COMPARISON INEQUALITIES FOR ORDER STATISTICS OF GAUSSIAN ARRAYS

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Abstract: Normal comparison lemma and Slepian's inequality are essential tools for the analysis of extremes of Gaussian processes. In this paper we show that the Normal comparison lemma for Gaussian vectors can be extended to order statistics of Gaussian arrays. Our applications include the derivation of mixed Gumbel limit laws for the order statistics of stationary Gaussian processes and the investigation of lower tail behaviour of order statistics of self-similar Gaussian processes.

Key Words: Slepian's inequality; conjunction probability; Normal comparison inequality; order statistics process; mixed Gumbel limit theorem; lower tail probability; self-similar Gaussian process.

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1. Introduction

In the recent contributions [11, 12, 13] order statistics of Gaussian and stationary processes are studied. Given a random process $\{X(t), t \geq 0\}$ with almost surely (a.s.) continuous trajectories, and $X_1, \ldots, X_n, n \in \mathbb{N}$ independent copies of X we define $X_{r:n}(t)$ generated by X as the rth lower order statistics of $X_1(t), \ldots, X_n(t)$ for any fixed $t \geq 0$, and thus $X_{1:n}(t) \leq \cdots \leq X_{n:n}(t), t \geq 0$. The calculation of the so-called r-th conjunction probability

$$p_{r:n}(u) = \mathbb{P}\left\{ \sup_{t \in [0,T]} X_{r:n}(t) > u \right\} \tag{1}$$

for fixed r, T and large u is of both theoretical and applied interest; see e.g., [3, 4, 25, 35].

Order statistics processes play a crucial role in various statistical applications, for instance in models concerned with the analysis of the surface roughness during all machinery processes and functional magnetic resonance imaging (FMRI) data. Given the fact that $p_{r:n}(u)$ cannot be in general calculated explicitly, asymptotic expansions as $u \to \infty$ and the so-called Gumbel limit results (with $u = u_T \to \infty$ as $T \to \infty$) are derived in [11, 13]. Indeed, such limit theorems have been in the focus of many theoretical and applied contributions, see e.g., [5, 7, 8, 29, 30] and the recent contributions [14, 20]. The crucial tool for establishing Gumbel limit theorems is the so-called Normal comparison lemma, which has been shown to be one of the most important tools in the study of Gaussian processes and random fields, see e.g., [7, 8, 21, 22, 23, 29]. The lack of a comparison lemma for order statistics processes has already been noted in [11]; therein some results are derived only for the minimum process.

In the simpler framework of two d-dimensional Gaussian distributions $\Phi_{\Sigma^{(1)}}$ and $\Phi_{\Sigma^{(0)}}$ with $N(0,1)$ marginal distributions, the normal comparison inequality gives explicit bounds for the difference

$$\Delta(\boldsymbol{u}) := \Phi_{\Sigma^{(1)}}(\boldsymbol{u}) - \Phi_{\Sigma^{(0)}}(\boldsymbol{u}), \quad \forall \boldsymbol{u} = (u_1, \ldots, u_d) \in \mathbb{R}^d$$

in terms of the covariance matrices $\Sigma^{(k)} = (\sigma_{ij}^{(k)})_{d \times d}, k = 0, 1$. The derivation of the bounds for $\Delta(\boldsymbol{u})$, by Slepian [33], Berman [6, 8] and Piterbarg [29, 31] relies strongly on Plackett's partial differential equation; see [32]. The most elaborate version of the normal comparison inequality is due to Li and Shao [22]. Specifically,

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Theorem 2.1 therein shows that

$$\Delta(\boldsymbol{u}) \le \frac{1}{2\pi} \sum_{1\le i<j\le d} \left(\arcsin(\sigma_{ij}^{(1)}) - \arcsin(\sigma_{ij}^{(0)})\right)_+ \exp\left(-\frac{u_i^2+u_j^2}{2(1+\rho_{ij})}\right), \quad \forall \boldsymbol{u}\in\mathbb{R}^d,$$

where $\rho_{ij} := \max(|\sigma_{ij}^{(0)}|, |\sigma_{ij}^{(1)}|)$ and $x_+ = \max(x,0)$. Clearly, if $\sigma_{ij}^{(0)} \ge \sigma_{ij}^{(1)}, 1\le i,j\le d$, then

$$\Phi_{\Sigma^{(1)}}(\boldsymbol{u}) \le \Phi_{\Sigma^{(0)}}(\boldsymbol{u}),$$

which is the well-known Slepian's inequality derived in [33]. Based on the results of Li and Shao [22], Yan [36] showed that for W an $N(0,1)$ random variable

$$(2)\; 1 \le \frac{\Phi_{\Sigma^{(1)}}(\boldsymbol{u})}{\Phi_{\Sigma^{(0)}}(\boldsymbol{u})} \le \exp\left(\frac{1}{\sqrt{2\pi}} \sum_{1\le i<j\le d} \frac{e^{-(u_i+u_j)^2/8}}{\mathbb{E}\left\{(W+(u_i+u_j)/2)_+\right\}} \ln\left(\frac{\pi-2\arcsin(\sigma_{ij}^{(0)})}{\pi-2\arcsin(\sigma_{ij}^{(1)})}\right)\right), \quad \boldsymbol{u}\in(0,\infty)^d,$$

provided that $0\le \sigma_{ij}^{(0)} \le \sigma_{ij}^{(1)} \le 1$. Recent extensions of the normal comparison inequalities are presented in [9, 11, 17, 18, 19, 26].

Our principal goal of this paper is the derivation of comparison inequalities for order statistics of Gaussian arrays, which are useful in several applications. In order to fix the notation, we denote by $\mathcal{X} = (X_{ij})_{d\times n}$ and $\mathcal{Y} = (Y_{ij})_{d\times n}$ two $d\times n$ random arrays with $N(0,1)$ components and jointly Gaussian (hereafter referred to as standard Gaussian arrays), and let $\Sigma^{(1)} = (\sigma_{ij,lk}^{(1)})_{dn\times dn}$ and $\Sigma^{(0)} = (\sigma_{ij,lk}^{(0)})_{dn\times dn}$ be the covariance matrices of $\mathcal{X}$ and $\mathcal{Y}$, respectively, with $\sigma_{ij,lk}^{(1)} := \mathbb{E}\{X_{ij}X_{lk}\}$ and $\sigma_{ij,lk}^{(0)} := \mathbb{E}\{Y_{ij}Y_{lk}\}$. Furthermore, define $\boldsymbol{X}_{(r)} = (X_{1(r)},\ldots,X_{d(r)}), 1\le r\le n$ to be the rth order statistics vector generated by $\mathcal{X}$ as follows

$$X_{i(1)} = \min_{1\le j\le n} X_{ij} \le \cdots \le X_{i(r)} \le \cdots \le \max_{1\le j\le n} X_{ij} = X_{i(n)}, \quad 1\le i\le d.$$

Similarly, we write $\boldsymbol{Y}_{(r)} = (Y_{1(r)},\ldots,Y_{d(r)})$ which is generated by $\mathcal{Y}$. Clearly, in case of independent rows of Gaussian arrays, the study of $\boldsymbol{X}_{(r)}$ reduces to that of the component-wise order statistics $X'_{i(r)}s$ for Gaussian random vector, see [9]. Our principal results, stated in Theorem 2.1 and Theorem 2.4, derive bounds for the difference

$$(3) \qquad \Delta_{(r)}(\boldsymbol{u}) := \mathbb{P}\left\{\boldsymbol{X}_{(r)} \le \boldsymbol{u}\right\} - \mathbb{P}\left\{\boldsymbol{Y}_{(r)} \le \boldsymbol{u}\right\}, \quad \boldsymbol{u}\in\mathbb{R}^d.$$

Two applications of those bounds are discussed in Section 3, including the study of the mixed Gumbel limit theorems for order statistics of stationary Gaussian processes and the lower tail probability of order statistics of self-similar Gaussian processes.

We organize this paper as follows. In Section 2 we display our main results. Section 3 is devoted to the applications. The proofs are relegated to Section 4 and Appendix.

