}.$$

From (4.3) it follows that $\bar{t}_1 < \bar{t}_2$. As we shall see later, the order between $\bar{t}_1, \bar{t}_2$ and t_* determines the asymptotics of $\mathcal{P}_T(u)$. Note in passing, that the variance of $\frac{B(t)}{\max(c_1 t+q_1, c_2 t+q_2)}$ is not smooth around t_* if (4.3) is satisfied. This observation does not allow us to obtain the asymptotics of $\mathcal{P}_T(u)$ straightforwardly by using the results of [15].

Define for any $L \geq 0$ and some function $h : \mathbb{R} \to \mathbb{R}$ constant

$$\mathcal{F}_L^h = \mathbb{E}\left\{\sup_{t\in\mathbb{R}} \inf_{s\in[t,t+L]} e^{\sqrt{2}B(s)-|s|+h(s)}\right\}$$

when the expectation above is finite. For the properties of $\mathcal{F}_L^h$ we refer to [15, 16]. Notice that $\mathcal{F}_0^h$ coincides with the Piterbarg constant introduced in [45]. For the properties of related Piterbarg constants see, e.g., [14, 59].

The next theorem derives the asymptotics of $\mathcal{P}_T(u)$ as $u \to \infty$:

Theorem 4.1.1 *Assume that* (4.3) *holds.*
1)If $t_* \notin (\bar{t}_1, \bar{t}_2)$, then as $u \to \infty$

$$\mathcal{P}_T(u) \sim \left(\frac{1}{2}\right)^{\mathbb{I}(t_*=\bar{t}_i)} \frac{e^{-c_i^2 T/2} - c_i\sqrt{2\pi T}\Phi(-c_i\sqrt{T})}{e^{-c_i^2 T/2} + c_i\sqrt{2\pi T}\Phi(c_i\sqrt{T})} e^{-2c_i q_i u}, \tag{4.5}$$

where $i = 1$ *if* $t_* \le \bar{t}_1$ *and* $i = 2$ *if* $t_* \ge \bar{t}_2$.

2)If $t_* \in (\bar{t}_1, \bar{t}_2)$, *then as* $u \to \infty$

$$\mathcal{P}_T(u) \sim \mathcal{F}^d_{T'} \overline{\Phi}\left((c_1 q_2 - c_2 q_1)\sqrt{\frac{q_2 - q_1}{c_1 - c_2}}\sqrt{u}\right),$$

where $\mathcal{F}^d_{T'} \in (0, \infty)$ *and*

$$T' = T\frac{(c_1q_2 - q_1c_2)^2}{2(c_1 - c_2)^2}, \ d(s) = s\frac{c_1q_2 + c_2q_1 - 2c_2q_2}{c_1q_2 - q_1c_2}\mathbb{I}(s < 0) + s\frac{2c_1q_1 - c_1q_2 - q_1c_2}{c_1q_2 - q_1c_2}\mathbb{I}(s \ge 0). \quad (4.6)$$

4.2 Main Results

In classical risk theory, the surplus process of an insurance company is modeled by the compound Poisson or the general compound renewal risk process, see, e.g., [31]. The calculation of the ruin probabilities is of a particular interest for both theoretical and applied domains. To avoid the technical issues and allow for dependence between claim sizes, these models are often approximated by the risk model (4.1), driven by B_H a standard fBM. Since the time spent by the surplus process below zero may depend on u, in the following we allow $T =: T_u$ in (4.2) to depend on u. As mentioned in [16], for the one-dimensional Parisian ruin probability we need to control the growth of T_u as $u \to \infty$. Namely, we impose the following condition:

$$\lim_{u\to\infty} T_u u^{1/H-2} = T \in [0, \infty), \ H \in (0, 1). \quad (4.7)$$

Note that if $H > 1/2$, then T_u may grow to infinity, while if $H < 1/2$, then T_u approaches zero as u tends to infinity. As we see later in Proposition 4.2.2, the condition above is necessary and the result does not hold without it. As for BM, by the self-similarity of fBM we obtain

$$\mathcal{P}_{T_u}(u) = \mathbb{P}\left\{\exists t \ge 0 : \inf_{s\in[t,t+T_u/u]} \frac{B_H(s)}{\max(c_1 s + q_1, c_2 s + q_2)} > u^{1-H}\right\}.$$

The variance of $\frac{B_H(t)}{\max(c_1 t+q_1, c_2 t+q_2)}$ can achieve its unique maxima only at one of the following points:

$$t_*, \quad t_1 := \frac{Hq_1}{(1-H)c_1}, \quad t_2 := \frac{Hq_2}{(1-H)c_2},. \quad (4.8)$$

From (4.3) it follows that $t_1 < t_2$. Again, the order between t_1, t_2 and t_* determines the asymptotics of $\mathcal{P}_{T_u}(u)$. Define for $H \in (0, 1)$ and $T \ge 0$ Parisian Pickands constant by

$$\mathcal{F}_{2H}(T) = \lim_{S\to\infty} \frac{1}{S}\mathbb{E}\left\{\sup_{t\in[0,S]} \inf_{s\in[0,T]} e^{\sqrt{2}B_H(t+s)-(t+s)^{2H}}\right\}.$$

It is shown in [15] that $\mathcal{F}_{2H}(T)$ is a finite and positive constant. Note that $\mathcal{F}_{2H}(0) = \mathcal{H}_{2H}$. Define for $i = 1, 2$ constants

$$\mathbb{D}_H = \frac{c_1 t_* + q_1}{t_*^H}, \ K_H = \frac{2^{\frac{1}{2}-\frac{1}{2H}}\sqrt{\pi}}{\sqrt{H(1-H)}}, \ \mathbb{C}_H^{(i)} = \frac{c_i^H q_i^{1-H}}{H^H(1-H)^{1-H}}, \ D_i = \frac{c_i^2(1-H)^{2-\frac{1}{H}}}{2^{\frac{1}{2H}}H^2}. \quad (4.9)$$

Now we are ready to give the asymptotics of $\mathcal{P}_{T_u}(u)$:

Theorem 4.2.1 *Assume that* (4.3) *holds and* T_u *satisfies* (4.7).
1)If $t_* \notin (t_1, t_2)$, *then as* $u \to \infty$

$$\mathcal{P}_{T_u}(u) \sim \left(\frac{1}{2}\right)^{\mathbb{I}(t_*=t_i)} \times \begin{cases} \frac{e^{-c_i^2 T/2} - c_i\sqrt{2\pi T}\Phi(-c_i\sqrt{T})}{e^{-c_i^2 T/2} + c_i\sqrt{2\pi T}\Phi(c_i\sqrt{T})} e^{-2c_i q_i u}, & H = 1/2, \\ K_H \mathcal{F}_{2H}(TD_i)(\mathbb{C}_H^{(i)} u^{1-H})^{\frac{1}{H}-1}\overline{\Phi}(\mathbb{C}_H^{(i)} u^{1-H}), & H \neq 1/2, \end{cases} \quad (4.10)$$

where $i = 1$ *if* $t_* \le t_1$ *and* $i = 2$ *if* $t_* \ge t_2$.
2)If $t_* \in (t_1, t_2)$ *and* $\lim_{u\to\infty} T_u u^{2-1/H} = 0$ *for* $H > 1/2$, *then*

$$\mathcal{P}_{T_u}(u) \sim \overline{\Phi}(\mathbb{D}_H u^{1-H}) \times \begin{cases} 1, & H > 1/2, \\ \mathcal{F}_{T'}^d, & H = 1/2, \\ \mathcal{F}_{2H}(\overline{D}T) A u^{(1-H)(1/H-2)}, & H < 1/2, \end{cases} \quad (4.11)$$

where $\mathcal{F}_{T'}^d \in (0, \infty)$ *with* T' *and* d *defined in* (4.6) *and*

$$A = \left(|H(c_1 t_* + q_1) - c_1 t_*|^{-1} + |H(c_2 t_* + q_2) - c_2 t_*|^{-1}\right) \frac{t_*^H \mathbb{D}_H^{\frac{1}{H}-1}}{2^{\frac{1}{2H}}}, \qquad \overline{D} = \frac{(c_1 t_* + q_1)^{\frac{1}{H}}}{2^{\frac{1}{2H}} t_*^2}. \quad (4.12)$$

The theorem above generalizes Theorem 4.1.1 and Theorem 3.1 in [45]. Note that if $T = 0$, then the result above reduces to Theorem 3.1 in [45].

As indicated in [16], it seems extremely difficult to find the exact asymptotics of the one-dimensional Parisian ruin probability if (4.7) does not hold. The initial reason is that the ruin happens over 'too long interval'. To illustrate difficulties arising in approximation of $\mathcal{P}_{T_u}(u)$ in this setup we consider a 'simple' scenario: let $T_u = T > 0$ and $H < 1/2$. In this case we have

Proposition 4.2.2 *If* $H < 1/2$, $T_u = T > 0$ *and* $t_* \in (t_1, t_2)$, *then*

$$\begin{aligned} \bar{C}\overline{\Phi}(\mathbb{D}_H u^{1-H}) e^{-C_{1,\alpha} u^{2-4H} - C_{2,\alpha} u^{2(1-3H)}} &\le \mathcal{P}_{T_u}(u) \\ &\le (2 + o(1))\overline{\Phi}(\mathbb{D}_H u^{1-H})\overline{\Phi}\left(u^{1-2H}\frac{T^H \mathbb{D}_H}{2t_*^H}\right), \end{aligned} \quad (4.13)$$

where $\bar{C} \in (0, 1)$ *is a fixed constant that does not depend on* u *and*

$$\alpha = \frac{T^{2H}}{2t_*^{2H}}, \qquad C_{i,\alpha} = \frac{\alpha^i}{i}\mathbb{D}_H^2, \quad i = 1, 2. \quad (4.14)$$

Note that the proposition above expands Theorem 3.2 in [16] for fBM case.

4.3 Simulation of Piterbarg & Pickands constants

In this section we give algorithms for simulations of Pickands and Piterbarg type constants appearing in Theorems 4.1.1 and 4.2.1 and study their properties relevant for simulations. Since the classical Pickands constant $\mathcal{H}_{2H}$ has been investigated in several contributions (see, e.g., [30]), later on we deal with $\mathcal{F}_L^h$ and $\mathcal{F}_{2H}(L)$.

Simulation of Piterbarg constant. In this subsection we always assume that

$$L \geq 0 \quad \text{and} \quad h(s) = bs\, \mathbb{I}(s<0) - as\, \mathbb{I}(s \geq 0), \quad s \in \mathbb{R},\ a,b>0.$$

To simulate $\mathcal{F}_L^h$ we use the approximation

$$\mathcal{F}_L^h \approx \mathbb{E}\left\{ \sup_{t\in[-M,M]_\tau} \inf_{s\in[t,t+L]_\tau} e^{\sqrt{2}B(s)-|s|+h(s)} \right\},$$

where M is sufficiently large and τ is sufficiently small. The approximation above has several errors: truncation error (i.e., choice of M), discretization error (i.e., choice of τ) and simulation error. It seems difficult to give a precise estimate of the discretization error, we refer to [30] for discussion of such problems. To take an appropriate M and give an upper bound of the truncation error we derive few lemmas. The first lemma provides us bounds for $\mathcal{F}_L^h$:

Lemma 4.3.1 *It holds that*

$$2e^{-L\min(a,b)}\overline{\Phi}(\sqrt{2L}) \leq \mathcal{F}_L^h \leq 1 + \frac{1}{a} + \frac{1}{b} - \frac{1}{a+b+1}.$$

Note that if $L=0$, then the upper bound becomes an equation (see the proof), and thus we obtain as a product the explicit expression for the two-sided Piterbarg constant introduced in [45]:

$$\mathbb{E}\left\{ \sup_{t\in\mathbb{R}} e^{\sqrt{2}B(t)-|t|(1+a\mathbb{I}(t>0)+b\mathbb{I}(t<0))} \right\} = 1 + \frac{1}{a} + \frac{1}{b} - \frac{1}{a+b+1}.$$

In the next lemma we focus on the truncation error:

Lemma 4.3.2 *For $M \geq 0$ it holds that*

$$\mathbb{E}\left\{ \sup_{t\in\mathbb{R}\setminus[-M,M]} \inf_{s\in[t,t+L]} e^{\sqrt{2}B(s)-|s|+h(s)} \right\} \leq e^{-aM}\left(1+\frac{1}{a}\right) + e^{-bM}\left(1+\frac{1}{b}\right). \tag{4.15}$$

Now we are ready to find an appropriate M. We have by Lemma 4.3.2 that

$$\begin{aligned} \left| \mathcal{F}_L^h - \mathbb{E}\left\{ \sup_{t\in[-M,M]} \inf_{s\in[t,t+L]} e^{\sqrt{2}B(s)-|s|+h(s)} \right\} \right| &\leq \mathbb{E}\left\{ \sup_{t\in\mathbb{R}\setminus[-M,M]} \inf_{s\in[t,t+L]} e^{\sqrt{2}B(s)-|s|+h(s)} \right\} \\ &\leq 2\left(1+\frac{1}{\min(a,b)}\right) e^{-M\min(a,b)} \end{aligned}$$

and on the other hand by Lemma 4.3.1

$$\mathcal{F}_L^h \geq 2e^{-L\min(a,b)}\overline{\Phi}(\sqrt{2L}),$$

hence to obtain a good accuracy we need that

$$\left(1+\frac{1}{\min(a,b)}\right)e^{-\min(a,b)M} << e^{-L\min(a,b)}\overline{\Phi}(\sqrt{2L}).$$

Assume for simulations that $\min(a,b)\geq 1$; otherwise special case $\min(a,b) << 1$ requires a choice of a large M implying very high level of computation capacity.
For simulations, we take $M=\frac{7+L(3+\min(a,b))}{\min(a,b)}$ providing us truncation error smaller than $3\cdot 10^{-3}$; we do not need to have better accuracy since there are also the errors of discretization and simulation. Since we cannot estimate the errors of discretization and simulation, we just take a 'small' τ and a 'big' number of simulation n. The above observations give us the following algorithm:

1) take $M=\frac{7+L(3+\min(a,b))}{\min(a,b)}$, $\tau=0.005$ and $n=10^4$;
2) simulate n times $B(t),\ t\in[-M,M]_\tau$, i.e., obtain $B_i(t),\ 1\leq i\leq n$;
3) compute

$$\widehat{\mathcal{F}_L^h} := \frac{1}{n}\sum_{i=1}^{n}\sup_{t\in[-M,M]_\tau}\inf_{s\in[t,t+L]_\tau} e^{\sqrt{2}B_i(s)-|s|+h(s)}.$$

Simulation of Picaknds constant. It seems difficult to simulate $\mathcal{F}_{2H}(L)$ relying straightforwardly on its definition. As follows from approach in [12, 30] for any $\eta>0$ with $W(t)=B_{2H}(t)-|t|^{2H}$

$$\mathcal{F}_{2H}(L)=\mathbb{E}\left\{\frac{\sup_{t\in\mathbb{R}}\inf_{s\in[t,t+L]} e^{W(t)}}{\eta\sum_{k\in\mathbb{Z}} e^{W(k\eta)}}\right\}.$$

The merit of the representation above is that there is no limit as is in the original definition and thus it is much easier to simulate $\mathcal{F}_{2H}(L)$ by the Monte-Carlo method. The second benefit is that there is a sum in the denominator, that can be simulated easily with a good accuracy. The only drawback is that the sup inf in the nominator is taken on the whole real line. Thus, we approximate $\mathcal{F}_{2H}(L)$ by discrete analog of the formula above:

$$\mathcal{F}_{2H}(L)\approx\mathbb{E}\left\{\frac{\sup_{t\in[-M,M]_\tau}\inf_{s\in[t,t+L]_\tau} e^{W(t)}}{\eta\sum_{k\in[-M,M]_\eta} e^{W(\eta k)}}\right\},$$

where big M and small τ,η are appropriately chosen positive numbers. In the following lemma we give a lower bound for $\mathcal{F}_{2H}(L)$.

Lemma 4.3.3 *For any $L \geq 0$ and $H \in (0,1)$ it holds that*

$$\mathcal{F}_{2H}(L) \geq Ce^{-L^{2H}} \sup_{n \geq 0.1} \left(e^{-\sqrt{2}nL^H} \mathbb{P}\left\{ \sup_{s\in[0,1]} B_H(s) < n \right\} \right).$$

Taking $m = 1/\sqrt{2}$ in the sup above we obtain a useful for large L estimate

$$\mathcal{F}_{2H}(L) \geq Ce^{-L^{2H}-L^H}, \quad L > 0$$

where C is some positive number that depends only on H. The following lemma provides us an upper bound for the truncation error:

Lemma 4.3.4 *For some fixed constant $c' > 0$ and $M, L > 0$ it holds that*

$$\left| \mathcal{F}_{2H}(L) - \mathbb{E}\left\{ \frac{\sup\limits_{t\in[-M,M]} \inf\limits_{s\in[t,t+L]} e^{W(t)}}{\int\limits_{[-M,M]} e^{W(t)} dt} \right\} \right| \leq e^{-c'M^{2H}}.$$

Based on 2 lemmas above we propose the following algorithm for simulation of $\mathcal{F}_{2H}(L)$:

1) Take $M = \max(10L, 5),\ \tau = \eta = 0.005$ and $n = 10^4$;
2) simulate n times $B_H(t),\ t \in [-M, M]_\tau$, i.e., obtain $B_H^{(i)}(t),\ 1 \leq i \leq n$;
3) calculate

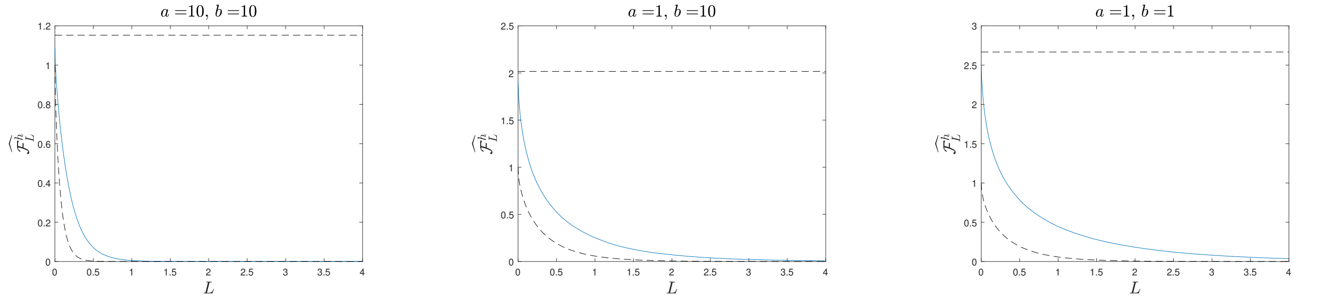
$$\widehat{\mathcal{F}_{2H}(L)} := \frac{1}{n}\sum_{i=1}^{n} \frac{\sup\limits_{t\in[-M,M]_\tau} \inf\limits_{s\in[t,t+L]_\tau} e^{\sqrt{2}B_H^{(i)}(s)-|s|^{2H}}}{\eta \sum\limits_{k\in[-M,M]_\eta} e^{\sqrt{2}B_H^{(i)}(k\eta)-|k\eta|^{2H}}}.$$

We give the proofs of all Lemmas above at the end of Section Proofs.

4.4 Approximate Values of Pickands & Piterbarg Constants

In this section we apply both algorithms introduced above and obtain approximate numerical values for some particular choices of parameters. To implement our approach, we use MATLAB software.

Piterbarg constant. We simulate several graphs of $\widehat{\mathcal{F}_L^h}$ for different choices of a and b.



(a) (b) (c)

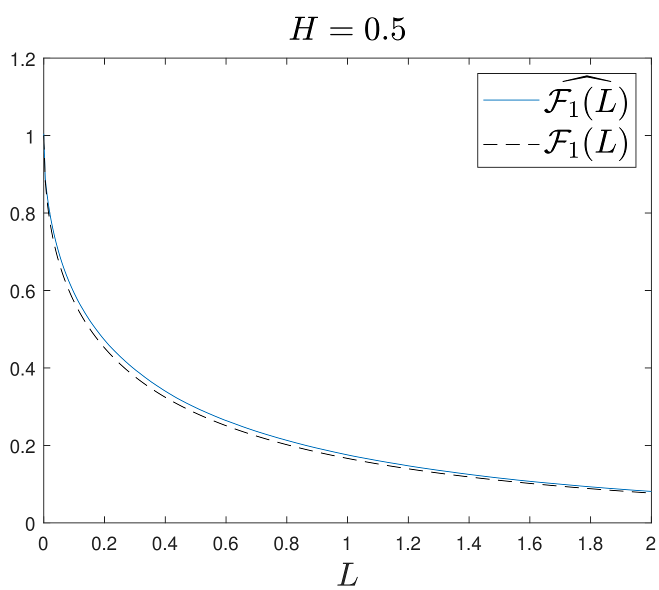
On each graph above the blue line is simulated value and the dashed lines are theoretical bounds given in Lemma 4.3.1. We observe that the simulated values are between the theoretical bounds, $\widehat{\mathcal{F}^h_L}$ is decreasing function and $\widehat{\mathcal{F}^h_L}$ tends to $1 + \frac{1}{a} + \frac{1}{b} - \frac{1}{a+b+1}$ as $L \to 0$.

Pickands constant. We simulate several graphs of $\widehat{\mathcal{F}_{2H}(L)}$ for different choices of H. We consider BM case $H = 0.5$, short-range dependence case $H < 0.5$ and the long-range dependence case $H > 0.5$. To simulate fBM we use Choleski method, (see, e.g., [29]).

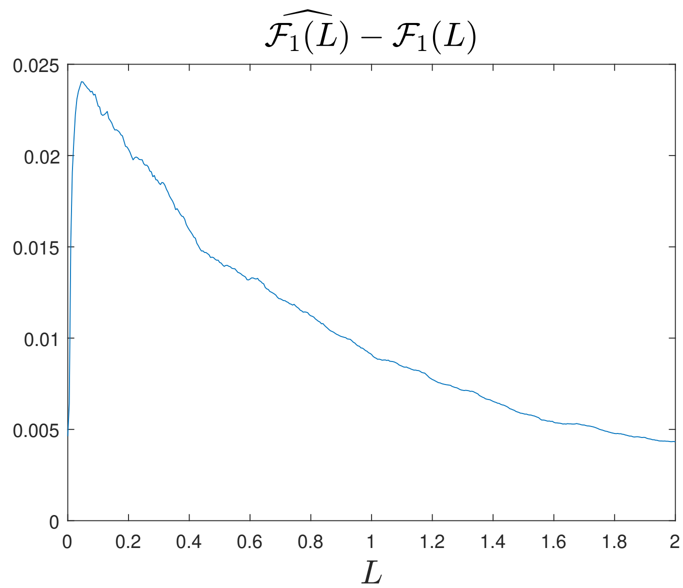
BM case. Here we plot $\widehat{\mathcal{F}_1(L)}$ and the explicit theoretical value given by

$$\mathcal{F}_1(L) = \frac{e^{-L/4} - \sqrt{\pi L}\Phi(-\sqrt{L/2})}{e^{-L/4} + \sqrt{\pi L}\Phi(\sqrt{L/2})}, \qquad L \geq 0,$$

(see, e.g., [53]). In the graph below the blue line corresponds to the simulated value and the dashed line represents the exact theoretical value.

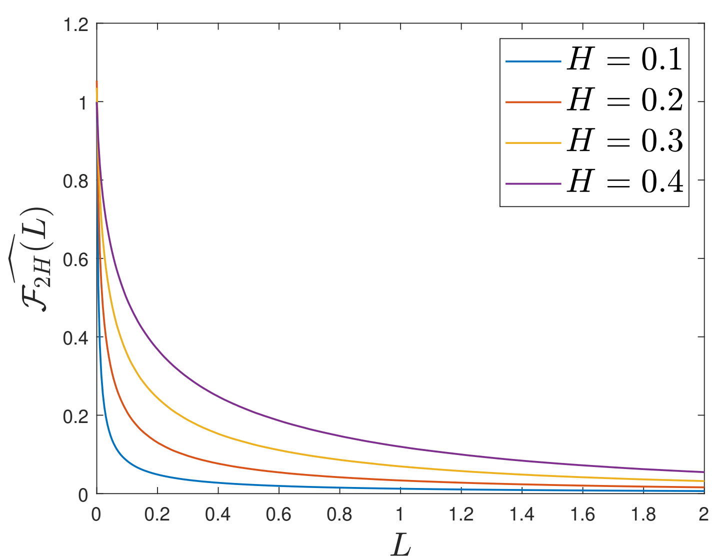


Observe the according to the picture $\widehat{\mathcal{F}_1(L)}$ is decreasing and does not drastically differ from $\mathcal{F}_1(L)$. We also point out that the theoretical value is smaller than the simulated one, that goes in a row with intuition that a discretization increases the value of the Parisian Pickands constant; we plot the difference between $\widehat{F_1(L)}$ and $\mathcal{F}_1(L)$:



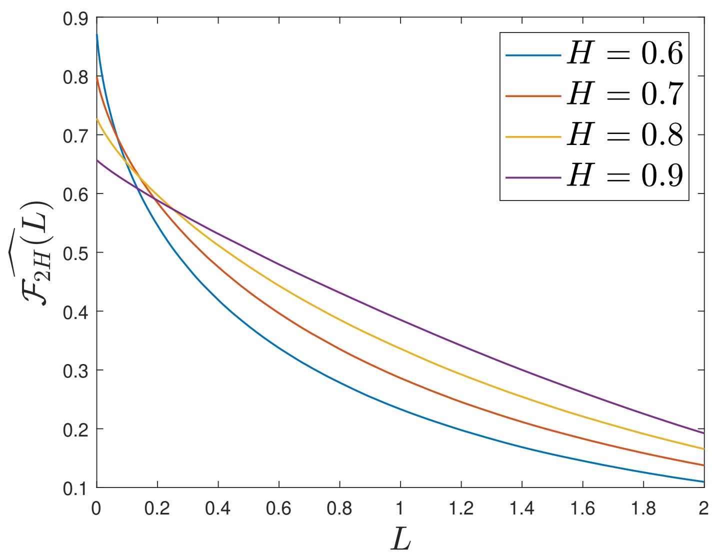
As seen from the plot above, our simulations do not contradict Conjecture 1 in [30], i.e., the error of the discretization may be of order $\sqrt{\tau}$ for small $\tau > 0$.

Short-range dependence case. Here we focus on the short-range dependent case. We consider two particular values of H, namely 0.1 and 0.3, and plot $\widehat{\mathcal{F}_{2H}(L)}$ for these values. The red line corresponds to case $H = 0.3$ while the blue line represents case $H = 0.1$.



Observe that $\widehat{\mathcal{F}_{2H}(L)}$ is a strictly decreasing function of L for both values of H.

Long-range dependence case. We take $H = 0.7$ and $H = 0.9$, and plot $\widehat{\mathcal{F}_{2H}(L)}$ for these values. The red and blue lines correspond to cases $H = 0.9$ and $H = 0.7$, respectively.



Observe that $\widehat{\mathcal{F}_{2H}(L)}$ is a strictly decreasing function of L for both values of H.

4.5 Proofs

Recall that K_H, D_1 and $\mathbb{C}_H^{(1)}$ are defined in (4.9). The following result immediately follows from [15, 53]:

Proposition 4.5.1 *Assume that T_u satisfies (4.7). Then as $u \to \infty$*

$$\mathbb{P}\left\{\sup_{t\geq 0}\inf_{[t,t+T_u]}(B_H(t) - c_1 t) > q_1 u\right\} \sim \begin{cases} \frac{e^{-c_1^2 T/2} - c_1\sqrt{2\pi T}\Phi(-c_1\sqrt{T})}{e^{-c_1^2 T/2} + c_1\sqrt{2\pi T}\Phi(c_1\sqrt{T})} e^{-2c_1 q_1 u}, & H = 1/2, \\ K_H \mathcal{F}_{2H}(TD_1)(\mathbb{C}_H^{(1)} u^{1-H})^{\frac{1}{H}-1}\overline{\Phi}(\mathbb{C}_H^{(1)} u^{1-H}), & H \neq 1/2. \end{cases}$$

Now we are ready to present our proofs.

Proof of Theorems 4.1.1 and 4.2.1. Since Theorem 4.1.1 follows immediately from Theorem 4.2.1, thus we prove Theorem 4.2.1 only.

Case (1). Assume that $t_* < t_1$. Let

$$\psi_i(T_u, u) = \mathbb{P}\left\{\sup_{t\geq 0}\inf_{[t,t+T_u]}(B_H(t) - c_i t) > q_i u\right\}, \quad i = 1, 2.$$

For $0 < \varepsilon < t_1 - t_*$ by the self-similarity of fBM we have

$$\psi_1(T_u, u) \geq \mathcal{P}_{T_u}(u) \geq \mathbb{P}\left\{\exists t \in (t_1 - \varepsilon, t_1 + \varepsilon) : \inf_{s\in[t,t+T_u/u]} V_1(t) > u^{1-H}, \inf_{s\in[t,t+T_u/u]} V_2(t) > u^{1-H}\right\}$$

$$= \mathbb{P}\left\{ \exists t \in (t_1 - \varepsilon, t_1 + \varepsilon) : \inf_{s \in [t, t+T_u/u]} V_1(t) > u^{1-H} \right\},$$

where

$$V_i(t) = \frac{B_H(t)}{c_i t + q_i}, \quad i = 1, 2.$$

We have by Borel-TIS inequality, see [59] (details are in the Appendix)

$$\psi_1(T_u, u) \sim \mathbb{P}\left\{ \exists t \in (t_1 - \varepsilon, t_1 + \varepsilon) : \inf_{s \in [t, t+T_u/u]} V_1(t) > u^{1-H} \right\}, \quad u \to \infty \tag{4.16}$$

implying $\mathcal{P}_{T_u}(u) \sim \psi_1(T_u, u)$ as $u \to \infty$. The asymptotics of $\psi_1(T_u, u)$ is given in Proposition 4.5.1, thus the claim follows.

Assume that $t_* = t_1$. We have

$$\begin{aligned}
& \mathbb{P}\left\{ \exists t \in [t_1, \infty) : \inf_{s \in [t, t+\frac{T_u}{u}]} V_1(s) > u^{1-H} \right\} \\
\leq\ & \mathcal{P}_{T_u}(u) \\
\leq\ & \mathbb{P}\left\{ \exists t \in [t_1, \infty) : \inf_{s \in [t, t+\frac{T_u}{u}]} V_1(s) > u^{1-H} \right\} + \mathbb{P}\left\{ \exists t \in [0, t_1] : V_2(t) > u^{1-H} \right\}.
\end{aligned}$$

From the proof of Theorem 3.1, case (4) in [45] it follows that the second term in the last line above is negligible comparing with the final asymptotics of $\mathcal{P}_{T_u}(u)$ given in (4.10), hence

$$\mathcal{P}_{T_u}(u) \sim \mathbb{P}\left\{ \exists t \in [t_1, \infty) : \inf_{s \in [t, t+\frac{T_u}{u}]} V_1(s) > u^{1-H} \right\}, \quad u \to \infty.$$

By the same arguments as in (4.16) it follows that for $\varepsilon > 0$ the last probability above is equivalent with

$$\mathbb{P}\left\{ \exists t \in [t_1, t_1 + \varepsilon] : \inf_{s \in [t, t+T_u/u]} V_1(s) > u^{1-H} \right\}, \quad u \to \infty.$$

Since $\mathcal{F}_1(T) = \frac{e^{-T/4} - \sqrt{\pi T}\Phi(-\sqrt{T/2})}{e^{-T/4} + \sqrt{\pi T}\Phi(\sqrt{T/2})}$, $T \geq 0$ (see [15]) applying Theorem 3.3 in [16] with parameters in the notation therein

$$\widetilde{\sigma} = \frac{t_1^H}{c_1 t + q_1}, \quad \beta_1 = 2, \quad D = \frac{1}{2t_1^{2H}}, \quad \alpha = 2H, \quad A = \frac{q_1^{H-3} H^{H-1} (1-H)^{4-H}}{2c_1^{H-2}}$$

we obtain as $u \to \infty$

$$\mathbb{P}\left\{ \exists t \in [t_1, t_1 + \varepsilon] : \inf_{s \in [t, t+T_u/u]} V_1(s) > u^{1-H} \right\} \sim K_H \mathcal{F}_{2H}(TD_1) (\mathbb{C}_H^{(1)} u^{1-H})^{\frac{1}{H}-1} \overline{\Phi}(\mathbb{C}_H^{(1)} u^{1-H})$$

and the claim is established. Case $t_* \geq t_2$ follows by the same arguments.

Case (2). Define

$$Z_H(t) = \frac{B_H(t)}{\max(c_1 t + q_1, c_2 t + q_2)}, \quad t \geq 0. \tag{4.17}$$

Similarly to the proof of (4.16) we have by Borell-TIS inequality for $\varepsilon > 0$ as $u \to \infty$

$$\begin{aligned}\mathcal{P}_{T_u}(u) &= \mathbb{P}\left\{\exists t \geq 0 : \inf_{s\in[t,t+T_u/u]} Z_H(t) > u^{1-H}\right\} \\ &\sim \mathbb{P}\left\{\exists t \in (t_* - \varepsilon, t_* + \varepsilon) : \inf_{s\in[t,t+T_u/u]} Z_H(t) > u^{1-H}\right\} \\ &=: p(u).\end{aligned}$$

Assume that $H < 1/2$. By "the double-sum" approach, see the proofs of Theorem 3.1, Case (3) $H < 1/2$ in [45] and Theorem 3.3. case i) in [16] we have as $u \to \infty$

$$p(u) \sim \mathbb{P}\left\{\exists t \in (t_*, t_*+\varepsilon) \colon \inf_{s\in[t,t+\frac{T_u}{u}]} V_1(t) > u^{1-H}\right\} + \mathbb{P}\left\{\exists t \in (t_*-\varepsilon, t_*) \colon \inf_{s\in[t,t+\frac{T_u}{u}]} V_2(t) > u^{1-H}\right\}. \tag{4.18}$$

To compute the asymptotics of each probability in the line above we apply Theorem 3.3 in [16]. For the first probability we have in the notation therein

$$\widetilde{\sigma} = \frac{t_*^H}{c_1 t_* + q_1}, \quad \beta_1 = 1, \quad D = \frac{1}{2t_*^{2H}}, \quad \alpha = 2H < 1, \quad A = \frac{t_*^{H-1}|H(c_1 t_* + q_1) - c_1 t_*|}{(c_1 t_* + q_1)^2}$$

implying as $u \to \infty$

$$\mathbb{P}\left\{\exists t \in (t_*, t_* + \varepsilon) : \inf_{s\in[t,t+\frac{T_u}{u}]} V_1(t) > u^{1-H}\right\} \sim \mathcal{F}_{2H}\left(\frac{(c_1 t_* + q_1)^{\frac{1}{H}}}{2^{\frac{1}{2H}} t_*^2} T\right) \frac{t_*^H \mathbb{D}_H^{\frac{1}{H}-1} u^{(1-H)(\frac{1}{H}-2)}}{|H(c_1 t_* + q_1) - c_1 t_*| 2^{\frac{1}{2H}}} \overline{\Phi}(\mathbb{D}_H u^{1-H}).$$

Applying again Theorem 3.3 in [16] we obtain the asymptotics of the second summand and the claim follows by (4.18).

