A queueing model with randomized depletion of inventory

Hansjörg Albrecher[*]   Onno Boxma[†]   Rim Essifi [†]   Richard Kuijstermans [†]

Abstract. In this paper we study an M/M/1 queue, where the server continues to work during idle periods and builds up inventory. This inventory is used for new arriving service requirements, but it is completely emptied at random epochs of a non-homogeneous Poisson process, whose rate depends on the current level of the acquired inventory. For several shapes of depletion rates, we derive differential equations for the stationary density of the workload and the inventory level and solve them explicitly. Finally numerical illustrations are given for some particular examples, and the effects of this depletion mechanism are discussed.

Keywords: single server queue, inventory, workload

1 Introduction

In this paper we combine queueing and inventory theory. We consider a classical single server queue, but with the special feature that the server keeps working even when there are no customers. It then builds up inventory. This inventory is used for new arriving customers (who are now possibly served instantaneously), but the built up inventory is removed at random epochs of a non-homogeneous Poisson process, whose rate depends on the current level of the acquired inventory. Our goal is to study the resulting two-sided stochastic model, obtaining both the steady-state workload level (when positive) and the steady-state inventory level (when positive) and their means. That will also yield other interesting performance measures like the probability that a service request is instantaneously satisfied (sojourn time zero). Our model is in particular of interest in situations where the service is homogeneous across customers (e.g., serving means producing some good).

This model obviously is closely related to classical queueing ($M/M/1$) and inventory models. A rich literature on queueing systems with inventory was developed in recent years, and with various approaches; but when it comes to computing the stationary distribution of inventory and/or

---

[*]Department of Actuarial Science, Faculty of Business and Economics, University of Lausanne, UNIL-Dorigny, CH-1015 Lausanne, Switzerland and Swiss Finance Institute.

[†]Department of Mathematics and Computer Science, Eindhoven University of Technology and EURANDOM, P.O. Box 513, 5600 MB Eindhoven, The Netherlands

workload states (which will interest us in this paper), the topic seems to be less investigated. We would like to mention the book of Sahin [17], and papers like [7, 12, 16, 18, 19], which are more oriented towards lost sale problems.

An important inventory model is the one with a basestock policy, in which a server produces products until the inventory has reached a certain basestock level, and in which requests for products arrive according to a Poisson process. If a request cannot be immediately satisfied, it joins a back-order queue. However, that model has a finite basestock level, and hence essentially differs from our model.

A production/inventory oriented paper that seems related to our study is [14] on sporadic clearing policies. That paper considers a production/inventory system that is continuously filled at fixed rate and satisfies demands at Poisson epochs. Under the sporadic clearing policy, clearing of all inventory takes place after a random time (which in [14] is independent of the content process). Explicit results are obtained for an expected discounted cost functional. See also [8], which considers the case of exponentially distributed demands, and derives the steady-state inventory level. Our model is motivated by some recent work in insurance risk theory, where the event of the declaration of bankruptcy was also modeled by an epoch of such a non-homogeneous Poisson process with rate $\omega(x)$ whenever the surplus process of the insurance portfolio is negative (see e.g. [2] and [4]). There are many interesting methodological links between the field of insurance risk theory and queueing, cf. Asmussen & Albrecher [6] for an overview. In the present context it also turns out that, although the model setups in [2] and the present paper are not identical, the formulas for workload densities and inventory densities of the resulting queueing/inventory model are of a similar complexity as the respective formulas in the risk setting, and we will obtain explicit expressions for those shapes of $\omega(x)$ for which this was also possible in the insurance model in [2].

The rest of the paper is organized as follows. In Section 2 we will introduce the model in detail. Section 3 then works out the respective quantities related to workload and inventory for the case of exponential service requirements. In particular, we consider a constant, a linear and an exponential depletion rate. Finally, Section 4 gives concrete numerical examples and studies the behavior of the considered quantities as a function of the model parameters. Section 5 concludes.
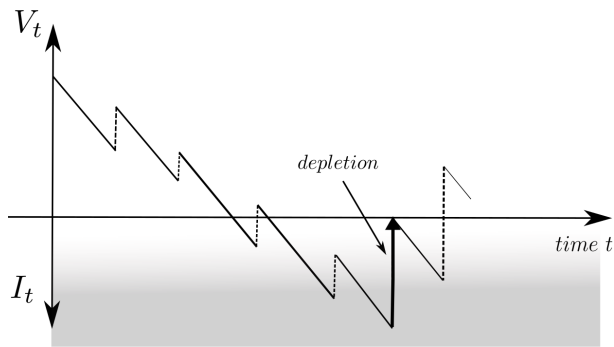
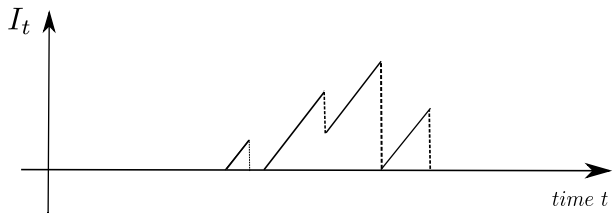Figure 1: Sample path of the workload V_t together with the inventory level I_t

Figure 2: Inventory level I_t of Figure 1 displayed separately

2 Model description

Consider the following queueing model. Customers arrive according to a Poisson process with rate λ. Their service requirements are independent, identically distributed random variables $B_1, B_2, \ldots$ with distribution $B(\cdot)$ and LST (Laplace-Stieltjes Transform) $\beta(\cdot)$. B will denote a generic service requirement. The server works continuously, at a fixed speed which is normalized to 1 – even if there are no service requirements. In the latter case, the server is building up inventory, which can be interpreted as negative workload. At random times, with an intensity $\omega(x) > 0$ when the inventory is at level $x > 0$, the present inventory is removed, instantaneously reducing the inventory to zero. Put differently, inventory is removed according to a Poisson process with a rate that depends on the amount of inventory present.