2. Main Results

This section is concerned with sharp bounds for $\Delta_{(r)}(\boldsymbol{u})$ defined in (3), which go in line with Li and Shao's [22] normal comparison inequality. For notational simplicity we set below

$$Q_{ij,lk} = \left|\arcsin(\sigma_{ij,lk}^{(1)}) - \arcsin(\sigma_{ij,lk}^{(0)})\right|, \quad Q_{ij,lk}^+ = (\arcsin(\sigma_{ij,lk}^{(1)}) - \arcsin(\sigma_{ij,lk}^{(0)}))_+.$$

Theorem 2.1. *If $\mathcal{X}$ and $\mathcal{Y}$ are two standard $d\times n$ Gaussian arrays, then for any $1\le r\le n$ we have*

$$(4) \quad \left|\Delta_{(r)}(\boldsymbol{u})\right| \le \frac{1}{2\pi}\left(\sum_{\substack{1\le i\le d\\1\le j<k\le n}} Q_{ij,ik}\exp\left(-\frac{u_i^2}{1+\rho_{ij,ik}}\right) + \sum_{\substack{1\le i<l\le d\\1\le j,k\le n}} Q_{ij,lk}\exp\left(-\frac{u_i^2+u_l^2}{2(1+\rho_{ij,lk})}\right)\right), \ \forall\boldsymbol{u}\in\mathbb{R}^d,$$

where $\rho_{ij,lk} = \max(|\sigma^{(0)}_{ij,lk}|, |\sigma^{(1)}_{ij,lk}|)$. *If further*

$$\sigma^{(1)}_{ij,ik} = \sigma^{(0)}_{ij,ik}, \quad 1 \le i \le d,\ 1 \le j,k \le n, \tag{5}$$

then

$$\Delta_{(r)}(\boldsymbol{u}) \le \frac{1}{2\pi} \sum_{\substack{1\le i<l\le d\\ 1\le j,k\le n}} Q^{+}_{ij,lk} \exp\left(-\frac{u_i^2+u_l^2}{2(1+\rho_{ij,lk})}\right), \quad \forall \boldsymbol{u} \in \mathbb{R}^d. \tag{6}$$

Remark 2.2. *For $r=1$ and $r=n$ the claims in (4) reduce to Lemma 11 in* [10]. *Note that for $1<r<n$ our results are derived using a different technique. Furthermore, using in addition similar arguments as in Theorem 1.2 in* [29], *one can establish for any $[\boldsymbol{a},\boldsymbol{b}] \subset [-\infty,\infty]^d$ the following comparison inequality*

$$\begin{aligned}
&\left|\mathbb{P}\left\{\boldsymbol{X}_{(r)} \in [\boldsymbol{a},\boldsymbol{b}]\right\} - \mathbb{P}\left\{\boldsymbol{Y}_{(r)} \in [\boldsymbol{a},\boldsymbol{b}]\right\}\right| \\
&\quad \le \frac{1}{\pi}\left(\sum_{\substack{1\le i\le d\\ 1\le j<k\le n}} Q_{ij,ik} \exp\left(-\frac{u_i^2}{1+\rho_{ij,ik}}\right) + \sum_{\substack{1\le i<l\le d\\ 1\le j,k\le n}} Q_{ij,lk} \exp\left(-\frac{u_i^2+u_l^2}{2(1+\rho_{ij,lk})}\right)\right),
\end{aligned}$$

with $u_i = \min(|a_i|,|b_i|), 1 \le i \le d$.

A direct consequence of Theorem 2.1 is the following Slepian's inequality for the order statistics of Gaussian arrays, which for $r=1$ is, however, weaker than Theorem 1.1 in [16].

Corollary 2.3. *Suppose that (5) is satisfied and $\sigma^{(0)}_{ij,lk} \ge \sigma^{(1)}_{ij,lk}$ holds for $1 \le i < l \le d, 1 \le j,k \le n$. Then*

$$\mathbb{P}\Big\{ \cup_{i=1}^d \{X_{i(r)} > u_i\}\Big\} \ge \mathbb{P}\Big\{ \cup_{i=1}^d \{Y_{i(r)} > u_i\}\Big\}, \quad \forall \boldsymbol{u} \in \mathbb{R}^d. \tag{7}$$

Note that the bounds in Theorem 2.1 do not depend on r, which indicates that in some cases they may not be sharp enough. Below we present a sharper result which holds under the assumption that the columns of both $\mathcal{X}$ and $\mathcal{Y}$ are mutually independent and identically distributed, i.e.,

$$\sigma^{(\kappa)}_{ij,lk} = \sigma^{(\kappa)}_{il} \mathbb{I}\{j=k\}, \quad 1 \le i,l \le d, 1 \le j,k \le n, \kappa = 0,1, \tag{8}$$

with some $\sigma^{(\kappa)}_{il} \in (-1,1), 1 \le i,l \le d, \kappa = 0,1$, where $\mathbb{I}\{\cdot\}$ stands for the indicator function. This result is useful for establishing mixed Gumbel limit theorems; see Section 3.

In order to simplify the presentation, we shall define

$$c_{n,r} = \frac{n!}{r!(n-r)!}, \ 0 \le r \le n, \quad \rho_{il} = \max(|\sigma^{(0)}_{il}|, |\sigma^{(1)}_{il}|), \quad 1 \le i,l \le d$$

and

$$A^{(r)}_{il} = \int_{\sigma^{(0)}_{il}}^{\sigma^{(1)}_{il}} \frac{(1+|h|)^{2(n-r)}}{(1-h^2)^{(n-r+1)/2}}\, dh, \quad 1 \le i,l \le d,\ 1 \le r \le n.$$

Theorem 2.4. *Under the assumptions of Theorem 2.1, if further (8) is satisfied, then for any $\boldsymbol{u} \in (0,\infty)^d$*

$$\Delta_{(r)}(\boldsymbol{u}) \le \frac{n(c_{n-1,r-1})^2}{(2\pi)^{n-r+1}} u^{-2(n-r)} \sum_{1\le i<l\le d} \left(A^{(r)}_{il}\right)_+ \exp\left(-\frac{(n-r+1)u^2}{1+\rho_{il}}\right) \tag{9}$$

and

$$\left|\Delta_{(r)}(\boldsymbol{u})\right| \le \frac{n(c_{n-1,r-1})^2}{(2\pi)^{n-r+1}} u^{-2(n-r)} \sum_{1\le i<l\le d} \left|A^{(r)}_{il}\right| \exp\left(-\frac{(n-r+1)u^2}{1+\rho_{il}}\right)$$

hold with $u = \min_{1\le i\le d} u_i$.

As in Theorem 2.4 we also have the following bounds, without introducing $u = \min_{1\le i\le d} u_i$.

Proposition 2.5. *Under the assumptions of Theorem 2.1, if further* (8) *is satisfied, then for any* $\boldsymbol{u} \in (0,\infty)^d$

$$\Delta_{(r)}(\boldsymbol{u}) \le \frac{n(c_{n-1,r-1})^2}{2(\sqrt{\pi})^{n-r+2}} \sum_{1\le i<l\le d} (B_{il}^{(r)})_+ \frac{1}{(u_i+u_l)^{n-r}} \exp\left(-\frac{(n-r)(u_i+u_l)^2 + 2(u_i^2+u_l^2)}{4(1+\rho_{il})}\right), \tag{10}$$

with

$$B_{il}^{(r)} = \int_{\sigma_{il}^{(0)}}^{\sigma_{il}^{(1)}} \frac{(1+|h|)^{(n-r)/2}}{(1-h^2)^{1/2}}\, dh, \quad 1\le i<l\le d,\ 1\le r\le n.$$

If additionally $D_{il} = \min(u_i - \rho_{il}u_l, u_l - \rho_{il}u_i) > 0$ *for all* $1 \le i < l \le d$, *then*

$$\Delta_{(r)}(\boldsymbol{u}) \le \frac{n(c_{n-1,r-1})^2}{(2\pi)^{n-r+1}} \sum_{1\le i<l\le d} (\widetilde{A}_{il}^{(r)})_+ \left(\frac{u_i+u_l}{2} D_{il}\right)^{-(n-r)} \exp\left(-\frac{(n-r+1)(u_i^2+u_l^2)}{2(1+\rho_{il})}\right), \tag{11}$$

where

$$\widetilde{A}_{il}^{(r)} = \int_{\sigma_{il}^{(0)}}^{\sigma_{il}^{(1)}} \frac{(1+|h|)^{2(n-r)}(1-|h|)^{(n-r)}}{(1-h^2)^{(n-r+1)/2}}\, dh, \quad 1\le i<l\le d,\ 1\le r\le n.$$

Motivated by, e.g., [22, 26, 36], we obtain next an upper bound for $\Theta_{(r)}(\boldsymbol{u}) := \mathbb{P}\left\{\boldsymbol{X}_{(r)} \le \boldsymbol{u}\right\} / \mathbb{P}\left\{\boldsymbol{Y}_{(r)} \le \boldsymbol{u}\right\}$.