Assume that $H = 1/2$. In order to compute the asymptotics of $p(u)$ applying Theorem 3.3 in [16] with parameters

$$\alpha = \beta_1 = \beta_2 = 1,\ A_\pm = \frac{q_1 - c_1 t_*}{q_1 + c_1 t_*},\ A = \frac{q_2 - c_2 t_*}{q_2 + c_2 t_*},\ \widetilde{\sigma} = \frac{\sqrt{t_*}}{c_1 t_* + q_1},\ D = \frac{1}{2t_*}$$

we obtain ($d(\cdot)$ and T' are defined in (4.6))

$$p(u) \sim \mathcal{F}^d_{T'} \overline{\Phi}(\mathbb{D}_{1/2}\sqrt{u}), \quad u \to \infty.$$

Assume that $H > 1/2$. Applying Theorem 3.3 in [16] with parameters $\alpha = 2H > 1 = \beta_1 = \beta_2$ we complete the proof since

$$p(u) \sim \overline{\Phi}(\mathbb{D}_H u^{1-H}), \quad u \to \infty.$$

Proof of Proposition 4.2.2.

Lower bound. Take $\kappa = 1 - 3H$ and recall that $\alpha = \frac{T^{2H}}{2t_*^{2H}}$. We have

$$\begin{aligned}\mathcal{P}_T(u) &\geq \mathbb{P}\left\{\forall t \in [t_* - T/u, t_*] V_2(t) > u^{1-H} \text{ and } V_2(t_*) > u^{1-H} + \alpha u^{\kappa}\right\} \\ &\geq \bar{C}\mathbb{P}\left\{V_2(t_*) > u^{1-H} + \alpha u^{\kappa}\right\} \qquad (4.19) \\ &\sim \bar{C}\overline{\Phi}(\mathbb{D}_H u^{1-H}) e^{-C_{1,\alpha} u^{1-H+\kappa} - C_{2,\alpha} u^{2\kappa}}, \quad u \to \infty,\end{aligned}$$

where $\bar{C}$ is a fixed positive constant that does not depend on u and $C_{1,\alpha}$ and $C_{2,\alpha}$ are defined in (4.14). Thus, to prove the lower bound we need to show (4.19). Note that (4.19) is the same as

$$\mathbb{P}\left\{\exists t \in [t_* - T/u, t_*] : V_2(t) \leq u^{1-H} \text{ and } V_2(t_*) > u^{1-H} + \alpha u^{\kappa}\right\} \leq \varepsilon' \mathbb{P}\left\{V_2(t_*) > u^{1-H} + \alpha u^{\kappa}\right\},$$

with some $\varepsilon' > 0$. The last line above is equivalent with

$$\begin{aligned}&\mathbb{P}\left\{\exists t \in [ut_* - T, ut_*] : B_H(t) - c_2 t \leq q_2 u \text{ and } B_H(ut_*) - c_2 u t_* > q_2 u + b\alpha u^{\kappa+H}\right\} \\ \leq\ & \varepsilon' \mathbb{P}\left\{B_H(ut_*) - c_2 u t_* > q_2 u + b\alpha u^{\kappa+H}\right\},\end{aligned}$$

where $b = c_2 t_* + q_2$. We have with $\varphi_u(x)$ the density of $B_H(ut_*)$ that the left part of the inequality above does not exceed

$$\begin{aligned}&\mathbb{P}\left\{\exists t \in [ut_* - T, ut_*] : B_H(ut_*) - B_H(t) > b\alpha u^{\kappa+H} \text{ and } B_H(ut_*) > bu\right\} \\ =\ & \int_{bu}^{\infty} \mathbb{P}\left\{\exists t \in [ut_* - T, ut_*] : x - B_H(t) > b\alpha u^{\kappa+H} | B_H(ut_*) = x\right\} \varphi_u(x) dx \\ \leq\ & \int_{bu}^{bu+1} \mathbb{P}\left\{\exists t \in [ut_* - T, ut_*] : x - B_H(t) > b\alpha u^{\kappa+H} | B_H(ut_*) = x\right\} \varphi_u(x) dx + \int_{bu+1}^{\infty} \varphi_u(x) dx.\end{aligned}$$

We also have that

$$\mathbb{P}\left\{B_H(ut_*) - c_2 u t_* > q_2 u\right\} = \int_{bu}^{\infty} \varphi_u(x) dx \geq \int_{bu}^{bu+1} \varphi_u(x) dx.$$

By (2.7) we have that $\int_{bu+1}^{\infty} \varphi_u(x) dx$ is negligible comparing with the last integral above. Thus, to prove (4.19) we need to show

$$\int_{bu}^{bu+1} \mathbb{P}\left\{\exists t \in [ut_* - T, ut_*] : x - B_H(t) > b\alpha u^{\kappa+H} | B_H(ut_*) = x\right\} \varphi_u(x) dx \leq \varepsilon' \int_{bu}^{bu+1} \varphi_u(x) dx, \qquad u \to \infty,$$

that follows from the inequality

$$\sup_{x \in [bu, bu+1]} \mathbb{P}\left\{\exists t \in [ut_* - T, ut_*] : x - B_H(t) > b\alpha u^{\kappa+H} | B_H(ut_*) = x\right\} \leq \varepsilon'', \quad u \to \infty, \qquad (4.20)$$

where $\varepsilon'' > 0$ is some number. We show the line above in the Appendix, thus the lower bound holds.

Upper bound. We have by the self-similarity of fBM

$$\mathcal{P}_T(u) = \mathbb{P}\left\{\sup_{t\ge 0} \inf_{s\in[t,t+T/u]} Z_H(s) > u^{1-H}\right\},$$

where Z_H is defined in (4.17). For $\varepsilon > 0$ by Borell-TIS inequality with $I(t_*) = (-u^{-\varepsilon}+t_*, t_*+u^{-\varepsilon})$ we have

$$\mathbb{P}\left\{\sup_{t\notin I(t_*)} \inf_{s\in[t,t+T/u]} Z_H(s) > u^{1-H}\right\} \le \mathbb{P}\left\{\sup_{t\notin I(t_*)} Z_H(t) > u^{1-H}\right\} \le \overline{\Phi}\left(\mathbb{D}_H u^{1-H}\right) e^{-u^{2-2H-2\varepsilon}}, \quad u\to\infty,$$

that is asymptotically smaller than the lower bound in (4.13) for sufficiently small ε. Thus, we shall focus on estimation of

$$q(u) := \mathbb{P}\left\{\sup_{t\in I(t_*)} \inf_{s\in[t,t+T/u]} Z_H(s) > u^{1-H}\right\}.$$

Denote $z^2(t) = \mathrm{Var}\{Z_H(t)\}$ and $\overline{Z}_H(t) = Z_H(t)/z(t)$. By Lemma 2.3 in [56] we have with $M = \max(z(t), z(t+T/u))$ (note, $1/M \ge \mathbb{D}_H$)

$$\begin{aligned}
q(u) &\le \mathbb{P}\left\{\exists t\in I(t_*): Z_H(t) > u^{1-H}, Z_H(t+T/u) > u^{1-H}\right\}\\
&= \mathbb{P}\left\{\exists t\in I(t_*): \overline{Z}_H(t) > u^{1-H}/z(t), \overline{Z}_H(t+T/u) > u^{1-H}/z(t+T/u)\right\}\\
&\le \mathbb{P}\left\{\exists t\in I(t_*): \overline{Z}_H(t) > u^{1-H}/M, \overline{Z}_H(t+T/u) > u^{1-H}/M\right\}\\
&\le 2(1+o(1))\overline{\Phi}\left(\frac{u^{1-H}}{M}\right)\overline{\Phi}\left(\frac{u^{1-H}}{M}\sqrt{\frac{1-r(t,t+T/u)}{1+r(t,t+T/u)}}\right)\\
&\le 2(1+o(1))\overline{\Phi}\left(\frac{u^{1-H}}{M}\right)\overline{\Phi}\left(\mathbb{D}_H u^{1-H}\sqrt{\frac{1-r(t,t+T/u)}{2}}\right),
\end{aligned} \tag{4.21}$$

where r is the correlation function of Z_H. Since $r(t,s) = corr(B_H(t), B_H(s))$ we have for all $t\in I(t_*)$

$$1-r(t,t+T/u) = \frac{T^{2H}}{2t_*^{2H}}u^{-2H} + O\left(u^{-2H}(|t-t_*|+|t+T/u-t_*|)+u^{-2}\right), \quad u\to\infty$$

implying

$$\mathbb{D}_H u^{1-H}\sqrt{\frac{1-r(t,t+T/u)}{2}} = u^{1-2H}\frac{T^H\mathbb{D}_H}{2t_*^H} + O(u^{1-2H}(|t-t_*|+|t+T/u-t_*|)+u^{-1}), \quad u\to\infty.$$

Thus, by (2.7) we obtain as $u\to\infty$

$$\overline{\Phi}\left(\mathbb{D}_H u^{1-H}\sqrt{\frac{1-r(t,t+T/u)}{2}}\right) \le \overline{\Phi}\left(u^{1-2H}\frac{T^H\mathbb{D}_H}{2t_*^H}\right)e^{Cu^{2-4H}(|t-t_*|+|t+T/u-t_*|)}. \tag{4.22}$$

Next we have as $u\to\infty$ for some $C_1>0$

$$\overline{\Phi}\left(\frac{u^{1-H}}{M}\right)\sim\overline{\Phi}(\mathbb{D}_H u^{1-H})e^{-C_1u^{2-2H}(|t-t_*|+|t+T/u-t_*|)}$$

and by (4.22) we have for all $t\in I(t_*)$ and large u

$$\begin{aligned}&\overline{\Phi}\left(\frac{u^{1-H}}{M}\right)\overline{\Phi}\left(\mathbb{D}_H u^{1-H}\sqrt{\frac{1-r(t,t+T/u)}{2}}\right)\\ \leq\;&\overline{\Phi}\left(\mathbb{D}_H u^{1-H}\right)\overline{\Phi}\left(u^{1-2H}\frac{T^H\mathbb{D}_H}{2t_*^H}\right)e^{(Cu^{2-4H}-C_1u^{2-2H})(|t-t_*|+|t+T/u-t_*|)}\end{aligned}$$

and the claim follows from the line above and (4.21). □

Proof of Lemma 4.3.1. *Lower bound.* We have

$$\sup_{t\in\mathbb{R}}\inf_{s\in[t,t+L]}e^{\sqrt{2}B(s)-|s|+h(s)}\geq\inf_{s\in[0,L]}e^{\sqrt{2}B(s)-(1+a)s}\geq e^{-(1+a)L}\inf_{s\in[0,L]}e^{\sqrt{2}B(s)}\overset{d}{=}e^{-(1+a)L}e^{-\sup_{s\in[0,L]}\sqrt{2}B(s)},$$

where the symbol '$\overset{d}{=}$' means equality in distribution between two rvs. Taking expectations of both sides in the line above we obtain

$$\mathcal{F}_L^h\geq e^{-L(1+a)}\mathbb{E}\left\{e^{-\sup_{s\in[0,L]}\sqrt{2}B(s)}\right\}$$

and our next step is to calculate the expectation above. It is known (see, e.g., Chapter 11.1 in [59]) that

$$\mathbb{P}\left\{\sup_{s\in[0,L]}\sqrt{2}B(s)>x\right\}=2\mathbb{P}\left\{\sqrt{2}B(L)>x\right\}=2\overline{\Phi}\left(\frac{x}{\sqrt{2L}}\right),\quad x>0$$

hence we obtain that $\frac{e^{-x^2/4L}}{\sqrt{\pi L}}$, $x>0$ is the density of $\sup_{s\in[0,L]}\sqrt{2}B(s)$. Thus, we have

$$\mathbb{E}\left\{e^{-\sup_{s\in[0,L]}\sqrt{2}B(s)}\right\}=\int_0^\infty e^{-x}\frac{e^{-x^2/4L}}{\sqrt{\pi L}}dx=\frac{e^L}{\sqrt{\pi L}}\int_0^\infty e^{-(\frac{x}{2\sqrt{L}}+\sqrt{L})^2}dx=\frac{2e^L}{\sqrt{\pi}}\int_{\sqrt{L}}^\infty e^{-z^2}dz=2e^L\overline{\Phi}(\sqrt{2L}),$$

and combining all calculations above we obtain

$$\mathcal{F}_L^h\geq 2e^{-La}\overline{\Phi}(\sqrt{2L}),\quad L\geq 0.$$

On the other hand, we have

$$\sup_{t\in\mathbb{R}}\inf_{s\in[t,t+L]}e^{\sqrt{2}B(s)-|s|+h(s)}\geq\inf_{s\in[-L,0]}e^{\sqrt{2}B(s)-(1+b)|s|}\overset{d}{=}\inf_{s\in[0,L]}e^{\sqrt{2}B(s)-(1+b)s}$$

and estimating $\inf_{s\in[0,L]}e^{\sqrt{2}B(s)-(1+b)s}$ as above we have $\mathcal{F}_L^h\geq 2e^{-Lb}\overline{\Phi}(\sqrt{2L})$, $L\geq 0$, that completes the proof of the lower bound.

Upper bound. Note that $\mathcal{F}_L^h \le \mathcal{F}_L^0$ and hence since a BM has independent branches for positive and negative time we have with B_* an independent BM

$$\begin{aligned}\mathcal{F}_{2H}^L \le \mathbb{E}\left\{\sup_{t\in\mathbb{R}} e^{\sqrt{2}B(t)-h(t)}\right\} &= \mathbb{E}\left\{\max\left(\sup_{t\ge 0} e^{\sqrt{2}B(t)-(a+1)t}, \sup_{t\le 0} e^{\sqrt{2}B(t)-(b+1)|t|}\right)\right\}\\ &= \mathbb{E}\left\{\max\left(\sup_{t\ge 0} e^{\sqrt{2}B(t)-(a+1)t}, \sup_{t\ge 0} e^{\sqrt{2}B^*(t)-(b+1)t}\right)\right\}\\ &= \mathbb{E}\left\{e^{\max(\xi_a,\xi_b)}\right\},\end{aligned}$$

where ξ_a and ξ_b are exponential rvs with survival functions $e^{-(a+1)x}$ and $e^{-(b+1)x}$, respectively, see [26]. Since ξ_a and ξ_b have exponential distributions the last expectation above can be easily calculated and we have finally

$$\mathbb{E}\left\{e^{\max(\xi_a,\xi_b)}\right\} = 1 + \frac{1}{a} + \frac{1}{b} - \frac{1}{a+b+1}$$

and the claim follows. □

Proof of Lemma 4.3.2. First we have

$$\mathbb{E}\left\{\sup_{t\in\mathbb{R}\setminus[-M,M]}\inf_{s\in[t,t+L]} e^{\sqrt{2}B(s)-|s|+h(s)}\right\} \le \mathbb{E}\left\{\sup_{s\in[M,\infty)} e^{\sqrt{2}B(s)-(a+1)s}\right\} + \mathbb{E}\left\{\sup_{s\in(-\infty,-M]} e^{\sqrt{2}B(s)-(b+1)|s|}\right\}.$$

Later on, we shall work with the first expectation above. We have

$$\begin{aligned}&\mathbb{E}\left\{\sup_{s\in[M,\infty)} e^{\sqrt{2}B(s)-(1+a)s}\right\}\\ =& \int_{\mathbb{R}} e^x \mathbb{P}\left\{\sup_{s\in[M,\infty)} (\sqrt{2}B(s)-(1+a)s) > x\right\} dx\\ =& \int_{\mathbb{R}} e^x \mathbb{P}\left\{\sup_{s\in[M,\infty)} (\sqrt{2}(B(s)-B(M))-(1+a)(s-M)) > x + M(1+a) - \sqrt{2}B(M)\right\} dx.\end{aligned}$$

Since a BM has independent increments we have with B^* an independent BM that the last integral above equals

$$\begin{aligned}&\int_{\mathbb{R}} e^x \mathbb{P}\left\{\sup_{s\in[0,\infty)} (\sqrt{2}B(s)-(1+a)s) > x + M(1+a) - \sqrt{2M}B^*(1)\right\} dx\\ =& \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\int_{\mathbb{R}} e^x e^{-z^2/2} \mathbb{P}\left\{\sup_{s\in[0,\infty)} (\sqrt{2}B(s)-(1+a)s) > x + M(1+a) - \sqrt{2M}z\right\} dxdz.\end{aligned}$$

We know that $\mathbb{P}\left\{\sup_{t\ge 0}(B(t)-ct)>x\right\}=\min(1,e^{-2cx})$ for $c>0$ and $x\in\mathbb{R}$, thus the expression above equals

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{x-z^2/2}\min(1,e^{-(1+a)(x+M(1+a)-\sqrt{2M}z)})dxdz\\
=&\ \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\int_{\frac{(1+a)M+x}{\sqrt{2M}}}^{\infty} e^{x-z^2/2}dzdx+\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\int_{-\infty}^{\frac{(1+a)M+x}{\sqrt{2M}}} e^{x-z^2/2-(1+a)(x+M(1+a)-\sqrt{2M}z)}dzdx\\
=&\ \int_{\mathbb{R}} e^{x}\overline{\Phi}(\frac{(1+a)M+x}{\sqrt{2M}})dx+\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ax}\int_{-\infty}^{\frac{(1+a)M+x}{\sqrt{2M}}} e^{-\frac{(z-\sqrt{2M}(1+a))^2}{2}}dzdx\\
=&\ \int_{\mathbb{R}} e^{x}\overline{\Phi}(\frac{(1+a)M+x}{\sqrt{2M}})dx+\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ax}\int_{-\infty}^{\frac{-(1+a)M+x}{\sqrt{2M}}} e^{-\frac{z^2}{2}}dzdx\\
=&\ \int_{\mathbb{R}} e^{x}\overline{\Phi}(\frac{(1+a)M+x}{\sqrt{2M}})dx+\int_{\mathbb{R}} e^{-ax}\Phi(\frac{-(1+a)M+x}{\sqrt{2M}})dx.
\end{aligned}$$

Integrating the first integral above by parts we have

$$\begin{aligned}
\int_{\mathbb{R}} e^{x}\overline{\Phi}(\frac{(1+a)M+x}{\sqrt{2M}})dx &= -\int_{\mathbb{R}}\left(\overline{\Phi}(\frac{(1+a)M+x}{\sqrt{2M}})\right)'e^{x}dx\\
&= \frac{1}{\sqrt{2\pi}\sqrt{2M}}\int_{\mathbb{R}} e^{-\frac{((1+a)M+x)^2}{4M}}e^{x}dx\\
&= \frac{e^{-aM}}{\sqrt{2\pi}\sqrt{2M}}\int_{\mathbb{R}} e^{-\frac{((a-1)M+x)^2}{4M}}dx\\
&= e^{-aM}.
\end{aligned}$$

For the second integral we have similarly

$$\begin{aligned}
\int_{\mathbb{R}} e^{-ax}\Phi(\frac{-(1+a)M+x}{\sqrt{2M}})dx &= -\frac{1}{a}\int_{\mathbb{R}}\Phi\big(\frac{-(1+a)M+x}{\sqrt{2M}}\big)'e^{-ax}dx\\
&= \frac{1}{a}\frac{1}{\sqrt{2\pi}\sqrt{2M}}\int_{\mathbb{R}} e^{-\frac{(-(1+a)M+x)^2}{4M}-ax}dx\\
&= \frac{e^{-aM}}{a\sqrt{2\pi}\sqrt{2M}}\int_{\mathbb{R}} e^{-\frac{((1-a)M+x)^2}{4M}}dx\\
&= \frac{e^{-aM}}{a}.
\end{aligned}$$

Summarizing all calculations above we obtain

$$\mathbb{E}\left\{\sup_{t\in[M,\infty)} e^{\sqrt{2}B(t)-(1+a)t}\right\}=e^{-aM}\left(1+\frac{1}{a}\right).$$

By the same approach and the symmetry of BM around zero we have

$$\mathbb{E}\left\{\sup_{t\in(-\infty,-M]} e^{\sqrt{2}B(t)-(1+b)|t|}\right\} = e^{-bM}\left(1+\frac{1}{b}\right)$$

and hence combining both equations above with the first inequality in the proof we obtain the claim. □

Proof of Lemma 4.3.3. From [30] it follows that for any $L \geq 0$

$$\mathcal{F}_{2H}(L) = \mathbb{E}\left\{\frac{\sup_{t\in\mathbb{R}} \inf_{s\in[t,t+L]} e^{W(s)}}{\int_{\mathbb{R}} e^{W(t)}dt}\right\}. \tag{4.23}$$

Observe that $\sup_{t\in\mathbb{R}} \inf_{s\in[t,t+L]} e^{W(s)} \geq \inf_{s\in[0,L]} e^{W(s)}$, hence

$$\mathcal{F}_{2H}(L) \geq \mathbb{E}\left\{\frac{\inf_{s\in[0,L]} e^{W(s)}}{\int_{\mathbb{R}} e^{W(t)}dt}\right\} \geq e^{-L^{2H}}\mathbb{E}\left\{\frac{e^{-\sqrt{2}\sup_{s\in[0,L]} B_H(s)}}{\int_{\mathbb{R}} e^{W(t)}dt}\right\}.$$

Let $\xi = \sup_{s\in[0,L]} B_H(s)$, $(\Omega, \mathbb{P})$ be the probability space where B_H is defined and $\Omega_m = \{\omega \in \Omega : \xi(\omega) < m\}$ for $m > 0$. The last expectation above equals

$$\begin{aligned}
\mathbb{E}\left\{\frac{e^{-\sqrt{2}\xi}}{\int_{\mathbb{R}} e^{W(t)}dt}\right\} &= \int_{\Omega} \frac{e^{-\sqrt{2}\xi(\omega)}}{\int_{\mathbb{R}} e^{\sqrt{2}B_H(t,\omega)-|t|^{2H}}dt} d\mathbb{P}(\omega) \\
&\geq \int_{\Omega_m} \frac{e^{-\sqrt{2}\xi(\omega)}}{\int_{\mathbb{R}} e^{\sqrt{2}B_H(t,\omega)-|t|^{2H}}dt} d\mathbb{P}(\omega) \\
&\geq \mathbb{P}\{\Omega_m\}\, e^{-\sqrt{2}m} \int_{\Omega_m} \frac{1}{\int_{\mathbb{R}} e^{W(t)}dt} d\mathbb{P}(\omega).
\end{aligned}$$

Taking $m = nL^H$ we obtain that the last line above equals

$$\mathbb{P}\left\{\sup_{s\in[0,L]} B_H(s) < nL^H\right\} e^{-\sqrt{2}nL^H} \int_{\Omega_m} \frac{1}{\int_{\mathbb{R}} e^{W(t)}dt} d\mathbb{P}(\omega) = \mathbb{P}\left\{\sup_{s\in[0,1]} B_H(s) < n\right\} e^{-\sqrt{2}nL^H} \int_{\Omega_m} \frac{1}{\int_{\mathbb{R}} e^{W(t)}dt} d\mathbb{P}(\omega).$$

Since uniformly for all $n, L > 1/10$ it holds that $\mathbb{P}\{\Omega_m\} > C$, we have that with some $\bar{C} > 0$ that does not depend on L

$$\int_{\Omega_m} \frac{1}{\int_{\mathbb{R}} e^{W(t)}dt} d\mathbb{P}(\omega) \geq \bar{C}.$$

Combining the lines above we have uniformly for $n, L > 1/10$ that

$$\mathbb{E}\left\{\frac{e^{-\sqrt{2}\xi}}{\int_{\mathbb{R}} e^{W(t)}dt}\right\} \geq \bar{C}\mathbb{P}\left\{\sup_{s\in[0,1]} B_H(s) < n\right\} e^{-\sqrt{2}nL^H}.$$

Since the inequality above holds for all large L, it holds also for all $L \geq 0$, maybe with different positive constant $\bar{C}$, this completes the proof. $\square$

Proof of Lemma 4.3.4. By (4.23) we have that

$$
\begin{aligned}
&\left|\mathcal{F}_{2H}(L) - \mathbb{E}\left\{\frac{\sup\limits_{t\in[-M,M]}\inf\limits_{s\in[t,t+L]} e^{W(s)}}{\int\limits_{[-M,M]} e^{W(t)}dt}\right\}\right| \\
&= \left|\left(\mathbb{E}\left\{\frac{\sup\limits_{t\in\mathbb{R}}\inf\limits_{s\in[t,t+L]} e^{W(s)}}{\int\limits_{\mathbb{R}} e^{W(t)}dt}\right\} - \mathbb{E}\left\{\frac{\sup\limits_{t\in[-M,M]}\inf\limits_{s\in[t,t+L]} e^{W(s)}}{\int\limits_{\mathbb{R}} e^{W(t)}dt}\right\}\right)\right. \\
&\quad \left. + \left(\mathbb{E}\left\{\frac{\sup\limits_{t\in[-M,M]}\inf\limits_{s\in[t,t+L]} e^{W(s)}}{\int\limits_{\mathbb{R}} e^{W(t)}dt}\right\} - \mathbb{E}\left\{\frac{\sup\limits_{t\in[-M,M]}\inf\limits_{s\in[t,t+L]} e^{W(s)}}{\int\limits_{[-M,M]} e^{W(t)}dt}\right\}\right)\right| \\
&\leq \mathbb{E}\left\{\frac{\sup\limits_{t\in\mathbb{R}\setminus[-M,M]} e^{W(t)}}{\int\limits_{\mathbb{R}} e^{W(t)}dt}\right\} + \mathbb{E}\left\{\sup_{t\in[-M,M]} e^{W(t)} \frac{\int\limits_{\mathbb{R}\setminus[-M,M]} e^{W(t)}dt}{\int\limits_{\mathbb{R}} e^{W(t)}dt \int\limits_{[-M,M]} e^{W(t)}dt}\right\}.
\end{aligned}
$$

As follows from Section 4 in [30], the last line above does not exceed $e^{-c'M^{2H}}$, and the claim holds. $\square$

4.6 Appendix

Proof of (4.16). To establish the claim we need to show, that

$$
\mathbb{P}\left\{\exists t \in \mathbb{R}\setminus[t_1-\varepsilon, t_1+\varepsilon] : \inf_{s\in[t,t+T/u]} V_1(s) > u^{1-H}\right\} = o(\psi_1(T_u, u)), \quad u \to \infty.
$$

Applying Borell-TIS inequality (see, e.g., [59]) we have as $u \to \infty$

$$
\begin{aligned}
\mathbb{P}\left\{\exists t \in \mathbb{R}\setminus[t_1-\varepsilon, t_1+\varepsilon] : \inf_{s\in[t,t+T/u]} V_1(s) > u^{1-H}\right\} &\leq \mathbb{P}\left\{\exists t \in \mathbb{R}\setminus[t_1-\varepsilon, t_1+\varepsilon] : V_1(t) > u^{1-H}\right\} \\
&\leq e^{-\frac{(u^{1-H}-M)^2}{2m^2}},
\end{aligned}
$$

where

$$
M = \mathbb{E}\left\{\sup_{\exists t\in\mathbb{R}\setminus[t_1-\varepsilon,t_1+\varepsilon]} V_1(t)\right\} < \infty, \quad m^2 = \max_{\exists t\in\mathbb{R}\setminus[t_1-\varepsilon,t_1+\varepsilon]} \mathrm{Var}\{V_1(t)\}.
$$

Since $\mathrm{Var}\{V_1(t)\}$ achieves its unique maxima at t_1 we obtain by (2.7) that

$$
e^{-\frac{(u^{1-H}-M)^2}{2m^2}} = o(\mathbb{P}\left\{V_1(t_1) < u^{1-H}\right\}), \quad u \to \infty
$$

and the claim follows from the asymptotics of $\psi_1(T_u, u)$ given in Proposition 4.5.1. $\square$

Proof of (4.20)**.** Define $X_{x,u}(t) = x - B_H(t)|B_H(ut_*) = x,\ t \in [ut_* - T, u]$. To calculate the covariance and expectation of $X_{x,u}$ we use the formulas

$$\mathrm{cov}((B,C)|A=x) = \mathrm{cov}(B,C) - \frac{\mathrm{cov}(A,B)\,\mathrm{cov}(A,C)}{\mathrm{Var}\{A\}} \quad \text{and} \quad \mathbb{E}\left\{B|A=x\right\} = x \cdot \frac{\mathrm{cov}(A,B)}{\mathrm{Var}\{A\}},$$

where A, B and C are centered Gaussian rvs and $x \in \mathbb{R}$. We have for $x \in [bu, bu+1]$ and $t, s \in [ut_* - T, ut_*]$ with $v = ut_*$, $y = 1 - \frac{t}{v}$ and $z = 1 - \frac{s}{v}$ as $u \to \infty$

$$\begin{aligned}
& \mathrm{cov}(X_{x,u}(t), X_{x,u}(s)) \\
= & \ \frac{t^{2H} + s^{2H} - |t-s|^{2H}}{2} - \frac{(t^{2H} + v^{2H} - |t-v|^{2H})(s^{2H} + v^{2H} - |s-v|^{2H})}{4v^{2H}} \\
= & \ \frac{v^{2H}}{4}\Big(2(\frac{t}{v})^{2H} + 2(\frac{s}{v})^{2H} - 2|\frac{t}{v} - \frac{s}{v}|^{2H} - ((\frac{t}{v})^{2H} + 1 - |\frac{t}{v} - 1|^{2H})((\frac{s}{v})^{2H} + 1 - |\frac{s}{v} - 1|^{2H})\Big) \\
= & \ \frac{v^{2H}}{4}\Big(2(1-y)^{2H} + 2(1-z)^{2H} - 2|y-z|^{2H} - ((1-y)^{2H} + 1 - y^{2H})((1-z)^{2H} + 1 - z^{2H})\Big) \\
= & \ \frac{v^{2H}}{4}\Big(2 - 4Hy + 2 - 4Hz + O(y^2 + z^2) - 2|y-z|^{2H} \\
& \quad - (2 - 2Hy - y^{2H} + O(y^2))(2 - 2Hz - z^{2H} + O(z^2))\Big) \\
= & \ \frac{v^{2H}}{4}\Big(2y^{2H} + 2z^{2H} - 2|y-z|^{2H} + O(y^2 + z^2 + z^{2H}y^{2H})\Big) \\
= & \ (1+o(1))\frac{(ut_* - t)^{2H} + (ut_* - s)^{2H} - |t-s|^{2H}}{2}. \qquad (4.24)
\end{aligned}$$

For the expectation we have as $u \to \infty$

$$\begin{aligned}
\mathbb{E}\left\{X_{x,u}(t)\right\} = x(1 - \frac{v^{2H} + t^{2H} - |v-t|^{2H}}{2v^{2H}}) &= \frac{x}{2}(1 - (t/v)^{2H} + (1 - t/v)^{2H}) \\
&\le \frac{1}{2}(bu+1)(1 - (1-y)^{2H} + y^{2H}) \\
&\le (bu/2 + 1)(1 - 1 + 2Hy - o(y) + y^{2H}) \\
&\le Hbuy + \frac{1}{2}buy^{2H} + o(1).
\end{aligned}$$

From the line above it follows that for some $C_* > 0$, $H < 1/2$, $x \in [bu, bu+1]$ and $t \in [ut_* - T, ut_*]$

$$\mathbb{E}\left\{X_{x,u}(t)\right\} \le C_* + \frac{u^{1-2H}b}{2t_*^{2H}}(ut_* - t)^{2H}.$$

We have

$$\begin{aligned}
& \sup_{x \in [bu, bu+1]} \mathbb{P}\left\{\exists t \in [ut_* - T, ut_*] : X_{x,u}(t) > u^{H+\kappa}\alpha b\right\} \\
= & \sup_{x \in [bu, bu+1]} \mathbb{P}\left\{\exists t \in [ut_* - T, ut_*] : X_{x,u}(t) - \mathbb{E}\left\{X_{x,u}(t)\right\} > u^{H+\kappa}\alpha b - \mathbb{E}\left\{X_{x,u}(t)\right\}\right\} \\
\le & \ \mathbb{P}\left\{\exists t \in [0, T] : Y_u(t) + f(t) > 0\right\},
\end{aligned}$$

where $Y_u(t) = X_{x,u}(ut_* - T + t) - \mathbb{E}\left\{X_{x,u}(ut_* - T + t)\right\}$, $t \in [0, T]$ and $f(t)$ is the linear function such that $f(T) = C_1$ and $f(0) = -C_* < 0$. Next we have by (4.24) for all large u and $t, s \in [0, T]$

$$\mathbb{E}\left\{(Y_u(t) + f(t) - Y_u(s) - f(s))^2\right\}$$

$$
\begin{aligned}
&= \mathbb{E}\left\{(Y_u(t)-Y_u(s))^2\right\}+C(t-s)^2\\
&\le C_1\Big((ut_*-t)^{2H}+(ut_*-s)^{2H}-(ut_*-t)^{2H}-(ut_*-s)^{2H}+|t-s|^{2H}\Big)+C(t-s)^2\\
&\le 2|t-s|^{2H}.
\end{aligned}
$$

Thus, by Proposition 9.2.4 in [59] the family $Y_u(t)+f(t),\ u>0,\ t\in[0,T]$ is tight in $\mathcal{B}(C([0,T]))$. As follows from (4.24), it holds that $\{Y_u(t)+f(t)\}_{t\in[0,T]}$ converges to $\{B_H(t)+f(t)\}_{t\in[0,T]}$ in the sense of convergence of finite-dimensional distributions as $u\to\infty$. Thus, by Theorems 4 and 5 in Chapter 5 in [8] the tightness and convergence of finite-dimensional distributions imply weak convergence

$$\{Y_u(t)+f(t)\}_{t\in[0,T]}\Rightarrow\{B(t)+f(t)\}_{t\in[0,T]}.$$

Since the functional $F(g)=\sup\limits_{t\in[0,T]} g(t)$ is continuous in the uniform metric we obtain

$$\mathbb{P}\left\{\exists t\in[0,T]:Y_u(t)+f(t)>0\right\}\to\mathbb{P}\left\{\exists t\in[0,T]:B_H(t)+f(t)>0\right\},\quad u\to\infty.$$

Thus, to prove the claim it is enough to show that

$$\mathbb{P}\left\{\exists t\in[0,T]:B_H(t)+f(t)>0\right\}<1. \tag{4.25}$$

We have for some large m with $l(s)$ the density of $B_H(T)$

$$
\begin{aligned}
&\mathbb{P}\left\{\sup_{t\in[0,T]}(B_H(t)+f(t))<0\right\}\\
&\ge \mathbb{P}\left\{\sup_{t\in[0,T]}(B_H(t)+f(t))<0 \text{ and } B_H(T)<-m\right\}\\
&= \int_{-\infty}^{-m}\mathbb{P}\left\{\sup_{t\in[0,T]}(B_H(t)+f(t))<0|B_H(T)=s\right\}l(s)ds.
\end{aligned} \tag{4.26}
$$

Define process $\widetilde{B}_s(t)=B_H(t)+f(t)|B_H(T)=s,\ t\in[0,T]$. We have for $s<-m$ and $t\in[0,T]$

$$
\begin{aligned}
\mathbb{E}\left\{\widetilde{B}_s(t)\right\} &= f(t)+s\frac{t^{2H}+T^{2H}-|T-t|^{2H}}{2T^{2H}}<-C_1/2,\\
\mathrm{Var}\{\widetilde{B}_s(t)\} &= t^{2H}-\frac{(T^{2H}+t^{2H}-|t-s|^{2H})^2}{4T^{2H}}<C_2
\end{aligned}
$$

and thus

$$\mathbb{P}\left\{\sup_{t\in[0,T]}(B_H(t)+f(t))<0|B_H(T)=s\right\}\ge\mathbb{P}\left\{\sup_{t\in[0,T]}\left(\widetilde{B}_s(t)-\mathbb{E}\left\{\widetilde{B}_s(t)\right\}\right)<C_1/2\right\}.$$

The last probability above is positive for any $s<-m$, see Chapters 10 and 11 in [51] and hence the integral in (4.26) is positive implying

$$\mathbb{P}\left\{\sup_{t\in[0,T]}(B_H(t)+f(t))<0\right\}>0.$$

Consequently (4.25) holds and the claim is established. □

Chapter 5

Two-Dimensional Fractional Brownian Motion Sojourn Ruin Problem

This chapter is based on G. Jasnovidov: Simultaneous Sojourn Fractional Brownian Motion Ruin, in progress, 2021.