Denote the required work per time unit by $\rho := \lambda \mathbb{E} B$. We assume that $\rho < 1$. This ensures that the steady-state workload distribution exists. Let $V_+(x)$, $x > 0$, denote this steady-state workload distribution, and $v_+(x)$ its density. During the times in which the inventory level is positive, there is an upward drift $1 - \rho$ of that inventory level; but when $\omega(x) > 0$ for x sufficiently large, the inventory level will always eventually return to zero, and the steady-state inventory distribution will exist. Let $V_-(x)$, $x > 0$, denote this steady-state inventory distribution, and $v_-(x)$ its density. Figure 1 displays the workload V_t as a function of time t (above the t-axis) and the corresponding

3

inventory level I_t (below the t-axis) in one figure. Figure 2 separately displays the inventory level. Notice that this is also the inventory level corresponding to the production/inventory system with sporadic clearing studied in [14]; but in that paper, clearing occurs according to a fixed rate, and the inventory level distribution is not derived. That distribution *is* derived in [8], where the case of exponentially distributed demands is considered. It should be noticed that their model is related to our model, in the sense that it is one-sided; in [8] only inventory is considered, there is no queue. The classical level crossing technique (see, e.g., [11]) yields the following integral equations for the workload and inventory densities:

$$v_+(x) = \lambda \int_0^x \mathbb{P}(B > x - y)v_+(y)\mathrm{d}y + \lambda \int_0^\infty \mathbb{P}(B > x + y)v_-(y)\mathrm{d}y, \quad x > 0, \tag{1}$$

$$v_-(x) = \lambda \int_x^\infty \mathbb{P}(B > y - x)v_-(y)\mathrm{d}y + \int_x^\infty \omega(y)v_-(y)\mathrm{d}y, \quad x > 0. \tag{2}$$

3 Exponential service requirements

In this paper we restrict ourselves to the case of exponential service requirements $\mathbb{P}(B > x) = \mathrm{e}^{-\mu x}$. Then (1) and (2) reduce to:

$$v_+(x) = \lambda \mathrm{e}^{-\mu x} \int_0^x \mathrm{e}^{\mu y}v_+(y)\mathrm{d}y + \lambda \mathrm{e}^{-\mu x} \int_0^\infty \mathrm{e}^{-\mu y}v_-(y)\mathrm{d}y, \quad x > 0, \tag{3}$$

$$v_-(x) = \lambda \mathrm{e}^{\mu x} \int_x^\infty \mathrm{e}^{-\mu y}v_-(y)\mathrm{d}y + \int_x^\infty \omega(y)v_-(y)\mathrm{d}y, \quad x > 0. \tag{4}$$

Let $z_+(x) := \mathrm{e}^{\mu x}v_+(x)$, $x > 0$, and $z_-(x) := \mathrm{e}^{-\mu x}v_-(x)$, $x > 0$, so that (3) and (4) become

$$z_+(x) = \lambda \int_0^x z_+(y)\mathrm{d}y + \lambda \int_0^\infty z_-(y)\mathrm{d}y, \quad x > 0, \tag{5}$$

$$z_-(x) = \lambda \int_x^\infty z_-(y)\mathrm{d}y + \mathrm{e}^{-\mu x} \int_x^\infty \omega(y)\mathrm{e}^{\mu y}z_-(y)\mathrm{d}y, \quad x > 0. \tag{6}$$

Differentiation of (5) readily leads to

$$z_+(x) = C\lambda \mathrm{e}^{\lambda x}, \quad x > 0, \tag{7}$$

4

with $C := \int_0^\infty z_-(y)\mathrm{d}y$, so

$$v_+(x) = C\lambda e^{-(\mu-\lambda)x}, \quad x > 0. \tag{8}$$

In particular, one has

$$\int_0^{+\infty} v_+(x)dx = C\frac{\rho}{1-\rho}. \tag{9}$$

It is not surprising that $v_+(x)$ is, up to a constant, equal to the density of the $M/M/1$ queue. When the workload becomes positive, it does so via a jump from below the t-axis; and (the excess of) that jump is $\mathrm{Exp}(\mu)$ distributed. And as long as the workload remains positive, it behaves exactly like the workload in an $M/M/1$ queue. We later turn to the determination of the missing constant $C = \int_0^\infty z_-(y)\mathrm{d}y$. Differentiation of (6) yields:

$$z'_-(x) = -\lambda z_-(x) - \omega(x)z_-(x) - \mu\left[z_-(x) - \lambda\int_x^\infty z_-(y)\mathrm{d}y\right], \tag{10}$$

and hence

$$z''_-(x) + (\lambda + \mu + \omega(x))z'_-(x) + (\lambda\mu + \omega'(x))z_-(x) = 0, \quad x > 0. \tag{11}$$

Below we discuss several choices of the rate function $\omega(\cdot)$ for which (11) can be solved explicitly.

3.1 Case 1: $\omega(x) \equiv \omega$, $x > 0$

When $\omega(x) \equiv \omega$, (11) reduces to:

$$z''_-(x) + (\lambda + \mu + \omega)z'_-(x) + \lambda\mu z_-(x) = 0, \quad x > 0. \tag{12}$$

Hence

$$z_-(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}, \quad x > 0,$$

with

$$r_{1,2} = \frac{1}{2}\left(-(\lambda+\mu+\omega) \mp \sqrt{(\lambda+\mu+\omega)^2 - 4\lambda\mu}\right),$$

and finally

$$v_-(x) = C_1 e^{s_1 x} + C_2 e^{s_2 x}, \quad x > 0, \tag{13}$$

with

$$s_{1,2} = r_{1,2} + \mu = \frac{1}{2}\left(-(\lambda-\mu+\omega) \mp \sqrt{(\lambda+\mu+\omega)^2 - 4\lambda\mu}\right). \tag{14}$$

5

The constants C (cf. (8)), C_1 and C_2 can be determined as follows. First observe that $s_2 > 0$, so that C_2 must be zero. Next observe that $C = \int_0^\infty z_-(y)\mathrm{d}y = \frac{C_1}{-r_1} = \frac{C_1}{\mu - s_1}$. Finally, use that

$$\int_0^\infty v_+(x)\mathrm{d}x + \int_0^\infty v_-(x)\mathrm{d}x = 1. \tag{15}$$

We thus find that

$$C_1 = \frac{(\mu - \lambda)(\mu - s_1)s_1}{\mu(s_1 - \mu + \lambda)}, \tag{16}$$

and

$$C = \frac{(\mu - \lambda)s_1}{\mu(s_1 - \mu + \lambda)}. \tag{17}$$

Finally, the solution reads

$$
\begin{aligned}
v_-(x) &= \frac{(\mu - \lambda)(\lambda - \mu + \omega + \sqrt{(\lambda + \mu + \omega)^2 - 4\lambda\mu})(\lambda + \mu + \omega + \sqrt{(\lambda + \mu + \omega)^2 - 4\lambda\mu})}{2\mu(-\lambda + \mu + \omega + \sqrt{(\lambda + \mu + \omega)^2 - 4\lambda\mu})} \\
&\quad e^{\frac{1}{2}(-\lambda + \mu - \omega - \sqrt{(\lambda + \mu + \omega)^2 - 4\lambda\mu})x}, \quad x > 0;
\end{aligned} \tag{18}
$$

$$v_+(x) = \frac{(\mu - \lambda)(\lambda - \mu + \omega + \sqrt{(\lambda + \mu + \omega)^2 - 4\lambda\mu})}{\mu(-\lambda + \mu + \omega + \sqrt{(\lambda + \mu + \omega)^2 - 4\lambda\mu})}\lambda e^{(\lambda - \mu)x}, \quad x > 0. \tag{19}$$