Proposition 2.6. *Under the assumptions of Theorem 2.1, if further* (5) *holds and* $0 \le \sigma_{ij,lk}^{(0)} \le \sigma_{ij,lk}^{(1)} < 1$ *for* $1 \le i < l \le d, 1 \le j,k \le n$, *then for any* $\boldsymbol{u} \in [0,\infty)^d$

$$1 \le \Theta_{(r)}(\boldsymbol{u}) \le \exp\left(\frac{1}{\sqrt{2\pi}} \sum_{\substack{1\le i<l\le d\\ 1\le j,k\le n}} \frac{C_{ij,lk}e^{-(u_i+u_l)^2/8}}{\mathbb{E}\left\{(W+(u_i+u_l)/2)_+\right\}}\right), \tag{12}$$

with W *an* $N(0,1)$ *random variable and*

$$C_{ij,lk} = \ln\left(\frac{\pi - 2\arcsin(\sigma_{ij,lk}^{(0)})}{\pi - 2\arcsin(\sigma_{ij,lk}^{(1)})}\right), \quad 1\le i<l\le d, 1\le j,k\le n.$$

3. Applications and Discussions

3.1. **Limit theorems for stationary order statistics processes.** When dealing with supremum of Gaussian processes on large intervals, the so-called Gumbel limit theorems are of interest for statistical applications, see e.g., [7, 8, 29, 30] and the recent contributions [14, 20]. Let $\{X_{n-r+1:n}(t), t\ge 0\}$ be the rth upper order statistics process generated by a centered stationary Gaussian process $\{X(t), t \ge 0\}$ with a.s. continuous sample paths, unit variance and correlation function $\rho(\cdot)$ satisfying

$$\rho(t) = 1 - |t|^\alpha + o(|t|^\alpha),\ t\to 0 \ \text{ for some } \alpha \in (0,2] \ \text{ and } \ \rho(t)<1,\ \forall t\ne 0. \tag{13}$$

From Theorem 1 in [11] or Theorem 2.2 in [12] for any $T>0$ we have

$$\mathbb{P}\left\{\sup_{t\in[0,T]} X_{n-r+1:n}(t) > u\right\} = T\mathcal{A}_{r,\alpha} c_{n,r} (2\pi)^{-\frac{r}{2}} u^{\frac{2}{\alpha}-r} \exp\left(-\frac{ru^2}{2}\right)(1+o(1)), \quad u\to\infty, \tag{14}$$

where $\mathcal{A}_{r,\alpha} \in (0,\infty)$ is given explicitly as a limit. As a continuation of [11] we establish below a limit theorem for the rth upper order statistics process $X_{n-r+1:n}$.

Theorem 3.1. *Let* $\{X_{n-r+1:n}(t), t\ge 0\}$ *be the* r*th upper order statistics process generated by* X*, a centered stationary Gaussian process with a.s. continuous sample paths. Suppose that* (13) *holds and further* $\lim_{t\to\infty} \rho(t)\ln t = \gamma \in [0,\infty]$.

a) If $\gamma = 0$, *then*

$$\lim_{T\to\infty} \sup_{x\in\mathbb{R}} \left| \mathbb{P}\left\{a_{r,T}\Big(\sup_{t\in[0,T]} X_{n-r+1:n}(t) - b_{r,T}\Big) \le x\right\} - \exp\left(-e^{-x}\right)\right| = 0,$$

where, with $D = c_{n,r}\mathcal{A}_{r,\alpha}(r/2)^{r/2-1/\alpha}(2\pi)^{-r/2}$

$$a_{r,T} = \sqrt{2r\ln T}, \quad b_{r,T} = \sqrt{(2/r)\ln T} + \frac{1}{\sqrt{2r\ln T}}\left(\left(\frac{1}{\alpha} - \frac{r}{2}\right)\ln\ln T + \ln D\right), \quad T > e. \tag{15}$$

b) If $\gamma = \infty$*, and* $\alpha \in (0, 1]$*,* $\rho(t)$ *is convex for* $t \geq 0$ *with* $\lim_{t\to\infty}\rho(t) = 0$ *and further* $\rho(t)\ln t$ *is monotone for large* t*, then with* $\Phi(\cdot)$ *the df of an* $N(0, 1)$ *random variable*

$$\lim_{T\to\infty}\sup_{x\in\mathbb{R}}\left|\mathbb{P}\left\{\frac{1}{\sqrt{\rho(T)}}\Big(\sup_{t\in[0,T]} X_{n-r+1:n}(t) - \sqrt{1-\rho(T)}b_{r,T}\Big) \leq x\right\} - \Phi(x)\right| = 0.$$

c) If $\gamma \in (0, \infty)$*, then, with* W *an* $N(0, 1)$ *random variable*

$$\lim_{T\to\infty}\sup_{x\in\mathbb{R}}\left|\mathbb{P}\left\{a_{r,T}\Big(\sup_{t\in[0,T]} X_{n-r+1:n}(t) - b_{r,T}\Big) \leq x\right\} - \mathbb{E}\left\{\exp\left(-e^{-(x+\gamma-\sqrt{2\gamma r}W)}\right)\right\}\right| = 0.$$

The proof of Theorem 3.1 is presented in Appendix.

3.2. **Lower tail probability for order statistics processes.** The seminal contributions [23, 24] show that the investigation of the lower tail probability of Gaussian processes is of special interest in many applied fields, including the study of real zeros of random polynomials, the study of Gaussian pursuit problem, and the study of the first-passage time for the Slepian process. In this section, we aim at generalizing some results in [23, 24], by considering order statistics processes instead of Gaussian processes.

Our first result is concerned with extension of the celebrated Slepian inequality for order statistics processes. Let $\{Y(t), t \geq 0\}$ and $\{Z(t), t \geq 0\}$ be two centered Gaussian processes with a.s. continuous sample paths, and $\{Y_{r:n}(t), t \geq 0\}$, $\{Z_{r:n}(t), t \geq 0\}$ be the corresponding rth lower order statistics processes. Applying the standard discrete-continuous approximation technique (cf. [1]) to Corollary 2.3 one can easily verify the following proposition.

Proposition 3.2. *If for all* $s, t \geq 0$

$$\mathbb{E}\left\{Y(t)^2\right\} = \mathbb{E}\left\{Z(t)^2\right\} \quad \text{and} \quad \mathbb{E}\left\{Y(s)Y(t)\right\} \leq \mathbb{E}\left\{Z(s)Z(t)\right\},$$

then for any $T > 0$ *and* $u \in \mathbb{R}$ *we have*

$$\mathbb{P}\left\{\sup_{t\in[0,T]} Y_{r:n}(t) > u\right\} \geq \mathbb{P}\left\{\sup_{t\in[0,T]} Z_{r:n}(t) > u\right\}.$$

Remark 3.3. *A straightforward application of Proposition 3.2 yields that, for any* $x \in \mathbb{R}$ *(cf.* [23]*)*

$$p_r(x) := \lim_{T\to\infty}\frac{1}{T}\ln\mathbb{P}\left\{\sup_{0\leq t\leq T} Y_{r:n}(t) \leq x\right\} = \sup_{T>0}\frac{1}{T}\ln\mathbb{P}\left\{\sup_{0\leq t\leq T} Y_{r:n}(t) \leq x\right\} \tag{16}$$

exists and $p_r(x), x \in \mathbb{R}$ *is left-continuous, provided that* $\{Y(t), t \geq 0\}$ *is a centered stationary Gaussian processes with* $\mathbb{E}\{Y(0)Y(t)\} \geq 0$ *for all* $t \geq 0$*.*

Next, let $\{X(t), t \geq 0\}$ be a centered self-similar Gaussian process with a.s. continuous sample paths and index $\alpha/2$ for some $\alpha > 0$, i.e.,

$$X(0) = 0, \quad \mathbb{E}\left\{X(1)^2\right\} = 1, \quad \{X(\lambda t), t \geq 0\} \stackrel{d}{=} \{\lambda^{\alpha/2}X(t), t \geq 0\}, \quad \forall\lambda > 0,$$

where $\stackrel{d}{=}$ denotes equality of the (finite-dimensional) distribution functions. It is well-known that, by Lamperti's transformation a dual stationary Gaussian process $\{X^*(t), t \geq 0\}$ can be defined as

$$X^*(t) = e^{-\alpha t/2}X(e^t), \quad t \geq 0.$$

Proposition 3.4. *Let $\{X(t), t \ge 0\}$ be the self-similar Gaussian process with self-similarity index $\alpha/2 \in (0, \infty)$ defined above. Suppose that $\mathbb{E}\{X(s)X(t)\} \ge 0$ for all $s, t \ge 0$, and $\rho(t) := \mathbb{E}\{X^*(0)X^*(t)\}$ is decreasing. If further for any $h \in (0, \infty)$ and $\theta \in (0, 1)$*

$$a_{h,\theta}^2 = \inf_{0<t\le h} \frac{\rho(\theta t) - \rho(t)}{1 - \rho(t)} > 0, \tag{17}$$

then

$$\lim_{x \downarrow 0} \frac{\ln \mathbb{P}\left\{\sup_{0\le t\le 1} X_{r:n}(t) \le x\right\}}{\ln x} = \frac{2}{\alpha} \pi_{r,\alpha}, \tag{18}$$

where

$$\pi_{r,\alpha} = -\lim_{T\to\infty} \frac{1}{T} \ln \mathbb{P}\left\{\sup_{0\le t\le T} X_{r:n}^*(t) \le 0\right\} \ge 0$$

is the Li-Shao type constant.

The proof of Proposition 3.4 is presented in Appendix.