5.1 Introduction & Preliminaries

Consider the risk model defined by

$$R(t) = u + \rho t - X(t), \qquad t \geq 0,\ \rho, u > 0, \tag{5.1}$$

where $X(t)$ is a centered Gaussian risk process with a.s. continuous sample paths. This model is very important for the theoretical and applied studies, we refer to Chapters 2, 3 and 4 and references therein for a list of possible applications. Some contributions (see, e.g., [21, 44]), extend the classical ruin problem to the sojourn ruin problem. Formally, the sojourn ruin time and ruin probability are defined by

$$\tau(u, T_u) = \Big\{ \inf t \geq 0 : \int_0^t \mathbb{I}(R(s) < 0) ds > T_u \Big\}$$

and

$$\mathbb{P}\left\{ \tau(u, T_u) < \infty \right\}, \tag{5.2}$$

where $T_u \geq 0$ is a measurable function of u. As in the classical case, only for X being a BM the probability above can be calculated explicitly (see [21])

$$\mathbb{P}\left\{ \int_0^\infty \mathbb{I}(B(s) - cs > u) > T \right\} ds = \Big(2(1 + c^2 T)\overline{\Phi}(c\sqrt{T}) - \frac{c\sqrt{2T}}{\sqrt{\pi}} e^{\frac{-c^2 T}{2}} \Big) e^{-2cu}, \quad c > 0,\ T, u \geq 0.$$

Motivated by [45] (see also [42, 43]), we study a generalization of the main problem in [45] for the sojourn ruin, i.e., we shall study the asymptotics of

$$\mathbb{C}_{T_u} := \mathbb{P}\left\{\int_0^\infty (B_H(s) - c_1 s > q_1 u, B_H(s) - c_2 s > q_2 u) ds > T_u\right\},$$

as $u \to \infty$. In order to prevent the problem for degenerating to the one-dimensional sojourn problem by the same reasons as in Chapter 4 we assume that

$$c_1 > c_2 > 0, \quad q_2 > q_1 > 0. \tag{5.3}$$

As in Chapter 4 by the self-similarity of fBM we obtain

$$\mathbb{C}_{T_u}(u) = \mathbb{P}\left\{\int_0^\infty \mathbb{I}\Big(\frac{B_H(t)}{\max(c_1 t + q_1, c_2 t + q_2)} > u^{1-H}\Big) dt > T_u/u\right\}.$$

The variance of the process above can achieve its unique maxima only at one of the following points:

$$t_*, \quad t_1 := \frac{Hq_1}{(1-H)c_1}, \quad t_2 := \frac{Hq_2}{(1-H)c_2}. \tag{5.4}$$

From (5.3) it follows that $t_1 < t_2$. As we shall see later, the order between t_1, t_2 and t_* determines the asymptotics of $\mathbb{C}_{T_u}$. As mentioned in [16], for the one-dimensional Parisian ruin probability we need to control the growth of T_u as $u \to \infty$. Namely, we impose the following condition:

$$\lim_{u\to\infty} T_u u^{1/H-2} = T \in [0, \infty), \ H \in (0, 1). \tag{5.5}$$

Define for some function h and $K \geq 0$ constant

$$\mathcal{B}_K^h = \int_{\mathbb{R}} \mathbb{P}\left\{\int_{-\infty}^\infty \mathbb{I}\Big(\sqrt{2}B(s) - |s| + h(s) > x\Big) ds > K\right\} e^x dx$$

when the integral above is finite and Berman's constant by

$$\mathcal{B}_{2H}(x) = \lim_{S\to\infty} \frac{1}{S} \int_{\mathbb{R}} \mathbb{P}\left\{\int_0^S \mathbb{I}(\sqrt{2}B_H(t) - t^{2H} + z > 0) dt > x\right\} e^{-z} dz, \quad x \geq 0.$$

It is known (see, e.g., [21]) that $\mathcal{B}_{2H}(x) \in (0, \infty)$ for all $x \geq 0$; we refer to [21] and references therein for the properties of relevant Berman's constants.

5.2 Main Result

Define for $i = 1, 2$

$$\mathbb{D}_H = \frac{c_1 t_* + q_1}{t_*^H}, \ K_H = \frac{2^{\frac{1}{2}-\frac{1}{2H}}\sqrt{\pi}}{\sqrt{H(1-H)}}, \ \mathbb{C}_H^{(i)} = \frac{c_i^H q_i^{1-H}}{H^H(1-H)^{1-H}}, \ D_i = \frac{c_i^2(1-H)^{2-\frac{1}{H}}}{2^{\frac{1}{2H}}H^2}. \quad (5.6)$$

Now we are ready to give the asymptotics of $\mathbb{C}_{T_u}(u)$:

Theorem 5.2.1 *Assume that* (5.3) *holds and* T_u *satisfies* (5.5).
1)If $t_* \notin (t_1, t_2)$, *then as* $u \to \infty$

$$\mathbb{C}_{T_u}(u) \sim (\frac{1}{2})^{\mathbb{I}(t_*=t_i)} \times \begin{cases} \left(2(1+c_i^2 T)\overline{\Phi}(c_i\sqrt{T}) - \frac{c_i\sqrt{2T}}{\sqrt{\pi}}e^{\frac{-c_i^2 T}{2}}\right)e^{-2c_i q_i u}, & H = 1/2 \\ K_H \mathcal{B}_{2H}(TD_i)(\mathbb{C}_H^{(i)} u^{1-H})^{\frac{1}{H}-1}\overline{\Phi}(\mathbb{C}_H^{(i)} u^{1-H}), & H \neq 1/2, \end{cases} \quad (5.7)$$

where $i = 1$ *if* $t_* \le t_1$ *and* $i = 2$ *if* $t_* \ge t_2$.
2) If $t_* \in (t_1, t_2)$ *and* $\lim_{u\to\infty} T_u u^{2-1/H} = 0$ *for* $H > 1/2$, *then as* $u \to \infty$

$$\mathbb{C}_{T_u}(u) \sim \overline{\Phi}(\mathbb{D}_H u^{1-H}) \times \begin{cases} 1, & H > 1/2 \\ \mathcal{B}_{T'}^d, & H = 1/2 \\ \mathcal{B}_{2H}(\overline{D}T)Au^{(1-H)(1/H-2)}, & H < 1/2, \end{cases} \quad (5.8)$$

where $\mathcal{F}_{T'}^d \in (0, \infty)$,

$$T' = T\frac{(c_1 q_2 - q_1 c_2)^2}{2(c_1 - c_2)^2}, \quad d(s) = s\frac{c_1 q_2 + c_2 q_1 - 2c_2 q_2}{c_1 q_2 - q_1 c_2}\mathbb{I}(s < 0) + s\frac{2c_1 q_1 - c_1 q_2 - q_1 c_2}{c_1 q_2 - q_1 c_2}\mathbb{I}(s \ge 0) \quad (5.9)$$

and

$$A = \left(|H(c_1 t_* + q_1) - c_1 t_*|^{-1} + |H(c_2 t_* + q_2) - c_2 t_*|^{-1}\right)\frac{t_*^H \mathbb{D}_H^{\frac{1}{H}-1}}{2^{\frac{1}{2H}}}, \qquad \overline{D} = \frac{(c_1 t_* + q_1)^{\frac{1}{H}}}{2^{\frac{1}{2H}} t_*^2}. \quad (5.10)$$

The theorem above generalizes Theorem 3.1 in [45]: if $T = 0$, then the result above reduces to Theorem 3.1 in [45].

As indicated in [16], it seems extremely difficult to find the exact asymptotics of the one-dimensional Parisian ruin probability if (5.5) does not hold. To illustrate difficulties arising in approximation of $\mathbb{C}_{T_u}(u)$ in this case we give

Proposition 5.2.2 *If* $H < 1/2$, $T_u = T > 0$ *and* $t_* \in (t_1, t_2)$, *then*

$$\begin{aligned} \bar{C}\overline{\Phi}(\mathbb{D}_H u^{1-H})e^{-C_{1,\alpha}u^{2-4H} - C_{2,\alpha}u^{2(1-3H)}} &\le \mathbb{C}_{T_u}(u) \\ &\le (2 + o(1))\overline{\Phi}(\mathbb{D}_H u^{1-H})\overline{\Phi}\left(u^{1-2H}\frac{T^H \mathbb{D}_H}{2t_*^H}\right), \end{aligned} \quad (5.11)$$

where $\bar{C} \in (0, 1)$ *is a fixed constant that does not depend on* u *and*

$$\alpha = \frac{T^{2H}}{2t_*^{2H}}, \qquad C_{i,\alpha} = \frac{\alpha^i}{i}\mathbb{D}_H^2, \quad i = 1, 2. \quad (5.12)$$

5.3 Proofs

Recall that K_H, D_1 and $\mathbb{C}_H^{(1)}$ are defined in (5.6). A proof of the proposition below is given in the Appendix.

Proposition 5.3.1 *Assume that* T_u *satisfies* (5.5). *Then as* $u \to \infty$

$$\mathbb{P}\left\{\int_0^\infty \mathbb{I}(B_H(t) - c_1 t > q_1 u) dt > T_u\right\} \sim \begin{cases} \left(2(1+c_1^2 T)\overline{\Phi}(c_1\sqrt{T}) - \frac{c_1\sqrt{2T}}{\sqrt{\pi}} e^{\frac{-c_1^2 T}{2}}\right) e^{-2c_1 q_1 u}, & H = 1/2, \\ K_H \mathcal{B}_{2H}(TD_1)(\mathbb{C}_H^{(1)} u^{1-H})^{\frac{1}{H}-1} \overline{\Phi}(\mathbb{C}_H^{(1)} u^{1-H}), & H \neq 1/2. \end{cases}$$

Proof of Theorem 5.2.1. Case (1). Assume that $t_* < t_1$. Let

$$V_i(t) = \frac{B_H(t)}{c_i t + q_i} \quad \text{and} \quad \psi_i(T_u, u) = \mathbb{P}\left\{\int_0^\infty \mathbb{I}(B_H(t) - c_i t > q_i u) ds > T_u\right\}, \quad i = 1, 2.$$

For $0 < \varepsilon < t_1 - t_*$ by the self-similarity of fBM we have

$$\begin{aligned} \psi_1(T_u, u) \geq \mathbb{C}_{T_u}(u) &\geq \mathbb{P}\left\{\int_{t_1-\varepsilon}^{t_1+\varepsilon} \mathbb{I}(V_1(t) > u^{1-H}, V_2(t) > u^{1-H}) dt > T_u/u\right\} \\ &= \mathbb{P}\left\{\int_{t_1-\varepsilon}^{t_1+\varepsilon} \mathbb{I}(V_1(t) > u^{1-H}) dt > T_u/u\right\}. \end{aligned}$$

We have by Borel-TIS inequality, see [59] (details are in the Appendix)

$$\psi_1(T_u, u) \sim \mathbb{P}\left\{\int_{t_1-\varepsilon}^{t_1+\varepsilon} \mathbb{I}(V_1(t) > u^{1-H}) ds > T_u/u\right\}, \quad u \to \infty \tag{5.13}$$

implying $\mathbb{C}_{T_u}(u) \sim \psi_1(T_u, u)$ as $u \to \infty$. The asymptotics of $\psi_1(T_u, u)$ is given in Proposition 5.3.1, thus the claim follows.

Assume that $t_* = t_1$. We have

$$\begin{aligned} \mathbb{P}\left\{\int_{t_1}^\infty \mathbb{I}(V_1(s) > u^{1-H}) ds > T_u\right\} &\leq \mathbb{C}_{T_u}(u) \\ &\leq \mathbb{P}\left\{\int_{t_1}^\infty \mathbb{I}(V_1(s) > u^{1-H}) ds > T_u\right\} + \mathbb{P}\left\{\exists t \in [0, t_1] : V_2(t) > u^{1-H}\right\}. \end{aligned}$$

From the proof of Theorem 3.1, case (4) in [45] it follows that the second term in the last line above is negligible comparing with the final asymptotics of $\mathbb{C}_{T_u}(u)$ given in (5.7), hence

$$\mathbb{C}_{T_u}(u) \sim \mathbb{P}\left\{\int_{t_1}^\infty \mathbb{I}(V_1(s) > u^{1-H}) ds > T_u\right\}, \quad u \to \infty.$$

Since t_1 is the unique maxima of $\mathrm{Var}\{V_1(t)\}$ from the proof of Theorem 2.1, case i) in [21] we have

$$\begin{aligned}\mathbb{P}\left\{\int_{t_1}^{\infty}\mathbb{I}(V_1(t)>u^{1-H})dt>T_u/u\right\} &\sim \frac{1}{2}\mathbb{P}\left\{\int_{0}^{\infty}\mathbb{I}(V_1(t)>u^{1-H})dt>T_u/u\right\}\\ &= \frac{1}{2}\mathbb{P}\left\{\int_{0}^{\infty}\mathbb{I}(B_H(t)-c_1t>q_1u)dt>T_u\right\}, \quad u\to\infty.\end{aligned}$$

The asymptotics of the last probability above is given in Proposition 5.3.1 establishing the claim. Case $t_*\ge t_2$ follows by the same arguments.

Case (2). Assume that $H>1/2$. We have by Theorem 2.1 in [43] and Theorem 3.1 in [45] with

$$\begin{aligned}\mathcal{R}_{T_u}(u) &= \mathbb{P}\left\{\exists t\ge 0: B_H(t)-c_1t>q_1u, B_H(t)-c_2t>q_2u\right\},\\ \mathcal{P}_{T_u}(u) &= \mathbb{P}\left\{\exists t\ge 0: \inf_{s\in[t,t+T_u]}(B_H(s)-c_1s)>q_1u, \inf_{s\in[t,t+T_u]}(B_H(s)-c_2s)>q_2u\right\}\end{aligned}$$

that

$$\overline{\Phi}(\mathbb{D}_H u^{1-H})\sim\mathcal{P}_{T_u}(u)\le\mathbb{C}_{T_u}(u)\le\mathcal{R}_{T_u}(u)\sim\overline{\Phi}(\mathbb{D}_H u^{1-H}),\quad u\to\infty,$$

and the claim follows.

Assume that $H=1/2$. First let (5.5) holds with $T_u=T>0$. We have as $u\to\infty$ and then $S\to\infty$ (proof is in the Appendix)

$$\mathbb{C}_{T_u}(u)\sim\mathbb{P}\left\{\int_{ut_*-S}^{ut_*+S}\mathbb{I}\Big(B(s)-c_1s>q_1u, B(s)-c_2s>q_2u\Big)ds>T\right\}:=\kappa_S(u). \tag{5.14}$$

Next with ϕ_u the density of $B(ut_*)$, $\eta=c_1t_*+q_1=c_2t_*+q_2$ and $\eta_*=\eta/t_*-c_2=q_2/t_*$ we have

$$\begin{aligned}&\kappa_S(u)\\ &= \int_{\mathbb{R}}\mathbb{P}\left\{\Big(\int_{ut_*-S}^{ut_*}\mathbb{I}(B(s)-c_2s>q_2u)ds+\int_{ut_*}^{ut_*+S}\mathbb{I}(B(s)-c_1s>q_1u)ds>T\Big)|B(ut_*)=\eta u-x\right\}\phi_u(\eta u-x)dx\\ &= \int_{\mathbb{R}}\mathbb{P}\Big\{\Big(\int_{ut_*-S}^{ut_*}\mathbb{I}(B(s)-c_2s>q_2u)ds\\ &\quad+\int_{ut_*}^{ut_*+S}\mathbb{I}(B(s)-B(ut_*)-c_1(s-ut_*)-c_1ut_*>q_1u-\eta u+x)ds>T\Big)|B(ut_*)=\eta u-x\Big\}\phi_u(\eta u-x)dx\\ &= \int_{\mathbb{R}}\mathbb{P}\left\{\Big(\int_{ut_*-S}^{ut_*}\mathbb{I}(B(s)-c_2s>q_2u)ds+\int_{0}^{S}\mathbb{I}(B_*(s)-c_1s>x)ds>T\Big)|B(ut_*)=\eta u-x\right\}\phi_u(\eta u-x)dx\end{aligned}$$

$$= \frac{e^{-\frac{\eta^2 u}{2t_*}}}{\sqrt{2\pi u t_*}} \int_{\mathbb{R}} \mathbb{P}\left\{ \int_{-S}^{0} \mathbb{I}\Big(Z_u(s) + \eta_* s > x\Big) ds + \int_0^S \mathbb{I}(B_*(s) - c_1 s > x) ds > T \right\} e^{\frac{\eta x}{t_*} - \frac{x^2}{2ut_*}} dx,$$

where $Z_u(t)$ is a Gaussian process with expectation and covariance defined below ($s \le t \le 0$):

$$\mathbb{E}\{Z_u(t)\} = \frac{-x}{ut_*} t, \qquad \mathrm{cov}(Z_u(s), Z_u(t)) = \frac{-st}{ut_*} - t. \tag{5.15}$$

Since $Z_u(t)$ converges to BM in the sense of convergence finite-dimension distributions for any fixed $x \in \mathbb{R}$ as $u \to \infty$ we have (details are in the Appendix)

$$\begin{aligned} & \int_{\mathbb{R}} \mathbb{P}\left\{ \int_{-S}^{0} \mathbb{I}\Big(Z_u(s) + \eta_* s > x\Big) ds + \int_0^S \mathbb{I}(B_*(s) - c_1 s > x) ds > T \right\} e^{\frac{\eta x}{t_*} - \frac{x^2}{2ut_*}} dx \\ \sim\ & \int_{\mathbb{R}} \mathbb{P}\left\{ \int_{-S}^{0} \mathbb{I}\Big(B(s) + \eta_* s > x\Big) ds + \int_0^S \mathbb{I}(B_*(s) - c_1 s > x) ds > T \right\} e^{\frac{\eta x}{t_*}} dx \\ =:\ & K(S). \end{aligned} \tag{5.16}$$

We have by the formula $\mathbb{P}\{\exists t \ge 0 : B(t) - ct > x\} = e^{-2cx},\ c, x > 0$ (see, e.g., [26])

$$\begin{aligned} K(S) &\le \int_0^\infty \Big(\mathbb{P}\{\exists s < 0 : B(s) + \eta_* s > x\} + \mathbb{P}\{\exists s \ge 0 : B_*(s) - c_1 s > x\}\Big) e^{\frac{\eta x}{t_*}} dx + \int_{-\infty}^0 e^{\frac{\eta x}{t_*}} dx \\ &= \int_0^\infty \Big(e^{(-2\eta_* + \eta/t_*)x} + e^{(-2c_1 + \eta/t_*)x}\Big) dx + t_*/\eta < \infty \end{aligned}$$

provided by $t_* \in (t_1, t_2)$. Since $K(S)$ is an increasing function and $\lim_{S\to\infty} K(S) < \infty$ we have as $S \to \infty$

$$\begin{aligned} K(S) &\to \int_{\mathbb{R}} \mathbb{P}\left\{ \int_0^\infty \mathbb{I}\Big(B(s) - \eta_* s > x\Big) ds + \int_0^\infty \mathbb{I}(B_*(s) - c_1 s > x) ds > T \right\} e^{\frac{\eta x}{t_*}} dx \\ &= \frac{t_*}{\eta} \int_{\mathbb{R}} \mathbb{P}\left\{ \int_0^\infty \mathbb{I}\Big(B(s) - \frac{\eta_* t_*}{\eta} s > x\Big) ds + \int_0^\infty \mathbb{I}(B_*(s) - \frac{c_1 t_*}{\eta} s > x) ds > \frac{\eta^2 T}{t_*^2} \right\} e^x dx \\ &= \int_{\mathbb{R}} \mathbb{P}\left\{ \int_{-\infty}^\infty \mathbb{I}\Big(\sqrt{2} B(s) - |s| + d(s) > x\Big) ds > \frac{\eta^2 T}{2t_*^2} \right\} e^x dx \\ &= \frac{t_*}{\eta} \mathcal{B}^d_{T'} \in (0, \infty), \end{aligned}$$

where T' and $d(s)$ are defined in (5.9). Finally, combining (5.16) with the line above we have as $u \to \infty$ and then $S \to \infty$

$$\kappa_S(u) \sim \mathcal{B}^d_{T'} \overline{\Phi}(\mathbb{D}_{1/2}\sqrt{u})$$

and by (5.14) the claim follows. If (5.5) holds with $T_u = 0$, then we obtain the claim immediately by Theorem 3.1 in [45] and observation that $\mathcal{B}_0^d$ coincides with the corresponding Piterbarg constant introduced in [45].

Now assume that (5.5) holds with any possible T_u. If (5.5) holds with $T > 0$, then for large u and any $\varepsilon > 0$ it holds that $\mathbb{C}_{(1+\varepsilon)T}(u) \le \mathbb{C}_{T_u}(u) \le \mathbb{C}_{(1-\varepsilon)T}(u)$ and hence

$$(1+o(1))\mathcal{B}^d_{T'(1+\varepsilon)}\overline{\Phi}(\mathbb{D}_{1/2}\sqrt{u}) \le \mathbb{C}_{T_u}(u) \le \mathcal{B}^d_{T'(1-\varepsilon)}\overline{\Phi}(\mathbb{D}_{1/2}\sqrt{u})(1+o(1)).$$

By Lemma 4.1 in [21] $\mathcal{B}_x^d$ is a continuous function with respect to x and thus letting $\varepsilon \to 0$ we obtain the claim. If (5.5) holds with $T = 0$, then for large u and any $\varepsilon > 0$ we have

$$\mathcal{B}_\varepsilon^d\overline{\Phi}(\mathbb{D}_{1/2}\sqrt{u}) \le \mathbb{C}_{T_u}(u) \le \mathcal{B}_0^d\overline{\Phi}(\mathbb{D}_{1/2}\sqrt{u})$$

and again letting $\varepsilon \to 0$ we obtain the claim by continuity of $\mathcal{B}^d_{(\cdot)}$.

Assume that $H < 1/2$. First we have with $\delta_u = u^{2H-2}\ln^2 u$ as $u \to \infty$ (proof is in Appendix)

$$\begin{aligned}\mathbb{C}_{T_u}(u) &\sim \mathbb{P}\left\{\int_{ut_*-u\delta_u}^{ut_*} \mathbb{I}(B_H(t) - c_2 t > q_2 u)dt > T_u\right\} + \mathbb{P}\left\{\int_{ut_*}^{ut_*+u\delta_u} \mathbb{I}(B_H(t) - c_1 t > q_1 u)dt > T_u\right\}\\ &=: g_1(u) + g_2(u).\end{aligned} \tag{5.17}$$

Assume that (5.5) holds with $T > 0$. Using the approach from [21] we have

$$\begin{aligned} g_2(u) &= \mathbb{P}\left\{\int_0^{\delta_u T_u^{-1} u} \mathbb{I}_{M(u)}\Big(\frac{B_H(ut_* + tT_u)}{u(q_1 + c_1 t_*) + c_1 t T_u} M(u)\Big)dt > 1\right\}\\ &=: \mathbb{P}\left\{\int_0^{\delta_u T_u^{-1} u} \mathbb{I}_{M(u)}(Z_u^{(1)}(t))dt > 1\right\}\\ &= \mathbb{P}\left\{\int_0^{\delta_u T_u^{-1} u K_1} \mathbb{I}_{M(u)}(Z_u^{(1)}(tK_1^{-1}))dt > K_1\right\}\\ &=: \mathbb{P}\left\{\int_0^{\delta_u T_u^{-1} u K_1} \mathbb{I}_{M(u)}(Z_u^{(2)}(t))dt > K_1\right\}, \end{aligned}$$

where $\mathbb{I}_a(b) = \mathbb{I}(b > a),\ a, b \in \mathbb{R}$ and

$$K_1 = \frac{T\mathbb{D}_H^{1/H}}{2^{\frac{1}{2H}} t_*}, \quad M(u) = \inf_{t\in[t_*,\infty)} \frac{u(c_1 t + q_1)}{\mathrm{Var}\{B_H(ut)\}} = \mathbb{D}_H u^{1-H}.$$

For variance $\sigma^2_{Z_u^{(2)}}(t)$ and correlation $r_{Z_u^{(2)}}(s,t)$ of $Z_u^{(2)}$ for $t,s \in [0, \delta_u T_u^{-1} u K_1]$ it holds that

$$\begin{aligned} 1-\sigma_{Z_u^{(2)}}(t) &= \frac{2^{\frac{1}{2H}} t_*^H \mathbb{D}_H^{1-1/H} |q_1 H-(1-H)c_1 t_*|}{(q_1+c_1 t_*)^2} t u^{1-1/H} + O(t^2 u^{2(1-1/H)}), \\ 1-r_{Z_u^{(2)}}(s,t) &= \mathbb{D}_H^{-2} u^{2H-2}|t-s|^{2H} + O(u^{2H-2}|t-s|^{2H}\delta_u). \end{aligned}$$

Now we apply Theorem 2.1 in [21]. We have that all conditions of the theorem are fulfilled with parameters

$$\begin{aligned} &\omega(x)=x,\ \overleftarrow{\omega}(x)=x,\ \beta=1,\ g(u)=\frac{2^{\frac{1}{2H}} t_*^H \mathbb{D}_H^{1-1/H} |q_1 H-(1-H)c_1 t_*|}{(q_1+c_1 t_*)^2} u^{1-1/H}, \\ &\eta_\varphi(t)=B_H(t),\ \sigma_\eta^2(t)=t^{2H},\ \Delta(u)=1,\ \varphi=1, \\ &n(u)=\mathbb{D}_H u^{1-H},\ a_1(u)=0,\ a_2(u)=\delta_u T_u^{-1} u K_1,\ \gamma=0,\ x_1=0,\ x_2=\infty,\ y_1=0,\ y_2=\infty,\ x=K_1, \\ &\theta(u)=u^{(1/H-2)(1-H)} \mathbb{D}_H^{-1+1/H} |q_1 H-(1-H)c_1 t_*|^{-1} t_*^H 2^{-\frac{1}{2H}}, \end{aligned}$$

and thus as $u \to \infty$

$$g_2(u)=\mathbb{P}\left\{\int_0^{\delta_u T_u^{-1} u K_1} \mathbb{I}_{M(u)}(Z_u^{(2)}(t))dt > K_1\right\} \sim \mathcal{B}_{2H}(\frac{T\mathbb{D}_H^{\frac{1}{H}}}{2^{\frac{1}{2H}} t_*}) u^{(\frac{1}{H}-2)(1-H)} \frac{t_*^H \mathbb{D}_H^{-1+1/H}}{2^{\frac{1}{2H}} |q_1 H-(1-H)c_1 t_*|} \overline{\Phi}(\mathbb{D}_H u^{1-H}).$$

Similarly we obtain

$$g_1(u) \sim \mathcal{B}_{2H}(\frac{T\mathbb{D}_H^{1/H}}{2^{\frac{1}{2H}} t_*}) u^{(1/H-2)(1-H)} \frac{t_*^H \mathbb{D}_H^{-1+1/H}}{2^{\frac{1}{2H}} |q_2 H-(1-H)c_2 t_*|} \overline{\Phi}(\mathbb{D}_H u^{1-H}), \quad u \to \infty$$

and the claim follows if in (5.5) $T>0$. Now let (5.5) holds with $T=0$. Since $\mathcal{P}_{T_u}(u) \le \mathbb{C}_{T_u}(u) \le \mathcal{R}_{T_u}(u)$ we obtain the claim by Theorem 2.1 in [43] and Theorem 3.1 in [45]. □

Proof of Proposition 5.2.2. The proof of this proposition is the same as the proof of Proposition 2.2 in [43], thus we refer to [43] for the proof. □

5.4 Appendix

Proof of (5.13). To establish the claim we need to show that

$$\mathbb{P}\left\{\int_{t_1-\varepsilon}^{t_1+\varepsilon} \mathbb{I}(V_1(s) > u^{1-H}) ds > T_u/u\right\} = o(\psi_1(T_u,u)), \quad u \to \infty.$$

Applying Borell-TIS inequality (see, e.g., [59]) we have as $u \to \infty$

$$\mathbb{P}\left\{\int_{[0,\infty)\setminus[t_1-\varepsilon,t_1+\varepsilon]} \mathbb{I}(V_1(s) > u^{1-H}) ds > T_u/u\right\} \le \mathbb{P}\left\{\exists t \in [0,\infty)\setminus[t_1-\varepsilon,t_1+\varepsilon] : V_1(t) > u^{1-H}\right\}$$

$$\leq \quad e^{-\frac{(u^{1-H}-M)^2}{2m^2}},$$

where

$$M = \mathbb{E}\left\{\sup_{\exists t\in[0,\infty)\setminus[t_1-\varepsilon,t_1+\varepsilon]} V_1(t)\right\} < \infty, \quad m^2 = \max_{\exists t\in[0,\infty)\setminus[t_1-\varepsilon,t_1+\varepsilon]} \mathrm{Var}\{V_1(t)\}.$$

Since $\mathrm{Var}\{V_1(t)\}$ achieves its unique maxima at t_1 we obtain by (2.7) that

$$e^{-\frac{(u^{1-H}-M)^2}{2m^2}} = o(\mathbb{P}\left\{V_1(t_1) < u^{1-H}\right\}), \quad u \to \infty$$

and the claim follows from the asymptotics of $\psi_1(T_u, u)$ given in Proposition 5.13. □

Proof of (5.14)**.** To prove the claim it is enough to show that as $u \to \infty$ and then $S \to \infty$

$$\mathbb{P}\left\{\int\limits_{[0,\infty)\setminus[ut_*-S,ut_*+S]} \mathbb{I}(B(t) - c_1 t > q_1 u, B(t) - c_2 t > q_2 u)dt > T\right\} = o(\mathbb{C}_{T_u}(u)), \quad u \to \infty.$$

We have that the probability above does not exceed

$$\mathbb{P}\left\{\exists t \in [0,\infty)\setminus[ut_* - S, ut_* + S] : B(t) - c_1 t > q_1 u, B(t) - c_2 t > q_2 u\right\}.$$

From the proof of Theorem 3.1 in [45], Case (3) and the final asymptotics of $\mathbb{C}_{T_u}(u)$ given in (5.8) it follows that the expression above equals $o(\mathbb{C}_{T_u}(u))$, as $u \to \infty$ and then $S \to \infty$. □

Proof of (5.16)**.** Define

$$G(u,x) = \mathbb{P}\left\{\int\limits_{-S}^{0} \mathbb{I}\Big(Z_u(s) + \eta_* s > x\Big)ds + \int\limits_{0}^{S} \mathbb{I}(B_*(s) - c_1 s > x)ds > T\right\}.$$

First we show that

$$\int\limits_{\mathbb{R}} G(u,x)e^{\frac{\eta x}{t_*}-\frac{x^2}{2ut_*}}dx = \int\limits_{-M}^{M} G(u,x)e^{\frac{\eta x}{t_*}}dx + A_{M,u}, \tag{5.18}$$

where $A_{M,u} \to 0$ as $u \to \infty$ and then $M \to \infty$. We have

$$\begin{aligned} |A_{M,u}| &= |\int\limits_{\mathbb{R}} G(u,x)e^{\frac{\eta x}{t_*}-\frac{x^2}{2ut_*}}dx - \int\limits_{-M}^{M} G(u,x)e^{\frac{\eta x}{t_*}}dx| \\ &\leq |\int\limits_{-M}^{M} G(u,x)(e^{\frac{\eta x}{t_*}-\frac{x^2}{2ut_*}} - e^{\frac{\eta x}{t_*}})dx| + \int\limits_{|x|>M} G(u,x)e^{\frac{\eta x}{t_*}}dx \\ &=: |I_1| + I_2. \end{aligned}$$

Since the variance of Z_u (see (5.15)) converges to those of BM we have by Borell-TIS inequality for $x>0$, large u and some $C>0$

$$\begin{aligned} G(u,x) &\leq \mathbb{P}\{\exists t\in[-S,0):(Z_u(t)+\eta_* t)>x\}+\mathbb{P}\{\exists t\in[0,S]:(B_*(t)-c_1t)>x\} \\ &\leq \mathbb{P}\{\exists t\in[-S,0]:(Z_u(t)-\mathbb{E}\{Z_u(t)\})>x\}+\mathbb{P}\{\exists t\in[0,S]:B_*(t)>x\} \quad (5.19)\\ &\leq e^{-x^2/C}. \end{aligned}$$

Let $u>M^4$. For $x\in[-M,M]$ it holds that $1-e^{-\frac{x^2}{2ut_*}}\leq\frac{x^2}{2ut_*}\leq\frac{1}{M}$ and hence for $u>M^4$ by (5.19) we have as $M\to\infty$

$$|I_1| \leq \int_{-M}^{0} e^{\frac{\eta x}{t_*}}(1-e^{-\frac{x^2}{2ut_*}})dx+\int_0^M e^{-x^2/C+\frac{\eta x}{t_*}}(1-e^{-\frac{x^2}{2ut_*}})dx\leq\frac{1}{M}\Big(\int_{-\infty}^0 e^{\frac{\eta x}{t_*}}+\int_0^\infty e^{-x^2/C+\frac{\eta x}{t_*}}\Big)\to 0.$$

For I_2 we have

$$I_2\leq\int_{-\infty}^{-M} e^{\frac{\eta x}{t_*}}dx+\int_M^\infty e^{-x^2/C}e^{\frac{\eta x}{t_*}}dx\to 0,\quad M\to\infty,$$

hence (5.18) holds. Next we show that

$$G(u,x) \to \mathbb{P}\left\{\int_{-S}^{0}\mathbb{I}\Big(B(s)+\eta_* s>x\Big)ds+\int_0^S\mathbb{I}(B_*(s)-c_1s>x)ds>T\right\},\quad u\to\infty$$

that is equivalent with

$$\lim_{u\to\infty}\mathbb{P}\left\{\int_{-S}^{S}\mathbb{I}\Big(X_u(s)>x\Big)ds>T\right\}=\mathbb{P}\left\{\int_{-S}^{S}\mathbb{I}\Big(B(s)+k(s)>x\Big)ds>T\right\},$$

where $k(s)=\mathbb{I}(s<0)\eta_* s-\mathbb{I}(s\geq 0)c_1 s$ and

$$X_u(t)=(Z_u(t)+\eta_* t)\mathbb{I}(t<0)+(B_*(t)-c_1t)\mathbb{I}(t\geq 0).$$

We have for large u

$$\mathbb{E}\left\{(X_u(t)-X_u(s))^2\right\}=\begin{cases}|t-s|+|t-s|^2 & t,s\geq 0\\ -\frac{(s-t)^2}{ut_*}+|t-s|+\frac{x^2(t-s)^2}{u^2t_*^2}-\frac{2x(t-s)^2\eta_*}{ut_*}+\eta_*^2(t-s)^2 & t,s\leq 0\\ |t-s|-\frac{s^2}{ut_*}+\frac{x^2s^2}{u^2t_*^2}-\frac{2xs(\eta_* s+c_1t)}{ut_*}+(\eta_* s+c_1t)^2 & s<0<t\end{cases}$$

implying for all u large enough, some $C>0$ and $t,s\in[-S,S+T]$

$$\mathbb{E}\left\{(X_u(t)-X_u(s))^2\right\}\leq C|t-s|.$$

Next, by Proposition 9.2.4 in [59] the family $X_u(t),\ u>0,\ t\in[-S,S+T]$ is tight in $\mathcal{B}(C([-S,S+T]))$ (Borell σ-algebra in the space of the continuous functions on $[-S,S+T]$ generated by the cylindric sets).