The mean inventory level is in this case

$$
\begin{aligned}
\mathbb{E}[I] &= \int_0^\infty x v_-(x)\mathrm{d}x \\
&= \frac{2(\mu - \lambda)\left(\sqrt{(\lambda + \mu + \omega)^2 - 4\lambda\mu} + \lambda + \mu + \omega\right)}{\mu\left(\sqrt{(\lambda + \mu + \omega)^2 - 4\lambda\mu} + \lambda - \mu + \omega\right)\left(\sqrt{(\lambda + \mu + \omega)^2 - 4\lambda\mu} - \lambda + \mu + \omega\right)}.
\end{aligned} \tag{20}
$$

The mean workload level is calculated as

$$\mathbb{E}[X] = \int_0^\infty x v_+(x)\mathrm{d}x = \frac{\lambda\left(\sqrt{(\lambda + \mu + \omega)^2 - 4\lambda\mu} + \lambda - \mu + \omega\right)}{\mu(\mu - \lambda)\left(\sqrt{(\lambda + \mu + \omega)^2 - 4\lambda\mu} - \lambda + \mu + \omega\right)}, \tag{21}$$

and the probability of finding no inventory in stationarity is

$$\int_0^\infty v_+(x)dx = \frac{\lambda\left(\sqrt{(\lambda + \mu + \omega)^2 - 4\lambda\mu} + \lambda - \mu + \omega\right)}{\mu\left(\sqrt{(\lambda + \mu + \omega)^2 - 4\lambda\mu} - \lambda + \mu + \omega\right)}. \tag{22}$$

Conversely, the probability for an arriving customer to be served immediately is given by

$$\int_0^\infty v_-(x)dx = 1 - \frac{\lambda\left(\sqrt{(\lambda+\mu+\omega)^2 - 4\lambda\mu} + \lambda - \mu + \omega\right)}{\mu\left(\sqrt{(\lambda+\mu+\omega)^2 - 4\lambda\mu} - \lambda + \mu + \omega\right)}.$$

Note that for $\omega \to \infty$ (immediate depletion of the inventory, i.e. no inventory) we retrieve the classical result that the probability to arrive in an idle period is $1 - \lambda/\mu$. On the other hand, for $\omega = 0$ we obtain $\int_0^\infty v_-(x)dx = 1$, as one should expect because the drift in the workload process is downward. The probability that due to the inventory an arriving customer can leave the system immediately again (i.e. his service requirement was available in the inventory, sojourn time 0) is

$$\int_0^\infty v_-(x)\,\mathbb{P}(B < x)dx = \int_0^\infty v_-(x)\left(1 - e^{-\mu x}\right)dx = \frac{2(\mu - \lambda)}{\sqrt{(\lambda+\mu+\omega)^2 - 4\lambda\mu} - \lambda + \mu + \omega}. \tag{23}$$

3.2   Case 2: $\omega(x) = ax$, $x > 0$, $a > 0$

Differentiating Equation (4), when $\omega(x) = ax$, one gets the following equation for v_-:

$$v_-'(x) + (\lambda + ax)v_-(x) - \lambda\mu e^{\mu x}\int_x^{+\infty} e^{-\mu y}v_-(y)dy = 0, \tag{24}$$

in particular, plugging $x = 0$ in Relation (24), we obtain

$$C = \frac{\lambda v_-(0) + v_-'(0)}{\lambda\mu}. \tag{25}$$

Now, differentiating the expression in Equation (24), the function v_- satisfies the following second order differential equation

$$v_-''(x) + (\lambda - \mu + ax)v_-'(x) + a(1 - \mu x)v_-(x) = 0. \tag{26}$$

Introduce the function $\theta(x) = v_-(x)e^{\frac{ax^2}{2} + \lambda x}$. The function v_- is a solution of Equation (26) if and only if the function θ is a solution of the following second order differential equation:

$$\theta''(x) - (\lambda + \mu + ax)\theta'(x) + \lambda\mu\theta(x) = 0. \tag{27}$$

One can check that θ is a solution of Equation (27) if and only if $\theta(x) = \mathcal{J}\left(\tilde{a}, \tilde{b}, \frac{a}{2}(x + \frac{\lambda+\mu}{a})^2\right)$, where $\tilde{a} = -\frac{\lambda\mu}{2a}$ and $\tilde{b} = \frac{1}{2}$ and $\mathcal{J}(\tilde{a}, \tilde{b}, \cdot)$ is a solution of the degenerate hypergeometric equation:

$$\xi y''(\xi) + (\tilde{b} - \xi)y'(\xi) - \tilde{a}y(\xi) = 0. \tag{28}$$

According to [13] (page 322) and [15] (page 143), Equation (28) has two standard solutions denoted by $\xi \mapsto M(\tilde{a}, \tilde{b}, \xi)$ and $\xi \mapsto U(\tilde{a}, \tilde{b}, \xi)$, the so called Kummer functions. Provided that $\tilde{b} \notin \{-1, -2, \cdots\}$, the function $\xi \mapsto M(\tilde{a}, \tilde{b}, \xi)$ is given by

$$M(\tilde{a}, \tilde{b}, \xi) = \sum_{0}^{+\infty} \frac{(\tilde{a})_s}{(\tilde{b})_s s!} \xi^s \quad \text{for all } \xi \in \mathbb{C};$$

where $(c)_s = c(c+1)\cdots(c+s-1)$. The function $\xi \mapsto U(\tilde{a}, \tilde{b}, \xi)$ is uniquely determined by the property $U(\tilde{a}, \tilde{b}, \xi) \sim \xi^{-a}$ when ξ goes to $+\infty$. In our case $(\tilde{b} = \frac{1}{2})$, one has

$$U(\tilde{a}, \tilde{b}, \xi) = \frac{\Gamma(\frac{1}{2})}{\gamma(\frac{1}{2} - \frac{\lambda\mu}{2a})} M(-\frac{\lambda\mu}{2a}, \frac{1}{2}, \xi) + \Gamma(-\frac{1}{2})\xi^{\frac{1}{2}} M(-\frac{\lambda\mu}{2a} + \frac{1}{2}, \frac{3}{2}, \xi). \tag{29}$$

The general solution of Equation (28) is given by:

$$y(\xi) = C_3 M(\tilde{a}, \tilde{b}, \xi) + C_4 \xi^{1-\tilde{b}} M(\tilde{a} - \tilde{b} + 1, 2 - \tilde{b}, \xi), \quad \text{where} \quad C_3, C_4 \in \mathbb{R}. \tag{30}$$

Thanks to Relation (29), the general solution of Equation (28) can also be expressed in the following way:

$$y(\xi) = K_1 M(\tilde{a}, \tilde{b}, \xi) + K_2 U(\tilde{a}, \tilde{b}, \xi), \quad \text{where} \quad K_1, K_2 \in \mathbb{R}. \tag{31}$$