Remark 3.5. *a) As discussed in [24], two examples of $\{X(t), t \ge 0\}$ that satisfy all conditions of Proposition 3.4 are the standard fBm B_α and the centered Gaussian process $\{X_\beta(t), t \ge 0\}, \beta > 0$ with*

$$\mathbb{E}\{X_\beta(s)X_\beta(t)\} = \frac{2^\beta (st)^{(1+\beta)/2}}{(s+t)^\beta}, \quad s, t > 0.$$

b) The positivity of the constants in the right-hand side of (18) is still an open problem. For the case $r = n$ and X is a standard fBm $B_\alpha, \alpha \in (0, 2)$, in view of Theorem 3.1 in [23] we have

$$\pi_{n,\alpha} = n\left(1 - \frac{\alpha}{2}\right) > 0.$$

4. Proofs

Hereafter, we write $\stackrel{d}{=}$ for equality of the distribution functions. A vector $\boldsymbol{z} = (z_1, \ldots, z_{dn})$ will also be denoted by

$$\boldsymbol{z} = (\boldsymbol{z}_1, \ldots, \boldsymbol{z}_d), \text{ with } \boldsymbol{z}_i = (z_{i1}, \ldots, z_{in}), \quad 1 \le i \le d,$$

where $z_{ij} = z_{(i-1)n+j}, 1 \le i \le d, 1 \le j \le n$. Note that for any $p = (i-1)n + j, q = (l-1)n + k, 1 \le i, l \le d, 1 \le j, k \le n$

$$\{p < q\} = \{i < l, \text{ or } i = l \text{ and } j < k\}.$$

Denote

$$\boldsymbol{z}/\boldsymbol{z}_i = (\boldsymbol{z}_1, \ldots, \boldsymbol{z}_{i-1}, \boldsymbol{z}_{i+1}, \ldots, \boldsymbol{z}_d), \quad 1 \le i \le d.$$

Furthermore, for any $\boldsymbol{x} \in \mathbb{R}^n$ we denote

$$\begin{aligned} \boldsymbol{x}/x_i &= (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), \\ \frac{d\boldsymbol{x}}{dx_i} &= dx_1 dx_2 \cdots dx_{i-1} dx_{i+1} \cdots dx_n, \quad 1 \le i \le n, \end{aligned}$$

and for $1 \le i < j \le n$

$$\frac{d\boldsymbol{x}}{dx_i dx_j} = dx_1 dx_2 \cdots dx_{i-1} dx_{i+1} \cdots dx_{j-1} dx_{j+1} \cdots dx_n.$$

Proof of Theorem 2.1: We shall first establish (4) by considering $r = 1$, $r = 2$ and $2 < r \le n$ separately.
Case $r = 1$. Note that $\mathcal{X} \stackrel{d}{=} -\mathcal{X}$ for the standard Gaussian array $\mathcal{X}$. It follows from Theorem 2.1 in [26] that

$$\begin{aligned} \left|\Delta_{(1)}(\boldsymbol{u})\right| &= \left|\mathbb{P}\left\{ \cup_{i=1}^d \cap_{j=1}^n \{-Y_{ij} < -u_i\}\right\} - \mathbb{P}\left\{ \cup_{i=1}^d \cap_{j=1}^n \{-X_{ij} < -u_i\}\right\}\right| \\ &\le \frac{1}{2\pi} \sum_{(i-1)n+j<(l-1)n+k} Q_{ij,lk} \exp\left(-\frac{u_i^2 + u_l^2}{2(1+\rho_{ij,lk})}\right) \end{aligned}$$

establishing (4) for $r = 1$.

Next, by a standard approximation procedure we may assume that both $\Sigma^{(1)}$ and $\Sigma^{(0)}$ are positive definite. Let further $\mathcal{Z} = (Z_{ij})_{d\times n}$ be a standard Gaussian array with covariance matrix

$$\Gamma^h = h\Sigma^{(1)} + (1-h)\Sigma^{(0)} = (\delta^h_{ij,lk})_{dn\times dn},$$

where by our notation $\delta^h_{ij,lk} = \mathbb{E}\{Z_{ij}Z_{lk}\}$. Clearly, Γ^h is also positive definite for any $h \in [0,1]$. Denote below by $g_h(\boldsymbol{z})$ the probability density function (pdf) of $\mathcal{Z}$. It is known that the Plackett's partial differential equation holds as (see e.g., [21], p. 82, or [26])

$$\frac{\partial g_h(\boldsymbol{z})}{\partial \delta^h_{ij,lk}} = \frac{\partial^2 g_h(\boldsymbol{z})}{\partial z_{ij}\partial z_{lk}}, \quad 1 \le i,l \le d, 1 \le j,k \le n, (i,j) \ne (l,k). \tag{19}$$

<u>Case $r = 2$</u>. Hereafter, we write $\boldsymbol{\lambda} = -\boldsymbol{u}$ and set

$$Q(\mathcal{Z};\Gamma^h) = \mathbb{P}\left\{\boldsymbol{Z}_{(n-1)} > \boldsymbol{\lambda}\right\} = \int_{\cap_{i=1}^d \cup_{j,j'=1;j\ne j'}^n \{z_{ij}>\lambda_i, z_{ij'}>\lambda_i\}} g_h(\boldsymbol{z})\, d\boldsymbol{z}. \tag{20}$$

Since $\boldsymbol{X}_{(2)} \overset{d}{=} -\boldsymbol{X}_{(n-1)}$ we have

$$\Delta_{(2)}(\boldsymbol{u}) = Q(\mathcal{Z};\Gamma^1) - Q(\mathcal{Z};\Gamma^0) = \int_0^1 \frac{\partial Q(\mathcal{Z};\Gamma^h)}{\partial h}\, dh. \tag{21}$$

Note that the quantities $Q(\mathcal{Z};\Gamma^h)$ and $g_h(\boldsymbol{z})$ depend on h only through the entries $\delta^h_{ij,lk}$ of Γ^h. Hence we have by (19)

$$\begin{aligned}\frac{\partial Q}{\partial h}(\mathcal{Z};\Gamma^h) &= \sum_{(i-1)n+j<(l-1)n+k} \frac{\partial Q(\mathcal{Z};\Gamma^h)}{\partial \delta^h_{ij,lk}} \frac{\partial \delta^h_{ij,lk}}{\partial h}\\ &= \sum_{(i-1)n+j<(l-1)n+k} (\sigma^{(1)}_{ij,lk} - \sigma^{(0)}_{ij,lk}) E_{il}(j,k),\end{aligned} \tag{22}$$

with

$$E_{il}(j,k) := \int_{\cap_{s=1}^d \cup_{t,t'=1;t\ne t'}^n \{z_{st}>\lambda_s, z_{st'}>\lambda_s\}} \frac{\partial^2 g_h(\boldsymbol{z})}{\partial z_{ij}\partial z_{lk}}\, d\boldsymbol{z}, \quad (i-1)n+j < (l-1)n+k. \tag{23}$$

Next, in order to establish (4) we shall show that

$$|E_{il}(j,k)| \le \varphi(\lambda_i, \lambda_l; \delta^h_{ij,lk}), \quad (i-1)n+j < (l-1)n+k, \tag{24}$$

where $\varphi(\cdot,\cdot;\cdot)$ is the pdf of (Z_{ij}, Z_{lk}), given by

$$\varphi(x,y;\delta^h_{ij,lk}) = \frac{1}{2\pi\sqrt{1-(\delta^h_{ij,lk})^2}} \exp\left(-\frac{x^2 - 2\delta^h_{ij,lk}xy + y^2}{2\left(1-(\delta^h_{ij,lk})^2\right)}\right), \quad x,y \in \mathbb{R}.$$

We consider below two sub-cases: a) $i = l$ and b) $i < l$.

<u>a) Proof of (24) for $i = l$</u>. Let

$$\begin{aligned} A_i' &= \cap_{s=1;s\ne i}^d \cup_{t,t'=1;t\ne t'}^n \{z_{st} > \lambda_s, z_{st'} > \lambda_s\} \\ &:= \{\boldsymbol{z}/\boldsymbol{z}_i \in \mathbb{R}^{(d-1)n} : \text{ for any } 1 \le s(\ne i) \le n \text{ there exist } 1 \le t,t' \le n : z_{st} > \lambda_s, z_{st'} > \lambda_s\},\end{aligned} \tag{25}$$

$$\begin{aligned} A_i &= \cup_{t,t'=1;t\ne t'}^n \{z_{it} > \lambda_i, z_{it'} > \lambda_i\} \\ &:= \{\boldsymbol{z}_i \in \mathbb{R}^n : \text{ there exist } 1 \le t,t' \le n : z_{it} > \lambda_i, z_{it'} > \lambda_i\}.\end{aligned} \tag{26}$$

We can rewrite $E_{ii}(j,k)$ as

$$E_{ii}(j,k) = \int_{A_i'}\int_{A_i} \frac{\partial^2 g_h(\boldsymbol{z})}{\partial z_{ij}\partial z_{ik}}\, d\boldsymbol{z}, \quad 1 \le i \le d,\ 1 \le j < k \le n. \tag{27}$$