As follows from (5.15), $Z_u(t)$ converges to $B(t)$ in the sense of convergence finite-dimensional distributions as $u \to \infty$, $t \in [-S, S+T]$. Thus, by Theorems 4 and 5 in Chapter 5 in [8] the tightness and convergence of finite-dimensional distributions imply weak convergence

$$X_u(t) \Rightarrow B(t) + k(t) =: W(t), \quad t \in [-S, S+T].$$

By Theorem 11 (Skorohod), Chapter 5 in [8] there exists a probability space Ω, where all random processes have the same distributions, while weak convergence becomes convergence almost sure. Thus, we assume that $X_u(t) \to W(t)$ a.s. as $u \to \infty$ as elements of $C[-S, S]$ space with the uniform metric. We shall prove that for all $x \in \mathbb{R}$

$$\mathbb{P}\left\{ \lim_{u\to\infty} \int_{-S}^{S} \mathbb{I}(X_u(t) > x)dt = \int_{-S}^{S} \mathbb{I}(W(t) > x)dt \right\} = 1. \tag{5.20}$$

Fix $x \in \mathbb{R}$. We shall show that as $u \to \infty$ with probability 1

$$\mu_\Lambda\{t \in [-S, S] : X_u(\omega, t) > x > W(\omega, t)\} + \mu_\Lambda\{t \in [-S, S] : W(\omega, t) > x > X_u(\omega, t)\} \to 0, \tag{5.21}$$

where μ_Λ is the Lebesgue measure. Since for any fixed $\varepsilon > 0$ for large u and $t \in [-S, S]$ with probability one $|W(t) - X_u(t)| < \varepsilon$ we have that

$$\begin{aligned} & \mu_\Lambda\{t \in [-S, S] : X_u(\omega, t) > x > W(\omega, t)\} + \mu_\Lambda\{t \in [-S, S] : W(\omega, t) > x > X_u(\omega, t)\} \\ \leq \; & \mu_\Lambda\{t \in [-S, S] : W(\omega, t) \in [-\varepsilon + x, \varepsilon + x]\}. \end{aligned}$$

Thus, (5.21) holds if

$$\mathbb{P}\left\{ \lim_{\varepsilon\to 0} \mu_\Lambda\{t \in [-S, S] : W(t) \in [-\varepsilon + x, x + \varepsilon]\} = 0 \right\} = 1. \tag{5.22}$$

Consider the subset $\Omega_* \subset \Omega$ consisting of all ω_* such that

$$\lim_{\varepsilon\to 0} \mu_\Lambda\{t \in [-S, S] : W(\omega_*, t) \in [-\varepsilon + x, x + \varepsilon]\} > 0.$$

Then for each ω_* there exists the set $\mathcal{A}(\omega_*) \subset [-S, S]$ such that $\mu_\Lambda\{\mathcal{A}(\omega_*)\} > 0$ and for $t \in \mathcal{A}(\omega_*)$ it holds that $W(\omega_*, t) = x$. Thus,

$$\mathbb{P}\{\Omega_*\} = \mathbb{P}\{\mu_\Lambda\{t \in [-S, S] : W(t) = x\} > 0\},$$

the right side of the equation above equals 0 by Lemma 5.4.1 below. Hence we conclude that (5.22) holds, consequently (5.21) and (5.20) are true. Since convergence almost sure implies convergence in distribution we have by (5.20) that for any fixed $x \in \mathbb{R}$

$$\lim_{u\to\infty} \mathbb{P}\left\{ \int_{-S}^{S} \mathbb{I}(X_u(t) > x)dt > T \right\} = \mathbb{P}\left\{ \int_{-S}^{S} \mathbb{I}(W(t) > x)dt > T \right\}.$$

By the dominated convergence theorem we obtain

$$\int_{-M}^{M} G(u,x)e^{\frac{\eta x}{t_*}}dx \to \int_{-M}^{M} \mathbb{P}\left\{\int_{-S}^{0} \mathbb{I}\Big(B(s)+\eta_* s > x\Big)ds + \int_{0}^{S} \mathbb{I}(B_*(s)-c_1 s > x)ds > T\right\} e^{\frac{\eta x}{t_*}}dx, \quad u\to\infty.$$

Thus, the claim follows from the line above and (5.18). □

Lemma 5.4.1 *For any $c>0$ and $x\in\mathbb{R}$*

$$\mathbb{P}\left\{\mu_\Lambda\{t\in[0,\infty) : B(t)-ct = x\} > 0\right\} = 0. \tag{5.23}$$

Proof of Lemma 5.4.1. Let for some fixed $x\in\mathbb{R}$ the assertion of the lemma does not hold. Thus, $\xi_x = \int_0^\infty \mathbb{I}(B(t)-ct>x)dt$ does not have a continuous df. That contradicts to the explicit expression of the df of ξ_x given in formula (3), p. 261 in [5]. □

Proof of (5.17)**.** We have by the proof of Theorem 3.1 in [45], Case (3) and the final asymptotics of $\mathbb{C}_{T_u}(u)$ given in (5.8)

$$\begin{aligned} &\mathbb{P}\left\{\int_{[0,\infty)\setminus[ut_*-u\delta_u,ut_*+u\delta_u]} \mathbb{I}(B_H(t)-c_1t>q_1u, B_H(t)-c_2t>q_2u)dt > T_u\right\}\\ \le\ & \mathbb{P}\left\{\exists t\in[0,\infty)\setminus[ut_*-u\delta_u,ut_*+u\delta_u] : B_H(t)-c_1t>q_1u, B_H(t)-c_2t>q_2u\right\}\\ =\ & o(\mathbb{C}_{T_u}(u)), \quad u\to\infty \end{aligned}$$

and hence

$$\mathbb{P}\left\{\int_{[ut_*-u\delta_u,ut_*+u\delta_u]} \mathbb{I}(B_H(t)-c_1t>q_1u, B_H(t)-c_2t>q_2u)dt > T_u\right\} \sim \mathbb{C}_{T_u}(u), \quad u\to\infty.$$

The last probability above is equivalent with $g_1(u)+g_2(u)$ as $u\to\infty$, this observation follows from the application of the double-sum method, see the proofs of Theorem 3.1, Case (3) $H<1/2$ in [45] and Theorem 2.1 in [21] case i). □

Proof of Proposition 5.3.1. If $H=1/2$, then an equality takes place, see [21], Eq. [5].

Assume that $H\neq 1/2$. First let (5.5) holds with $T>0$. We have for $c>0$ with $\widetilde{M}(u) = u^{1-H}\frac{c^H}{(1-H)^{1-H}H^H}$ (recall, $\mathbb{I}_a(b)=\mathbb{I}(b>a),\ a,b\in\mathbb{R}$)

$$h_{T_u}(u) \ := \ \mathbb{P}\left\{\int_0^\infty \mathbb{I}(B_H(t)-ct>u)dt > T_u\right\}$$

$$= \quad \mathbb{P}\left\{ u(u^{\frac{1}{H}-2}\frac{c^2(1-H)^{2-\frac{1}{H}}}{2^{\frac{1}{2H}}H^2}) \int_0^\infty \mathbb{I}_{\widetilde{M}(u)}(\frac{B_H(tu)\widetilde{M}(u)}{u(1+ct)})dt > T\frac{c^2(1-H)^{2-\frac{1}{H}}}{2^{\frac{1}{2H}}H^2} \right\}.$$

Next we apply Theorem 3.1 in [21] to calculate the asymptotics of the last probability above as $u \to \infty$. For the parameters in the notation therein we have

$$\alpha_0 = \alpha_\infty = H,\ \sigma(t) = t^H,\ \overleftarrow{\sigma}(t) = t^{\frac{1}{H}},\ t^* = \frac{H}{c(1-H)},\ A = \frac{c^H}{H^H(1-H)^{1-H}},\ x = T\frac{c^2(1-H)^{2-\frac{1}{H}}}{2^{\frac{1}{2H}}H^2}$$

$$B = \frac{c^{2+H}(1-H)^{2+H}}{H^{H+1}},\ M(u) = u^{1-H}\frac{c^H}{(1-H)^{1-H}H^H},\ v(u) = u^{\frac{1}{H}-2}\frac{c^2(1-H)^{2-\frac{1}{H}}}{2^{\frac{1}{2H}}H^2}.$$

and hence we obtain

$$h_{T_u}(u) \sim K_H \mathcal{B}_{2H}(TD)(C_H u^{1-H})^{\frac{1}{H}-1}\overline{\Phi}(C_H u^{1-H}), \quad u \to \infty, \tag{5.24}$$

where

$$C_H = \frac{c^H}{H^H(1-H)^{1-H}} \quad \text{and} \quad D = 2^{-\frac{1}{2H}}c^2H^{-2}(1-H)^{2-1/H}.$$

Assume that (5.5) holds with $T = 0$. For $\varepsilon > 0$ for all large u we have $h_{\varepsilon u^{1/H-2}}(u) \le h_{T_u}(u) \le h_0(u)$ and thus

$$K_H\mathcal{B}_{2H}(\varepsilon D)(C_H u^{1-H})^{\frac{1}{H}-1}\overline{\Phi}(C_H u^{1-H}) \le h_{T_u}(u) \le K_H\mathcal{B}_{2H}(0)(C_H u^{1-H})^{\frac{1}{H}-1}.$$

Since $\mathcal{B}_{2H}(\cdot)$ is a continuous function (Lemma 4.1 in [21]) letting $\varepsilon \to 0$ we obtain (5.24) for any T_u satisfying (5.5). Replacing in (5.24) u and c by $q_1 u$ and c_1 we obtain the claim. □

Chapter 6

Extremes of Reflecting Gaussian Processes on Discrete Grid

This chapter is based on K. Dębicki and G. Jasnovidov: Extremes of reflecting Gaussian processes on discrete Grid, in progress, 2021.

6.1 Introduction

For $X(t), t \ge 0$ a centered Gaussian process with a.s. continuous sample paths, stationary increments and variance function $\sigma^2(t) := \mathrm{Var}(X(t))$ such that $\sigma^2(0) = 0$, consider the *reflected* (at 0) process

$$\hat{Q}_X(t) = X(t) - ct + \max\left(\hat{Q}_X(0), -\inf_{s\in[0,t]}(X(s) - cs)\right), \ t \ge 0, \tag{6.1}$$

where $c > 0$. The motivation for the investigation of properties of $\hat{Q}_X(t)$ stems from its relation with the solution of the *Skorokhod problems* and their applications to queueing theory, ruin theory and financial mathematics. In particular, the behavior of the buffer content in a fluid queueing model fed by X and emptied at rate c evolves according to (6.1).

Distributional properties of the unique stationary solution of (6.1), which has the following representation

$$Q_X(t) = \sup_{t\le s}(X(s) - X(t) - c(s-t)), \tag{6.2}$$

were intensively analyzed in, e.g., [11, 27, 36, 37]. The extremes of (6.2) were investigated in [19, 20, 58].

From the point of view of the stochastic modelling, discrete-time models frequently appear to be more natural. However, despite of its relevance in modelling of, e.g., queueing systems, much less is known on distributional properties of the discrete counterpart of (6.2), i.e.,

$$Q_{\delta,X}(t) = \sup_{s\in[t,\infty)\cap G_\delta}\big(X(s) - X(t) - c(s-t)\big), \ t \in G_\delta. \tag{6.3}$$

A notable exception is a recent work [42], where the exact asymptotics of $\mathbb{P}\{Q_{\delta,B_H}(0) > u\}$, as $u \to \infty$, was derived for $H \in (0,1)$.

In this contribution we extend the findings of [40, 42, 47] to a more general class of Gaussian processes with stationary increments and derive the exact asymptotics of

$$\psi_{T,\delta}^{\sup}(u) := \mathbb{P}\left\{\sup_{t\in[0,T]_\delta} Q_{\delta,X}(t) > u\right\}, \quad \psi_{T,\delta}^{\inf}(u) := \mathbb{P}\left\{\inf_{t\in[0,T]_\delta} Q_{\delta,X}(t) > u\right\}, \tag{6.4}$$

as $u \to \infty$, for $T > 0$ and $\delta > 0$, complementing results for continuous time given in [19, 58]. It appears that the influence of the grid size δ in (6.4) strongly depends on

$$\varphi := \lim_{u\to\infty} \frac{\sigma^2(u)}{u} \in [0, \infty],$$

leading to three scenarios: $\varphi = 0$, $\varphi \in (0, \infty)$ and $\varphi = \infty$. The case $\varphi = \infty$ leads to the same asymptotics as its continuous-time counterpart, which reflects the *long-range dependance property* of X when its variance σ^2 is superlinear.

6.2 Notation and preliminary results

Let $X(t), t \in \mathbb{R}$ be a centered Gaussian process with stationary increments, as introduced in Section 6.1. Suppose that

A: σ^2 is regularly varying at ∞ with index $2\alpha \in (0, 2)$ and $\sigma^2(t)$ is twice continuously differentiable for any $t \in (0, \infty)$. Further, the first and second derivatives of σ^2 are ultimately monotone;

B: σ^2 satisfies

$$\varphi := \lim_{u\to\infty} \frac{\sigma^2(u)}{u} \in [0, \infty];$$

Note that if $\alpha < 1/2$, then $\varphi = 0$ and if $\alpha > 1/2$, then $\varphi = \infty$. If $\alpha = 1/2$, then φ can be either 0, ∞ or finite and positive constant.

C: if $\varphi = 0$ and $\alpha = \frac{1}{2}$, then for $\kappa = \sqrt{c \inf_{t\in\{\delta,2\delta,\dots\}} \sigma(t)} - \varepsilon$, with sufficiently small $\varepsilon > 0$,

$$\sigma(u) \le \kappa \frac{\sqrt{u}}{\ln^{1/4} u}, \qquad u \to \infty. \tag{6.5}$$

Conditions **A** and **B** are satisfied for a wide class of Gaussian processes with stationary increments, including family of fractional Brownian motions and integrated stationary Gaussian processes; see Section 6.4 for details. We note that condition **B** already appeared in [27], where it was observed that the form of the asymptotic behavior, as $u \to \infty$, of $\mathbb{P}\{Q_X(0) > u\}$ introduced in (6.2) is determined by the value of φ. It appears that cases where σ^2 is asymptotically close to a linear function need particularly precise analysis, for which condition **C** is a tractable assumption.

Let

$$\mathcal{H}_\xi(M) = \mathbb{E}\left\{\sup_{t\in M} e^{\sqrt{2}\xi(t)-\mathrm{Var}(\xi(t))}\right\} \in (0,\infty), \quad \mathcal{H}_\xi^{\inf}(M) = \mathbb{E}\left\{\inf_{t\in M} e^{\sqrt{2}\xi(t)-\mathrm{Var}(\xi(t))}\right\} \in (0,\infty), \quad (6.6)$$

where M is a compact subset of $\mathbb{R}$ and $\xi(t),\ t\in\mathbb{R}$ is a Gaussian field with stationary increments and a.s. continuous sample paths. Define Pickands constant by

$$\mathcal{H}_\xi^\delta = \lim_{S\to\infty} \frac{\mathcal{H}_\xi([0,S]\cap\delta\mathbb{Z})}{S}, \qquad \delta \geq 0,$$

where we set $\delta\mathbb{Z}=\mathbb{R}$ if $\delta=0$. From [11] it follows that $\mathcal{H}_\xi^0 \in (0,\infty)$ under **A** and some additional smoothness conditions on ξ, while for $\delta>0$ we prove in Lemma 6.5.4 that it is sufficient to suppose that ξ satisfies **A** to claim that $\mathcal{H}_\xi^\delta \in (0,\infty)$. Later on, for $\delta=0$ we simply write $\mathcal{H}_\xi$ instead of $\mathcal{H}_\xi^0$.

The following result from [27][Proposition 2] (see also [20] [Theorems 3.1-3.3]) will be a useful reference to the new results presented in the next section. Let $\overleftarrow{\sigma}(t), t\geq 0$ stands for the asymptotic inverse function of σ, i.e., $\overleftarrow{\sigma}(x) = \inf\{y\in[0,\infty) : f(y) > x\}$ (for details and properties of the asymptotic inverse functions see, e.g., [49]) and let

$$t_* = \frac{\alpha}{c(1-\alpha)}, \qquad m(u) = \inf_{t>0}\frac{u(1+ct)}{\sigma(ut)}, \qquad \Delta(u) = \begin{cases} \overleftarrow{\sigma}\left(\frac{\sqrt{2}\sigma^2(ut_*)}{u(1+ct_*)}\right), & \varphi\notin(0,\infty) \\ 1, & \varphi\in(0,\infty). \end{cases} \qquad (6.7)$$

Let for X such that $\varphi\in(0,\infty)$,

$$\eta(t) = \frac{c\sqrt{2}}{\varphi}X(t), \qquad t\geq 0. \qquad (6.8)$$

As shown in [27][Proposition 2], if σ^2 is regularly varying at 0 with index $2\alpha_0\in(0,2]$ and **A** is satisfied, then the following result holds:

$$\mathbb{P}\{Q_X(0)>u\} \sim f(u)\overline{\Phi}(m(u)) \times \begin{cases} \mathcal{H}_{B_\alpha}, & \varphi=\infty \\ \mathcal{H}_\eta, & \varphi\in(0,\infty)\ , \quad u\to\infty, \\ \mathcal{H}_{B_{\alpha_0}}, & \varphi=0 \end{cases} \qquad (6.9)$$

where

$$f(u) = \sqrt{\frac{2\pi A}{B}}\frac{u}{m(u)\Delta(u)}, \qquad A = \frac{1}{(1-\alpha)t_*^\alpha}, \qquad B = \frac{\alpha}{t_*^{\alpha+2}}. \qquad (6.10)$$

The following result establishes the asymptotics of $\mathbb{P}\{Q_{\delta,X}(0)>u\}$ for $\delta>0$ as $u\to\infty$. It generalizes the findings of [42, 47].

Theorem 6.2.1 *Let $X(t), t\geq 0$ be a centered Gaussian process with continuous trajectories and stationary increments satisfying* **A, B, C**. *Then, for $\delta>0$, as $u\to\infty$, it holds that*

$$\mathbb{P}\{Q_{\delta,X}(0)>u\} \sim \overline{\Phi}(m(u)) \times \begin{cases} \frac{\sqrt{2\pi\alpha}u}{\delta c(1-\alpha)^{3/2}m(u)}, & \varphi=0 \\ \mathcal{H}_\eta^\delta f(u), & \varphi\in(0,\infty) \\ \mathcal{H}_{B_\alpha}f(u), & \varphi=\infty. \end{cases} \qquad (6.11)$$

Remarks 6.2.2 *Comparing the asymptotics in* (6.9) *and* (6.11) *we have under* **A-C** *and the smooth conditions on* σ^2 *necessary for* (6.9), *with* $\tilde{C} = \frac{\mathcal{H}_\eta^\delta}{\mathcal{H}_\eta} \in (0,1)$

$$\lim_{u\to\infty} \frac{\mathbb{P}\left\{Q_{\delta,X}(0) > u\right\}}{\mathbb{P}\left\{Q_X(0) > u\right\}} = \begin{cases} 0, & \varphi = 0, \\ \tilde{C}, & \varphi \in (0,\infty), \\ 1, & \varphi = \infty. \end{cases}$$

6.3 Main Results

In this section we derive the exact asymptotics of

$$\psi_{T,\delta}^{\sup}(u) := \mathbb{P}\left\{ \sup_{t\in[0,T]_\delta} Q_{\delta,X}(t) > u \right\}, \quad \psi_{T,\delta}^{\inf}(u) := \mathbb{P}\left\{ \inf_{t\in[0,T]_\delta} Q_{\delta,X}(t) > u \right\}, \tag{6.12}$$

as $u \to \infty$, for $T > 0$ and $\delta > 0$.

Theorem 6.3.1 *Let* $X(t), t \ge 0$ *be a centered Gaussian process with continuous trajectories and stationary increments satisfying* **A-C**. *Then for* $\delta > 0$ *as* $u \to \infty$ *it holds that*

$$\psi_{T,\delta}^{\sup}(u) \sim \overline{\Phi}(m(u)) \times \begin{cases} (1 + [\frac{T}{\delta}])\frac{\sqrt{2\pi\alpha u}}{\delta c(1-\alpha)^{3/2} m(u)}, & \varphi = 0 \\ \mathcal{H}_\eta([0,T]_\delta)\mathcal{H}_\eta^\delta f(u), & \varphi \in (0,\infty) \\ \mathcal{H}_{B_\alpha} f(u), & \varphi = \infty. \end{cases}$$

Theorem 6.3.2 *Let* $X(t), t \ge 0$ *be a centered Gaussian process with continuous trajectories and stationary increments satisfying* **A-C**. *Then for* $\delta > 0$ *as* $u \to \infty$ *it holds that*

$$\psi_{T,\delta}^{\inf}(u) \sim f(u)\overline{\Phi}(m(u)) \times \begin{cases} \mathcal{H}_\eta^{\inf}([0,T]_\delta)\mathcal{H}_\eta^\delta, & \varphi \in (0,\infty) \\ \mathcal{H}_{B_\alpha}, & \varphi = \infty. \end{cases}$$

If $\varphi = 0$, then for the non-degenerated scenario (when set $[0,T]_\delta$ consists of more than 1 element, i.e., for $T \ge \delta$) it seems difficult to derive even logarithmic asymptotics of $\psi_{T,\delta}^{\inf}(u)$. One can argue that $\psi_{T,\delta}^{\inf}(u)$ is exponentially smaller than $\mathbb{P}\left\{Q_{\delta,X}(0) > u\right\}$ in this case, as $u \to \infty$. We have the following proposition giving an upper bound for $\psi_{T,\delta}^{\inf}(u)$.

Proposition 6.3.3 *If* $\varphi = 0$ *and for some small* $\varepsilon > 0$

$$\sigma(u) \le \frac{\sqrt{u}}{\ln^{1/4+\varepsilon} u}, \qquad u \to \infty, \tag{6.13}$$

then for $T \ge \delta$ *with any* $\widetilde{\mathbb{C}} < \frac{1+ct_*}{2t_*^{2\alpha}} \sup_{t\in[0,T]_\delta} \sigma(t)$ *it holds that*

$$\psi_{T,\delta}^{\inf}(u) \le \overline{\Phi}(m(u))\overline{\Phi}(\widetilde{\mathbb{C}}\frac{u}{\sigma^2(u)}), \qquad u \to \infty.$$

All our results hold without additional smoothness assumptions in contrary to the continuous time case studied in [20, 27]. Intuition behind absence of known behavior σ^2 around zero is that one can consider a random process on a discrete grid with step $\delta > 0$ as a sequence of correlated Gaussian random variables. The covariance of the sequence is determined only by its values on the grid, hence assumptions concerning behavior of σ at 0 are not necessary.

Remarks 6.3.4 *Comparing the asymptotics in Theorems 6.2.1, 6.3.1 and 6.3.2 for case $\varphi = \infty$ we observe the so-called strong Piterbarg property for the storage process:*

$$\psi^{\sup}_{T,\delta}(u) \sim \mathbb{P}\left\{Q_{\delta,X}(0) > u\right\} \sim \psi^{\inf}_{T,\delta}(u), \qquad T \geq 0,\ u \to \infty.$$

The statement above agrees with Theorem 1 in [19].

Remarks 6.3.5 *If $\varphi = 0$, $\alpha = 1/2$ in the theorem above and* (6.5) *does not hold, then from the proof it follows that* (6.11) *reduces to the upper bound.*

6.4 Examples

Fractional Brownian motion. Let

$$C_H = \frac{c^H}{H^H(1-H)^{1-H}}, \quad D_H = \frac{\sqrt{2\pi}H^{H+1/2}}{c^{H+1}(1-H)^{H+1/2}}, \quad E_H = \frac{2^{\frac{1}{2}-\frac{1}{2H}}\sqrt{\pi}}{H^{1/2}(1-H)^{1/2}}.$$

Applying Theorems 6.2.1, 6.3.1 and 6.3.2 for X being a standard fBM we obtain the following results:

Corollary 6.4.1 *As $u \to \infty$ it holds that*

$$\mathbb{P}\left\{Q_{\delta,B_H}(0) > u\right\} \sim \begin{cases} \frac{D_H u^H}{\delta}\overline{\Phi}(C_H u^{1-H}), & H < 1/2 \\ \mathcal{H}^{2c^2\delta}_{B_{1/2}} e^{-2cu}, & H = 1/2 \\ \mathcal{H}_{B_H} E_H (C_H u^{1-H})^{1/H-1}\overline{\Phi}(C_H u^{1-H}), & H > 1/2. \end{cases}$$

Corollary 6.4.2 *For $T, \delta > 0$ as $u \to \infty$ it holds that*

$$\mathbb{P}\left\{\sup_{t\in[0,T]_\delta} Q_{\delta,B_H}(t) > u\right\} \sim \begin{cases} (1+[\frac{T}{\delta}])\frac{D_H u^H}{\delta}\overline{\Phi}(C_H u^{1-H}), & H < 1/2 \\ \mathcal{H}_{B_{1/2}}([0,2c^2T]_{2c^2\delta})\mathcal{H}^{2c^2\delta}_{B_{1/2}} e^{-2cu}, & H = 1/2 \\ \mathcal{H}_{B_H} E_H (C_H u^{1-H})^{1/H-1}\overline{\Phi}(C_H u^{1-H}), & H > 1/2. \end{cases}$$

Corollary 6.4.3 *For $T, \delta > 0$ as $u \to \infty$ it holds that*

$$\mathbb{P}\left\{\inf_{t\in[0,T]_\delta} Q_{\delta,B_H}(t) > u\right\} \sim \begin{cases} \mathcal{H}^{\inf}_{B_{1/2}}([0,2c^2T]_{2c^2\delta})\mathcal{H}^{2c^2\delta}_{B_{1/2}} e^{-2cu}, & H = 1/2 \\ \mathcal{H}_{B_H} E_H (C_H u^{1-H})^{1/H-1}\overline{\Phi}(C_H u^{1-H}), & H > 1/2. \end{cases}$$

Note that Corollary 6.4.1 intersects with the results in [40, 42, 47] while Corollaries 6.4.2 and 6.4.3 are discrete counterparts of Theorems 5-7 in [58] and Theorem 1 in [19], respectively.

Gaussian integrated process of SRD and LRD type. For a stationary centered Gaussian process with a.s. continuous sample paths $\zeta(s),\ s\geq 0$ define the integrated process by

$$Z(t)=\int_0^t \zeta(s)ds, \qquad t\geq 0. \tag{6.14}$$

This process is also Gaussian, has a.s. continuous sample paths and stationary increments. In what follows we consider two classes of processes Z, which differ by property of the correlation function $R(t):=\mathbb{E}\left\{\zeta(0)\zeta(t)\right\}$ of ζ as $t\to\infty$.

SRD case. Following, e.g., [11] (see also [24]), we impose the following conditions on the correlation of ζ:

R1: $R(t)\in C([0,\infty)),\ \lim\limits_{t\to\infty} tR(t)=0$;

R2: $\int_0^t R(s)ds>0$ for all $t\in(0,\infty]$;

R3: $\int_0^\infty t^2|R(t)|dt<\infty$.

The above assertions imply the existence of the first and second derivatives of $\sigma_Z^2(t)=Var(Z(t))$ and establish the asymptotic behavior of $\sigma_Z^2(t)$ at ∞ (see e.g., Remark 6.1 in [11]):

$$\sigma_Z^2(t)=\frac{2}{G}t-2D+o(t^{-1}), \qquad t\to\infty,$$

where $G=1/\int_0^\infty R(t)dt$ and $D=\int_0^\infty tR(t)dt$. Thus, σ_Z^2 satisfies **A** with $\alpha=1/2$ and applying Theorems 6.2.1, 6.3.1 and 6.3.2 for scenario $\varphi\in(0,\infty)$ we have

Corollary 6.4.4 *If $Z(t)$ is an integrated process defined in* (6.14) *and $R(t)$ satisfies* ***R1-R3****, then for $T\geq 0$ and $\delta>0$ as $u\to\infty$*

$$\begin{aligned}
\mathbb{P}\left\{Q_{\delta,Z}(0)>u\right\} &\sim \mathcal{A}\mathcal{H}_\xi^\delta e^{-cGu},\\
\mathbb{P}\left\{\sup_{t\in[0,T]_\delta} Q_{\delta,Z}(t)>u\right\} &\sim \mathcal{A}\mathcal{H}_\xi([0,T]_\delta)\mathcal{H}_\xi^\delta e^{-cGu},\\
\mathbb{P}\left\{\inf_{t\in[0,T]_\delta} Q_{\delta,Z}(t)>u\right\} &\sim \mathcal{A}\mathcal{H}_\xi^{\inf}([0,T]_\delta)\mathcal{H}_\xi^\delta e^{-cGu},
\end{aligned}$$

where $\mathcal{A}=\frac{1}{c^2Ge^{c^2G^2D}}$ and $\xi(t)=cGZ(t)/\sqrt{2}$.

Note that the first asymptotics in Corollary 6.4.4 differs from its continuous-time analog (Theorem 5.1 in [11]) only by the corresponding Pickands constants.

LRD case. Following, e.g., [27, 36] we characterize LRD case by the following assumptions on $R(t)$:

L1: $R(t)$ is a continuous strictly positive function for $t \geq 0$;

L2: $R(t)$ is regularly varying at ∞ with index $2\alpha - 2$, $\alpha \in (1/2, 1)$.

Under the above assumptions, by Karamata's theorem, σ_Z^2 is regularly varying at ∞ with index 2α. Since $2\alpha > 1$ we are in $\varphi = \infty$ scenario. Hence, applying Theorems 6.2.1, 6.3.1 and 6.3.2 we immediately obtain the following result.

Corollary 6.4.5 *If $Z(t)$ is an integrated process defined in* (6.14) *and $R(t)$ satisfies* ***L1-L2****, then as $u \to \infty$ it holds that*

$$\mathbb{P}\left\{\inf_{t\in[0,T]_\delta} Q_{\delta,Z}(t) > u\right\} \sim \mathbb{P}\left\{Q_{\delta,Z}(0) > u\right\} \sim \mathbb{P}\left\{\sup_{t\in[0,T]_\delta} Q_{\delta,Z}(t) > u\right\}$$
$$\sim \mathcal{H}_{B_{2\alpha}} f(u)\Psi(m(u)),$$

where $m(u)$ and $f(u)$ are defined in (6.7) *and* (6.10)*, respectively. In this setup of the problem we observe that the strong Piterbarg's property holds.*

6.5 Proofs

In this section we give proofs of all results. Hereafter, denote by $\overline{X} := \frac{X}{\sqrt{Var(X)}}$ for any nontrivial random variable X. For any $u > 0$ we have

$$\mathbb{P}\left\{Q_{\delta,X}(0) > u\right\} = \mathbb{P}\left\{\sup_{t\in G_\delta}(X(t) - ct) > u\right\} = \mathbb{P}\left\{\sup_{t\in G_{\delta/u}} X_u(t) > m(u)\right\},$$

where $m(u)$ is defined in (6.7) and

$$X_u(t) = \frac{X(ut)}{u(1+ct)} m(u).$$

Denote by $\sigma_{X_u}^2$ the variance function of $X_u(t)$, $t \geq 0$. In the next lemma we focus on asymptotic properties of the variance and correlation functions of $X_u(t)$; we refer to, e.g., [20] for the proof.