Then, in our case, one gets:

$$\mathcal{J}(-\frac{\lambda\mu}{2a}, \frac{1}{2}, \xi) = K_1 M(-\frac{\lambda\mu}{2a}, \frac{1}{2}, \xi) + K_2 U(-\frac{\lambda\mu}{2a}, \frac{1}{2}, \xi), \quad \text{where} \quad K_1, K_2 \in \mathbb{R}. \tag{32}$$

Remembering that for all $x \geq 0$, one has $v_-(x) = e^{-\frac{a}{2}x^2 - \lambda x} \mathcal{J}\left(-\frac{\lambda\mu}{2a}, \frac{1}{2}, \frac{a}{2}(x + \frac{\lambda+\mu}{a})^2\right)$, one obtains for all $x \geq 0$,

$$v_-(x) = e^{-\frac{a}{2}x^2 - \lambda x}\left[K_1 M\left(-\frac{\lambda\mu}{2a}, \frac{1}{2}, \frac{a}{2}(x + \frac{\lambda+\mu}{a})^2\right) + K_2 U\left(-\frac{\lambda\mu}{2a}, \frac{1}{2}, \frac{a}{2}(x + \frac{\lambda+\mu}{a})^2\right)\right], \tag{33}$$

where $K_1, K_2 \in \mathbb{R}$. Now the plan is to determine the constants K_1 and K_2. To do so, we will use the well known asymptotic behavior of the functions $\xi \mapsto M(\tilde{a}, \tilde{b}, \xi)$ and $\xi \mapsto U(\tilde{a}, \tilde{b}, \xi)$. We will distinguish two cases.

Case 1: $\tilde{a} = -\frac{\lambda\mu}{2a} \notin \mathbb{Z}$, i.e $\frac{\lambda\mu}{2a} \notin \mathbb{N}$. Denote by $\nu := \frac{\lambda\mu}{a}$. In this case, according to [13] (page 323), one knows that $M(\tilde{a}, \tilde{b}, \xi) \sim \frac{e^\xi \xi^{\tilde{a}-\tilde{b}}}{\Gamma(\tilde{a})}$, when ξ goes to $+\infty$, and so,

$$M(-\frac{\lambda\mu}{2a}, 1/2, \frac{a}{2}(x + \frac{\lambda+\mu}{a})^2) \sim \frac{\frac{a}{2}^{\frac{a-1}{2}}}{\Gamma(-\frac{\lambda\mu}{2a})} e^{\frac{a}{2}(x+\frac{\lambda+\mu}{a})^2}(x + \frac{\lambda+\mu}{a})^{-\frac{\lambda\mu}{a}-1}, \quad \text{when } x \to +\infty. \quad (34)$$

Now, according to [13] (page 322), one knows that

$$U(\tilde{a}, \tilde{b}, \xi) \sim \xi^{-\tilde{a}} \quad \text{when } \xi \to +\infty, \quad (35)$$

and then

$$U(-\frac{\lambda\mu}{2a}, \frac{1}{2}, \frac{a}{2}(x + \frac{\lambda+\mu}{a})^2) \sim a^{\frac{\lambda\mu}{2a}}(x + \frac{\lambda+\mu}{a})^{\frac{\lambda\mu}{a}}, \quad \text{when } x \to +\infty. \quad (36)$$

Since the function v_- is integrable, Equation (33) and Relations (34), (36) imply that the constant K_1 should be necessarily equal to zero and then, one has

$$v_-(x) = K_2 e^{-\frac{a}{2}x^2 - \lambda x} U\left(-\frac{\nu}{2}, \frac{1}{2}, \frac{a}{2}(x + \frac{\lambda+\mu}{a})^2\right), \quad \forall\, x \geq 0. \quad (37)$$

Now coming back to Equation (25), one obtains

$$C = \frac{K_2(\lambda + \mu)U'(-\frac{\nu}{2}, 1/2, \frac{(\lambda+\mu)^2}{2a})}{\lambda\mu}. \quad (38)$$

On the other hand, using the relation $\int_0^{+\infty} v_+(x)dx + \int_0^{+\infty} v_-(x)dx = 1$, and Equations (9), (33), one gets:

$$C\frac{\rho}{1-\rho} + K_2 \int_0^{+\infty} e^{-\frac{a}{2}x^2 - \lambda x} U(-\frac{\nu}{2}, 1/2, a/2(x + \frac{\lambda+\mu}{a})^2)dx = 1. \quad (39)$$

Finally, Relations (38) and (39) allow us to determine the constants K_2 and C.

In particular, if ν is odd, i.e. $\nu = 2n + 1$, where $n \in \mathbb{N}$, then from [13] (Page 325), one knows that

$$U(\tilde{a}, \tilde{b}, z) = z^{1-\tilde{b}}U(\tilde{a} - \tilde{b} + 1, 2 - \tilde{b}, z), \quad z \in \mathbb{C}. \quad (40)$$

For $x \geq 0$, set $z^2 = \frac{a}{2}(x + \frac{\lambda+\mu}{a})^2$, relation (40) implies

$$U\left(-\frac{\nu}{2}, \frac{1}{2}, \frac{a}{2}(x + \frac{\lambda+\mu}{a})^2\right) = U(-\frac{\nu}{2}, \frac{1}{2}, z^2) = zU(-\frac{\nu}{2} + \frac{1}{2}, \frac{3}{2}, z^2). \tag{41}$$

For $m \in \mathbb{Z}$, let $H_m(\cdot)$ be the Hermite polynomial of order m (see [1], Page 775) given by the formula $H_m(x) = (-1)^m e^{\frac{1}{2}x^2}(\frac{d}{dx})^m[e^{\frac{-1}{2}x^2}]$. According to [13] (Page 328) (notice that we are using in this paper the notation H_m, which in [13] is denoted by He_m), for $z \in \mathbb{C}$, one has

$$U(-\frac{\nu}{2} + \frac{1}{2}, \frac{3}{2}, z^2) = 2^{-\frac{\nu}{2}} z^{-1} H_\nu(\sqrt{2}z). \tag{42}$$

Combining relations (37), (41), and (42), one gets

$$v_-(x) = 2^{-\frac{\nu}{2}} K_2 e^{-\frac{a}{2}x^2 - \lambda x} H_\nu\left(\sqrt{a}x + \frac{\lambda+\mu}{\sqrt{a}}\right), \quad \forall\, x \geq 0. \tag{43}$$