Next, we decompose the integral region A_i according to

a_1) $\{z_{ij} > \lambda_i, z_{ik} > \lambda_i\} := \{\boldsymbol{z}_i \in \mathbb{R}^n :\ z_{ij} > \lambda_i, z_{ik} > \lambda_i\}$;

a_2) $\{z_{ij}>\lambda_i, z_{ik}\le\lambda_i\}:=\{\boldsymbol{z}_i\in\mathbb{R}^n:\ z_{ij}>\lambda_i, z_{ik}\le\lambda_i\}$;
a_3) $\{z_{ij}\le\lambda_i, z_{ik}>\lambda_i\}:=\{\boldsymbol{z}_i\in\mathbb{R}^n:\ z_{ij}\le\lambda_i, z_{ik}>\lambda_i\}$;
a_4) $\{z_{ij}\le\lambda_i, z_{ik}\le\lambda_i\}:=\{\boldsymbol{z}_i\in\mathbb{R}^n:\ z_{ij}\le\lambda_i, z_{ik}\le\lambda_i\}$.
For case a_1) we have

$$\int_{A_i\cap\{z_{ij}>\lambda_i,z_{ik}>\lambda_i\}}\frac{\partial^2 g_h(\boldsymbol{z})}{\partial z_{ij}\partial z_{ik}}\,d\boldsymbol{z}_i=\int_{\mathbb{R}^{n-2}}g_h(z_{ij}=\lambda_i,z_{ik}=\lambda_i)\,\frac{d\boldsymbol{z}_i}{dz_{ij}dz_{ik}},\tag{28}$$

where $g_h(z_{ij}=\lambda_i,z_{ik}=\lambda_i)$ denotes a function of $dn-2$ variables formed from $g_h(\boldsymbol{z})$ by putting $z_{ij}=\lambda_i, z_{ik}=\lambda_i$. Similarly, for cases a_2) and a_3)

$$\begin{aligned}\int_{A_i\cap\{z_{ij}>\lambda_i,z_{ik}\le\lambda_i\}}\frac{\partial^2 g_h(\boldsymbol{z})}{\partial z_{ij}\partial z_{ik}}\,d\boldsymbol{z}_i&=\int_{A_i\cap\{z_{ij}\le\lambda_i,z_{ik}>\lambda_i\}}\frac{\partial^2 g_h(\boldsymbol{z})}{\partial z_{ij}\partial z_{ik}}\,d\boldsymbol{z}_i\\&=-\int_{\cup_{t=1;t\ne j,k}^n\{z_{it}>\lambda_i\}}g_h(z_{ij}=\lambda_i,z_{ik}=\lambda_i)\,\frac{d\boldsymbol{z}_i}{dz_{ij}dz_{ik}},\end{aligned}\tag{29}$$

where

$$\cup_{t=1;t\ne j,k}^n\{z_{it}>\lambda_i\}:=\{\boldsymbol{z}_i/(z_{ij}z_{ik})\in\mathbb{R}^{n-2}:\ \text{it exists } 1\le t(\ne j,k)\le n \text{ such that } z_{it}>\lambda_i\}.$$

Finally, for case a_4)

$$\int_{A_i\cap\{z_{ij}\le\lambda_i,z_{ik}\le\lambda_i\}}\frac{\partial^2 g_h(\boldsymbol{z})}{\partial z_{ij}\partial z_{ik}}\,d\boldsymbol{z}_i=\int_{\cup_{t,t'=1;t,t'\ne j,k;t\ne t'}^n\{z_{it}>\lambda_i,z_{it'}>\lambda_i\}}g_h(z_{ij}=\lambda_i,z_{ik}=\lambda_i)\,\frac{d\boldsymbol{z}_i}{dz_{ij}dz_{ik}},$$

where

$$\begin{aligned}&\cup_{t,t'=1;t,t'\ne j,k;t\ne t'}^n\{z_{it}>\lambda_i,z_{it'}>\lambda_i\}\\&\quad:=\{\boldsymbol{z}_i/(z_{ij}z_{ik})\in\mathbb{R}^{n-2}:\ \text{there exist } 1\le t,t'(\ne j,k)\le n \text{ such that } z_{it}>\lambda_i,z_{it'}>\lambda_i\}.\end{aligned}$$

This together with (27)–(29) yields

$$\begin{aligned}E_{ii}(j,k)&=\int_{A_i'}\int_{\mathbb{R}^{n-2}-\cup_{t=1;t\ne j,k}^n\{z_{it}>\lambda_i\}}g_h(z_{ij}=\lambda_i,z_{ik}=\lambda_i)\,\frac{d\boldsymbol{z}}{dz_{ij}dz_{ik}}\\&\quad-\int_{A_i'}\int_{\cup_{t=1;t\ne j,k}^n\{z_{it}>\lambda_i\}-\cup_{t,t'=1;t,t'\ne j,k;t\ne t'}^n\{z_{it}>\lambda_i,z_{it'}>\lambda_i\}}g_h(z_{ij}=\lambda_i,z_{ik}=\lambda_i)\,\frac{d\boldsymbol{z}}{dz_{ij}dz_{ik}}\\&=\varphi(\lambda_i,\lambda_i;\delta^h_{ij,ik})\Big(\mathbb{P}\Big\{(\cap_{s=1,s\ne i}^d\{Z_{s(n-1)}>\lambda_s\})\cap\{\boldsymbol{Z}_i''\in\{w_{i1}''=\infty\}\}\Big|\{Z_{ij}=\lambda_i,Z_{ik}=\lambda_i\}\Big\}\\&\quad-\mathbb{P}\Big\{\big(\cap_{s=1,s\ne i}^d\{Z_{s(n-1)}>\lambda_s\}\big)\cap\{\boldsymbol{Z}_i''\in\{w_{i1}''\le n,w_{i2}''=\infty\}\}\Big|\{Z_{ij}=\lambda_i,Z_{ik}=\lambda_i\}\Big\}\Big),\end{aligned}\tag{30}$$

with $\boldsymbol{Z}_i''$ the $(n-2)$-dimensional components of $\boldsymbol{Z}_i$ obtained by deleting Z_{ij} and Z_{ik}, and w_{i1}'',w_{i2}'' given by

$$w_{i1}''=\inf\{t:1\le t(\ne j,k)\le n,z_{it}>\lambda_i\},\quad w_{i2}''=\inf\{t:w_{i1}''<t(\ne j,k)\le n,z_{it}>\lambda_i\}.\tag{31}$$

Hereafter we use the convention that $\inf\{\emptyset\}=\infty$. For instance,

$$\begin{aligned}&\{w_{i1}''=\infty\}=\{\boldsymbol{z}_i/(z_{ij}z_{ik})\in\mathbb{R}^{n-2}:z_{it}\le\lambda_i \text{ for all } 1\le t(\ne j,k)\le n\},\\&\{w_{i1}''\le n,w_{i2}''=\infty\}=\Big\{\boldsymbol{z}_i/(z_{ij}z_{ik})\in\mathbb{R}^{n-2}:\text{it exists } 1\le l(\ne j,k)\le n \text{ such that}\\&\quad z_{il}>\lambda_i,\ \text{and } z_{it}\le\lambda_i \text{ for all } 1\le t(\ne j,k,l)\le n\Big\}.\end{aligned}$$

Consequently, it follows thus from (30) that (24) holds for $i=l$.
b) Proof of (24) for $i<l$. Denote $A_{il}''=\cap_{s=1;s\ne i,l}^d\cup_{t,t'=1;t\ne t'}^n\{z_{st}>\lambda_s,z_{st'}>\lambda_s\}\subset\mathbb{R}^{(d-2)n}$ parallel to (25) and recall A_i in (26). We have

$$E_{il}(j,k)=\int_{A_{il}''}\int_{A_l}\int_{A_i}\frac{\partial^2 g_h(\boldsymbol{z})}{\partial z_{ij}\partial z_{lk}}\,d\boldsymbol{z}.\tag{32}$$

Next, we decompose the integral region A_i according to $\{\boldsymbol{z}_i \in \mathbb{R}^n : z_{ij} > \lambda_i\}$ and $\{\boldsymbol{z}_i \in \mathbb{R}^n : z_{ij} \le \lambda_i\}$. We have

$$\begin{aligned}
&\int_{A_i \cap \{\boldsymbol{z}_i \in \mathbb{R}^n : z_{ij} > \lambda_i\}} \frac{\partial^2 g_h(\boldsymbol{z})}{\partial z_{ij} \partial z_{lk}} \, d\boldsymbol{z}_i + \int_{A_i \cap \{\boldsymbol{z}_i \in \mathbb{R}^n : z_{ij} \le \lambda_i\}} \frac{\partial^2 g_h(\boldsymbol{z})}{\partial z_{ij} \partial z_{lk}} \, d\boldsymbol{z}_i \\
&\quad = -\int_{\cup_{t=1;t\neq j}^n \{z_{it} > \lambda_i\} - \cup_{t,t'=1;t,t'\neq j;t\neq t'}^n \{z_{it} > \lambda_i, z_{it'} > \lambda_i\}} \frac{\partial g_h(z_{ij} = \lambda_i)}{\partial z_{lk}} \frac{d\boldsymbol{z}_i}{dz_{ij}} \\
&\quad = -\int_{\{w'_{i1} \le n, w'_{i2} = \infty\}} \frac{\partial g_h(z_{ij} = \lambda_i)}{\partial z_{lk}} \frac{d\boldsymbol{z}_i}{dz_{ij}},
\end{aligned}$$

where w'_{i1}, w'_{i2} are defined by (similar notation below for w'_{l1}, w'_{l2} with respect to k instead of j)

$$w'_{i1} = \inf\{t : 1 \le t(\neq j) \le n, z_{it} > \lambda_i\}, \quad w'_{i2} = \inf\{t : w'_{i1} < t(\neq j) \le n, z_{it} > \lambda_i\}. \tag{33}$$