Lemma 6.5.1 *Suppose that* **A** *is satisfied. For u large enough the maximizer t_u of σ_{X_u} is unique and $t_u \to t_* = \frac{\alpha}{c(1-\alpha)}$ as $u \to \infty$. Moreover, for $\delta_u > 0$ satisfying $\lim_{u\to\infty} \delta_u = 0$ (A, B are defined in* (6.10)*)*

$$\lim_{u\to\infty} \sup_{t\in(t_u-\delta_u, t_u+\delta_u)\setminus\{t_u\}} \left| \frac{1-\sigma_{X_u}(t)}{\frac{B}{2A}(t-t_u)^2} - 1 \right| = 0$$

and (recall, σ^2 is the variance of X)

$$\lim_{u\to\infty} \sup_{s\neq t, s,t\in(t_u-\delta_u, t_u+\delta_u)} \left| \frac{1 - Cor\left(X(us), X(ut)\right)}{\frac{\sigma^2(u|s-t|)}{2\sigma^2(ut_*)}} - 1 \right| = 0.$$

By Lemma 6.5.1 we have that t_u is the unique minimizer of $\frac{u(1+ct)}{\sigma(ut)}$ for large u and hence by Potter's theorem (Theorem 1.5.6 in [4]) we obtain useful in the following proofs asymptotics of $m(u)$

$$m(u)=\frac{u(1+ct_u)}{\sigma(ut_u)}\sim\frac{u(1+ct_*)}{\sigma(ut_*)}\cdot\frac{\sigma(ut_*)}{\sigma(ut_u)}\sim\frac{u(1+ct_*)}{t_*^{\alpha}\sigma(u)},\quad u\to\infty. \tag{6.15}$$

Observe that

$$\psi_{T,\delta}^{\sup}(u)=\mathbb{P}\left\{\sup_{t\in[0,T/u]_{\delta/u},t\le s\in G_{\delta/u}} Z_u(t,s)>m(u)\right\},$$

where

$$Z_u(t,s)=\frac{X(us)-X(ut)}{u(1+c(s-t))}m(u). \tag{6.16}$$

Notice that for the variance $\sigma^2_{Z_u}$ of Z_u it holds, that $\sigma^2_{Z_u}(s,t)=\sigma^2_{X_u}(s-t)$ and for correlation r_{Z_u} we have for $\delta_u>0$ satisfying $\lim_{u\to\infty}\delta_u=0$ (Lemma 5.4. in [20])

$$\lim_{u\to\infty}\sup_{|t-t_1|<\delta_u,s-t,s_1-t_1\in(-\delta_u+t_u,t_u+\delta_u),(s,t)\neq(s_1,t_1)}\left|\frac{1-r_{Z_u}(s,t,s_1,t_1)}{\frac{\sigma^2(u|s-s_1|)+\sigma^2(u|t-t_1|)}{2\sigma^2(ut_*)}}-1\right|=0. \tag{6.17}$$

To the rest of the paper we suppose that

$$\delta_u=\begin{cases}u^{-1/2}\ln u, & \varphi<\infty\\ u^{-1}\ln(u)\sigma(u), & \varphi=\infty\end{cases}$$

and set

$$I(t_u)=G_{\frac{\delta}{u}}\bigcap(-\delta_u+t_u,t_u+\delta_u)$$

for $u>0$. The following lemma allow us to extract the main area contributing in the asymptotics of $\psi_{T,\delta}^{\sup}(u)$, $\psi_{T,\delta}^{\inf}(u)$ and $\mathbb{P}\{M_\delta>u\}$ as $u\to\infty$:

Lemma 6.5.2 *For any $T\ge 0$ it holds, that as $u\to\infty$*

$$\begin{aligned}\psi_{T,\delta}^{\sup}(u) &\sim \mathbb{P}\left\{\sup_{t\in[0,T/u]_{\delta/u},s\in I(t_u)} Z_u(t,s)>m(u)\right\}\\ \psi_{T,\delta}^{\inf}(u) &\sim \mathbb{P}\left\{\inf_{t\in[0,T/u]_{\delta/u},s\in I(t_u)} Z_u(t,s)>m(u)\right\}.\end{aligned}$$

The lemma below allows us to give upper bounds for the double-sum terms appearing in case $\varphi=0$ in Theorems 6.2.1 and 6.3.2.

Lemma 6.5.3 *Assume that $\varphi=0$. Then uniformly for $t\neq s\in I(t_u)$ and all large u with some $\varepsilon>0$ it holds that*

$$\mathbb{P}\{X_u(t)>m(u),X_u(s)>m(u)\}\le u^{-1/2-\varepsilon}\overline{\Phi}(m(u)).$$

In the next lemma we prove that the discrete Pickands constant appearing in Theorems 6.2.1, 6.3.1 and 6.3.2 is well defined, positive and finite.

Lemma 6.5.4 *For any $\delta \geq 0$ and η a centered Gaussian process with stationary increments, a.s. continuous sample paths and variance satisfying* $\boldsymbol{A}$ *it holds, that*

$$\lim_{S\to\infty} \frac{\mathcal{H}_\eta(\{0,\delta,...,S\})}{S} = \mathcal{H}_\eta^\delta \in (0,\infty). \tag{6.18}$$

We give the proofs of Lemmas 6.5.2, 6.5.3 and 6.5.4 at the end of this section. Now we are ready to prove main findings of this contribution.

Proof of Theorem 6.2.1: Taking $T = 0$ in Lemma 6.5.2 we obtain that

$$\mathbb{P}\left\{Q_{\delta_X}(0) > u\right\} \sim \mathbb{P}\left\{\sup_{t\in I(t_u)} X_u(t) > m(u)\right\}, \quad u\to\infty. \tag{6.19}$$

Next we consider 3 cases: case i) when $\varphi = 0$, case ii) when $\varphi \in (0,\infty)$ and case iii) when $\varphi = \infty$.

Case i). We have by Bonferroni inequality

$$\begin{aligned}\sum_{t\in I(t_u)} \mathbb{P}\left\{X_u(t) > m(u)\right\} &\geq \mathbb{P}\left\{\sup_{t\in I(t_u)} X_u(t) > m(u)\right\} \\ &\geq \sum_{t\in I(t_u)} \mathbb{P}\left\{X_u(t) > m(u)\right\} - \sum_{t\neq s\in I(t_u)} \mathbb{P}\left\{X_u(t) > m(u), X_u(s) > m(u)\right\}.\end{aligned} \tag{6.20}$$

There are less then $\mathbb{C}u\ln^2 u$ summands in the double-sum above, hence by Lemma 6.5.3 we have

$$\sum_{t\neq s\in I(t_u)} \mathbb{P}\left\{X_u(t) > m(u), X_u(s) > m(u)\right\} \leq \mathbb{C}\ln^2(u)u^{1/2-\varepsilon'}\overline{\Phi}(m(u)), \quad u\to\infty. \tag{6.21}$$

Next we focus on calculation of the single sum in (6.20). Since by Lemma 6.5.1 $\sup_{t\in I(t_u)} |\sigma_{X_u}(t)-1| \to 0$ as $u\to\infty$ (2.7) implies as $u\to\infty$

$$\begin{aligned}\sum_{t\in I(t_u)} \mathbb{P}\left\{X_u(t) > m(u)\right\} &= \sum_{t\in I(t_u)} \overline{\Phi}\left(\frac{m(u)}{\sigma_{X_u}(t)}\right) \\ &\sim \sum_{t\in I(t_u)} \frac{\sigma_{X_u}(t)}{\sqrt{2\pi}m(u)} e^{-\frac{m^2(u)}{2\sigma^2_{X_u}(t)}} \\ &\sim \frac{e^{-\frac{m^2(u)}{2}}}{\sqrt{2\pi}m(u)} \sum_{t\in I(t_u)} e^{-\frac{m^2(u)}{2\sigma^2_{X_u}(t)}+\frac{m^2(u)}{2}} \\ &\sim \overline{\Phi}(m(u)) \sum_{t\in I(t_u)} e^{-\frac{m^2(u)}{2}\frac{(1-\sigma^2_{X_u}(t))}{\sigma^2_{X_u}(t)}}.\end{aligned}$$

By Lemma 6.5.1 we have that as $u \to \infty$ the last sum above is equivalent to

$$
\begin{aligned}
\sum_{t \in I(t_u)} e^{-m^2(u)\frac{B}{2A}(t-t_u)^2} &= \sum_{t \in (-\frac{\ln u}{\sqrt{u}}, \frac{\ln u}{\sqrt{u}})_{\delta/u}} e^{-m^2(u)\frac{B}{2A}t^2} \\
&= \frac{u}{\delta m(u)} \Big(\frac{\delta m(u)}{u} \sum_{t \in (-\frac{m(u)\ln u}{\sqrt{u}}, \frac{m(u)\ln u}{\sqrt{u}})_{\delta m(u)/u}} e^{-\frac{B}{2A}t^2} \Big) \\
&\sim \frac{u}{\delta m(u)} \int_{\mathbb{R}} e^{-\frac{B}{2A}t^2} dt, \quad u \to \infty \qquad (6.22) \\
&= \frac{u}{\delta m(u)} \sqrt{\frac{2\pi A}{B}} \\
&= \frac{u}{\delta m(u)} \frac{\sqrt{2\pi\alpha}}{c(1-\alpha)^{3/2}},
\end{aligned}
$$

where the asymptotic equivalence in (6.22) holds since by (6.15) $\frac{\delta m(u)}{u} \to 0$ and $\frac{m(u)\ln u}{\sqrt{u}} \to \infty$ as $u \to \infty$. Thus,

$$
\sum_{t \in I(t_u)} \mathbb{P}\{X_u(t) > m(u)\} \sim \frac{\sqrt{2\pi\alpha} u \overline{\Phi}(m(u))}{\delta c(1-\alpha)^{3/2} m(u)}, \quad u \to \infty \qquad (6.23)
$$

and hence by (6.15),(6.20) and (6.21) we have that

$$
\mathbb{P}\left\{ \sup_{t \in I(t_u)} X_u(t) > m(u) \right\} \sim \frac{\sqrt{2\pi\alpha} u \overline{\Phi}(m(u))}{\delta c(1-\alpha)^{3/2} m(u)}, \quad u \to \infty
$$

and the claim follows by (6.19).

Cases ii-iii). For any fixed $u > 0$ and $S \in \{0, \delta, 2\delta, ...\}$ denote

$$
N_u = \lceil \frac{u\delta_u}{S\Delta(u)} \rceil, \quad t_j = \frac{\Delta(u) j S}{u}, \quad \Delta_{j,S,u} = [t_u + t_j, t_u + t_{j+1}]_{\delta/u}, \quad j \in [-N_u - 1, N_u],
$$

where $\lceil \cdot \rceil$ is the ceiling function. We have by Bonferroni inequality that

$$
\sum_{-N_u \le j \le N_u - 1} p_{j,S,u} - \sum_{-N_u - 1 \le i \ne j \le N_u} p_{i,j;S,u} \le \mathbb{P}\left\{ \sup_{t \in I(t_u)} X_u(t) > m(u) \right\} \le \sum_{-N_u - 1 \le j \le N_u} p_{j,S,u}, \qquad (6.24)
$$

where

$$
p_{j,S,u} = \mathbb{P}\left\{ \sup_{t \in \Delta_{j,S,u}} X_u(t) > m(u) \right\} \text{ and } p_{i,j;S,u} = \mathbb{P}\left\{ \sup_{t \in \Delta_{j,S,u}} X_u(t) > m(u), \sup_{t \in \Delta_{i,S,u}} X_u(t) > m(u) \right\}.
$$

By [20] we have that the double-sum term above is $o(\mathbb{P}\{M_\delta > u\})$ as $u \to \infty$. Hence from the asymptotics of $\sum_{-N_u \le j \le N_u - 1} p_{j,S,u}$ given in (6.25) and (6.26) later on we obtain that the double-sum term is negligible and hence as $u \to \infty$ and then $S \to \infty$

$$
\mathbb{P}\left\{ \sup_{t \in I(t_u)} X_u(t) > m(u) \right\} \sim \sum_{-N_u \le j \le N_u} p_{j,S,u}
$$

and we need to calculate the asymptotics of the sum above. That can be done via uniform approximation of $p_{j,S,u}$ for all $-N_u-1 \le j \le N_u$. As shown for cases $ii)$ and $iii)$ separately below.

In case $ii)$ we have $\Delta(u)=1$, $N_u=\lceil \frac{\sqrt{u}\ln u}{S}\rceil$, $t_j=\frac{jS}{u}$ and $\Delta_{j,S,u}=[t_u+t_j, t_u+t_{j+1}]$. We have by Lemma 6.5.1 for any $\varepsilon>0,\ 0\le j\le N_u$ for all u large enough with $m_j^-(u)=\frac{m(u)}{1-(1-\varepsilon)\frac{B}{2A}(\frac{jS}{u})^2}$

$$
\begin{aligned}
p_{j,S,u}=\mathbb{P}\left\{\exists t\in\Delta_{j,S,u}:\overline{X}_u(t)>\frac{m(u)}{\sigma_u(t)}\right\} &\le \mathbb{P}\left\{\exists t\in\Delta_{j,S,u}:\overline{X}_u(t)>\frac{m(u)}{1-(1-\varepsilon)\frac{B}{2A}(t-t_u)^2}\right\}\\
&\le \mathbb{P}\left\{\sup_{t\in\Delta_{j,S,u}}\overline{X}_u(t)>m_j^-(u)\right\}\\
&\le \mathbb{P}\left\{\sup_{t\in[0,S]_\delta}\overline{X}'_u(t)>m_j^-(u)\right\},
\end{aligned}
$$

where $\overline{X}'_u(t),\ t\in[0,S]$ is some centered Gaussian process with unit variance and correlation satisfying $1-r_{\overline{X}'_u}(t,s)\sim\frac{\sigma^2(|t-s|)}{2\sigma^2(ut_*)},\ u\to\infty\ t,s\in[0,S]$. By Lemma 1 in [19] the last probability above as $u\to\infty$ is equivalent to $\mathcal{H}_{\eta'}(\{0,\delta,...,S\})\overline{\Phi}(m_j^-(u))$, where η' is a centered Gaussian process with stationary increments, a.s. continuous sample paths and variance (asymptotics of $m(u)$ is given in (6.15))

$$
\sigma^2_{\eta'}(t)=\lim_{u\to\infty}\frac{m^2(u)}{2\sigma^2(ut_*)}\sigma^2(t)=\lim_{u\to\infty}\frac{u^2(1+ct_*)^2}{2t_*\sigma^2(ut_*)\sigma^2(u)}\sigma^2(t)=\frac{2c^2}{\varphi^2}\sigma^2(t).
$$

Note that η' and η defined in (6.8) have the same distributions. Thus,

$$
\sum_{0\le j\le N_u}p_{j,S,u}\le\mathcal{H}_\eta(\{0,\delta,...,S\})\sum_{0\le j\le N_u}\overline{\Phi}(m_j^-(u)),\quad u\to\infty.
$$

Next as $u\to\infty$ (set $C_-=\frac{(1-\varepsilon)B}{2A}$) similarly to *case i)* we have

$$
\begin{aligned}
\frac{\sum\limits_{0\le j\le N_u}\overline{\Phi}(m_j^-(u))}{\overline{\Phi}(m(u))} &\sim \sum_{0\le j\le N_u} e^{-\frac{m^2(u)}{2}\left(\frac{1}{(1-C_-(\frac{jS}{u})^2)^2}-1\right)}\\
&\sim \sum_{0\le j\le N_u} e^{-\frac{m^2(u)}{2}2C_-(\frac{jS}{u})^2}\\
&= \sum_{\frac{Sm(u)}{u}\in[0,\frac{N_uSm(u)}{u}]_{\frac{Sm(u)}{u}}} e^{-C_-(\frac{jSm(u)}{u})^2}\\
&= \frac{u}{Sm(u)}\Big(\frac{Sm(u)}{u}\sum_{t\in[0,\frac{m(u)\ln u}{\sqrt{u}}]_{\frac{Sm(u)}{u}}} e^{-C_-t^2}\Big).
\end{aligned}
$$

Since by (6.15) $\frac{u}{Sm(u)}\to0$ and $\frac{m(u)\ln u}{\sqrt{u}}\to\infty$ as $u\to\infty$ we have that the sum above converges to $\int\limits_0^\infty e^{-C_-t^2}dt=\frac{\sqrt{\pi}}{2\sqrt{C_-}}$ as $u\to\infty$. Similar calculation can be done for $j<0$, hence summarizing all calculations above we have as $u \to \infty$ and then $S \to \infty$

$$\sum_{-N_u \le j \le N_u} p_{j,S,u} \le \mathcal{H}_\eta(\{0,\delta,...,S\}) \frac{u\overline{\Phi}(m(u))}{Sm(u)} \frac{\sqrt{\pi}}{\sqrt{C_-}}.$$

By Lemma 6.5.4 $\frac{\mathcal{H}_\eta(\{0,\delta,...,S\})}{S} \to \mathcal{H}_\eta^\delta \in (0,\infty)$ as $S \to \infty$, hence letting $S \to \infty$ we have

$$\sum_{-N_u \le j \le N_u} p_{j,S,u} \le \mathcal{H}_\eta^\delta \frac{u\overline{\Phi}(m(u))}{m(u)} \frac{\sqrt{\pi}}{\sqrt{C_-}}(1+o(1)), \quad u \to \infty.$$

By the same arguments we have the lower bound

$$\sum_{-N_u \le j \le N_u} p_{j,S,u} \ge \mathcal{H}_\eta^\delta \frac{u\overline{\Phi}(m(u))}{m(u)} \frac{\sqrt{\pi}}{\sqrt{C_+}}(1+o(1)), \quad u \to \infty,$$

with $C_+ = \frac{(1+\varepsilon)B}{2A}$. Hence letting $\varepsilon \to 0$ we have that as $S \to \infty$ and then $u \to \infty$

$$\sum_{-N_u \le j \le N_u} p_{j,S,u} \sim \mathcal{H}_\eta^\delta \frac{u}{m(u)} \sqrt{\frac{2\pi A}{B}} \overline{\Phi}(m(u)). \tag{6.25}$$

For case $iii)$ we note that $\Delta(u), N_u \to \infty$ and $\Delta(u)/u \to 0$ as $u \to \infty$. We have with $m_k(u) = \frac{m(u)}{1-C_+|t_k|^2}$, $\ -N_u \le k \le N_u$ for large S, u

$$\begin{aligned}
p_{k,S,u} \ge \mathbb{P}\left\{\sup_{t\in[t_k,t_{k+1}]_{\delta/u}} \overline{X}_u(t) > m_k(u)\right\} &\ge \mathbb{P}\left\{\sup_{t\in[t_k,t_{k+1}]_{\frac{\Delta(u)}{u}\frac{\delta[\varepsilon_1\Delta(u)]}{\Delta(u)}}} \overline{X}_u(t) > m_k(u)\right\} \\
&\ge \mathbb{P}\left\{\sup_{t\in[0,S-\frac{\delta[\varepsilon_1\Delta(u)]}{\Delta(u)}]_{\frac{\delta[\varepsilon_1\Delta(u)]}{\Delta(u)}}} Y_u'(t) > m_k(u)\right\} \\
&\ge \mathbb{P}\left\{\sup_{t\in[0,S(1-\varepsilon_2)]_{\delta\varepsilon_1}} Y_u'(t\frac{[\Delta(u)\varepsilon_1]}{\Delta(u)\varepsilon_1}) > m_k(u)\right\} \\
&= \mathbb{P}\left\{\sup_{t\in[0,S(1-\varepsilon_2)]_{\delta\varepsilon_1}} Y_u(t) > m_k(u)\right\}, \quad u \to \infty,
\end{aligned}$$

where $\varepsilon_1, \varepsilon_2$ are any small positive numbers and $Y_u(t), Y_u'(t)$ are some centered Gaussian processes with unit variances and correlation functions having expansions as $u \to \infty$

$$\begin{aligned}
r_{Y_u}(t,s) &= 1 - \frac{\sigma^2(\Delta(u))|t-s|^{2\alpha}}{2\sigma^2(u)t_*^{2\alpha}} + o(\frac{\sigma^2(\Delta(u))|t-s|^{2\alpha}}{\sigma^2(u)}), \\
r_{Y_u'}(t,s) &= 1 - \frac{\sigma^2(\Delta(u))|t-s|^{2\alpha}}{2\sigma^2(u)t_*^{2\alpha}} + o(\frac{\sigma^2(\Delta(u))|t-s|^{2\alpha}}{\sigma^2(u)}).
\end{aligned}$$

Next by Lemma 5.1 in [20] with (in their notation) index set K consisting of 1 element and

$$g(u) = m_k(u), \quad \theta(u,s,t) = |t-s|^{2\alpha}, \quad V = B_\alpha, \quad \sigma_V(t) = |t|^{2\alpha}$$

we have uniformly for $-N_u \le k \le N_u$

$$\mathbb{P}\left\{ \sup_{t\in[0,(1-\varepsilon_2)S]_{\delta\varepsilon_1}} Y_u(t) > m_k(u) \right\} \sim \mathcal{H}_{B_\alpha}([0, S(1-\varepsilon_2)]_{\delta\varepsilon_1})\overline{\Phi}(m_k(u)), \quad u\to\infty.$$

Thus, for large S as $u\to\infty$

$$\sum_{-N_u\le k\le N_u} p_{k,S,u} \ge \mathcal{H}_{B_\alpha}([0,(1-\varepsilon_2)S]_{\delta\varepsilon_1}) \sum_{-N_u\le k\le N_u} \overline{\Phi}(m_k(u))(1+o(1)).$$

Next we calculate the sum above. Similarly to *cases i-ii)* we have as $u\to\infty$ with $\widehat{C}_+ = \frac{C_+(1+ct_*)^2}{(t_*)^{2\alpha}}$ and $l_u = \frac{S\Delta(u)}{\sigma(u)} \to 0$

$$\begin{aligned}
\frac{\sum_{k=-N_u}^{N_u} \overline{\Phi}(m_k(u))}{\overline{\Phi}(m(u))} &\sim \sum_{k=-N_u}^{N_u} e^{-\frac{m^2(u)}{2}\left(\frac{1}{(1-C_+t_k^2)^2}-1\right)} \\
&\sim \sum_{k=-N_u}^{N_u} e^{-C_+m^2(u)t_k^2} \\
&\sim \sum_{k=-\frac{\sigma(u)\ln u}{S\Delta(u)}}^{\frac{\sigma(u)\ln u}{S\Delta(u)}} e^{-\frac{C_+(1+ct_*)^2}{t_*^{2\alpha}}\left(\frac{kS\Delta(u)}{\sigma(u)}\right)^2} \\
&= \sum_{kl_u\in(-\ln u,\ln u)_{l_u}} e^{-\widehat{C}_+(kl_u)^2} \\
&= \frac{1}{l_u}\Big(l_u \sum_{t\in(-\ln u,\ln u)_{l_u}} e^{-\widehat{C}_+t^2}\Big).
\end{aligned}$$

Since $l_u\to 0$ as $u\to\infty$ the expression in the parentheses above converges as $u\to\infty$ to

$$\int_{\mathbb{R}} e^{-\widehat{C}_+t^2}dt = \frac{\sqrt{\pi}}{\sqrt{\widehat{C}_+}} = \sqrt{\frac{2A\pi}{B}}\frac{t_*^\alpha}{1+ct_*}\frac{1}{\sqrt{1+\varepsilon}}.$$

Thus, summarizing the calculations above we have as $u\to\infty$ for large S

$$\sum_{-N_u\le k\le N_u} p_{k,S,u} \;\ge\; \frac{1}{\sqrt{1+\varepsilon}}\frac{1}{S}\mathcal{H}_{B_\alpha}([0,(1-\varepsilon_2)S]_{\delta\varepsilon_1})\overline{\Phi}(m(u))\frac{\sigma(u)}{\Delta(u)}\sqrt{\frac{2A\pi}{B}}\frac{t_*^\alpha}{1+ct_*}(1+o(1)).$$

Letting $S\to\infty$ then $\varepsilon_2\to 0$ and then $\varepsilon_1\to 0$ in view of Lemma 12.2.7 ii) and Remark 12.2.10 in [49] we obtain $\frac{1}{S}\mathcal{H}_{B_\alpha}([0,(1-\varepsilon_2)S]_{\delta\varepsilon_1}) \to \mathcal{H}_{B_\alpha}$. Letting then $\varepsilon\to 0$ we obtain the lower bound

$$\sum_{-N_u\le k\le N_u} p_{k,S,u} \;\ge\; \mathcal{H}_{B_\alpha}\overline{\Phi}(m(u))\frac{\sigma(u)}{\Delta(u)}\sqrt{\frac{2A\pi}{B}}\frac{t_*^\alpha}{1+ct_*}(1+o(1)), \quad u\to\infty. \tag{6.26}$$

Similarly to the calculation of the lower bound we have for $-N_u \le k \le N_u$ and $\varepsilon > 0$ with $m_k^-(u) = \frac{m(u)}{1-C_-|t_k|^2}$ (recall, $C_- = \frac{B(1-\varepsilon)}{2A}$) as $u\to\infty$ and then $S\to\infty$

$$p_{k,S,u} \le \mathcal{H}_{B_\alpha}([0,S])\overline{\Phi}(m_k^-(u))(1+o(1)).$$

Summing $p_{k,S,u}$ and letting $S\to\infty$ as above we have the upper bound

$$\sum_{-N_u\le k\le N_u} p_{k,S,u} \le \frac{1}{\sqrt{1-\varepsilon}}\mathcal{H}_{B_\alpha}\overline{\Phi}(m(u))\frac{\sigma(u)}{\Delta(u)}\sqrt{\frac{2A\pi}{B}}\frac{t_*^\alpha}{1+ct_*}(1+o(1)), \quad u\to\infty.$$

Letting $\varepsilon\to 0$ we obtain the right side in (6.26), thus the claim is established. □

Proof of Theorem 6.3.1. By Lemma 6.5.2 we have

$$\psi_{T,\delta}^{\sup}(u) \sim \mathbb{P}\left\{\sup_{t\in[0,T/u]_{\delta/u}, s-t\in I(t_u)} Z_u(t,s) > m(u)\right\}, \quad u\to\infty. \tag{6.27}$$

As in the proof of Theorem (6.2.1) next we consider three cases: case i) when $\varphi=0$, case ii) when $\varphi\in(0,\infty)$ and case iii) when $\varphi=\infty$.

Case i). First we have by Bonferroni inequality

$$\begin{aligned}\sum_{t\in[0,\frac{T}{u}]_{\frac{\delta}{u}}}\sum_{s\in I(t_u)}\mathbb{P}\{Z_u(t,s)>m(u)\} &\ge \mathbb{P}\left\{\sup_{t\in[0,\frac{T}{u}]_{\frac{\delta}{u}}, s\in I(t_u)} Z_u(t,s)>m(u)\right\}\\ &\ge \sum_{t\in[0,\frac{T}{u}]_{\frac{\delta}{u}}}\sum_{s\in I(t_u)}\mathbb{P}\{Z_u(t,s)>m(u)\}\\ &\quad -\sum_{t\in[0,\frac{T}{u}]_{\frac{\delta}{u}}}\sum_{\substack{s_1,s_2\in I(t_u)\\ s_1\ne s_2}}\mathbb{P}\{Z_u(t,s_1),Z_u(t,s_2)>m(u)\}.\end{aligned}$$

For the double-sum above we have

$$\begin{aligned}&\sum_{t\in[0,\frac{T}{u}]_{\frac{\delta}{u}}}\sum_{\substack{s_1,s_2\in I(t_u)\\ s_1\ne s_2}}\mathbb{P}\{Z_u(t,s_1),Z_u(t,s_2)>m(u)\}\\ \le\ &(1+[\frac{T}{\delta}])\sup_{t\in[0,\frac{T}{u}]_{\frac{\delta}{u}}}\sum_{\substack{s_1,s_2\in I(t_u)\\ s_1\ne s_2}}\mathbb{P}\{X_u(s_1-t),X_u(s_2-t)>m(u)\}\\ \le\ &\mathbb{C}(1+[\frac{T}{\delta}])\ln^2(u)u^{1/2-\varepsilon'}\overline{\Phi}(m(u)),\end{aligned} \tag{6.28}$$

where the last inequality above follows from (6.21). For the asymptotics of the single sum we have

$$\begin{aligned}\sum_{t\in[0,\frac{T}{u}]_{\frac{\delta}{u}}}\sum_{s\in I(t_u)}\mathbb{P}\{Z_u(t,s)>m(u)\} &= \sum_{t\in[0,\frac{T}{u}]_{\frac{\delta}{u}}}\sum_{s\in I(t_u)}\mathbb{P}\{X_u(s-t)>m(u)\}\\ &\sim (1+[\frac{T}{\delta}])\sum_{\tau\in I(t_u)}\mathbb{P}\{X_u(\tau)>m(u)\}, \quad u\to\infty.\end{aligned}$$

The last sum above was calculated in the proof of Theorem 6.2.1, hence

$$\sum_{t\in[0,\frac{T}{u}]_{\frac{\delta}{u}}}\sum_{s_1\ne s_2\in I(t_u)}\mathbb{P}\{Z_u(t,s)>m(u)\}\sim(1+[\frac{T}{\delta}])\frac{\sqrt{2\pi\alpha}u\overline{\Phi}(m(u))}{\delta c(1-\alpha)^{3/2}m(u)}, \quad u\to\infty.$$

By the line above combined with (6.28) we obtain

$$\mathbb{P}\left\{\sup_{t\in[0,\frac{T}{u}]_{\frac{\delta}{u}},s\in I(t_u)} Z_u(t,s) > m(u)\right\} \sim (1+[\frac{T}{\delta}])\frac{\sqrt{2\pi\alpha}u\overline{\Phi}(m(u))}{\delta c(1-\alpha)^{3/2}m(u)}, \qquad u\to\infty$$

and the claim follows by (6.27).

Case ii). With the notation of Theorem 6.2.1 we have by Bonferroni inequality for $u>0$

$$\sum_{j=-N_u}^{N_u} q_{j,S,u} \geq \mathbb{P}\left\{\sup_{t\in[0,\frac{T}{u}]_{\frac{\delta}{u}},s\in I(t_u)} Z_u(t,s) > m(u)\right\} \geq \sum_{j=-N_u}^{N_u} q_{j,S,u} - \sum_{-N_u\leq i<j\leq N_u} q_{i,j;S,u},$$

where

$$q_{j,S,u} = \mathbb{P}\left\{\sup_{t\in[0,\frac{T}{u}]_{\frac{\delta}{u}},s\in\Delta_{j,S,u}} Z_u(t,s) > m(u)\right\}$$

and

$$q_{i,j;S,u} = \mathbb{P}\left\{\exists t\in[0,\frac{T}{u}]_{\frac{\delta}{u}} : \sup_{s\in\Delta_{j,S,u}} Z_u(t,s) > m(u), \sup_{s\in\Delta_{i,S,u}} Z_u(t,s) > m(u)\right\}.$$

By [20] we have that the double-sum term above is $o(\mathbb{P}\{M_\delta > u\})$ as $u\to\infty$. Hence from the asymptotics of $\sum_{j=-N_u}^{N_u} q_{j,S,u}$ given in Theorem 6.3.1 we obtain that the double-sum term is negligible and hence as $u\to\infty$ and then $S\to\infty$

$$\mathbb{P}\left\{\sup_{t\in[0,\frac{T}{u}]_{\frac{\delta}{u}},s\in I(t_u)} Z_u(t,s) > m(u)\right\} \sim \sum_{j=-N_u}^{N_u} q_{j,S,u}$$

and we need to calculate the asymptotics of the sum above. Next we uniformly approximate each summand in the sum above. For $\varepsilon>0, j\geq 1, S>T$ and u large enough we have (recall, $m_{j-1}^-(u) = \frac{m(u)}{1-(1-\varepsilon)\frac{B}{2A}(\frac{(j-1)S}{u})^2}$)

$$\begin{aligned}
q_{j,S,u} &= \mathbb{P}\left\{\exists(t,s)\in[0,\frac{T}{u}]_{\delta/u}\times\Delta_{j,S,u} : \overline{Z}_u(t,s) > \frac{m(u)}{\sigma_u(s-t)}\right\}\\
&\leq \mathbb{P}\left\{\exists(t,s)\in[0,\frac{T}{u}]_{\delta/u}\times\Delta_{j,S,u} : \overline{Z}_u(t,s) > \frac{m(u)}{1-(1-\varepsilon)\frac{B}{2A}(s-t-t_u)^2}\right\}\\
&\leq \mathbb{P}\left\{\sup_{t\in[0,T]_\delta,s\in[0,S]_\delta} \overline{Z}'_u(t,s) > \frac{m(u)}{1-C_-(\frac{(j-1)S}{u})^2}\right\}\\
&= \mathbb{P}\left\{\sup_{t\in[0,T]_\delta,s\in[0,S]_\delta} \overline{Z}'_u(t,s) > m_{j-1}^-(u)\right\},
\end{aligned}$$

where $\overline{Z}'_u(t,s)$ is some centered Gaussian process with unit variance and correlation $r_{\overline{Z}'_u}$ having an expansion

$$1 - r_{\overline{Z}'_u}(t,s,t_1,s_1) \sim \frac{\sigma(|s-s_1|)^2 + \sigma^2(|t-t_1|)}{2\sigma^2(ut_*)}, \quad (t,s) \in [0,T] \times [0,S], \quad u \to \infty. \tag{6.29}$$

Applying Lemma 5.1 in [20] with parameters (with $X^{(1)}, X^{(2)}$ being independent copies of X)

$$\Phi = \sup, \quad \theta(u,s,t) = \frac{2c^2}{\varphi^2}(\sigma^2(|s-s_1|) + \sigma^2(|t-t_1|)), \quad V(t,s) = \frac{\sqrt{2}c}{\varphi}(X^{(1)}(t) + X^{(2)}(s))$$

we have

$$\mathbb{P}\left\{ \sup_{t \in [0,T]_\delta, s \in [0,S]_\delta} \overline{Z}'_u(t,s) > m^-_{j-1}(u) \right\} \sim \mathcal{H}_{V(t,s)}([0,T]_\delta \times [0,S]_\delta)\overline{\Phi}(m^-_{j-1}(u)).$$

Since $X^{(1)}(t)$ and $X^{(2)}(s)$ are independent we have

$$\mathcal{H}_{V(t,s)}([0,T]_\delta \times [0,S]_\delta) = \mathcal{H}_{\frac{\sqrt{2}c}{\varphi}X}([0,T]_\delta)\mathcal{H}_{\frac{\sqrt{2}c}{\varphi}X}([0,S]_\delta).$$

Finally, for $\varepsilon > 0, j \geq 1, S > T$ and u large we have

$$\mathbb{P}\left\{ \sup_{t \in [0,\frac{T}{u}]_{\delta/u}, s \in \Delta_{j,S,u}} Z_u(t,s) > m(u) \right\} \leq \mathcal{H}_{\frac{\sqrt{2}c}{\varphi}X}([0,T]_\delta)\mathcal{H}_{\frac{\sqrt{2}c}{\varphi}X}([0,S]_\delta)\overline{\Phi}(m^-_{j-1}(u))(1+o(1)).$$

The rest of the proof is the same as in Theorem 6.2.1 case ii), thus the claim is established. □

Case iii). By Theorem 6.2.1 we have

$$\psi^{\sup}_{T,\delta}(u) \geq \mathbb{P}\{M_\delta > u\} \sim \mathcal{H}_{B_\alpha} f(u)\overline{\Phi}(m(u)), \quad u \to \infty.$$

By (6.27) we have

$$\psi^{\sup}_{T,\delta}(u) \leq \mathbb{P}\left\{ \sup_{t \in [0,T/u], s-t \in (-\delta_u + t_u, t_u + \delta_u)} Z_u(t,s) > m(u) \right\}(1+o(1)), \quad u \to \infty.$$

From the proof of Theorem 3.1 in [20] it follows that the last probability above does not exceed $(1+o(1))\mathcal{H}_{B_\alpha} f(u)\overline{\Phi}(m(u)),\ u \to \infty$. Combining both bounds above we obtain the claim. □

Proof of Theorem 6.3.2. Assume that $\varphi \in (0,\infty)$. First by Lemma 6.5.2 we have

$$\psi^{\inf}_{T,\delta}(u) \sim \mathbb{P}\left\{ \inf_{t \in [0,\frac{T}{u}]_{\delta/u}} \sup_{s \in I(t_u)} Z_u(t,s) > m(u) \right\}, \quad u \to \infty.$$

With notation of Theorem 6.2.1 in view of the final asymptotics of $\psi^{\inf}_{\delta,T}(u)$ given in Theorem 6.3.2 repeating the proof of Theorem 6.3.1 we have as $u \to \infty$ and then $S \to \infty$

$$\mathbb{P}\left\{ \inf_{t \in [0,\frac{T}{u}]_{\delta/u}} \sup_{s \in I(t_u)} Z_u(t,s) > m(u) \right\} \sim \sum_{j=-N_u}^{N_u} \mathbb{P}\left\{ \inf_{t \in [0,\frac{T}{u}]_{\delta/u}} \sup_{s \in \Delta_{j,S,u}} Z_u(t,s) > m(u) \right\}.$$

Next we uniformly approximate each summand in the sum above. For $\varepsilon > 0, j \geq 1, S > T$ and u large enough similarly to the proof of Theorem 6.3.1 we obtain (recall, $m_{j-1}^{-}(u) = \frac{m(u)}{1-(1-\varepsilon)\frac{B}{2A}(\frac{(j-1)S}{u})^2}$)

$$\mathbb{P}\left\{\inf_{t\in[0,\frac{T}{u}]_{\delta/u}}\sup_{s\in\Delta_{j,S,u}} Z_u(t,s) > m(u)\right\} \leq \mathbb{P}\left\{\inf_{t\in[0,T]_\delta}\sup_{s\in[0,S]_\delta} \overline{Z}'_u(t,s) > m_{j-1}^{-}(u)\right\},$$

where $\overline{Z}'_u(t,s)$ is a centered Gaussian process with unit variance and correlation satisfying (6.29). Applying Lemma 5.1 in [20] with parameters (with $X^{(1)}, X^{(2)}$ being independent copies of X)

$$\Phi = \inf\sup, \quad \theta(u,s,t) = \frac{2c^2}{\varphi^2}(\sigma^2(|s-s_1|) + \sigma^2(|t-t_1|)), \quad V(t,s) = \frac{\sqrt{2}c}{\varphi}(X^{(1)}(t) + X^{(2)}(s))$$

we have as $u \to \infty$

$$\mathbb{P}\left\{\inf_{t\in[0,T]_\delta}\sup_{s\in[0,S]_\delta} \overline{Z}'_u(t,s) > m_{j-1}^{-}(u)\right\} \sim \mathbb{E}\left\{\inf_{t\in[0,T]_\delta}\sup_{s\in[0,S]_\delta} e^{\sqrt{2}V(t,s)-\mathrm{Var}(V(t,s))}\right\}\overline{\Phi}(m_{j-1}^{-}(u)).$$

Since $X^{(1)}(t)$ and $X^{(2)}(s)$ are independent we have

$$\mathbb{E}\left\{\inf_{t\in[0,T]_\delta}\sup_{s\in[0,S]_\delta} e^{\sqrt{2}V(t,s)-\mathrm{Var}(V(t,s))}\right\} = \mathcal{H}^{\inf}_{\frac{\sqrt{2}c}{\varphi}X}([0,T]_\delta)\mathcal{H}_{\frac{\sqrt{2}c}{\varphi}X}([0,S]_\delta).$$

Thus, as $u \to \infty$

$$\mathbb{P}\left\{\inf_{t\in[0,\frac{T}{u}]_{\delta/u}}\sup_{s\in\Delta_{j,S,u}} Z_u(t,s) > m(u)\right\} \leq \mathcal{H}^{\inf}_{\frac{\sqrt{2}c}{\varphi}X}([0,T]_\delta)\mathcal{H}_{\frac{\sqrt{2}c}{\varphi}X}([0,S]_\delta)\overline{\Phi}(m_{j-1}^{-}(u)).$$

The rest of the proof is the same as in Theorem 6.2.1, thus the claim is established.