Case 2: $\frac{\lambda\mu}{2a} = n \in \mathbb{N}$. In this case, thanks to Equation (30), one has, for all $x \geq 0$,

$$v_-(x) = e^{-\frac{a}{2}x^2 - \lambda x}\left[K_1 M\left(-n, \frac{1}{2}, \frac{a}{2}(x + \frac{\lambda+\mu}{a})^2\right) + K_2\sqrt{\frac{a}{2}}(x + \frac{\lambda+\mu}{a})M\left(-n + \frac{1}{2}, \frac{3}{2}, \frac{a}{2}(x + \frac{\lambda+\mu}{a})^2\right)\right], \tag{44}$$

where $K_1, K_2 \in \mathbb{R}$. According to [13] (Page 328), for $z \in \mathbb{C}$, one has

$$M(-n, \frac{1}{2}, \frac{z^2}{2}) = (-1)^n \frac{n!}{(2n)!} 2^n H_{2n}(\sqrt{2}z) = (-1)^{\frac{\nu}{2}} \frac{(\frac{\nu}{2})!}{\nu!} 2^{\frac{\nu}{2}} H_\nu(\sqrt{2}z), \tag{45}$$

and,

$$z^{\frac{1}{2}} M(-n + \frac{1}{2}, \frac{3}{2}, z) \sim \frac{z^{-n-1}e^z}{\Gamma(-n + \frac{1}{2})} \quad \text{when } z \to +\infty. \tag{46}$$

Therefore, the integrability condition satisfied by the function v_- implies that the constant K_2 must be zero in this case, and one obtains

$$v_-(x) = (-1)^{\frac{\nu}{2}} \frac{(\frac{\nu}{2})!}{\nu!} 2^{\frac{\nu}{2}} K_1 e^{-\frac{a}{2}x^2 - \lambda x} H_\nu\left(\sqrt{a}x + \frac{\lambda+\mu}{\sqrt{a}}\right), \quad x \geq 0. \tag{47}$$

Following the same line of reasoning as previously, one can find two linear equations involving the unknowns C and K_1 and then determine them.

3.3 Case 3: $\omega(x) = e^{bx}$, $x > 0$, $b > 0$

If $\omega(x) = e^{bx}$, (11) reduces to

$$z_-''(x) + (\lambda + \mu + e^{bx})z_-'(x) + (\lambda\mu + be^{bx})z_-(x) = 0, \quad x > 0. \tag{48}$$

Below we show how, via a substitution, one can arrive at exactly the same Kummer differential equation (28) as we found for the case $\omega(x) = ax$ (there we had to use other substitutions). Substituting $z_-(x) = e^{-\frac{e^{bx}}{b}}g(x)$, we get

$$g''(x) + (\lambda + \mu - e^{bx})g'(x) + (\lambda\mu - (\lambda + \mu)e^{bx})g(x) = 0, \quad x > 0. \tag{49}$$

Now introduce $u(x) := e^{bx}/b$ and substitute $g(x) = e^{-\mu x}y(u(x)) = (b\,u(x))^{-\frac{\mu}{b}}y(u(x))$ in (49). This leads to the Kummer differential equation

$$uy''(u) + \left(\frac{\lambda - \mu}{b} + 1 - u\right)y'(u) - \frac{\lambda}{b}y(u) = 0, \quad u > 0, \tag{50}$$

with solution (cf. [1, Ch.13])

$$y(u) = C_5\,U\left(\frac{\lambda}{b}, 1 + \frac{\lambda - \mu}{b}, u\right) + C_6\,F_{1,1}\left(\frac{\lambda}{b}, 1 + \frac{\lambda - \mu}{b}, u\right), \tag{51}$$

where this time we use the notation $F_{1,1}(.,.,.)$ for the hypergeometric function, instead of $M(.,.,.)$, to be consistent with [2]. Hence

$$g(x) = e^{-\mu x}\left(C_5\,U\left(\frac{\lambda}{b}, 1 + \frac{\lambda - \mu}{b}, \frac{e^{bx}}{b}\right) + C_6\,F_{1,1}\left(\frac{\lambda}{b}, 1 + \frac{\lambda - \mu}{b}, \frac{e^{bx}}{b}\right)\right), \tag{52}$$

and subsequently

$$z_-(x) = e^{-\frac{e^{bx}}{b}}e^{-\mu x}\left(C_5\,U\left(\frac{\lambda}{b}, 1 + \frac{\lambda - \mu}{b}, \frac{e^{bx}}{b}\right) + C_6\,F_{1,1}\left(\frac{\lambda}{b}, 1 + \frac{\lambda - \mu}{b}, \frac{e^{bx}}{b}\right)\right), \tag{53}$$

$$v_-(x) = e^{-\frac{e^{bx}}{b}}\left(C_5\,U\left(\frac{\lambda}{b}, 1 + \frac{\lambda - \mu}{b}, \frac{e^{bx}}{b}\right) + C_6\,F_{1,1}\left(\frac{\lambda}{b}, 1 + \frac{\lambda - \mu}{b}, \frac{e^{bx}}{b}\right)\right), \tag{54}$$

with $C_5, C_6 \in \mathbb{R}$. In addition, (8) still holds. The constants C, C_5 and C_6 follow from the conditions (15),

$$C = \frac{z'_-(0) + (\lambda + \mu + 1)z_-(0)}{\lambda\mu} \tag{55}$$

and $\lim_{x\to\infty} v_-(x) = 0$. We have, from (34),

$$\lim_{x\to\infty} e^{-\frac{e^{bx}}{b}} U\left(\frac{\lambda}{b}, 1 + \frac{\lambda-\mu}{b}, \frac{e^{bx}}{b}\right) = \lim_{x\to\infty} e^{-\frac{e^{bx}}{b}} \left(\frac{e^{bx}}{b}\right)^{-\frac{\lambda}{b}} = \lim_{x\to\infty} e^{-\frac{e^{bx}}{b}} \left(\frac{1}{b}\right)^{-\frac{\lambda}{b}} e^{-\lambda x} = 0.$$

Furthermore, from (35),

$$\lim_{x\to\infty} e^{-\frac{e^{bx}}{b}} F_{1,1}\left(\frac{\lambda}{b}, 1 + \frac{\lambda-\mu}{b}, \frac{e^{bx}}{b}\right) = \lim_{x\to\infty} \frac{\Gamma(1 + \frac{\lambda-\mu}{b})}{\Gamma(\frac{\lambda}{b})} \left(\frac{1}{b}\right)^{\frac{\mu}{b}-1} e^{(\mu-b)x} = \begin{cases} +\infty &, \text{ if } \mu > b, \\ 1 &, \text{ if } \mu = b, \\ 0 &, \text{ if } \mu < b. \end{cases}$$

Consequently, if $\mu \geq b$, then $C_6 = 0$ (in contrast to the risk model in [2], we can not a priori exclude the case $\mu < b$ here, but we nevertheless assume $\mu \geq b$ for convenience of the present context). Then the over-all solution reads:

$$\begin{cases} v_-(x) = e^{-\frac{e^{bx}}{b}} C_5 \, U\left(\frac{\lambda}{b}, 1 + \frac{\lambda-\mu}{b}, \frac{e^{bx}}{b}\right); \\ v_+(x) = C\lambda e^{(\lambda-\mu)x}, \end{cases}$$

with the constants C and C_5 still to be determined. From (55)

$$C = C_5 \frac{e^{-\frac{1}{b}}\left(b\, U\left(\frac{\lambda}{b}, 1 + \frac{\lambda-\mu}{b}, \frac{1}{b}\right) - U\left(1 + \frac{\lambda}{b}, 2 + \frac{\lambda-\mu}{b}, \frac{1}{b}\right)\right)}{\mu b}. \tag{56}$$