Using similar arguments for the integral with region A_l, we have by (32)

$$\begin{aligned}
E_{il}(j,k) &= \int_{A''_{il}} \int_{\{w'_{i1} \le n, w'_{i2} = \infty\}} \int_{\{w'_{l1} \le n, w'_{l2} = \infty\}} g_h(z_{ij} = \lambda_i, z_{lk} = \lambda_l) \frac{d\boldsymbol{z}}{dz_{ij} dz_{lk}} \\
&= \varphi(\lambda_i, \lambda_l; \delta^h_{ij,lk}) \mathbb{P}\Big\{\cap_{s=1;s\neq i,l}^d \{Z_{s(n-1)} > \lambda_s\} \cap \{\boldsymbol{Z}'_i \in \{w'_{i1} \le n, w'_{i2} = \infty\}\} \\
&\qquad \cap \{\boldsymbol{Z}'_l \in \{w'_{l1} \le n, w'_{l2} = \infty\}\} \Big| \{Z_{ij} = \lambda_i, Z_{lk} = \lambda_l\}\Big\},
\end{aligned} \tag{34}$$

where $\boldsymbol{Z}'_i$ and $\boldsymbol{Z}'_l$ are the $(n-1)$-dimensional components of $\boldsymbol{Z}_i$ and $\boldsymbol{Z}_l$ obtained by deleting Z_{ij} and Z_{lk}, respectively. Consequently, by (30) and (34) the validity of (24) follows.

Next, by combining (21)–(24), the claim in (4) for $r = 2$ follows by the fact that (see [22])

$$\int_0^1 \varphi(\lambda_i, \lambda_l; \delta^h_{ij,lk}) \, dh \le \frac{\arcsin(\sigma^{(1)}_{ij,lk}) - \arcsin(\sigma^{(0)}_{ij,lk})}{2\pi(\sigma^{(1)}_{ij,lk} - \sigma^{(0)}_{ij,lk})} \exp\left(-\frac{\lambda_i^2 + \lambda_l^2}{2(1 + \rho_{ij,lk})}\right). \tag{35}$$

<u>Case $2 < r \le n$</u>. Letting $\widetilde{Q}(\mathcal{Z}; \Gamma^h) = \mathbb{P}\left\{\boldsymbol{Z}_{(n-r+1)} > \boldsymbol{\lambda}\right\}$ we have

$$\Delta_{(r)}(\boldsymbol{u}) = \int_0^1 dh \left(\sum_{(i-1)n+j < (l-1)n+k} (\sigma^{(1)}_{ij,lk} - \sigma^{(0)}_{ij,lk}) \widetilde{E}_{il}(j,k)\right), \tag{36}$$

where

$$\widetilde{E}_{il}(j,k) := \int_{\cap_{s=1}^d \cup_{t_1,\ldots,t_r=1;t_l\neq t_j}^n \{z_{st_1} > \lambda_s, \ldots, z_{st_r} > \lambda_s\}} \frac{\partial^2 g_h(\boldsymbol{z})}{\partial z_{ij} \partial z_{lk}} \, d\boldsymbol{z}.$$

With the aid of (35), it suffices to show that

$$\left|\widetilde{E}_{il}(j,k)\right| \le \varphi(\lambda_i, \lambda_l; \delta^h_{ij,lk}), \quad (i-1)n + j < (l-1)n + k. \tag{37}$$

Similarly as above, two sub-cases : a) $i = l$ and b) $i < l$ need to be considered separately.

<u>a) Proof of (37) for $i = l$</u>. Similarly to $E_{ii}(j,k)$, we rewrite $\widetilde{E}_{ii}(j,k)$ as

$$\widetilde{E}_{ii}(j,k) = \int_{\widetilde{A}'_i} \int_{\widetilde{A}_i} \frac{\partial^2 g_h(\boldsymbol{z})}{\partial z_{ij} \partial z_{ik}} \, d\boldsymbol{z}, \tag{38}$$

with

$$\begin{aligned}
\widetilde{A}'_i &= \cap_{s=1;s\neq i}^d \cup_{t_1,\ldots,t_r=1;t_l\neq t_j}^n \{\boldsymbol{z}/\boldsymbol{z}_i \in \mathbb{R}^{(d-1)n} : z_{st_1} > \lambda_s, \ldots, z_{st_r} > \lambda_s\}, \\
\widetilde{A}_i &= \cup_{t_1,\ldots,t_r=1;t_l\neq t_j}^n \{\boldsymbol{z}_i \in \mathbb{R}^n : z_{it_1} > \lambda_i, \ldots, z_{it_r} > \lambda_i\}.
\end{aligned}$$

Next, we decompose the integral region $\widetilde{A}_i$ according to the four cases a_1)–a_4) as introduced for A_i (see the lines right above (28)).

For case a_1)

$$\int_{\widetilde{A}_i \cap \{z_{ij} > \lambda_i, z_{ik} > \lambda_i\}} \frac{\partial^2 g_h(\boldsymbol{z})}{\partial z_{ij} \partial z_{ik}} \, d\boldsymbol{z}_i = \int_{\{w''_{i,r-2} \le n\}} g_h(z_{ij} = \lambda_i, z_{ik} = \lambda_i) \frac{d\boldsymbol{z}_i}{dz_{ij} dz_{ik}}, \tag{39}$$

where w''_{i1} is given by (31) and (notation: $w''_{i,t} = w''_{it}$)

$$w''_{it} = \inf\{s : w''_{i,t-1} < s(\neq j,k) \le n, z_{is} > \lambda_i\}, \quad 2 \le t \le r,\ 1 \le i \le d.$$

Next, for cases a_2) and a_3)

$$\begin{aligned}\int_{\widetilde{A}_i \cap \{z_{ij}>\lambda_i, z_{ik}\le \lambda_i\}} \frac{\partial^2 g_h(\boldsymbol{z})}{\partial z_{ij}\partial z_{ik}}\, d\boldsymbol{z}_i &= \int_{\widetilde{A}_i \cap \{z_{ij}\le\lambda_i, z_{ik}> \lambda_i\}} \frac{\partial^2 g_h(\boldsymbol{z})}{\partial z_{ij}\partial z_{ik}}\, d\boldsymbol{z}_i \\ &= -\int_{\{w''_{i,r-1}\le n\}} g_h(z_{ij}=\lambda_i, z_{ik}=\lambda_i)\, \frac{d\boldsymbol{z}_i}{dz_{ij}dz_{ik}}. \end{aligned} \tag{40}$$

Finally, for case a_4)

$$\int_{\widetilde{A}_i \cap \{z_{ij}\le\lambda_i, z_{ik}\le \lambda_i\}} \frac{\partial^2 g_h(\boldsymbol{z})}{\partial z_{ij}\partial z_{ik}}\, d\boldsymbol{z}_i = \int_{\{w''_{ir}\le n\}} g_h(z_{ij}=\lambda_i, z_{ik}=\lambda_i)\, \frac{d\boldsymbol{z}_i}{dz_{ij}dz_{ik}}.$$

This together with (38)–(40) yields that

$$\begin{aligned}\widetilde{E}_{ii}(j,k) &= \int_{\widetilde{A}'_i}\int_{\{w''_{i,r-2}\le n, w''_{i,r-1}=\infty\}} g_h(z_{ij}=\lambda_i, z_{ik}=\lambda_i)\, \frac{d\boldsymbol{z}}{dz_{ij}dz_{ik}} \\ &\quad - \int_{\widetilde{A}'_i}\int_{\{w''_{i,r-1}\le n, w''_{ir}=\infty\}} g_h(z_{ij}=\lambda_i, z_{ik}=\lambda_i)\, \frac{d\boldsymbol{z}}{dz_{ij}dz_{ik}} \\ &= \varphi(\lambda_i,\lambda_i;\delta^h_{ij,ik}) \\ &\quad \times \Big(\mathbb{P}\Big\{\cap_{s=1,s\neq i}^d \{Z_{s(n-r+1)}>\lambda_s\} \cap \{\boldsymbol{Z}''_i \in \{w''_{i,r-2}\le n, w''_{i,r-1}=\infty\}\}\Big| \{Z_{ij}=\lambda_i, Z_{ik}=\lambda_i\}\Big\} \\ &\quad - \mathbb{P}\Big\{\cap_{s=1,s\neq i}^d \{Z_{s(n-r+1)}>\lambda_s\} \cap \{\boldsymbol{Z}''_i \in \{w''_{i,r-1}\le n, w''_{i,r}=\infty\}\}\Big| \{Z_{ij}=\lambda_i, Z_{ik}=\lambda_i\}\Big\}\Big) \end{aligned} \tag{41}$$

establishing (37) for $i=l$.