Assume that $\varphi = \infty$. Let $R(s,t) = X(s) - X(t) - c(s-t),\ t,s \geq 0$. Using the idea from [19], (the next equation after (2)) we write

$$\begin{aligned}\frac{\psi^{\inf}_{T,\delta}(u)}{\psi^{\sup}_{T,\delta}(u)} &= \mathbb{P}\left\{\inf_{t\in[0,T]_\delta}\sup_{t\leq s\in G_\delta} R(s,t) > u \,\Big|\, \sup_{t\in[0,T]_\delta}\sup_{t\leq s\in G_\delta} R(s,t) > u\right\} \\ &\geq 1 - \sum_{t\in[0,T]_\delta}\left(1 - \frac{\mathbb{P}\left\{\sup_{t\leq s\in G_\delta} R(s,t) > u\right\}}{\mathbb{P}\left\{\sup_{a\in[0,T]_\delta}\sup_{a\leq b\in G_\delta} R(a,b) > u\right\}}\right).\end{aligned}$$

By Theorems 6.2.1 and 6.3.1 we have that the right part of the expression above tends to 1 as $u \to \infty$. Thus, we have $\psi^{\inf}_{T,\delta}(u) \sim \psi^{\sup}_{T,\delta}(u) \sim \mathcal{H}_{B_\alpha} f(u)\overline{\Phi}(m(u)),\ u \to \infty$ and the claim follows. $\square$

Proof of Proposition 6.3.3. Since $T \geq \delta$ with any $K \in [\delta, T]_\delta$, $a \in (0, t_*), b > 0$ and $J(t_u) = [-a + t_u, t_u + b]$ we have

$$\psi^{\inf}_{\delta,T}(u) \quad \leq \quad \psi^{\inf}_{\delta,K}(u)$$

$$
\begin{aligned}
&\le \mathbb{P}\left\{\exists s_1,s_2\in J(t_u)\cap G_{\frac{\delta}{u}} : Z_u(0,s_1)>m(u), Z_u(\frac{K}{u},s_2)>m(u)\right\}\\
&\quad +\mathbb{P}\left\{\exists s\notin J(t_u) : Z_u(0,s)>m(u)\right\}\\
&=: p_1(u)+p_2(u).
\end{aligned}
$$

Estimation of $p_1(u)$. Fix some $s_1,s_2\in J(t_u)\cap G_{\frac{\delta}{u}}$ and let $(W_1,W_2)=(Z_u(0,s_1),Z_u(\frac{K}{u},s_2))$. We have that (W_1,W_2) is a centered Gaussian vector with $Var(W_1),Var(W_2)\le 1$ and correlation $r_{W_u}(s_1,s_2)$ satisfying (see (6.17))

$$
1-r_{W_u}(s_1,s_2)\ge \frac{\sigma^2(u|s_1-s_2|)+\sigma^2(K)}{2\sigma^2(ut_*)}(1+o(1))\ge (1+o(1))\frac{\sigma^2(K)}{2t_*^{2\alpha}\sigma^2(u)}, \qquad s_1,s_2\in J(t_u).
$$

Thus, by Lemma 2.3 in [56]

$$
\begin{aligned}
\overline{\Phi}(m(u))^{-1}\mathbb{P}\left\{W_1>m(u),W_2(u)>m(u)\right\} &\le 3\overline{\Phi}\Big(m(u)\sqrt{\frac{1-r_{W_u}(s_1,s_2)}{2}}\Big)\\
&= 3\overline{\Phi}\Big((1+o(1))\frac{\sigma(K)m(u)}{2t_*^{\alpha}\sigma(u)}\Big)\\
&= 3\overline{\Phi}\Big((1+o(1))\frac{\sigma(K)(1+ct_*)u}{2t_*^{2\alpha}\sigma^2(u)}\Big).
\end{aligned}
$$

Note that $\varphi=0$ implies $\frac{u}{\sigma^2(u)}\to\infty$ as $u\to\infty$. Thus, since there are less than $Cu^2\ln^2 u$ points in $(J(t_u)\cap G_{\frac{\delta}{u}})\times(J(t_u)\cap G_{\frac{\delta}{u}})$ we have with any $\mathbb{C}_K<\frac{\sigma(K)(1+ct_*)}{2t_*^{2\alpha}}$ by (6.13) as $u\to\infty$

$$
p_1(u) \le Cu^2(\ln^2 u)\cdot 3\overline{\Phi}(m(u))\overline{\Phi}\Big((1+o(1))\frac{\sigma(K)(1+ct_*)u}{2t_*^{2\alpha}\sigma^2(u)}\Big)\le \overline{\Phi}(m(u))\overline{\Phi}\Big(\mathbb{C}_K\frac{u}{\sigma^2(u)}\Big).
$$

Estimation of $p_2(u)$. Since $Z_u(0,s)\stackrel{d}{=}X_u(s),\ s\ge 0$, it follows from the estimation of $R_1(u)$ and $R_2(u)$ in the proof of Lemma 6.5.2 (see (6.32) and (6.33), respectively) that for appropriately chosen $a\in(0,t_*), b>0$ and small $\varepsilon>0$

$$
p_2(u)\le \mathbb{P}\left\{\exists s\in[0,a] : X_u(s)>m(u)\right\}+\mathbb{P}\left\{\exists s\in[t_*+b,\infty) : X_u(s)>m(u)\right\}\le \overline{\Phi}(m(u))\mathbb{C}e^{-\frac{m^2(u)}{u^{2\varepsilon}}}.
$$

Combing this inequality with the upper bound of $p_1(u)$ we obtain that for any $K\in[\delta,T]_\delta$ it holds that

$$
\psi^{\inf}_{\delta,T}(u)\le \overline{\Phi}(m(u))\overline{\Phi}\Big(\mathbb{C}_K\frac{u}{\sigma^2(u)}\Big), \quad u\to\infty
$$

and taking the supremum with respect to K we obtain the claim. □

Proof of Corollary 6.4.4. First we give a proof of the first statement. Since $Z(t)$ is a Gaussian process with stationary increments, a.s. sample paths and variance satisfying **A** applying Theorem 6.2.1 with parameters

$$
\varphi=\lim_{u\to\infty}\frac{\sigma_\zeta^2(u)}{u}=\frac{2}{G}\in(0,\infty),\ \alpha=1/2,\ t_*=1/c,\ \Delta(u)=1,\ A=2\sqrt{C},\ D=c^2\sqrt{c}/2,
$$

$$m(u)=\sqrt{2Gc}\sqrt{u}+\frac{c^{3/2}G^{3/2}D}{\sqrt{2}}u^{-1/2}+o(u^{-1/2}),\ f(u)=\frac{2\sqrt{\pi}}{c\sqrt{cG}}\sqrt{u}+O(u^{-1}),\ \eta(t)=\frac{cG}{\sqrt{2}}Z(t)$$

we have $\overline{\Phi}(m(u))\sim e^{-ucG-c^2G^2D}\frac{1}{2\sqrt{\pi Gcu}}, u\to\infty$ implying

$$\mathbb{P}\left\{\exists t\in[0,\infty): Z(t)-ct>u\right\}\sim \mathcal{H}^{\delta}_{cGZ(t)/\sqrt{2}}\frac{1}{c^2G}e^{-ucG-c^2G^2D}, \qquad u\to\infty$$

and the first claim follows. Applying Theorems 6.3.1 and 6.3.2 with the same parameters we obtain the second and third claims, respectively. □

Proof of Lemma 6.5.2. First we show the first claim. We have

$$\begin{aligned} &\mathbb{P}\left\{\sup_{t\in[0,\frac{T}{u}]_{\frac{\delta}{u}}}\sup_{s\in I(t_u)} Z_u(t,s)>m(u)\right\}\\ \le\ &\psi^{\sup}_{\delta,T}(u) &(6.30)\\ \le\ &\mathbb{P}\left\{\sup_{t\in[0,\frac{T}{u}]_{\frac{\delta}{u}}}\sup_{s\in I(t_u)} Z_u(t,s)>m(u)\right\}+\mathbb{P}\left\{\sup_{t\in[0,\frac{T}{u}]_{\frac{\delta}{u}}}\sup_{s\in(G_{\frac{\delta}{u}}\setminus I(t_u))} Z_u(t,s)>m(u)\right\}.\end{aligned}$$

Our first aim is to show that

$$\mathbb{P}\left\{\sup_{t\in[0,\frac{T}{u}]_{\frac{\delta}{u}}}\sup_{s\in(G_{\frac{\delta}{u}}\setminus I(t_u))} Z_u(t,s)>m(u)\right\}=o(\overline{\Phi}(m(u))),\qquad u\to\infty. \tag{6.31}$$

Since for any fixed $0\le t\le s$ random variables $Z_u(t,s)$ and $X_u(s-t)$ have the same distributions we have with $I'(t_u)=(-\frac{\delta_u}{2}+t_*,\frac{\delta_u}{2}+t_*)\cap G_{\delta/u}$

$$\begin{aligned}\mathbb{P}\left\{\sup_{t\in[0,T/u]_{\delta/u}}\sup_{s\in(G_{\delta/u}\setminus I(t_u))} Z_u(t,s)>m(u)\right\} &\le \sum_{t\in[0,T/u]_{\delta/u}}\mathbb{P}\left\{\sup_{s\in(G_{\delta/u}\setminus I(t_u))} Z_u(t,s)>m(u)\right\}\\ &= \sum_{t\in[0,T/u]_{\delta/u}}\mathbb{P}\left\{\sup_{s\in(G_{\delta/u}\setminus I(t_u))} X_u(s-t)>m(u)\right\}\\ &\le (1+[\frac{T}{\delta}])\mathbb{P}\left\{\sup_{s\in(G_{\delta/u}\setminus I'(t_u))} X_u(s)>m(u)\right\}.\end{aligned}$$

We have that for any chosen small ε and large M the last probability above does not exceed

$$\begin{aligned} &\sum_{t\in(G_{\frac{\delta}{u}}\setminus I'(t_u))}\mathbb{P}\left\{X_u(t)>m(u)\right\}\\ =\ &\sum_{t\in(G_{\delta/u}\setminus I'(t_u))}\overline{\Phi}(\frac{m(u)}{\sigma_{X_u}(t)})\\ \le\ &2\overline{\Phi}(m(u))\Big(\sum_{t\in[0,\varepsilon]_{\frac{\delta}{u}}} e^{-\frac{m^2(u)}{2}(\frac{1}{\sigma^2_{X_u}(t)}-1)}+\sum_{t\in[M,\infty)_{\frac{\delta}{u}}} e^{-\frac{m^2(u)}{2}(\frac{1}{\sigma^2_{X_u}(t)}-1)}+\sum_{t\in([\varepsilon,M]_{\frac{\delta}{u}}\setminus I'(t_u))} e^{-\frac{m^2(u)}{2}(\frac{1}{\sigma^2_{X_u}(t)}-1)}\Big)\end{aligned}$$

$$=: \quad 2\overline{\Phi}(m(u))(R_1(u)+R_2(u)+R_3(u)).$$

Thus, to establish (6.31) we need to prove that $R_1(u)+R_2(u)+R_3(u)\to 0$ as $u\to\infty$. By Lemma 6.5.1 t_u is unique for large u and we have

$$\sigma_{X_u}(t)=\frac{\sigma(ut)}{u(1+ct)}m(u)=\frac{\sigma(ut)}{\sigma(ut_u)}\frac{1+ct_u}{1+ct}.$$

Estimation of $R_1(u)$. We have for all large u and $t\in[0,\varepsilon]_{\delta/u}$

$$\sigma_{X_u}(t)\le \mathbb{C}\frac{\sigma(ut)}{\sigma(ut_u)}.$$

i) Assume that $ut\ge \ln u$. Then with h being a slowly varying at ∞ function and $0<\epsilon<\alpha$ by Potter's theorem (Theorem 1.5.6 in [4]) we have

$$\frac{\sigma(ut)}{\sigma(ut_u)}=(\frac{t}{t_u})^{\alpha}\frac{h(ut)}{h(ut_u)}\le \mathbb{C}t^{\alpha}(\frac{t_u}{t})^{\epsilon}\le \mathbb{C}t^{\alpha-\epsilon}.$$

ii) Assume that $ut<\ln u$. Since $t\in[0,\varepsilon]_{\delta/u}$ we have $ut\ge\delta$ for $t\neq 0$. Then for $\epsilon\in(0,\alpha)$ and large u

$$\frac{\sigma(ut)}{\sigma(ut_u)}\le u^{-(\alpha-\epsilon)}\sup_{t\in[\delta,\ln u]}\sigma(t)\le u^{-(\alpha-\epsilon)}\ln u.$$

Combining the above inequalities we have that for sufficiently small ε and for all $t\in[0,\varepsilon]_{\delta/u}$ uniformly for large u it holds that $\frac{1}{\sigma^2_{X_u}(t)}-1\ge 2$. Thus, for small enough $\varepsilon>0$

$$R_1(u)\le \mathbb{C}ue^{-m^2(u)}\to 0,\quad u\to\infty. \tag{6.32}$$

Estimation of $R_2(u)$. By Potter's theorem we have for M large enough and $0<\epsilon'<1-\alpha$

$$\frac{\sigma(ut)}{\sigma(ut_u)}\le \mathbb{C}(\frac{t}{t_u})^{\alpha+\epsilon'}.$$

Since $t_u\to t^*$ as $u\to\infty$ we have for some small $\epsilon>0$

$$\sigma_{X_u}(t)\le \mathbb{C}\frac{t^{\alpha+\epsilon'}}{1+ct}\le t^{-\epsilon},$$

hence for all $t>M$ uniformly for u large it holds that $\frac{1}{\sigma^2_{X_u}(t)}-1\ge 2t^{\epsilon}$. Choosing M large enough and ε sufficiently small we have as $u\to\infty$

$$R_2(u)\le \sum_{t\in[M,\infty)_{\delta/u}} e^{-t^{\epsilon}m^2(u)}=\sum_{t\in[M,\infty)_{\delta}} e^{-t^{\epsilon}\frac{m^2(u)}{u^{\varepsilon}}}\le \mathbb{C}e^{-\frac{m^2(u)}{u^{\varepsilon}}}\to 0. \tag{6.33}$$

Estimation of $R_3(u)$. We have by Lemma 6.5.1 that with some $\mathbb{C}>0$

$$1-\sigma_{X_u}(t)\ge \mathbb{C}(t-t_u)^2,\quad t\in[\varepsilon,M]$$

and hence by (6.15) for $t\in[\varepsilon,M]_{\frac{\delta}{u}}\backslash I(t_u)$ it holds that

$$m^2(u)(\frac{1}{\sigma^2_{X_u}(t)}-1)\ge m^2(u)(1-\sigma_{X_u}(t))\ge \mathbb{C}(t-t_u)^2m^2(u)\ge \mathbb{C}\ln^2 u.$$

Thus, for $t \in [\varepsilon, M]_{\frac{\delta}{u}} \backslash I(t_u)$ it holds that $e^{-\frac{m^2(u)}{2}(\frac{1}{\sigma_u^2(t)}-1)} \le u^{-\mathbb{C}\ln u}$, and we obtain

$$R_3(u) \le u\mathbb{C}_1 u^{-\mathbb{C}_2 \ln u} \to 0, \quad u \to \infty.$$

Combining the estimate above with (6.32) and (6.33) obtain (6.31). It follows from the calculations in Theorem 6.3.1 that

$$\mathbb{P}\left\{\sup_{t\in[0,\frac{T}{u}]_{\frac{\delta}{u}}} \sup_{s\in I(t_u)} Z_u(t,s) > m(u)\right\} \ge \overline{\Phi}(m(u)), \quad u \to \infty$$

and the first claim follows by (6.30) and (6.31).

Next we show the second statement of the lemma. Again, by Bonferroni inequality we have

$$\begin{aligned} & \mathbb{P}\left\{\inf_{t\in[0,\frac{T}{u}]_{\frac{\delta}{u}}} \sup_{s\in I(t_u)} Z_u(t,s) > m(u)\right\} \\ \le\ & \psi^{\inf}_{\delta,T}(u) \\ \le\ & \mathbb{P}\left\{\inf_{t\in[0,\frac{T}{u}]_{\frac{\delta}{u}}} \sup_{s\in I(t_u)} Z_u(t,s) > m(u)\right\} + \mathbb{P}\left\{\inf_{t\in[0,\frac{T}{u}]_{\frac{\delta}{u}}} \sup_{s\in(G_{\frac{\delta}{u}}\backslash I(t_u))} Z_u(t,s) > m(u)\right\}. \end{aligned}$$

It follows from the calculations in Theorem 6.3.1 that

$$\mathbb{P}\left\{\inf_{t\in[0,\frac{T}{u}]_{\frac{\delta}{u}}} \sup_{s\in I(t_u)} Z_u(t,s) > m(u)\right\} \ge \overline{\Phi}(m(u)), \quad u \to \infty$$

and by (6.31) we obtain that

$$\mathbb{P}\left\{\inf_{t\in[0,\frac{T}{u}]_{\frac{\delta}{u}}} \sup_{s\in(G_{\frac{\delta}{u}}\backslash I(t_u))} Z_u(t,s) > m(u)\right\} = o(\overline{\Phi}(m(u))), \quad u \to \infty.$$

Combining both statements above we obtain the second claim of the lemma. □

Proof of Lemma 6.5.3. Fix some $t \neq s \in I(t_u)$. Since $\sigma_{X_u}(t), \sigma_{X_u}(s) \le 1$ we have by Lemma 2.3 in [56] with $r_u(t,s) = \mathrm{Corr}(\overline{X}_u(t), \overline{X}_u(s))$

$$\begin{aligned} \mathbb{P}\{X_u(t) > m(u), X_u(s) > m(u)\} &\le \mathbb{P}\{\overline{X}_u(t) > m(u), \overline{X}_u(s) > m(u)\} \\ &\le \overline{\Phi}(m(u))\overline{\Phi}(m(u)\sqrt{\frac{1-r_u(t,s)}{2}}). \end{aligned}$$

If $\alpha < 1/2$, then by Lemma 6.5.1 as $u \to \infty$ for some $\epsilon > 0$ it holds that $m(u)\sqrt{\frac{1-r_u(t,s)}{2}} \ge u^{\epsilon}$ and thus uniformly for $t \neq s \in I(t_u)$

$$\overline{\Phi}(m(u)\sqrt{\frac{1-r_u(t,s)}{2}}) \le e^{-\frac{1}{3}u^{2\epsilon}}, \quad u \to \infty. \tag{6.34}$$

If $\alpha = 1/2$, then $t_* = 1/c$ and by Lemma 6.5.1 and (6.5) we have that for some $\varepsilon' > 0$ as $u \to \infty$

$$\frac{m^2(u)}{2}\frac{1-r_u(t,s)}{2} \sim \frac{u^2(1+ct_*)^2\sigma^2(u|s-t|)}{8\sigma^4(ut_*)} \sim \frac{u^2c^2\sigma^2(u|s-t|)}{2\sigma^4(u)} \geq (1/2+\varepsilon')\ln u$$

implying as $u \to \infty$

$$\overline{\Phi}(m(u)\sqrt{\frac{1-r_u(t,s)}{2}}) \leq u^{-1/2-\varepsilon'}.$$

Combining the line above with (6.34) we obtain the claim. □

Proof of Lemma 6.5.4. Let $a = K\delta$, where K is a large natural number that we shall choose later on. By the proofs of Theorem 15 and Lemma 16 in [23] we have with σ_η^2 being the variance of η that

$$\liminf_{S\to\infty} \frac{\mathcal{H}_\eta(\{0,a,2a,...,S\})}{S} \geq \frac{1}{a}\Big(1-\frac{2}{a}\int_0^\infty e^{-\frac{\sigma_\eta^2(t)}{4}}dt\Big).$$

We have that for all u large enough $\sigma_\eta^2(t) \geq \mathbb{C}t$ implying that $\int_0^\infty e^{-\frac{\sigma_\eta^2(t)}{4}}dt < \infty$. Choosing sufficiently large K we have

$$\liminf_{S\to\infty} \frac{\mathcal{H}_\eta(\{0,\delta,2\delta,...,S\})}{S} \geq \liminf_{S\to\infty} \frac{\mathcal{H}_\eta(\{0,a,2a,...,S\})}{S} \geq \frac{1}{a}\cdot\frac{1}{2} > 0.$$

Next we prove that for $I(S) := \frac{\mathcal{H}_\eta(\{0,\delta,2\delta,...,S\})}{S}$ it holds that for large $S \in G_\delta$

$$I(S) \geq I(S+\delta). \tag{6.35}$$

We have

$$\begin{aligned}(S+\delta)I(S+\delta) &\leq \mathbb{E}\left\{\sup_{t\in\{0,\delta,...,S\}} e^{\sqrt{2}\eta(t)-\sigma_\eta^2(t)}\right\} + \mathbb{E}\left\{e^{\sqrt{2}\eta(S+\delta)-\sigma_\eta^2(S+\delta)}\right\}F(S+\delta)\\ &= SI(S)+F(S+\delta),\end{aligned}$$

where $F(M) = \mathbb{P}\left\{\underset{t\in\{0,\delta,...,M\}}{\operatorname{argmax}}(\sqrt{2}\eta(t)-\sigma_\eta^2(t)) = M\right\}$ for $M \in G_\delta$. Thus, to claim (6.35) we need to show that for large S

$$\delta I(S) \geq F(S+\delta). \tag{6.36}$$

Since $\liminf_{S\to\infty} I(S) > 0$, we have that $\delta I(S) > \varepsilon$ for all S and some positive ε, but on the other hand as $S \to \infty$ it holds that

$$\begin{aligned}F(S+\delta) &\leq \mathbb{P}\left\{\sqrt{2}\eta(S+\delta)-\sigma_\eta^2(S+\delta) \geq \sqrt{2}\eta(0)-\sigma_\eta^2(0)\right\}\\ &= \mathbb{P}\left\{\sqrt{2}\eta(S+\delta)-\sigma_\eta^2(S+\delta) \geq 0\right\} \to 0,\end{aligned}$$

consequently (6.36) holds and hence $I(S)$ is non-increasing for large S. Thus, $\lim_{S\to\infty} I(S) \in (0,\infty)$ and the claim holds. □

Chapter 7

Properties of Pickands Constants

In this chapter we give new results on Pickands-type constants.

7.1 Introduction

The classical Pickands constant $\mathcal{H}_{2H}$ plays an important role in the theory of Gaussian process and appears in many asymptotics of the ruin probabilities, see, e.g., [59]. However, the exact value of the classical Pickands constant is known only for $H = 1/2$. Thus, naturally arises the question of approximation of the classical Pickands constants. The main difficulty in approximation of $\mathcal{H}_{2H}$ by using the classical definition is the limit. The other technical issue is simulation of fBM on continuous interval $[0, S]$.

In practice, $\mathcal{H}_{2H}$ is approximated by the discrete Pickands constant. It is known, see [12], that $\lim_{\delta \to 0} \mathcal{H}_{2H}^{\delta} = \mathcal{H}_{2H}$. Therefore, to approximate $\mathcal{H}_{2H}$ we need to answer two main questions:

Q1: How to approximate/simulate $\mathcal{H}_{2H}^{\delta}$?
Q2: What is the error of discretization, i.e., what is the speed of convergence of $\mathcal{H}_{2H} - \mathcal{H}_{2H}^{\delta}$ to 0 as $\delta \to 0$?

The first question is answered in [30], the approximation involves an alternative representation of $\mathcal{H}_{2H}^{\delta}$ without limit. We also give Lemma 7.2.5 that may help to estimate an error of simulation in approach of [30].

For the second question, the following conjecture is formulated in [30]:

Conjecture 7.1.1 *It holds that*

$$\mathcal{A}_H := \lim_{\delta \to 0} \frac{\mathcal{H}_{2H} - \mathcal{H}_{2H}^{\delta}}{\delta^H} \in (0, \infty).$$

We think that the above conjecture is true, but it seems very difficult to prove it for $H \neq 1/2$. In Theorem 7.2.1 we give a relatively precise upper bound for $\mathcal{H}_{2H} - \mathcal{H}^{\delta}_{2H}$ and all small $\delta > 0$, while in Theorem 7.2.2 we prove the conjecture for $H = 1/2$ and calculate $\mathcal{A}_{1/2}$ explicitly.

The question of speed of convergence of the discrete Pickands constants to continuous ones is related to the estimation of

$$\sup_{t\in[0,1]} B_H(t) - \sup_{t\in[0,1]_\delta} B_H(t).$$

as $\delta \to 0$. We refer to [6, 7] for the interesting analysis of the expression above. For BM case, (i.e., when $H = 1/2$) we refer to [28] for the survey of the known results for the current moment.

7.2 Main Results

The theorem below gives an upper bound for $\mathcal{H}_{2H} - \mathcal{H}^{\delta}_{2H}$ for all small δ.

Theorem 7.2.1 *It holds that for $H \in (0, 1)$ and all sufficiently small $\delta > 0$*

$$\mathcal{H}_{2H} - \mathcal{H}^{\delta}_{2H} \leq -C\delta^H \ln \delta.$$

Next we focus on BM case. We start with the theorem providing us an alternative representation of $\mathcal{H}^{\delta}$.

Theorem 7.2.2 *For any $\delta > 0$ it holds that*

$$\mathcal{H}^{\delta} = \big(\delta \exp(2 \sum_{k=1}^{\infty} \frac{\overline{\Phi}(\sqrt{\delta k/2})}{k})\big)^{-1}. \tag{7.1}$$

The theorem above follows by Lemma 5.16 and Remark 5.17 in [46], in Section 7.4 we present an alternative proof. Differentiating the formula above with respect to δ we obtain

Corollary 7.2.3 *$\mathcal{H}^{\delta}$ is a strictly decreasing function with respect to δ for all $\delta \geq 0$.*

Now we can prove Conjecture 7.1.1 for $H = 1/2$ relying on Theorem 7.2.2 and calculate $\mathcal{A}_{1/2}$.

Theorem 7.2.4 *It holds with ζ being the Euler-Riemann zeta function that*

$$\lim_{\delta\to 0} \frac{\mathcal{H}_1 - \mathcal{H}^{\delta}_1}{\sqrt{\delta}} = -\frac{\zeta(1/2)}{\sqrt{\pi}} > 0.$$

It is interesting, that the constant $-\frac{\zeta(1/2)}{\sqrt{\pi}}$ is the limit in the theorem above. This constant appears in many problems concerning the difference between supremum of BM on a continuous and discrete grids, see [28].

Next we present a lemma establishing tail properties of the rv appearing in the Dieker's representation of the classical Pickands constant, see Proposition 1 in [30]. Namely,

$$\mathcal{H}_{2H} = \mathbb{E}\left\{ \frac{\sup_{t\in\mathbb{R}} e^{(\sqrt{2}B_H(t)-|t|^{2H})}}{\int_{\mathbb{R}} e^{\sqrt{2}B_H(t)-|t|^{2H}}dt} \right\} =: \mathbb{E}\left\{\xi_H\right\} \in (0,\infty). \tag{7.2}$$

Since this representation is useful for approximations of $\mathcal{H}_{2H}$ via Monte-Carlo simulations (see [30]) it is worth to know the heaviness of tail of ξ_H. In the following lemma we prove that ξ_H has a light tail.

Lemma 7.2.5 *For any large x with ξ_H defined in (7.2) it holds that*

$$\mathbb{P}\left\{\xi_H > x\right\} \le e^{-C\ln^2 x}.$$

7.3 Technical Lemmas

In this section we present and prove four lemmas needed for our proof of Theorem 7.2.1. Let

$$Z(t) = \sqrt{2}B_{2H}(t) - |t|^{2H}, \quad t\in\mathbb{R}.$$

The first lemma allows us to estimate the difference between the continuous and discrete time supremum on a finite interval.

Lemma 7.3.1 *For sufficiently small $\delta > 0$ it holds that*

$$\mathbb{P}\left\{ \sup_{t\in[0,1]} Z(t) - \sup_{t\in[0,1]_\delta} Z(t) < -\delta^H \ln\delta \right\} \ge 1 - e^{-C\ln^2\delta}.$$

Proof of Lemma 7.3.1. Note that for any $b > 0$ it holds that

$$\begin{aligned}
&\mathbb{P}\left\{ \sup_{t\in[0,1]} Z(t) - \sup_{t\in[0,1]_\delta} Z(t) > b \right\}\\
\le\ & \mathbb{P}\big(\exists t\in[0,1]_\delta, s\in[t,t+\delta] : Z(s)-Z(t) > b\big)\\
\le\ & \sum_{t\in[0,1]_\delta} \mathbb{P}\left\{\exists s\in[t,t+\delta] : Z(s)-Z(t) > b\right\}\\
=\ & \sum_{t\in[0,1]_\delta} \mathbb{P}\left\{\exists s\in[t,t+\delta] : \sqrt{2}(B_H(s)-B_H(t)) - s^{2H} + t^{2H} > b\right\}.
\end{aligned}$$

Since for any considered s,t in the sum above it holds, that $-s^{2H}+t^{2H}\le 0$, the last sum above does not exceed

$$\sum_{t\in[0,1]_\delta} \mathbb{P}\left\{\exists s\in[t,t+\delta] : \sqrt{2}(B_H(s)-B_H(t)) > b\right\}.$$

Next, by the stationarity of increments of fBM we have

$$\begin{aligned}\sum_{t\in[0,1]_\delta} \mathbb{P}\left\{\exists s\in[t,t+\delta]:\sqrt{2}(B_H(s)-B_H(t))>b\right\} &\le \frac{1}{\delta}\mathbb{P}\left\{\exists t\in[0,\delta]:\sqrt{2}B_H(t)>b\right\}\\ &= \frac{1}{\delta}\mathbb{P}\left\{\exists t\in[0,1]:\sqrt{2}B_H(t)>b\delta^{-H}\right\}.\end{aligned}$$

Taking $b=-\delta^H\ln\delta$ we finally have for sufficiently small δ

$$\begin{aligned}\mathbb{P}\left\{\sup_{t\in[0,1]} Z(t)-\sup_{t\in[0,1]_\delta} Z(t)>-\delta^H\ln\delta\right\} &\le \frac{1}{\delta}\mathbb{P}\left\{\exists t\in[0,1]:B_H(t)>-\frac{\ln\delta}{\sqrt{2}}\right\}\\ &\le e^{-C\ln^2\delta},\end{aligned}$$

where the last line above follows from Borell-TIS inequality (see [50]). Rewriting the statement above we have

$$\mathbb{P}\left\{\sup_{t\in[0,1]} Z(t)-\sup_{t\in[0,1]_\delta} Z(t)<-\delta^H\ln\delta\right\}\ge 1-e^{-C\ln^2\delta}$$

and the claim follows. □

Lemma 7.3.2 *For any large x it holds that*

$$\mathbb{P}\left\{\sup_{t\in[0,1]} e^{Z(t)}>x\right\}\le e^{-C\ln^2 x}.$$

Proof of Lemma 7.3.2. Observe that

$$\mathbb{P}\left\{\sup_{t\in[0,1]} e^{Z(t)}>x\right\}=\mathbb{P}\left\{\sup_{t\in[0,1]}(\sqrt{2}B_H(t)-t^{2H})>\ln x\right\}\le\mathbb{P}\left\{\sup_{t\in[0,1]} B_H(t)>\ln x/\sqrt{2}\right\}\le e^{-C\ln^2 x},$$

where the last line follows by Borell-TIS inequality. □

The following lemma provides us a crucial bound for $\mathcal{H}_{2H}-\mathcal{H}_{2H}^\delta$.