From (15) we then obtain

$$C_5 = \left(\int_0^\infty e^{-\frac{e^{by}}{b}} U\left(\frac{\lambda}{b}, 1 + \frac{\lambda-\mu}{b}, \frac{e^{by}}{b}\right) \mathrm{d}y \right.$$
$$\left. + \frac{e^{-\frac{1}{b}}\left(b\, U\left(\frac{\lambda}{b}, 1 + \frac{\lambda-\mu}{b}, \frac{1}{b}\right) - U\left(1 + \frac{\lambda}{b}, 2 + \frac{\lambda-\mu}{b}, \frac{1}{b}\right)\right)}{\mu b} \frac{\lambda}{\mu - \lambda}\right)^{-1},$$

and finally, for $x > 0$,

$$v_+(x) = \left(\frac{\frac{\mu b}{\lambda} e^{\frac{1}{b}} \int_0^\infty e^{-\frac{e^{by}}{b}} U\left(\frac{\lambda}{b}, 1 + \frac{\lambda-\mu}{b}, \frac{e^{by}}{b}\right) \mathrm{d}y}{b\, U\left(\frac{\lambda}{b}, 1 + \frac{\lambda-\mu}{b}, \frac{1}{b}\right) - U\left(1 + \frac{\lambda}{b}, 2 + \frac{\lambda-\mu}{b}, \frac{1}{b}\right)} + \frac{1}{\mu - \lambda} \right)^{-1} e^{(\lambda-\mu)x}.$$

4 Numerical results

In this subsection we give some numerical illustrations for the case $\omega(x) \equiv \omega$.

4.1 Mean inventory level

The mean inventory level $\mathbb{E}[I] = \int_0^\infty x v_-(x)\mathrm{d}x$ is given in Figure 3 as a function of the Poisson rate λ (keeping the other parameters fixed) when the function ω is constant ($\omega(x) = \omega > 0$), linear ($\omega(x) = ax$) and exponential ($\omega(x) = e^{bx}$). Figure 4 depicts $\mathbb{E}[I]$ as a function of the exponential rate μ of the service requirements.
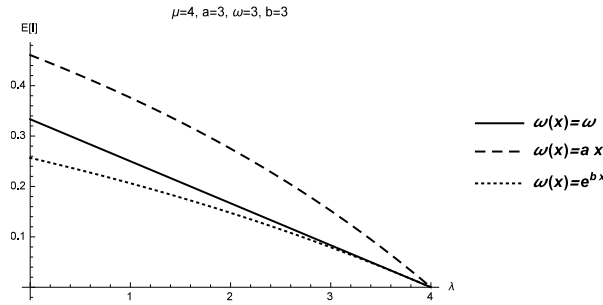
Figure 3: Mean inventory level, with $\mu = 4$, $a = 3$, $\omega = 3$ and $b = 3$
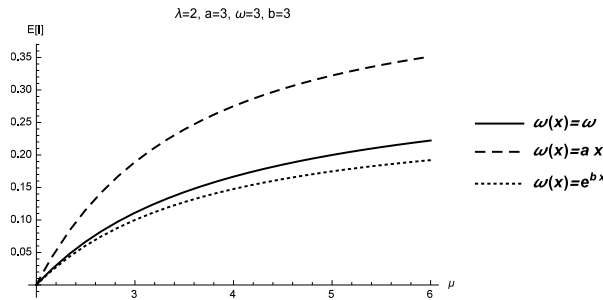
Figure 4: Mean inventory level, with $\lambda = 2$, $a = 3$, $\omega = 3$ and $b = 3$

In Figure 3, we see that if λ increases and approaches μ, the mean inventory level decreases (in fact, almost linearly). If $\rho = \frac{\lambda}{\mu}$ approaches 1, it is hardly possible to build up an inventory. In Figure 4, we see that if μ increases, the mean inventory level also increases. This is obvious, as the service requirements become smaller. For the displayed cases, $\omega(x) = e^{bx} \geq 1 + bx > ax$, for all

$x \geq 0$ and accordingly $\mathbb{E}I$ for $\omega(x) = e^{bx}$ is smaller than $\mathbb{E}I$ for $\omega(x) = ax$.

4.2 Mean workload

Next, consider the mean workload $\int_0^\infty x v_+(x)\mathrm{d}x$. Figures 5, 6 and 7 depict the mean workload as a function of λ for different values of the parameters μ, a, ω and b when the function ω is constant $(\omega(x) = \omega > 0)$, linear $(\omega(x) = ax)$ and exponential $(\omega(x) = e^{bx})$. Similarly, Figures 8, 9 and 10 depict the mean workload as a function of μ.

From the figures below, we see that the various ω-choices lead to similar results in heavy traffic. This makes sense because in this case it is hardly relevant what happens during periods of a positive inventory; those periods are too rare. In the case $a = b = \omega = 3$, one has $\omega(x) = e^{3x} \geq 1 + 3x > 3x$, and it makes sense that the mean workload for $\omega(x) = e^{3x}$ is larger than the mean workload for $\omega(x) = 3x$; of course $\omega(x) = +\infty$ gives the largest mean workload.
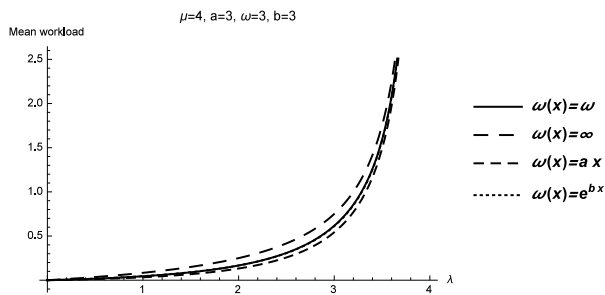
Figure 5: Mean workload, with $\mu = 4$, $a = 3$, $\omega = 3$ and $b = 3$
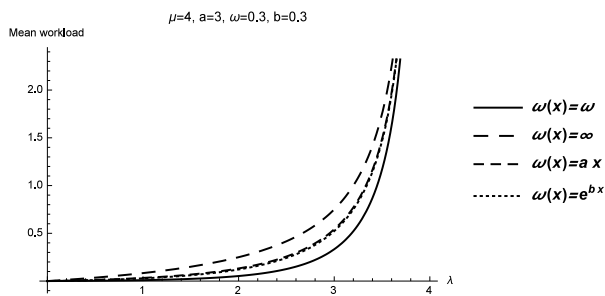
Figure 6: Mean workload, with $\mu = 4$, $a = 3$, $\omega = 0.3$ and $b = 0.3$