b) Proof of (37) for $i<l$. By $\widetilde{A}''_{il} = \cap_{s=1;s\neq i,l}^d \cup_{t_1,\ldots,t_r=1;t_l\neq t_j}^n \{\boldsymbol{z}/(\boldsymbol{z}_i\boldsymbol{z}_l) : z_{st_1}>\lambda_s,\ldots,z_{st_r}>\lambda_s\}$ and $\widetilde{A}_i$ as in (38) we have

$$\widetilde{E}_{il}(j,k) = \int_{\widetilde{A}''_{il}}\int_{\widetilde{A}_i}\int_{\widetilde{A}_l} \frac{\partial^2 g_h(\boldsymbol{z})}{\partial z_{ij}\partial z_{lk}}\, d\boldsymbol{z}. \tag{42}$$

By decomposing the integral regions $\widetilde{A}_i$ and $\widetilde{A}_l$ according to $z_{ij} >, \le \lambda_i$ and $z_{lk} >, \le \lambda_l$ in $\mathbb{R}^n$, respectively, we obtain by similar arguments as for $E_{il}(j,k)$ that

$$\begin{aligned}\widetilde{E}_{il}(j,k) = \varphi(\lambda_i,\lambda_l;\delta^h_{ij,lk})\mathbb{P}\Big\{&\cap_{s=1;s\neq i,l}^d \{Z_{s(n-r+1)}>\lambda_s\} \cap \{\boldsymbol{Z}'_i \in \{w'_{i,r-1}\le n, w'_{ir}=\infty\}\} \\ &\cap(\boldsymbol{Z}'_l \in \{w'_{l,r-1}\le n, w'_{lr}=\infty\})\Big| \{Z_{ij}=\lambda_i, Z_{lk}=\lambda_l\}\Big\}, \end{aligned} \tag{43}$$

where w'_{i1} is introduced in (33) and (similar notation for w'_{lt})

$$w'_{it} = \inf\{s : w'_{i,t-1} < s(\neq j) \le n, z_{is} > \lambda_i\}, \quad 2 \le t \le r,\ 1 \le i \le d.$$

It follows then from (43) that (37) holds. Consequently, the desired result (4) follows for $2<r\le n$.
Finally, in view of (5) we see that the indices over the sum in (22) and (36) are simplified to $1\le i<l\le d, 1\le j,k\le n$. Then the claim in (6) follows immediately from (34), (35) and (43). This completes the proof of Theorem 2.1. □

Proof of Theorem 2.4: It is sufficient to prove (9) since it implies the second result of Theorem 2.4. It follows from the argument of Lemma 12 in [10] that (9) holds for $r=1$. We shall present next the proofs for a) $r=2$ and b) $2<r\le n$.

a) Proof of (9) for $r=2$. It follows from (8), (21) and (22) that

$$\Delta_{(2)}(\boldsymbol{u}) = n\sum_{1\le i<l\le d} (\sigma_{il}^{(1)} - \sigma_{il}^{(0)}) \int_0^1 E_{il}\, dh, \tag{44}$$

where $E_{il} := E_{il}(1,1)$; $E_{il}(1,1)$ is defined in (23). Further, by (8) and (34) we have, with $\delta_{il}^h := \delta_{i1,l1}^h$ (recall $\lambda_i := -u_i, 1 \le i \le d$)

$$0 \le \frac{E_{il}}{\varphi(-u_i,-u_l;\delta_{il}^h)} \le \mathbb{P}\Big\{ \boldsymbol{Z}_i' \in \{w_{i1}' \le n, w_{i2}' = \infty\}, \boldsymbol{Z}_l' \in \{w_{l1}' \le n, w_{l2}' = \infty\} \Big\}. \tag{45}$$

Note that hereafter w_{i1}', w_{i2}' and w_{l1}', w_{l2}' are defined as in (33) with respect to $j = k = 1$.
Next, let $(\widetilde{Z}_i, \widetilde{Z}_l)$ be a bivariate standard normal random vector with correlation $\left|\delta_{il}^h\right|$ and $u = \min_{1 \le i \le d} u_i > 0$. It follows by Slepian's inequality in [33] and Lemma 2.3 in [28] that

$$\begin{aligned} \mathbb{P}\Big\{ Z_{ij} < -u_i, Z_{lk} < -u_l \Big\} \quad &\le \quad \mathbb{P}\left\{ \widetilde{Z}_i < -u_i, \widetilde{Z}_l < -u_l \right\} \\ &\le \quad \mathbb{P}\left\{ -\widetilde{Z}_i > u, -\widetilde{Z}_l > u \right\} \le \frac{(1+\left|\delta_{il}^h\right|)^2}{u^2} \varphi(u,u;\left|\delta_{il}^h\right|), \quad j,k \le n, \end{aligned} \tag{46}$$

implying thus

$$\begin{aligned} &\mathbb{P}\Big\{ \boldsymbol{Z}_i' \in \{(w_{i1}', w_{i2}') = (2,\infty)\}, \boldsymbol{Z}_l' \in \{(w_{l1}', w_{l2}') = (2,\infty)\} \Big\} \\ &\quad = \mathbb{P}\Big\{ Z_{i2} > -u_i, Z_{l2} > -u_l \Big\} \prod_{j=3}^n \mathbb{P}\Big\{ Z_{ij} \le -u_i, Z_{lj} \le -u_l \Big\} \le \Big(\frac{(1+|\delta_{il}^h|)^2}{u^2} \varphi(u,u;|\delta_{il}^h|) \Big)^{n-2} \end{aligned}$$

and

$$\begin{aligned} &\mathbb{P}\Big\{ \boldsymbol{Z}_i' \in \{(w_{i1}', w_{i2}') = (3,\infty)\}, \boldsymbol{Z}_l' \in \{(w_{l1}', w_{l2}') = (2,\infty)\} \Big\} \\ &\quad = \mathbb{P}\Big\{ Z_{i2} < -u_i, Z_{l2} > -u_l, Z_{i3} > -u_i, Z_{l3} < -u_l \Big\} \prod_{j=4}^n \mathbb{P}\Big\{ Z_{ij} < -u_i, Z_{lj} < -u_l \Big\} \\ &\quad \le \Big(\frac{(1+|\delta_{il}^h|)^2}{u^2} \varphi(u,u;|\delta_{il}^h|) \Big)^{n-2}. \end{aligned}$$

Similarly, we may consider all $(n-1)^2$ cases in (45) for $w_{i1}' = w_{l1}'$ and $w_{i1}' \ne w_{l1}'$. Therefore, using further (4.6) in [22] we have

$$\begin{aligned} E_{il} &\le (n-1)^2 \Big(\frac{(1+|\delta_{il}^h|)^2}{u^2} \varphi(u,u;|\delta_{il}^h|) \Big)^{n-2} \varphi(-u_i,-u_l;\delta_{il}^h) \\ &\le \frac{(n-1)^2}{(2\pi)^{n-1}} u^{-2(n-2)} \frac{(1+|\delta_{il}^h|)^{2(n-2)}}{(1-|\delta_{il}^h|^2)^{(n-1)/2}} \exp\left(-\frac{(n-1)u^2}{1+|\delta_{il}^h|} \right). \end{aligned}$$

Consequently, by (44) we have

$$\begin{aligned} \Delta_{(2)}(\boldsymbol{u}) \quad &\le \quad n \sum_{1 \le i < l \le d} (\sigma_{il}^{(1)} - \sigma_{il}^{(0)})_+ \int_0^1 E_{il}\, dh \\ &\le \quad \frac{n(n-1)^2}{(2\pi)^{n-1}} u^{-2(n-2)} \sum_{1 \le i < l \le d} (\sigma_{il}^{(1)} - \sigma_{il}^{(0)})_+ \exp\left(-\frac{(n-1)u^2}{1+\rho_{il}} \right) \int_0^1 \frac{(1+|\delta_{il}^h|)^{2(n-2)}}{(1-|\delta_{il}^h|^2)^{(n-1)/2}}\, dh \\ &= \quad \frac{n(n-1)^2}{(2\pi)^{n-1}} u^{-2(n-2)} \sum_{1 \le i < l \le d} (A_{il}^{(2)})_+ \exp\left(-\frac{(n-1)u^2}{1+\rho_{il}} \right). \end{aligned}$$