Lemma 7.3.3 *For sufficiently small $\delta>0$ it holds that*

$$\mathcal{H}_{2H}-\mathcal{H}_{2H}^\delta\le 2\Big(\mathbb{E}\left\{\sup_{t\in[0,1]} e^{Z(t)}\right\}-\mathbb{E}\left\{\sup_{t\in[0,1]_\delta} e^{Z(t)}\right\}\Big).$$

Proof of Lemma 7.3.3. As follows from the proof of Theorem 1 in [33], the first equation on p.12 with $c_\delta=[1/\delta]\delta$ ($[\cdot]$ is the integer part) it holds that

$$\begin{aligned}\mathcal{H}_{2H}-\mathcal{H}_{2H}^\delta &\le c_\delta^{-1}\mathbb{E}\left\{\sup_{t\in[0,c_\delta]} e^{Z(t)}-\sup_{t\in[0,c_\delta]_\delta} e^{Z(t)}\right\}\\ &\le 2\mathbb{E}\left\{\sup_{t\in[0,c_\delta]} e^{Z(t)}-\sup_{t\in[0,c_\delta]_\delta} e^{Z(t)}\right\}\le 2\mathbb{E}\left\{\sup_{t\in[0,1]} e^{Z(t)}-\sup_{t\in[0,1]_\delta} e^{Z(t)}\right\}\end{aligned}$$

and the claim follows. □

The next lemma is a general observation on properties of a random variable with a light tail.

Lemma 7.3.4 *For any $p > 0$, non-negative rv ξ such that $\mathbb{P}\{\xi > x\} \le e^{-C\ln^2 x}$ and $N_\delta \subset \Omega$ such that $\mathbb{P}\{N_\delta\} \le e^{-C\ln^2\delta}$ it holds that*

$$\int_{N_\delta} \xi(w) d\mathbb{P}(w) \le \delta^p, \qquad \delta \to 0.$$

Proof of Lemma 7.3.4. We have

$$\begin{aligned}\int_{N_\delta} \xi(\omega) d\mathbb{P}(\omega) &= \mathbb{E}\left\{\mathbb{I}(\omega \in N_\delta)\xi(\omega)\right\} \\ &\le \mathbb{E}\left\{\xi(\omega)\mathbb{I}(\xi(\omega) > a_\delta)\right\},\end{aligned}$$

where $a > 0$ is such number that $\mathbb{P}\{\xi > a_\delta\} = e^{-C\ln^2\delta}$. Next with any $p > 0$ for sufficiently small $\delta > 0$ we have

$$\begin{aligned}\mathbb{E}\left\{\xi(\omega)\mathbb{I}(\xi > a_\delta)\right\} &= \int_0^\infty \mathbb{P}\left\{\xi\mathbb{I}(\xi > a_\delta) > x\right\} dx \\ &= \int_0^{a_\delta} \mathbb{P}\left\{\xi\mathbb{I}(\xi > a_\delta) > x\right\} dx + \int_{a_\delta}^\infty \mathbb{P}\left\{\xi\mathbb{I}(\xi > a_\delta) > x\right\} dx \\ &\le a_\delta \mathbb{P}\left\{\xi > a_\delta\right\} + \int_{a_\delta}^\infty \mathbb{P}\left\{\xi > x\right\} dx.\end{aligned}$$

Since $\mathbb{P}\{\xi > x\}$ is a decreasing function with respect to x for the integral above we have

$$\begin{aligned}\int_{a_\delta}^\infty \mathbb{P}\left\{\xi > x\right\} dx &= \sum_{k=1}^\infty \int_{ka_\delta}^{(k+1)a_\delta} \mathbb{P}\left\{\xi > x\right\} dx \\ &\le a_\delta \sum_{k=1}^\infty \mathbb{P}\left\{\xi > ka_\delta\right\}.\end{aligned}$$

By the lines above in order to prove the lemma it is sufficient to show that for any $p > 0$ uniformly for small δ it holds that

$$a_\delta \sum_{k=1}^\infty \mathbb{P}\left\{\xi > ka_\delta\right\} < \delta^p \tag{7.3}$$

We have that

$$\begin{aligned}a_\delta \sum_{k=1}^\infty \mathbb{P}\left\{\xi > ka_\delta\right\} &\le a_\delta \Big(\sum_{k=1}^{a_\delta/\delta} \mathbb{P}\left\{\xi > a_\delta\right\} + \sum_{k=a_\delta/\delta}^\infty e^{-C\ln^2 k} \Big) \\ &\le \frac{a_\delta^2}{\delta} \mathbb{P}\left\{\xi > a_\delta\right\} + a_\delta \sum_{k=a_\delta/\delta}^\infty e^{-C\ln^2 k}.\end{aligned}$$

For the sum above we have

$$\sum_{k=a_\delta/\delta}^{\infty} e^{-C\ln^2 k} \le e^{-C\ln^2(a_\delta/\delta)} = o(\delta^p), \qquad \delta \to 0$$

and hence to prove (7.3) it is sufficient to show that

$$\frac{a_\delta^2}{\delta}\mathbb{P}\left\{\xi > a_\delta\right\} = o(\delta^p), \quad \delta \to 0. \tag{7.4}$$

By the choice of a_δ we have

$$\frac{a_\delta^2}{\delta}\mathbb{P}\left\{\xi > a_\delta\right\} \le \frac{a_\delta^2}{\delta}e^{-C\ln^2\delta}$$

and thus (7.4) does not follow from the line above only if $a_\delta > e^{C\ln^2\delta}$. In this case (7.4) follows by the assertion that $\mathbb{P}\left\{\xi > x\right\} \ge e^{-C\ln^2 x}$ and the claim is established. □

7.4 Proofs

Now we are ready to perform our proofs.

Proof of Theorem 7.2.1. Set $\lambda = \sup\limits_{t\in[0,1]} Z(t)$ and $\lambda_\delta = \sup\limits_{t\in[0,1]_\delta} Z(t)$. We have by Lemma 7.3.3

$$\mathcal{H}_{2H} - \mathcal{H}_{2H}^\delta \le 2\mathbb{E}\left\{e^\lambda - e^{\lambda_\delta}\right\} = 2\mathbb{E}\left\{e^{\lambda_\delta}\left(e^{\lambda-\lambda_\delta} - 1\right)\right\}.$$

Let $A_\delta \subset \Omega$ consists of $\omega \in \Omega$ such that $\lambda - \lambda_\delta \le -\delta^H \ln\delta$ and $B_\delta = \Omega\backslash A_\delta$. We have for small $\delta > 0$

$$\begin{aligned}
\mathbb{E}\left\{e^{\lambda_\delta}\left(e^{\lambda-\lambda_\delta} - 1\right)\right\} &= \int\limits_{A_\delta} e^{\lambda_\delta}\left(e^{\lambda-\lambda_\delta} - 1\right)d\mathbb{P}(w) + \int\limits_{B_\delta} e^{\lambda_\delta}\left(e^{\lambda-\lambda_\delta} - 1\right)d\mathbb{P}(w) \\
&\le 2\int\limits_{A_\delta} e^{\lambda_\delta}\left(\lambda - \lambda_\delta\right)d\mathbb{P}(w) + \int\limits_{B_\delta}(e^\lambda - e^{\lambda_\delta})d\mathbb{P}(w) \\
&\le 2\sup_{\omega\in A_\delta}\left(\lambda - \lambda_\delta\right)\int\limits_{A_\delta} e^\lambda d\mathbb{P}(w) + \int\limits_{B_\delta} e^\lambda d\mathbb{P}(w) \\
&\le -2\delta^H \ln\delta \int\limits_{\Omega} e^\lambda d\mathbb{P}(w) + \int\limits_{B_\delta} e^\lambda d\mathbb{P}(w) \\
&\le -C\delta^H \ln\delta + \int\limits_{B_\delta} e^\lambda d\mathbb{P}(w),
\end{aligned}$$

where the second line from the end follows by Lemma 7.3.1. By Lemma 7.3.2 e^λ satisfies conditions of Lemma 7.3.4 and hence we have

$$\int\limits_{B_\delta} e^\lambda d\mathbb{P}(w) = o(\delta^H), \qquad \delta \to 0.$$

and the claim follows. □

Proof of Theorem 7.2.2. We shall compute the asymptotics of

$$\psi_\delta(u) := \mathbb{P}\left\{\exists t \in G(\delta) : B(t) - ct > u\right\}, \quad u \to \infty$$

in two different ways and then compare the answers.

First approach. In [40, 47] it is shown that

$$\psi_\delta(u) \sim \mathcal{H}_{2c^2\delta} e^{-2cu}, \quad u \to \infty. \tag{7.5}$$

Second approach. Let

$$X_t = B(\delta\Pi(t)) - c\delta\Pi(t), \quad t \geq 0,$$

where $\Pi(t)$ is a standard Poisson process with intensity 1 and $\delta > 0$. Since a trajectory of $\Pi(t),\ t \geq 0$ with probability one is $\{0, 1, 2, ...\}$ we have for all $u > 0$

$$\psi_\delta(u) = \mathbb{P}\left\{\exists t \geq 0 : X_t > u\right\},$$

and later on we focus on the computation of the right side in the line above. Since $X_t, t \geq 0$ is a Lévy process we have by Theorem 8.2 in [26]

$$\mathbb{P}\left\{\exists t \geq 0 : X_t > u\right\} \sim e^{-\omega u} \frac{1}{\omega k(0,0)} \frac{1}{l'(0,-\omega)},$$

where the definitions of ω, $k(a,v)$ and $l'(0,-\omega)$ are given in p.116: third line in paragraph "General case", p.37: (3.10) and p.117 in [26], respectively. In our case $\omega > 0$ solves the equation $\mathbb{E}\left\{e^{\omega\mathcal{N}(-c\delta,\delta)}\right\} = 1$ implying $\omega = 2c$. Next we calculate $k(0,a)$. For $F_t(x)$ be the df of X_t we have

$$F_t(x) = e^{-t}\frac{t^0}{0!}\mathbb{P}\left\{B(0) - 0 * c < x\right\} + e^{-t}\frac{t^1}{1!}\mathbb{P}\left\{B(\delta) - \delta c < x\right\} + e^{-t}\frac{t^2}{2!}\mathbb{P}\left\{B(2\delta) - 2\delta c < x\right\} + ..., \ x \in \mathbb{R}.$$

Differentiating we obtain the density $g_t(x)$

$$g_t(x) = e^{-t}\left(\frac{t^1}{1!\sqrt{\delta}}\phi(\frac{x+\delta c}{\sqrt{\delta}}) + \frac{t^2}{2!\sqrt{2\delta}}\phi(\frac{x+2\delta c}{\sqrt{2\delta}}) + ...\right), \quad x \in \mathbb{R},$$

where ϕ is the density of a standard Gaussian rv. To compute

$$\int_0^\infty \int_0^\infty \frac{1}{t}(e^{-t} - e^{-ax}) g_t(x) dx dt$$

we calculate this integral for each summand in $g_t(x)$.

$$\int_0^\infty \int_0^\infty \frac{1}{t}(e^{-t} - e^{-ax}) e^{-t} \frac{t^k}{k!\sqrt{\delta k}}\phi(\frac{x+k\delta c}{\sqrt{\delta k}}) dx dt$$

$$
\begin{aligned}
&= \int_0^\infty e^{-2t}\frac{t^{k-1}}{k!\sqrt{\delta k}}dt\int_0^\infty \phi(\frac{x+\delta kc}{\sqrt{\delta k}})dx - \int_0^\infty \frac{1}{t}e^{-t}\frac{t^k}{k!\sqrt{\delta k}}dt\int_0^\infty \phi(\frac{x+\delta kc}{\sqrt{\delta k}})e^{-ax}dx \\
&= \frac{1}{k2^k}\int_0^\infty e^{-2t}\frac{(2t)^{k-1}}{(k-1)!}d(2t)\int_0^\infty \phi(\frac{x+\delta kc}{\sqrt{\delta k}})d(\frac{x}{\sqrt{\delta k}}) - \int_0^\infty e^{-t}\frac{t^{k-1}}{k!}dt\int_0^\infty \phi(\frac{x+\delta kc}{\sqrt{\delta k}})e^{-a\sqrt{\delta k}\frac{x}{\sqrt{\delta k}}}d(\frac{x}{\sqrt{\delta k}}) \\
&= \frac{1}{k2^k}\overline{\Phi}(c\sqrt{\delta k}) - \frac{1}{k}\int_0^\infty \phi(z+c\sqrt{\delta k})e^{-a\sqrt{\delta k}z}dz \\
&= \frac{1}{k2^k}\overline{\Phi}(c\sqrt{\delta k}) - \frac{1}{k}\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{1}{2}\left((z+(c+a)\sqrt{\delta k})^2+c^2\delta k-k\delta(c+a)^2\right)}dz \\
&= \frac{1}{k2^k}\overline{\Phi}(c\sqrt{\delta k}) - \frac{1}{k}\overline{\Phi}((c+a)\sqrt{\delta k})e^{\frac{\delta k(a^2+2ac)}{2}}.
\end{aligned}
$$

Since $\sum_{k=1}^{n}\frac{1}{t}(e^{-t}-e^{-ax})e^{-t}\frac{t^k}{k!\sqrt{\delta k}}\phi(\frac{x+k\delta c}{\sqrt{\delta k}})$ converges uniformly to $\sum_{k=1}^{\infty}\frac{1}{t}(e^{-t}-e^{-ax})e^{-t}\frac{t^k}{k!\sqrt{\delta k}}\phi(\frac{x+k\delta c}{\sqrt{\delta k}}),\ x,t\geq 0$ as $n\to\infty$ we can change the order of integration and summation and write

$$
k(0,a) = \exp\Big(\sum_{k=1}^{\infty}(\frac{1}{k}\overline{\Phi}((c+a)\sqrt{\delta k})e^{\frac{\delta k(a^2+2ac)}{2}} - \frac{1}{k2^k}\overline{\Phi}(c\sqrt{\delta k}))\Big).
$$

Thus, we obtain

$$
k(0,0) = \exp\Big(\sum_{k=1}^{\infty}\overline{\Phi}(c\sqrt{\delta k})\frac{2^k-1}{k2^k}\Big). \tag{7.6}
$$

Next we have

$$
l(0,a) = \frac{1}{k(0,a)} = \exp\Big(-\sum_{k=1}^{\infty}\frac{1}{k}\overline{\Phi}((c+a)\sqrt{\delta k})e^{\frac{\delta k(a^2+2ac)}{2}}\Big)\exp\Big(\sum_{k=1}^{\infty}\frac{1}{k2^k}\overline{\Phi}(c\sqrt{\delta k})\Big).
$$

We show in the Appendix that

$$
\frac{\partial}{\partial a}\Big(\exp\Big(-\sum_{k=1}^{\infty}\frac{1}{k}\overline{\Phi}((c+a)\sqrt{\delta k})e^{\frac{\delta k(a^2+2ac)}{2}}\Big)\Big)|_{a=-\omega} = c\delta\exp(\sum_{k=1}^{\infty}\frac{1}{k}\overline{\Phi}(c\sqrt{\delta k})) \tag{7.7}
$$

and hence

$$
l'(0,a)|_{a=-\omega} = c\delta\exp(\sum_{k=1}^{\infty}\frac{1}{k}\overline{\Phi}(c\sqrt{\delta k}))\exp\Big(\sum_{k=1}^{\infty}\frac{1}{k2^k}\overline{\Phi}(c\sqrt{\delta k})\Big).
$$

Hence by (7.6) and the line above

$$
\begin{aligned}
\omega k(0,0)l(0,-\omega)' &= 2c^2\delta\exp(\sum_{k=1}^{\infty}\frac{\overline{\Phi}(c\sqrt{\delta k})}{k})\exp\Big(\sum_{k=1}^{\infty}\frac{1}{k2^k}\overline{\Phi}(c\sqrt{\delta k})\Big)\exp\Big(\sum_{k=1}^{\infty}\overline{\Phi}(c\sqrt{\delta k})\frac{2^k-1}{k2^k}\Big) \\
&= 2c^2\delta\exp(2\sum_{k=1}^{\infty}\frac{\overline{\Phi}(c\sqrt{\delta k})}{k}).
\end{aligned}
$$

Finally we have

$$\mathbb{P}\left\{\exists t \geq 0 : X_t > u\right\} \sim e^{-2cu}\Big(2c^2\delta \exp(2\sum_{k=1}^{\infty}\frac{\overline{\Phi}(c\sqrt{\delta k})}{k})\Big)^{-1}, \quad u \to \infty.$$

Since $\psi_\delta(u) = \mathbb{P}\left\{\exists t \geq 0 : X_t > u\right\}$ by (7.5) we obtain

$$\mathcal{H}_{2c^2\delta} = \Big(2c^2\delta \exp(2\sum_{k=1}^{\infty}\frac{\overline{\Phi}(c\sqrt{\delta k})}{k})\Big)^{-1}.$$

Putting $\eta = 2c^2\delta$ in the formula above we obtain the claim. □

Proof of Corollary 7.2.3. First we show that $v(\eta) = \eta \exp(2\sum_{k=1}^{\infty}\frac{\overline{\Phi}(\sqrt{\eta k/2})}{k})$ is an increasing function for $\eta > 0$. We have for any $0 < a < b$ and $\eta \in [a, b]$

$$\begin{aligned} v'(\eta) &= \exp(2\sum_{k=1}^{\infty}\frac{\overline{\Phi}(\sqrt{\frac{\eta k}{2}})}{k})\Big(1 - 2\eta\sum_{k=1}^{\infty}\frac{\phi(\sqrt{\frac{\eta k}{2}})}{k}\frac{\sqrt{k}}{\sqrt{2}}\frac{1}{2\sqrt{\eta}}\Big) \\ &= \exp(2\sum_{k=1}^{\infty}\frac{\overline{\Phi}(\sqrt{\frac{\eta k}{2}})}{k})\Big(1 - \frac{\sqrt{\eta}}{2\sqrt{\pi}}\sum_{k=1}^{\infty}\frac{e^{-\frac{\eta k}{4}}}{\sqrt{k}}\Big), \end{aligned} \tag{7.8}$$

justification of the differentiating in (7.8) is in the Appendix. We have

$$\frac{\sqrt{\eta}}{2\sqrt{\pi}}\sum_{k=1}^{\infty}\frac{e^{-\frac{\eta k}{4}}}{\sqrt{k}} < \frac{1}{\sqrt{\pi}}\sqrt{\frac{\eta}{4}}\int_0^{\infty} e^{-\frac{\eta z}{4}} z^{-1/2}dz = \frac{1}{\sqrt{\pi}}\int_0^{\infty} e^{-\frac{\eta z}{4}}(\frac{\eta z}{4})^{-1/2}d(\frac{\eta z}{4}) = \frac{1}{\sqrt{\pi}}\Gamma(1/2) = 1$$

and we obtain by (7.8) that $v'(\eta) > 0$ for any $\eta > 0$. Thus, $\mathcal{H}_\eta$ is decreasing for $\eta > 0$, and since by the classical definition $\mathcal{H}_0 > \mathcal{H}_\eta$ for any $\eta > 0$ we obtain the claim. □

Proof of Theorem 7.2.4. Since $\mathcal{H}_0 = 1$ (see, e.g., [26]) by Theorem 7.2.2 we obtain that

$$\mathcal{A} := \lim_{\eta\to 0}\frac{\mathcal{H}_0 - \mathcal{H}_\eta}{\sqrt{\eta}} = \lim_{\eta\to 0}\frac{1 - 1/v(\eta)}{\sqrt{\eta}},$$

where (recall) $v(\eta) = \eta \exp(2\sum_{k=1}^{\infty}\frac{\overline{\Phi}(\sqrt{\eta k/2})}{k})$. Since $\mathcal{H}_\eta = v(\eta)^{-1} \to 1$ as $\eta \to 0$ (see, e.g., [30]) we conclude that $\lim_{\eta\to 0} v(\eta) = 1$ and hence

$$\mathcal{A} = \lim_{\eta\to 0}\frac{v(\eta) - 1}{\sqrt{\eta}}.$$

Implementing the L'Hôpital's rule we obtain by (7.8)

$$\mathcal{A} = \lim_{\eta\to 0}\frac{v'(\eta)}{1/(2\sqrt{\eta})} = 2\lim_{\eta\to 0}\Big(\sqrt{\eta}\exp(2\sum_{k=1}^{\infty}\frac{\overline{\Phi}(\sqrt{\frac{\eta k}{2}})}{k})\Big(1 - \frac{\sqrt{\eta}}{2\sqrt{\pi}}\sum_{k=1}^{\infty}\frac{e^{-\frac{\eta k}{4}}}{\sqrt{k}}\Big)\Big).$$

Note that by the definition of $v(\eta)$ observation that $\lim_{\eta\to 0} v(\eta) = 1$ implies

$$\sqrt{\eta}\exp(2\sum_{k=1}^{\infty}\frac{\overline{\Phi}(\sqrt{\frac{\eta k}{2}})}{k}) \sim \frac{1}{\sqrt{\eta}}, \quad \eta \to 0$$

and hence

$$\mathcal{A} = \lim_{\eta\to 0}\frac{2}{\sqrt{\eta}}\Big(1 - \frac{\sqrt{\eta}}{2\sqrt{\pi}}\sum_{k=1}^{\infty}\frac{e^{-\frac{\eta k}{4}}}{\sqrt{k}}\Big)\Big).$$

Let $x = \sqrt{\eta}/2$, thus

$$\mathcal{A} = \lim_{x\to 0}\frac{1}{x}\Big(1 - \frac{x}{\sqrt{\pi}}\sum_{k=1}^{\infty}\frac{e^{-x^2}}{\sqrt{k}}\Big)\Big) = \lim_{x\to 0}\frac{1}{x}\Big(1 - \frac{x}{\sqrt{\pi}}\mathrm{Li}_{\frac{1}{2}}(e^{-x^2})\Big),$$

where $\mathrm{Li}_{\frac{1}{2}}$ is the polylogarithm function, see, e.g., [9]. As follows from equation (9.3) in [62]

$$\begin{aligned}\lim_{x\to 0}\frac{1}{x}\Big(1 - \frac{x}{\sqrt{\pi}}\mathrm{Li}_{\frac{1}{2}}(e^{-x^2})\Big) &= \lim_{x\to 0}\frac{1}{x}\Big(1 - \frac{x}{\sqrt{\pi}}\Big(\Gamma(1/2)(x^2)^{-1/2} + \zeta(1/2) + \sum_{k=1}^{\infty}\zeta(1/2-k)\frac{(-x^2)^k}{k!}\Big)\Big) \\ &= \frac{\zeta(1/2)}{\sqrt{\pi}} - \frac{1}{\sqrt{\pi}}\lim_{x\to 0}\Big(\sum_{k=1}^{\infty}\zeta(1/2-k)\frac{x^{2k}(-1)^k}{k!}\Big).\end{aligned}$$

Thus, to prove the claim is it enough to show that

$$\lim_{x\to 0}\sum_{k=1}^{\infty}\zeta(1/2-k)\frac{x^{2k}(-1)^k}{k!} = 0. \tag{7.9}$$

By the Riemann functional equation (equation (2.3) in [32]) and observation that $\zeta(s)$ is strictly decreasing for real $s > 1$ we have for any natural number k

$$\begin{aligned}|\zeta(1/2-k)| &\leq 2^{1/2-k}\pi^{-1/2-k}\Gamma(1/2+k)\zeta(1/2+k) \\ &\leq 2^{-k}\Gamma(k+1)\zeta(3/2) \\ &= \frac{\zeta(3/2)k!}{2^k}.\end{aligned}$$

Thus, for $|x| < 1$ we have

$$\begin{aligned}\Big|\sum_{k=1}^{\infty}\zeta(1/2-k)\frac{x^{2k}(-1)^k}{k!}\Big| &\leq x^2\sum_{k=1}^{\infty}\frac{|\zeta(1/2-k)|}{k!} \\ &\leq x^2\zeta(3/2)\sum_{k=1}^{\infty}2^{-k} = \zeta(3/2)x^2\end{aligned}$$

and (7.9) follows, this completes the proof. □

Proof of Lemma 7.2.5. We have for any $M > 0$

$$\begin{aligned}
\mathbb{P}\left\{\frac{e^{\sup_{t\in\mathbb{R}} Z(t)}}{\int_{\mathbb{R}} e^{Z(t)}dt} > x\right\} &= \mathbb{P}\left\{\frac{e^{\sup_{t\in[-M,M]} Z(t)}}{\int_{\mathbb{R}} e^{Z(t)}dt} > x \text{ and } Z(t) \text{ achieves its maxima at } t \in [-M,M]\right\} \\
&\quad + \mathbb{P}\left\{\frac{e^{\sup_{t\in\mathbb{R}\setminus[-M,M]} Z(t)}}{\int_{\mathbb{R}} e^{Z(t)}dt} > x \text{ and } Z(t) \text{ achieves its maxima at } t \in \mathbb{R}\setminus[-M,M]\right\} \\
&\le \mathbb{P}\left\{\frac{e^{\sup_{t\in[-M,M]} Z(t)}}{\int_{\mathbb{R}} e^{Z(t)}dt} > x\right\} + \mathbb{P}\left\{\exists t \in \mathbb{R}\setminus[-M,M] : Z(t) > 0\right\} \\
&\le \mathbb{P}\left\{\frac{e^{\sup_{t\in[-M,M]} Z(t)}}{\int_{[-M,M]} e^{Z(t)}dt} > x\right\} + 2\mathbb{P}\left\{\exists t \ge M : Z(t) > 0\right\} \\
&=: p_1(M,x) + p_2(M). \qquad (7.10)
\end{aligned}$$

Estimation of $p_2(M)$. We have for all $M \ge 1$

$$\begin{aligned}
p_2(M) &\le \sum_{k=1}^{\infty} \mathbb{P}\left\{\exists t \in [kM,(k+1)M] : \sqrt{2}B_H(t) - t^{2H} > 0\right\} \\
&= \sum_{k=1}^{\infty} \mathbb{P}\left\{\exists t \in [1, 1+\frac{1}{k}] : \sqrt{2}B_H(t)(kM)^H > (kM)^{2H}t^{2H}\right\} \\
&\le \sum_{k=1}^{\infty} \mathbb{P}\left\{\exists t \in [1,2] : \sqrt{2}B_H(t) > (kM)^H\right\} \\
&\le \sum_{k=1}^{\infty} Ce^{-\frac{(kM)^{2H}}{10}} \\
&\le Ce^{-\frac{M^{2H}}{10}}
\end{aligned}$$

and hence for any $M \ge 1$ we have

$$p_2(M) \le Ce^{-\frac{M^{2H}}{10}}. \qquad (7.11)$$

Estimation of $p_1(M,x)$. Observe that for any sufficiently large M

$$p_1(M,x) \le \mathbb{P}\left\{\frac{\sum_{k=-M}^{M-1} e^{\sup_{t\in[k,k+1]} Z(t)}}{\sum_{k=-M}^{M-1} \int_{[k,k+1]} e^{Z(t)}dt} > x\right\} =: \mathbb{P}\left\{\frac{\sum_{k=-M}^{M-1} a_k(\omega)}{\sum_{k=-M}^{M-1} b_k(\omega)} > x\right\}.$$

Since event

$$\frac{\sum_{k=-M}^{M-1} a_k(\omega)}{\sum_{k=-M}^{M-1} b_k(\omega)} > x$$

implies that for some $k \in [-M, M-1]_1$ event

$$a_k(\omega)/b_k(\omega) > x$$

happens, we have

$$\mathbb{P}\left\{\frac{\sum\limits_{k=-M}^{M-1} a_k(\omega)}{\sum\limits_{k=-M}^{M-1} b_k(\omega)} > x\right\} \le \sum_{k=-M}^{M-1} \mathbb{P}\left\{\frac{a_k(\omega)}{b_k(\omega)} > x\right\} \le 2M \sup_{k\in[-M,M]} \mathbb{P}\left\{\frac{\sup\limits_{t\in[k,k+1]} e^{Z(t)}}{\int\limits_{[k,k+1]} e^{Z(t)}dt} > x\right\}$$

and thus we obtain that

$$p_1(M,x) \le 2M \sup_{k\in[-M,M]} \mathbb{P}\left\{\frac{\sup\limits_{t\in[k,k+1]} e^{Z(t)}}{\int\limits_{[k,k+1]} e^{Z(t)}dt} > x\right\}.$$

Since $e^x, x \in \mathbb{R}$ is a convex function we have by the Jensen's inequality

$$\int\limits_{[k,k+1]} e^{Z(t)}dt \ge \exp\Big(\int\limits_{[k,k+1]} Z(t)dt\Big)$$

implying

$$\begin{aligned}
\mathbb{P}\left\{\frac{\sup\limits_{t\in[k,k+1]} e^{Z(t)}}{\int\limits_{[k,k+1]} e^{Z(t)}dt} > x\right\} &\le \mathbb{P}\left\{\frac{e^{\sup\limits_{t\in[k,k+1]} Z(t)}}{\exp(\int\limits_{[k,k+1]} Z(t)dt)} > x\right\} \\
&= \mathbb{P}\left\{\sup_{t\in[k,k+1]} Z(t) - \int\limits_{[k,k+1]} Z(t)dt > \ln x\right\} \\
&= \mathbb{P}\left\{\int\limits_{[k,k+1]} \big(\sup_{t\in[k,k+1]} Z(t) - Z(s)\big)ds > \ln x\right\} \\
&\le \mathbb{P}\left\{\exists t, s \in [k, k+1] : Z(t) - Z(s) > \ln x\right\} \\
&\le \mathbb{P}\left\{\exists t, s \in [k, k+1] : B_H(t) - B_H(s) > \frac{\ln x - \sup\limits_{t,s\in[k,k+1]} (|t|^{2H} - |s|^{2H})}{\sqrt{2}}\right\} \\
&\le \mathbb{P}\left\{\exists t \in [0,1] : B_H(t) > \frac{\ln x - CM^{2H-1}}{\sqrt{2}}\right\}
\end{aligned}$$

Thus, for all sufficiently large M

$$p_1(M,x) \le 2M\mathbb{P}\left\{\exists t \in [0,1] : B_H(t) > \frac{\ln x - CM^{2H-1}}{\sqrt{2}}\right\}$$

and combining the statement above with (7.10) and (7.11) we have

$$\mathbb{P}\left\{\frac{\sup\limits_{t\in[k,k+1]} e^{Z(t)}}{\int\limits_{\mathbb{R}} e^{Z(t)}dt} > x\right\} \le \widetilde{C}e^{-\frac{M^{2H}}{10}} + 2M\mathbb{P}\left\{\exists t \in [0,1] : B_H(t) > \frac{\ln x - CM^{2H-1}}{\sqrt{2}}\right\}.$$

Assume that $H \le 1/2$. Then choosing $M = x$ in the inequality above we have for all large x

$$\mathbb{P}\left\{\frac{\sup\limits_{t\in[k,k+1]} e^{Z(t)}}{\int\limits_{\mathbb{R}} e^{Z(t)}dt} > x\right\} \le e^{-C\ln^2 x}$$

Assume that $H > 1/2$. Taking $M = C'(\ln x)^{\frac{1}{2H-1}}$ with sufficiently small $C' > 0$ we obtain

$$\mathbb{P}\left\{\frac{\sup\limits_{t\in[k,k+1]} e^{Z(t)}}{\int\limits_{\mathbb{R}} e^{Z(t)}dt} > x\right\} \le \widetilde{C}e^{-C''(\ln x)^{2\frac{H}{2H-1}}} + e^{-C\ln^2 x} \le e^{-C\ln^2 x}$$

and the claim follows. □

7.5 Appendix

Proof of (7.7)**.** By the definition of derivative we have

$$\begin{aligned} L &:= \frac{\partial}{\partial a}\Big(\lim_{n\to\infty} \exp\Big(-\sum_{k=1}^{n}\frac{1}{k}\overline{\Phi}((c+a)\sqrt{\delta k})e^{\frac{\delta k(a^2+2ac)}{2}}\Big)\Big)|_{a=-2c} \\ &= \lim_{\Delta\to 0}\Big(\frac{1}{\Delta}\Big(\lim_{n\to\infty}\exp\Big(-\sum_{k=1}^{n}\frac{1}{k}\overline{\Phi}((\Delta-c)\sqrt{\delta k})e^{\frac{-\delta k(2c-\Delta)\Delta}{2}}\Big) - \lim_{n\to\infty}\exp\Big(-\sum_{k=1}^{n}\frac{1}{k}\overline{\Phi}(-c\sqrt{\delta k})\Big)\Big)\Big). \end{aligned}$$

Since $\sum\limits_{k=1}^{n}\frac{1}{k}\overline{\Phi}(-c\sqrt{\delta k}) \ge \frac{1}{2}\sum\limits_{k=1}^{n}\frac{1}{k} \ge \frac{\ln n}{2}$ we have

$$\lim_{n\to\infty}\exp\Big(-\sum_{k=1}^{n}\frac{1}{k}\overline{\Phi}(-c\sqrt{\delta k})\Big) \le \lim_{n\to\infty} n^{-1/2} = 0.$$