14
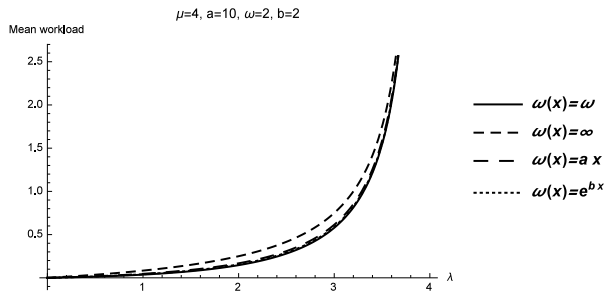
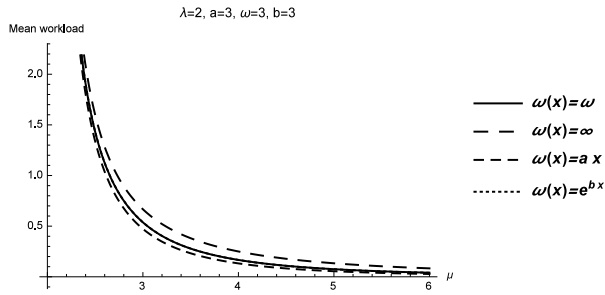Figure 7: Mean workload, with $\mu = 4$, $a = 10$, $\omega = 2$ and $b = 2$

Figure 8: Mean workload, with $\lambda = 2$, $a = 3$, $\omega = 3$ and $b = 3$
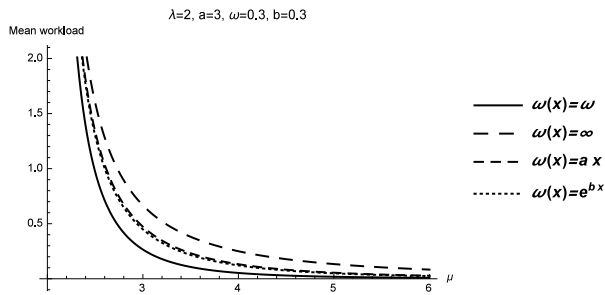

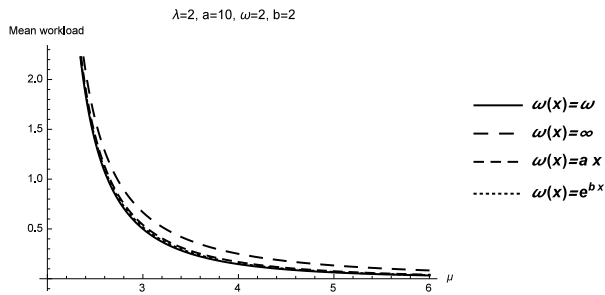Figure 9: Mean workload, with $\lambda = 2$, $a = 3$, $\omega = 0.3$ and $b = 0.3$

Figure 10: Mean workload, with $\lambda = 2$, $a = 10$, $\omega = 2$ and $b = 2$

4.3 Probability of having no inventory

Figures 11, 12 and 13 depict the probability of finding no inventory in the stationary state $\int_{0}^{+\infty} v_+(x)dx$ as a function of λ for different values of the parameters μ, a, ω and b when the function ω is constant ($\omega(x) = \omega > 0$), linear ($\omega(x) = ax$) and exponential ($\omega(x) = e^{bx}$).
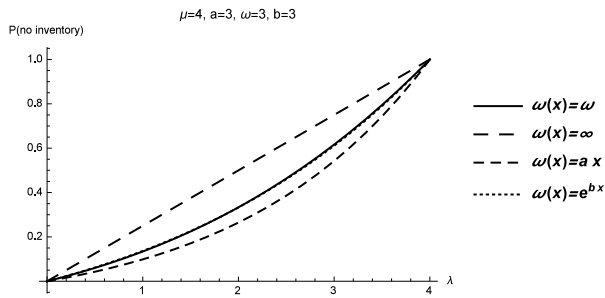
15

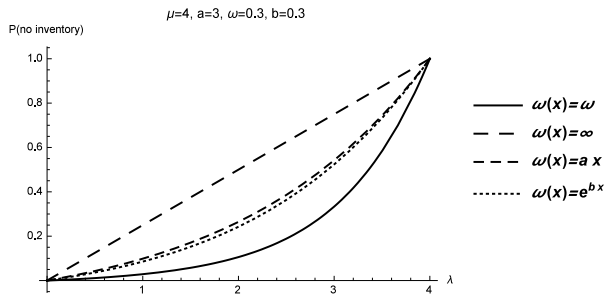Figure 11: Probability of no inventory, with $\mu = 4$, $a = 3$, $\omega = 3$ and $b = 3$

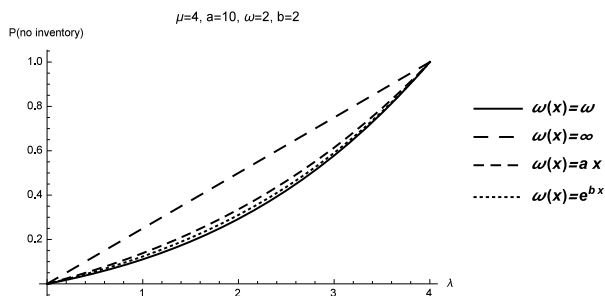Figure 12: Probability of no inventory, with $\mu = 4$, $a = 3$, $\omega = 0.3$ and $b = 0.3$

Figure 13: Probability of no inventory, with $\mu = 4$, $a = 10$, $\omega = 2$ and $b = 2$

Similarly, Figures 14, 15 and 16 depict the probability of finding no inventory in the stationary state as a function of μ instead of λ. The resulting shapes are quite intuitive.

5 Conclusion

In this paper we considered a queueing model with inventory, where the inventory is depleted at random times according to a Poisson process with surplus-dependent intensity. For a number of
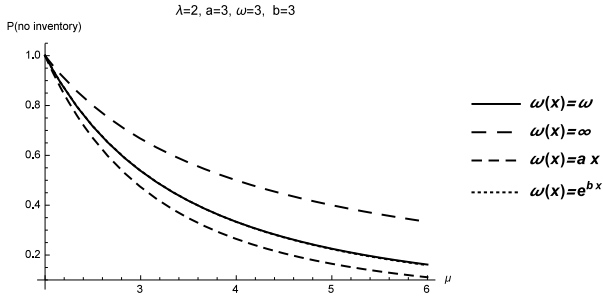
Figure 14: Probability of no inventory, with $\lambda = 2$, $a = 3$, $\omega = 3$ and $b = 3$
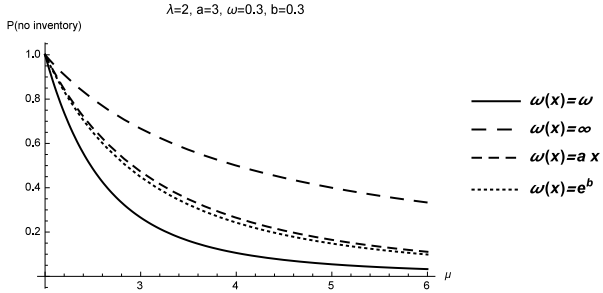

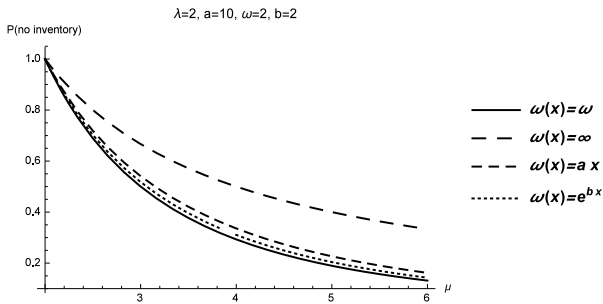Figure 15: Probability of no inventory, with $\lambda = 2$, $a = 3$, $\omega = 0.3$ and $b = 0.3$

Figure 16: Probability of no inventory, with $\lambda = 2$, $a = 10$, $\omega = 2$ and $b = 2$

intensity shapes this leads to explicit formulas, and for constant depletion rate the expressions become particularly simple.

There are a number of directions for possible future research. Such a random depletion mechanism with its simple formulas may serve as an approximation of depletion mechanisms at deterministic time points. In fact, if the depletion mechanism according to a homogeneous Poisson process is extended to a renewal process with Erlang-distributed inter-occurrence times, an analysis in the spirit of [3] may still be possible in the present setup. We leave this extension for future research. For surplus-dependent Poisson rates, we have seen that the expressions, although explicit, quickly become quite involved. One alternative may be to consider piecewise constant upper and lower bounds for $\omega(x)$ and determine upper and lower bounds for the quantities under consideration

17

through smooth-pasting of tractable expressions at the respective layer boundaries (see [2] for such a mechanism in the risk theory context).

It may be also worthwhile to explore the following idea. Application of the transformation

$$\theta(x) = e^{\lambda x + \int_0^x \omega(y)\mathrm{d}y} \; v_-(x) \tag{57}$$

to (4) gives

$$\theta''(x) - (\lambda + \mu + \omega(x))\theta'(x) + \lambda\mu\theta(x) = 0. \tag{58}$$

Notice that this transformation was applied below Equation(26) for the case $\omega(x) = ax$, and is closely related to the transformation applied below Equation (48) for the case $\omega(x) = e^{bx}$. It could be very interesting to solve Equation (58) for other $\omega(x)$, or to find a function transformation that even provides a solution to (58) for general $\omega(\cdot)$.

Another direction for future research is optimization. One might like to choose $\omega(x)$ in such a way that the profit (for example, income from serving customers and from selling inventory at depletion epochs, minus waiting costs and holding costs) is maximized. One might also try to optimally choose the server speed during the periods in which there is no customer; an analysis of the model with different server speeds during periods with and without customers does not pose any problem.

An important extension is to allow generally distributed service times. The analysis then becomes considerably more involved, but for $\omega(x) \equiv \omega$ an approach via Wiener-Hopf factorization appears possible, see e.g. [9].

References

[1] M. Abramowitz and I.A. Stegun (1964). *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*. Dover Publications, New York.

[2] H. Albrecher and V. Lautscham (2013). From ruin to bankruptcy for compound Poisson surplus processes. ASTIN Bulletin **43**, 213-243.

[3] H. Albrecher, E.C.K. Cheung and S. Thonhauser (2013). Randomized observation periods for the compound Poisson risk model: the discounted penalty function. *Scandinavian Actuarial Journal* **6**, 424-452.

[4] H. Albrecher, H.U. Gerber and E.S.W. Shiu (2011). The optimal dividend barrier in the Gamma-Omega model. *European Actuarial Journal* **1**, 43-55.

[5] H. Albrecher, H.U. Gerber and H. Yang (2011). A direct approach to the discounted penalty function. *North American Actuarial Journal* **14**, 420-447.

[6] S. Asmussen and H. Albrecher (2010). *Ruin Probabilities*. World Scientific, Hackensack, New Jersey.

[7] O. Berman and E. Kim (2001). Dynamic order replenishment policy in internet-based supply chains. *Math. Methods Oper. Res.* **53**, 371-390.

[8] O. Berman, M. Parlar, D. Perry and M.J.M. Posner (2005). Production/clearing models under continuous and sporadic review. *Methodology and Computing in Applied Probability* **7**, 203-224.

[9] O.J. Boxma, R. Essifi and A.J.E.M. Janssen (2015). A queueing/inventory and an insurance risk model. *To appear in J. Applied Probability*.

[10] J.W. Cohen (1982). *The Single Server Queue*. North-Holland Publ. Cy., Amsterdam, 2nd ed.

[11] B.T. Doshi (1992). *Level crossing analysis of queues*. In: U.N. Bhat and I.V. Basawa (eds.), *Queueing and Related Models*, pp. 3-33 (Clarendon Press, Oxford).

[12] E. Mohebbi and M.J.M. Posner (1998). A continuous-review inventory system with lost sales and variable lead time. *Naval Res. Logist* **45**, 259-278.

[13] F.W.J. Olver, D.W. Lozier, R.F. Boisvert and C.W. Clark (2010). *NIST Handbook of Mathematical Functions.* NIST, Cambridge University Press.

[14] D. Perry, W. Stadje and S. Zacks (2005). Sporadic and continuous clearing policies for a production/inventory system under an M/G demand process. *Mathematics of Operations Research* **30**, 354-368.

[15] A.D. Polyanin and V.F. Zaitsev (2003). *Handbook of Exact Solutions for Ordinary Differential Equations*, 2nd ed. CRC Press, Boca Raton.

[16] M. Saffari, S. Asmussen and R. Haji (2013). The M/M/1 queue with inventory, lost sale, and general lead times. *Queueing Systems* **75**, 65-77.

[17] I. Sahin (1990). *Regenerative Inventory Systems, Operating characteristics and optimization*, Springer, New York.

[18] M. Schwarz and H. Daduna (2006). Queueing systems with inventory management with random lead times and with backordering. *Math. Methods Oper. Res.* **64**, 383-414.

[19] M. Schwarz, C. Sauer, H. Daduna, R. Kulik and R. Szekli (2006). M/M/1 queueing systems with inventory. *Queueing Systems* **54**, 55-78.

[20] K. Sigman and D. Simchi-Levi (1992). Light traffic heuristic for an M/G/1 queue with limited inventory. *Ann. Oper. Res.* **40**, 371-380.

[21] E.C. Titchmarsh (1968). *The Theory of Functions, 2nd. ed.* Oxford University Press, London.

[22] R.W. Wolff (1982). Poisson arrivals see time averages. *Oper. Res.* **30**, 223-231.