Thus, since $\Phi(x) = 1-\Phi(-x),\ x\in\mathbb{R}$ we have

$$\begin{aligned} L &= \lim_{\Delta\to 0}\Big(\frac{1}{\Delta}\lim_{n\to\infty}\exp\Big(-\sum_{k=1}^{n}\frac{1}{k}\overline{\Phi}((\Delta-c)\sqrt{\delta k})e^{\frac{-\delta k(2c-\Delta)\Delta}{2}}\Big)\Big) \\ &= \lim_{\Delta\to 0}\Big(\frac{1}{\Delta}\lim_{n\to\infty}\Big(\exp\Big(-\sum_{k=1}^{n}\frac{1}{k}e^{\frac{-\delta k(2c-\Delta)\Delta}{2}}\Big)\exp\Big(\sum_{k=1}^{n}\frac{1}{k}\overline{\Phi}((c-\Delta)\sqrt{\delta k})e^{\frac{-\delta k(2c-\Delta)\Delta}{2}}\Big)\Big)\Big) \\ &= \lim_{\Delta\to 0}\Big(\frac{1}{\Delta}\lim_{n\to\infty}\Big(\exp\Big(-\sum_{k=1}^{n}\frac{1}{k}e^{\frac{-\delta k(2c-\Delta)\Delta}{2}}\Big)\Big)\Big) \times \lim_{\Delta\to 0}\Big(\lim_{n\to\infty}\exp\Big(\sum_{k=1}^{n}\frac{1}{k}\overline{\Phi}((c-\Delta)\sqrt{\delta k})e^{\frac{-\delta k(2c-\Delta)\Delta}{2}}\Big)\Big) \\ &=: L_1\times L_2, \end{aligned}$$

if L_1 and L_2 are finite limits, that we shall show later on. Note that for any $A>0$ by Taylor's formula $\sum\limits_{k=1}^{\infty}\frac{e^{-kA}}{k} = -\ln(1-e^{-A})$ and hence

$$L_1 = \lim_{\Delta\to 0}\Big(\frac{1}{\Delta}\exp\Big(\ln(1-e^{\frac{-\delta(2c-\Delta)\Delta}{2}})\Big)\Big) = \lim_{\Delta\to 0}\frac{1-e^{\frac{-\delta(2c-\Delta)\Delta}{2}}}{\Delta} = \delta c \in (0,\infty).$$

Since for any small $\Delta > 0$

$$\sum_{k=1}^{n} \frac{1}{k}\overline{\Phi}((c-\Delta)\sqrt{\delta k})e^{\frac{-\delta k(2c-\Delta)\Delta}{2}} \le \sum_{k=1}^{n} \overline{\Phi}(c\sqrt{\delta k}/2) \le \sum_{k=1}^{n} e^{-c^2\delta k/8} < \frac{1}{1-e^{-c^2\delta/8}}$$

we have that

$$L_2 = \exp\left(\lim_{\Delta\to 0}\sum_{k=1}^{\infty}\frac{1}{k}\overline{\Phi}((c-\Delta)\sqrt{\delta k})e^{\frac{-\delta k(2c-\Delta)\Delta}{2}}\right) \in (0,\infty).$$

Next we focus on the expression in the exponent above. We have

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{k}\overline{\Phi}((c-\Delta)\sqrt{\delta k})e^{\frac{-\delta k(2c-\Delta)\Delta}{2}} &= \sum_{k=1}^{\frac{1}{\sqrt{\Delta}}}\frac{1}{k}\overline{\Phi}((c-\Delta)\sqrt{\delta k})e^{\frac{-\delta k(2c-\Delta)\Delta}{2}} + \sum_{k=\frac{1}{\sqrt{\Delta}}}^{\infty}\frac{1}{k}\overline{\Phi}((c-\Delta)\sqrt{\delta k})e^{\frac{-\delta k(2c-\Delta)\Delta}{2}} \\ &=: S_1 + S_2.\end{aligned}$$

For S_2 we have

$$S_2 \le \sum_{k=\frac{1}{\sqrt{\Delta}}}^{\infty}\overline{\Phi}(c\sqrt{\delta k}/2) \le Ce^{-c^2\delta\sqrt{\Delta}/8}. \tag{7.12}$$

Next we have for $k \le 1/\sqrt{\Delta}$

$$|\overline{\Phi}((c-\Delta)\sqrt{\delta k}) - \overline{\Phi}(c\sqrt{\delta k})| \le C\Delta\sqrt{k} \le C\sqrt{\Delta}, \quad |1-e^{\frac{-\delta k(2c-\Delta)\Delta}{2}}| \le Ck\Delta \le C\sqrt{\Delta}$$

implying

$$\frac{1}{k}|\overline{\Phi}((c-\Delta)\sqrt{\delta k})e^{\frac{-\delta k(2c-\Delta)\Delta}{2}} - \overline{\Phi}(c\sqrt{\delta k})| \le C\sqrt{\Delta}/k.$$

Hence

$$|S_1 - \sum_{k=1}^{\frac{1}{\sqrt{\Delta}}}\frac{1}{k}\overline{\Phi}(c\sqrt{\delta k})| \le C\sqrt{\Delta}\sum_{k=1}^{\frac{1}{\sqrt{\Delta}}}\frac{1}{k} \le C\sqrt{\Delta}\ln\Delta.$$

We have by the line above and (7.12)

$$|\sum_{k=1}^{\infty}\frac{1}{k}\overline{\Phi}((c-\Delta)\sqrt{\delta k})e^{\frac{-\delta k(2c-\Delta)\Delta}{2}} - \sum_{k=1}^{\frac{1}{\sqrt{\Delta}}}\frac{1}{k}\overline{\Phi}(c\sqrt{\delta k})| \le C\sqrt{\Delta}\ln\Delta$$

and hence letting $\Delta \to 0$ we obtain $L_2 = \exp(\sum_{k=1}^{\infty}\frac{1}{k}\overline{\Phi}(c\sqrt{\delta k}))$. Finally we have

$$L = L_1 \times L_2 = c\delta\exp(\sum_{k=1}^{\infty}\frac{1}{k}\overline{\Phi}(c\sqrt{\delta k})).$$

$\square$

Proof of (7.8)**.** To prove the claim we need to show that for any $0 < a < b$ we can differentiate

$$f(z) = \sum_{k=1}^{\infty} \overline{\Phi}(z\sqrt{k}), \;\; z \in [a, b]$$

by terms, i.e., switch order of differentiation and integration. According to paragraph 3.1 p. 385 in [48] it is enough to show that with $f_n(z) = \sum_{k=1}^{n} \overline{\Phi}(z\sqrt{k}), \;\; z \in [a, b]$

1) exists $z_0 \in [a, b]$ such that the sequence $\{f_n(z_0)\}_{n\in\mathbb{N}}$ converges to a finite limit,
2) $f_n'(z), z \in [a, b]$ converge uniformly to some function.

The first condition holds since $\overline{\Phi}(x) < e^{-x^2/2}$ for $x > 0$. For the second condition we need to prove that uniformly for all $z \in [a, b]$ it holds that $\sum_{k=n+1}^{\infty} f_k'(z) \to 0$ as $n \to \infty$. We have

$$\sum_{k=n+1}^{\infty} f_k'(z) = \sum_{k=n+1}^{\infty} \frac{\phi(z\sqrt{k})}{\sqrt{k}} = \sum_{k=n+1}^{\infty} \frac{e^{-z^2k/2}}{\sqrt{2\pi k}} \leq Ce^{-na^2/2} \to 0, \quad n \to \infty$$

and the claim holds.

Chapter 8

Alternative Proofs of Some Results

In this chapter we give alternative proofs of some results of the previous chapters.

8.1 Proof of (2.9).

We give an elementary proof based on some elegant properties of BM. We use the independence of the increments property and the explicit formulas for ruin probability over a finite and the infinite horizons.

Observe that for any $u > 0$

$$\begin{aligned}\mathbb{P}\left\{\sup_{t\notin[T_u^-,T_u^+]} Z(t) > u\right\} &\le \mathbb{P}\left\{\sup_{t\in[0,T_u^-]} Z(t) > u\right\} + \mathbb{P}\left\{\sup_{t\in[T_u^+,\infty)} Z(t) > u\right\}\\ &=: p_1(u) + p_2(u).\end{aligned}$$

Estimation of $p_1(u)$. By the following explicit expression of the ruin probability over a finite horizon (see [26])

$$\mathbb{P}\left\{\sup_{t\in[0,T]} (B(t) - ct) > u\right\} = \overline{\Phi}\left(\frac{u}{\sqrt{T}} + c\sqrt{T}\right) + e^{-2cu}\overline{\Phi}\left(\frac{u}{\sqrt{T}} - c\sqrt{T}\right), \qquad T, c, u > 0 \qquad (8.1)$$

and (2.7) we obtain that

$$p_1(u) \le e^{-2cu - C\ln^2 u}, \qquad u \to \infty.$$

Estimation of $p_2(u)$. Using the independence of the increments of BM and that $Z(T_u^+)$ is a Gaussian rv with mean $-cT_u^+$ and variance T_u^+ we have

$$p_2(u) = \int_{\mathbb{R}} \mathbb{P}\left\{\sup_{t\ge T_u^+} [(Z(t) - Z(T_u^+)) + x] > u \,\Big|\, Z(T_u^+) = x\right\} dF_u(x)$$

$$= \int_{\mathbb{R}} \mathbb{P}\left\{\sup_{t\geq T_u^+} Z(t-T_u^+) > u-x\right\} dF_u(x)$$

$$= \frac{1}{\sqrt{2\pi T_u^+}} \int_{\mathbb{R}} \mathbb{P}\left\{\sup_{t\geq 0} Z(t) > u-x\right\} e^{-\frac{(x+cT_u^+)^2}{2T_u^+}} dx$$

$$= \frac{1}{\sqrt{2\pi T_u^+}} \int_{-\infty}^{u} e^{-2cu} e^{2cx} e^{-\frac{(x+cT_u^+)^2}{2T_u^+}} dx + \frac{1}{\sqrt{2\pi T_u^+}} \int_{u}^{\infty} e^{-\frac{(x+cT_u^+)^2}{2T_u^+}} dx$$

$$= \overline{\Phi}\left(\frac{u+cT_u^+}{\sqrt{T_u^+}}\right) + e^{-2cu}\Phi\left(\frac{u-cT_u^+}{\sqrt{T_u^+}}\right),$$

where F_u is the df of $Z(T_u^+)$. By (2.7) for all large u the expression above does not exceed $e^{-2cu-C\ln^2 u}$, thus we conclude that

$$p_2(u) \leq e^{-2cu-C\ln^2 u}.$$

Combining both estimates above we obtain that as $u \to \infty$

$$\mathbb{P}\left\{\sup_{t\notin[T_u^-,T_u^+]} Z(t) > u\right\} \leq e^{-2cu-C\ln^2 u}$$

and since $e^{-C\ln^2 u}$ decreases faster than any power function as $u \to \infty$ we obtain the claim. □

8.2 Proof of (3.18).

Here we present an elementary proof of (3.18) based on the independence of the increments of BM.

We have by Borell-TIS inequality that

$$\bar{\psi}_\delta(u) \sim \mathbb{P}\left\{\exists t \in [t^* - \frac{\ln u}{\sqrt{u}}, t^* + \frac{\ln u}{\sqrt{u}}]_{\frac{\delta}{u}} : Z(t) > \sqrt{u}\right\}, \quad u \to \infty, \tag{8.2}$$

for details see proof of (3.12) in Appendix in Chapter 3. Next for any fixed $S, u > 0$ we consider the intervals

$$\Delta_{j,S,u} = [t_u + jSu^{-1}, t_u + (j+1)Su^{-1}]_{\frac{\delta}{u}}, \qquad -N_u \leq j \leq N_u,$$

where $N_u = \lfloor S^{-1}\ln(u)\sqrt{u}\rfloor$ and $\lfloor\cdot\rfloor$ is the ceiling function. Let

$$p_{j,S,u} = \mathbb{P}\left\{\sup_{t\in\Delta_{j,S,u}} \frac{B(t)}{c_1 t + q_1} > \sqrt{u}\right\} \text{ for } j \geq 0, \qquad p_{j,S,u} = \mathbb{P}\left\{\sup_{t\in\Delta_{j,S,u}} \frac{B(t)}{c_2 t + q_2} > \sqrt{u}\right\} \text{ for } j < 0.$$

Note that $\Delta = \Delta_{-1} \bigcup \Delta_0$. We have

$$\mathbb{P}\left\{\sup_{t\in\Delta} Z(t) > \sqrt{u}\right\} \leq \mathbb{P}\left\{\exists t \in [t^* - \frac{\ln u}{\sqrt{u}}, t^* + \frac{\ln u}{\sqrt{u}}]_{\frac{\delta}{u}} : Z(t) > \sqrt{u}\right\}$$

$$\leq \sum_{j=-N_u}^{-2} p_{j,S,u} + \sum_{j=1}^{N_u} p_{j,S,u} + \mathbb{P}\left\{\sup_{t\in\Delta} Z(t) > \sqrt{u}\right\} \tag{8.3}$$

and thus to prove the claim we need to derive a sufficiently accurate asymptotic upper bound for the sums in the line above.

Approximation of $\sum_{j=1}^{N_u} p_{j,S,u}$. We have

$$p_{j,S,u} = \mathbb{P}\left\{\sup_{t\in\Delta_{j,S,u}} \frac{B(t)}{c_1 t + q_1} > \sqrt{u}\right\} = \mathbb{P}\left\{\sup_{t\in\Delta_{j,S,u}} (B(t) - \sqrt{\alpha}\mu t) > \sqrt{\alpha}\right\},$$

where

$$\sqrt{\alpha} = q_1\sqrt{u} \ \text{ and } \ \mu = \frac{c_1}{q_1}.$$

By the independence of the increments of BM with $\alpha = v^2$, $c_{j,S,v} = t_u + jSv^{-2}$, $\hat{S} = Sq_1^2$, $\hat{\delta} = \delta q_1^2$ and $\varphi_{v,j}$ the df of $\sqrt{c_{j,S,v}}\mathcal{N}$ we have

$$\begin{aligned}
& p_{j,S,u} \\
= \ & \mathbb{P}\left\{\exists_{t\in\Delta_{j,S,u}} : (B(t) - \sqrt{\alpha}\mu t) > \sqrt{\alpha}\right\} \\
= \ & \mathbb{P}\left\{\exists t \in [t_u + \frac{jS}{u}, t_u + \frac{(j+1)S}{u}]_{\frac{\delta}{u}} : B(t) - B(c_{j,S,v}) - \sqrt{\alpha}\mu(t - c_{j,S,v}) + B(c_{j,S,v}) - \sqrt{\alpha}\mu c_{j,S,v} > \sqrt{\alpha}\right\} \\
= \ & \int_{\mathbb{R}} \mathbb{P}\left\{\exists t \in [0, \frac{S}{u}]_{\frac{\delta}{u}} : B(t) - \sqrt{\alpha}\mu t - \sqrt{\alpha}\mu c_{j,S,v} > \sqrt{\alpha} - x | \sqrt{c_{j,S,v}}\mathcal{N} = x\right\} \varphi_{v,j}(x)dx \\
= \ & \int_{\mathbb{R}} \mathbb{P}\left\{\exists tv^2 \in [0, Sq_1^2]_{q_1^2\delta} : \frac{B(tv^2)}{v} - \frac{\sqrt{\alpha}\mu tv^2}{v^2} - \sqrt{\alpha}\mu c_{j,S,v} > \sqrt{\alpha} - x\right\} \varphi_{v,j}(x)dx \\
= \ & \int_{\mathbb{R}} \mathbb{P}\left\{\exists t \in [0, \hat{S}]_{\hat{\delta}} : (B(t)/v - v\mu(c_{j,S,v} + t/v^2) > v - x\right\} \varphi_{v,j}(x)dx \\
= \ & \frac{1}{v}\int_{\mathbb{R}} \mathbb{P}\left\{\exists t \in [0, \hat{S}]_{\hat{\delta}} : (B(t)/v - v\mu(c_{j,S,v} + t/v^2) > v - (v - x/v)\right\} \varphi_{v,j}(v - x/v)dx \\
= \ & \frac{1}{v}\int_{\mathbb{R}} \mathbb{P}\left\{\exists t \in [0, \hat{S}]_{\hat{\delta}} : (B(t) - \mu t) > x + \mu c_{j,S,v}v^2\right\} \varphi_{v,j}(v - x/v)dx \\
= \ & \frac{1}{v}\int_{\mathbb{R}} \mathbb{P}\left\{\exists t \in [0, \hat{S}]_{\hat{\delta}} : (B(t) - \mu t) > x\right\} \varphi_{v,j}(v(1 + \mu c_{j,S,v}) - x/v)dx \\
= \ & \frac{e^{-v^2(1+\mu c_{j,S,v})^2/(2c_{j,S,v})}}{v\sqrt{2\pi c_{j,S,v}}} \int_{\mathbb{R}} \mathbb{P}\left\{\exists t \in [0, \hat{S}]_{\hat{\delta}} : (B(t) - \mu t) > x\right\} e^{x(1+\mu c_{j,S,v})/c_{j,S,v} - x^2/(2c_{j,S,v}v^2)}dx.
\end{aligned}$$

By the same arguments as in the proof of Theorem 2.1.1 in Chapter 2 we have with $\chi = \frac{1+\mu t^*}{t^*}$ as $u \to \infty$

$$\begin{aligned}
& \int_{\mathbb{R}} \mathbb{P}\left\{\exists_{t\in[0,\hat{S}]_{\hat{\delta}}} : (B(t) - \mu t) > x\right\} e^{x(1+\mu c_{j,S,v})/c_{j,S,v} - x^2/(2c_{j,S,v}v^2)}dx \\
\sim \ & \int_{\mathbb{R}} \mathbb{P}\left\{\exists t \in [0, \hat{S}]_{\hat{\delta}} : (B(t) - \mu t) > x\right\} e^{\chi x}dx
\end{aligned}$$

$$=: \quad J(S).$$

Clearly, $J(S)$ is a non-decreasing function and

$$J(S) \le \int_{-\infty}^{0} e^{\chi x} dx + \int_{0}^{\infty} \mathbb{P}\left\{\exists t \ge 0 : B(t) - \mu t > x\right\} e^{\chi x} dx = \frac{1}{\chi} + \int_{0}^{\infty} e^{(-2\mu+\chi)x} dx < \infty,$$

provided that $\frac{\chi}{\mu} < 2$, which follows from $t_1 < t^*$. Thus, we have that

$$\lim_{S\to\infty} J(S) \in (0, \infty).$$

Hence we have as $u \to \infty$ and then $S \to \infty$

$$\sum_{j=1}^{N_u} p_{j,S,u} \le C\frac{1}{v}\sum_{j=1}^{N_u} e^{\frac{-v^2(1+\mu c_{j,S,v})^2}{2c_{j,S,v}}} = \frac{C}{v} e^{\frac{-v^2(1+\mu t^*)^2}{2t^*}} \sum_{j=1}^{N_u} e^{-v^2\left(\frac{(1+\mu c_{j,S,v})^2}{2c_{j,S,v}} - \frac{(1+\mu t^*)^2}{2t^*}\right)}.$$

Setting

$$a(t) = (1+\mu t)^2/2t = 1/(2t) + \mu + \mu^2 t/2, \quad a'(t) = (-1/t^2 + \mu^2)/2$$

we have $a(t^* + x) \ge a(t^*) + \frac{1}{2} x a'(t^*)$ as $x \to +0$. Since jS/u for $1 \le j \le N_u$ uniformly tends to $+0$ as $u \to \infty$ we have

$$\sum_{j=1}^{N_u} e^{-v^2\left(\frac{(1+\mu c_{j,S,v})^2}{2c_{j,S,v}} - \frac{(1+\mu t^*)^2}{2t^*}\right)} \le \sum_{j=1}^{N_u} e^{-\frac{v^2 a'(t^*)}{2}\left(\frac{\theta_u + jS}{u}\right)}, \quad u \to \infty.$$

We have with $\omega = a'(t^*) q_1^2/2 > 0$

$$\sum_{j=1}^{N_u} e^{-v^2 a'(t^*)\left(\frac{\theta_u + jS}{u}\right)} = e^{-\omega\theta_u} \sum_{j=1}^{N_u} e^{-jS\omega} \le Ce^{-\omega S}, \quad S \to \infty.$$

In the light of the calculations above, we have

$$\sum_{j=1}^{N_u} p_{j,S,u} \le C\frac{1}{v} e^{\frac{-v^2(1+\mu t^*)^2}{2t^*}} e^{-\omega S} \le C\overline{\Phi}(\mathbb{D}_{1/2}\sqrt{u}) e^{-\omega S}.$$

Similarly we obtain that

$$\sum_{j=-N_u}^{-2} p_{j,S,u} \le \overline{\Phi}(\mathbb{D}_{1/2}\sqrt{u}) e^{-CS}.$$

Combining both bounds above we have

$$\sum_{j=1}^{N_u} p_{j,S,u} + \sum_{j=-N_u}^{-2} p_{j,S,u} \le \overline{\Phi}(\mathbb{D}_{1/2}\sqrt{u}) e^{-CS}$$

and letting $S \to \infty$ we obtain in view of bounds for $\mathbb{P}\left\{\sup_{t\in\Delta} Z(t) > \sqrt{u}\right\}$ given in (3.22) that as $u \to \infty$ and then $S \to \infty$

$$\sum_{j=1}^{N_u} p_{j,S,u} + \sum_{j=-N_u}^{-2} p_{j,S,u} = o\left(\mathbb{P}\left\{\sup_{t\in\Delta} Z(t) > \sqrt{u}\right\}\right).$$

Thus, the claim follows by the line above, (8.2) and (8.3).

8.3 Proof of Theorem 4.2.1, Case (2), $H = 1/2$.

Here we give a proof of Theorem 4.2.1, Case (2), $H = 1/2$ using the independence of the increments of BM. The proof is based on the same ideas as the proof of Theorem 5.2.1.

By the same arguments as in the proof of Theorem 5.2.1 we have as $u \to \infty$ and then $S \to \infty$

$$p(u) \sim \mathbb{P}\left\{\exists t \in \Delta : \inf_{s\in[t,t+\frac{T}{u}]} Z(s) > \sqrt{u}\right\} =: p_S(u), \tag{8.4}$$

where $\Delta = [-S/u + t_*, t_* + S/u]$. Denote

$$\Delta_1 = [-S + ut_*, -T + ut_*), \quad \Delta_2 = [-T + ut_*, ut_*), \quad \Delta_3 = [ut_*, ut_* + S].$$

Let $\phi_u(x)$ be the density of $B(ut_*)$ and B_* be an independent copy of the same BM. For $S > T$ with

$$\eta = c_1t_* + q_1 = c_2t_* + q_2 = \frac{c_1q_2 - c_2q_1}{c_1 - c_2}, \ \eta_* = \frac{\eta}{t_*} - c_2 = \frac{q_2(c_1 - c_2)}{q_2 - q_1}, \ B_2(s) = B(s) - c_2s$$

we have

$$\begin{aligned}
& p_S(u) \\
= \ & \mathbb{P}\left\{\exists t \in [-S + ut_*, ut_* + S] : \inf_{s\in[t,t+T]} (B(s) - \max(c_1s + q_1u, c_2s + q_2u)) > 0\right\} \\
= \ & \mathbb{P}\Big\{\exists t_1 \in \Delta_1 : \inf_{s\in[t_1,t_1+T]} B_2(s) > q_2u \\
& \text{or } \exists t_2 \in \Delta_2 : \inf_{s\in[t_2,ut_*]} B_2(s) > q_2u, \inf_{s\in[ut_*,t_2+T]} (B(s) - B(ut_*) - c_1s) > q_1u - B(ut_*) \\
& \text{or } \exists t_3 \in \Delta_3 : \inf_{s\in[t_3,t_3+T]} (B(s) - B(ut_*) - c_1s) > q_1u - B(ut_*)\Big\} \\
= \ & \int_{\mathbb{R}} \phi_u(\eta u - x) \times \mathbb{P}\Big\{\exists t_1 \in \Delta_1 : \inf_{s\in[t_1,t_1+T]} B_2(s) > q_2u \\
& \text{or } \exists t_2 \in \Delta_2 : \inf_{s\in[t_2,ut_*]} B_2(s) > q_2u, \inf_{s\in[ut_*,t_2+T]} (B_*(s - ut_*) - c_1(s - ut_*) - c_1t_*u) > q_1u - (\eta u - x) \\
& \text{or } \exists t_3 \in \Delta_3 : \inf_{s\in[t_3,t_3+T]} (B_*(s - ut_*) - c_1(s - ut_*) - c_1t_*u) > q_1u - (\eta u - x) | B(ut_*) = \eta u - x\Big\} dx \\
= \ & \int_{\mathbb{R}} \phi_u(\eta u - x) \times \mathbb{P}\Big\{\exists t_1 \in \Delta_1 : \inf_{s\in[t_1,t_1+T]} B_2(s) > q_2u \\
& \text{or } \exists t_2 \in \Delta_2 : \inf_{s\in[t_2,ut_*]} B_2(s) > q_2u, \inf_{s\in[0,t_2+T-ut_*]} (B_*(s) - c_1s) > x \\
& \text{or } \exists t_3 \in \Delta_3 : \inf_{s\in[t_3-ut_*,t_3+T-ut_*]} (B_*(s) - c_1s) > x | B(ut_*) = \eta u - x\Big\} dx \\
= \ & \frac{e^{-\frac{\eta^2 u}{2t_*}}}{\sqrt{2\pi ut_*}} \int_{\mathbb{R}} \mathbb{P}\Big\{\exists t_1 \in [-S, -T) : \inf_{s\in[t_1,t_1+T]} (Z_u(s) + \eta_*s) > x \\
& \text{or } \exists t_2 \in [-T, 0) : \inf_{s\in[t_2,0]} (Z_u(s) + \eta_*s) > x, \inf_{s\in[0,t_2+T]} (B_*(s) - c_1s) > x
\end{aligned}$$

$$\text{or } \exists t_3 \in [0,S] : \inf_{s\in[t_3,t_3+T]} (B_*(s) - c_1 s) > x \Big\} e^{\frac{\eta x}{t_*} - \frac{x^2}{2ut_*}} dx,$$

$$=: \frac{e^{-\frac{\eta^2 u}{2t_*}}}{\sqrt{2\pi u t_*}} \int_{\mathbb{R}} G(u,x) e^{\frac{\eta x}{t_*} - \frac{x^2}{2ut_*}} dx,$$

where $Z_u(t)$ is a Gaussian process with expectation and covariance defined by

$$\mathbb{E}\{Z_u(t)\} = \frac{-x}{ut_*} t, \qquad \operatorname{cov}(Z_u(s), Z_u(t)) = \frac{-st}{ut_*} - t, \qquad s \le t \le 0.$$

Next, since Z_u weakly converges to BM (this is shown in the proof of Theorem 5.2.1) and by the independence of $B(x)$ and $B(y)$ for $x > 0 > y$ we write as $u \to \infty$

$$\begin{aligned}
\int_{\mathbb{R}} G(u,x) e^{\frac{\eta x}{t_*} - \frac{x^2}{2ut_*}} dx \quad \sim \quad & \int_{\mathbb{R}} \mathbb{P}\Big\{ \exists t_1 \in [-S,-T) : \inf_{s\in[t_1,t_1+T]} (B(s) + \eta_* s) > x \\
& \text{or } \exists t_2 \in [-T,0) : \inf_{s\in[t_2,0]} (B(s)+\eta_* s) > x, \inf_{s\in[0,t_2+T]} (B(s) - c_1 s) > x \\
& \text{or } \exists t_3 \in [0,S] : \inf_{s\in[t_3,t_3+T]} (B(s) - c_1 s) > x \Big\} e^{\frac{\eta x}{t_*}} dx \\
=: \quad & I(S).
\end{aligned} \tag{8.5}$$

From the proof of Theorem 2.1, Case (2), $H = 1/2$ in [42] we have that $\lim_{S\to\infty} I(S) \in (0,\infty)$. Denote

$$\theta = \frac{\eta^2}{t_*^2}, \quad k(s) = \eta_* s \mathbb{I}(s<0) - c_1 s \mathbb{I}(s \ge 0), \quad \hat{k}(s) = \frac{t_*}{\eta} k(s).$$

We have

$$\begin{aligned}
I(S) &= \int_{\mathbb{R}} \mathbb{P}\left\{ \exists t \in [-S,S] : \inf_{s\in[t,t+T]} (B(s) + k(s)) > x \right\} e^{\frac{\eta x}{t_*}} dx \\
&= \frac{t_*}{\eta} \int_{\mathbb{R}} \mathbb{P}\left\{ \exists t \in [-S,S] : \inf_{s\in[t,t+T]} \left(B(\frac{s\eta^2}{t_*^2}) + \frac{t_*}{\eta} k(\frac{s\eta^2}{t_*^2}) \right) > x \right\} e^x dx \\
&= \frac{t_*}{\eta} \int_{\mathbb{R}} \mathbb{P}\left\{ \sup_{t\in[-\theta S,\theta S]} \inf_{s\in[t,t+\theta T]} (B(s) + \hat{k}(s)) > x \right\} e^x dx \\
&= \frac{t_*}{\eta} \int_{\mathbb{R}} \mathbb{P}\left\{ \sup_{t\in[-\theta S/2,\theta S/2]} \inf_{s\in[t,t+\theta T/2]} (\sqrt{2}B(s) - |s| + \hat{k}(2s) + |s|) > x \right\} e^x dx \\
&= \frac{t_*}{\eta} \mathbb{E}\left\{ \sup_{t\in[-\theta S/2,\theta S/2]} \inf_{s\in[t,t+\theta T/2]} e^{\sqrt{2}B(s) - |s| + d(s)} \right\},
\end{aligned}$$

recall that $d(s)$ is defined in (4.6). Thus, we have that

$$I(S) \to \frac{t_*}{\eta} \mathbb{E}\left\{ \sup_{t\in\mathbb{R}} \inf_{s\in[t,t+\theta T/2]} e^{\sqrt{2}B(s)-|s|+d(s)} \right\} = \frac{t_*}{\eta} \mathcal{H}^d_{\theta T/2} \in (0,\infty), \quad S \to \infty.$$

Finally combining (8.5) with the line above we have that as $u \to \infty$ and then $S \to \infty$

$$p_S(u) \sim \frac{e^{-\frac{\eta^2 u}{2t_*}}}{\sqrt{2\pi u t_*}} \frac{t_*}{\eta} \mathcal{H}^d_{\theta T/2} \sim \overline{\Phi}(\mathbb{D}_{1/2}\sqrt{u})\mathcal{H}^d_{T'},$$

hence the claim follows by (8.4).

8.4 Proof of Lemma 5.4.1.

Here we present an elementary proof of Lemma 5.4.1 that does not require specific knowledge about the distribution of $\int\limits_0^\infty \mathbb{I}(B(t) - ct > x)dt$. The key-properties for our proof is the independence and stationarity of the increments of BM.

Denote for $0 \le a \le b$

$$f([a,b]) = \mathbb{P}\left\{\exists x \in \mathbb{R} : \mu_\Lambda\{t \in [a,b] : B(t) - ct = x\} > 0\right\}, \qquad f(S) := f([0,S]), \quad S \ge 0.$$

We shall prove that

$$f(S) = 1 - e^{-kS}, \quad k \in [0, \infty]. \tag{8.6}$$

We have for any $0 \le a < b$ with $\mathcal{N}$ being a Gaussian rv independent of B with variance a and zero expectation

$$\begin{aligned}
f([a,b]) &= \mathbb{P}\left\{\exists x \in \mathbb{R} : \mu_\Lambda\{t \in [a,b] : B(t) - ct = x\} > 0\right\} \\
&= \mathbb{P}\left\{\exists x \in \mathbb{R} : \mu_\Lambda\{t \in [a,b] : B(t) - B(a) - c(t-a) + B(a) = x + ac\} > 0\right\} \\
&= \mathbb{P}\left\{\exists x \in \mathbb{R} : \mu_\Lambda\{t \in [0,b-a] : B(t) - ct + \mathcal{N} = x + ac\} > 0\right\} \\
&= \mathbb{P}\left\{\exists y \in \mathbb{R} : \mu_\Lambda\{t \in [0,b-a] : B(t) - ct = y\} > 0\right\} \\
&= f(b-a).
\end{aligned}$$

By the inclusion-exclusion principle for any $0 \le a < c < b$ it holds, that

$$f([a,b]) = 1 - (1 - f([a,c]))(1 - f([c,b])).$$

By the last two equations above it follows, that for any $S_1, S_2 > 0$

$$f(S_1 + S_2) = 1 - (1 - f(S_1))(1 - f(S_2)). \tag{8.7}$$

Since $f(S)$ is non-decreasing and non-negative, then exists $\lim\limits_{S \to 0} f(S) \ge 0$.

i) Assume that $\lim_{S\to 0} f(S) = \varepsilon > 0$. Then by (8.7) we have that for any $n \in \mathbb{N}$

$$f(S) = 1 - (1 - f(\frac{S}{n}))^n \geq 1 - (1-\varepsilon)^n \to 1, \quad n \to \infty,$$

hence $f(S) = 1$ for all $S > 0$ and (8.6) holds for $k = \infty$.

ii) Assume that $\lim_{S\to 0} f(S) = 0$. We have that $f(S)$ is continuous: by (8.7) for any $\varepsilon > 0$

$$f(S+\varepsilon) - f(S) = 1 - (1 - f(S))(1 - f(\varepsilon)) - f(S) = f(\varepsilon) - f(S)f(\varepsilon) \leq f(\varepsilon) \to 0, \quad \varepsilon \to 0.$$

Denote $g(S) = 1 - f(S)$. Then we have that $0 \leq g(S) \leq 1$ is a continuous function and by (8.7)

$$g(S_1 + S_2) = g(S_1)g(S_2), \qquad S_1, S_2 > 0.$$

By Chapter 8, exercise 6 in [61] this yields that $g(S) = e^{-kS}$ for $k \in [0, \infty]$ and (8.6) holds.

Assume, that the assertion of the lemma does not hold. Then by (8.6) we have

$$\lim_{S\to\infty} f(S) = 1.$$

We have for $S > 1$ and some family of positive numbers $A_{S,x}$ such that $A_{S_1,x} \leq A_{S_2,x}$ for $S_1 < S_2$ with $A_0 = A_{1,x} > 0$

$$\begin{aligned} f(S) &= \mathbb{P}\left\{\mu_\Lambda\{t :\in [0,S] : B(t) - ct = x\} = A_{S,x} > 0\right\} \\ &\leq \mathbb{P}\left\{\int_0^S \mathbb{I}(B(t) - ct \geq x)dt \geq A_{S,x}\right\} \\ &\leq \mathbb{P}\left\{\int_0^\infty \mathbb{I}(B(t) - ct \geq x)dt \geq A_0\right\}. \end{aligned}$$

Thus, for $S \geq 1$

$$f(S) \leq \mathbb{P}\left\{\int_0^\infty \mathbb{I}(B(t) - ct \geq x)dt \geq A_0\right\}, \qquad c, A_0 > 0, x \in \mathbb{R}.$$

The probability above is strictly less than one, that contradicts $\lim_{S\to\infty} f(S) = 1$. $\square$

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