The last step follows since for $\delta_{il}^h = h(\sigma_{il}^{(1)} - \sigma_{il}^{(0)}) + \sigma_{il}^{(0)}$ we have $\rho_{il} = \max(|\sigma_{il}^{(0)}|, |\sigma_{il}^{(1)}|) \ge \delta_{il}^h$ and

$$\int_0^1 \frac{(1+|\delta_{il}^h|)^{2(n-2)}}{(1-|\delta_{il}^h|)^2)^{(n-1)/2}}\, dh = \frac{1}{\sigma_{il}^{(1)} - \sigma_{il}^{(0)}} \int_{\sigma_{il}^{(0)}}^{\sigma_{il}^{(1)}} \frac{(1+|h|)^{2(n-2)}}{(1-h^2)^{(n-1)/2}}\, dh. \tag{47}$$

<u>b) Proof of (9) for $2 < r \le n$.</u> By (8) and (36)

$$\Delta_{(r)}(\boldsymbol{u}) = n \sum_{1 \le i < l \le d} (\sigma_{il}^{(1)} - \sigma_{il}^{(0)}) \int_0^1 \widetilde{E}_{il}\, dh,$$

where $\widetilde{E}_{il} := \widetilde{E}_{il}(1,1)$. Clearly, from (43) we have $\widetilde{E}_{il} \ge 0$. Further, similar arguments as for E_{il} (consider the number of $w'_{it} = w'_{ls}, s,t < r$) yield that

$$\begin{aligned} \frac{\widetilde{E}_{il}}{\varphi(-u_i, -u_l; \delta^h_{il})} &\le \mathbb{P}\Big\{ \boldsymbol{Z}'_i \in \{w'_{i,r-1} \le n, w'_{ir} = \infty\}, \boldsymbol{Z}'_l \in \{w'_{l,r-1} \le n, w'_{lr} = \infty\} \Big\} \\ &\le (c_{n-1,r-1})^2 \Big(\frac{(1+|\delta^h_{il}|)^2}{u^2} \varphi(u,u;|\delta^h_{il}|) \Big)^{n-r}. \end{aligned}$$

Consequently, the claim in (9) for $2 < r \le n$ follows, establishing the proof. □

We give next a result which extends Lemma 2.3 in [28] needed for the proof of Proposition 2.5.

Lemma 4.1. *Let (X,Y) be a bivariate standard normal random vector with correlation $\rho \in (-1,1)$. For any $x, y > 0$, if $\rho < \max(x/y, y/x)$, then*

$$\mathbb{P}\{X > x, Y > y\} \le \frac{2(1+\rho)^2(1-\rho)}{(x+y)\min(x-\rho y, y - \rho x)} \varphi(x,y;\rho). \tag{48}$$

Proof of Lemma 4.1: The proof follows with similar arguments as in [28]. By a change of variable $x' = x + u/x, y' = y + v/y$, we have

$$\begin{aligned} \mathbb{P}\{X > x, Y > y\} &= \int_x^\infty \int_y^\infty \varphi(x',y';\rho)\, dx'dy' \\ &= \frac{\varphi(x,y;\rho)}{xy} \int_0^\infty \int_0^\infty \exp\left(-\frac{u(1-\rho y/x) + v(1-\rho x/y)}{1-\rho^2} \right) \exp\left(-\frac{(u/x)^2 - 2\rho(u/x)(v/y) + (y/v)^2}{2(1-\rho^2)} \right) dudv \\ &\le \frac{\varphi(x,y;\rho)}{xy} \int_0^\infty \int_0^\infty \exp\left(-\frac{(u/x)(x-\rho y) + (v/y)(y-\rho x)}{1-\rho^2} \right) \exp\left(-\frac{(u/x - y/v)^2}{2(1-\rho^2)} \right) dudv \\ &=: \frac{\varphi(x,y;\rho)}{xy} J(x,y,\rho). \end{aligned}$$

Next, let $s = \big((u/x)(x-\rho y) + (v/y)(y - \rho x)\big)/(1-\rho^2), t = (u/x - v/y)/\sqrt{1-\rho^2}$. Clearly,

$$\begin{vmatrix} \frac{\partial s(u,v)}{\partial u} & \frac{\partial s(u,v)}{\partial v} \\ \frac{\partial t(u,v)}{\partial u} & \frac{\partial t(u,v)}{\partial v} \end{vmatrix} = -\frac{x+y}{xy(1+\rho)\sqrt{1-\rho^2}}. \tag{49}$$

Further, since $\rho < \max(x/y, y/x)$, we have

$$\frac{s(1-\rho^2)}{\min(x-\rho y, y-\rho x)} \ge \frac{u}{x} + \frac{v}{y} \ge |t|\sqrt{1-\rho^2}, \quad -\infty < t < \infty.$$

Consequently, with $m_{x,y} := \min(x-\rho y, y - \rho x)/\sqrt{1-\rho^2}$

$$\begin{aligned} J(x,y,\rho) &\le \frac{xy(1+\rho)\sqrt{1-\rho^2}}{x+y} \int_{-\infty}^\infty e^{-t^2/2} \int_{|t|m_{x,y}}^\infty e^{-s}\, dsdt \\ &= \frac{2xy(1+\rho)\sqrt{1-\rho^2}}{x+y} \int_0^\infty \exp\left(-\frac{t^2}{2} - m_{x,y} t \right) dt \\ &= \frac{2xy(1+\rho)\sqrt{1-\rho^2}}{x+y} \frac{[1-\Phi(m_{x,y})]}{\varphi(m_{x,y})}, \end{aligned}$$

where $\varphi(x)$ and $\Phi(x)$ are the pdf and df of the standard normal random variable, respectively. Hence the well-known inequality

$$1 - \Phi(x) \le \varphi(x)/x, \quad x > 0 \tag{50}$$

establishes the proof. □

Proof of Proposition 2.5: We adopt the same notation as in Theorem 2.4. We have by Slepian's inequality

$$
\begin{aligned}
\mathbb{P}\Big\{Z_{ij} < -u_i, Z_{lk} < -u_l\Big\} &\le \mathbb{P}\Big\{\widetilde{Z}_i < -u_i, \widetilde{Z}_l < -u_l\Big\} \\
&\le \mathbb{P}\left\{\frac{\widetilde{Z}_i + \widetilde{Z}_l}{\sqrt{2(1+|\delta_{il}^h|)}} > \frac{u_i + u_l}{\sqrt{2(1+|\delta_{il}^h|)}}\right\} \\
&\le \frac{\sqrt{2(1+|\delta_{il}^h|)}}{u_i+u_l}\varphi\left(\frac{u_i+u_l}{\sqrt{2(1+|\delta_{il}^h|)}}\right). \qquad (51)
\end{aligned}
$$

Furthermore, it follows from Lemma 4.1 that

$$
\begin{aligned}
\mathbb{P}\Big\{Z_{ij} < -u_i, Z_{lk} < -u_l\Big\} &\le \mathbb{P}\Big\{-\widetilde{Z}_i > u_i, -\widetilde{Z}_l > u_l\Big\} \\
&\le \frac{2(1+|\delta_{il}^h|)^2(1-|\delta_{il}^h|)}{(u_i+u_l)\min(u_i - |\delta_{il}^h|u_l, u_l - |\delta_{il}^h|u_i)}\varphi(u_i, u_l; |\delta_{il}^h|) \\
&\le \frac{2(1+|\delta_{il}^h|)^2(1-|\delta_{il}^h|)}{(u_i+u_l)\min(u_i - \rho_{il}u_l, u_l - \rho_{il}u_i)}\varphi(u_i, u_l; |\delta_{il}^h|). \qquad (52)
\end{aligned}
$$

Hence (10) and (11) are established by replacing (46) with (51) and (52), respectively, and utilising similar arguments as in the proof of Theorem 2.4. □

Proof of Proposition 2.6: The lower bound follows directly from Corollary 2.3. Next we focus on the upper bound. We present below the proof for $r = 2$. Hereafter, we adopt the same notation as in the proof of Theorem 2.1. Further, define

$$
f(h) = \exp\left(\sum_{\substack{1\le i<l\le d\\ 1\le j,k\le n}} \frac{1}{H\big((u_i+u_l)/2\big)}\mathcal{C}_{ij,lk}^h\right), \quad h \in [0,1],
$$

where

$$
\mathcal{C}_{ij,lk}^h = \ln\left(\frac{\pi - 2\arcsin(\sigma_{ij,lk}^{(0)})}{\pi - 2\arcsin(\delta_{ij,lk}^h)}\right), \quad H(x) = \sqrt{2\pi}e^{x^2/2}\mathbb{E}\left\{(W+x)_+\right\},
$$

with W an $N(0,1)$ random variable. It suffices to show that $Q(\mathcal{Z};\Gamma^h)/f(h)$ is non-increasing in h, i.e.,

$$
\frac{\partial Q(\mathcal{Z};\Gamma^h)/\partial h}{Q(\mathcal{Z};\Gamma^h)} \le \frac{\partial f(h)/\partial h}{f(h)}, \quad h \in [0,1]. \qquad (53)
$$

We have

$$
\frac{\partial f(h)/\partial h}{f(h)} = \sum_{\substack{1\le i<l\le d\\ 1\le j,k\le n}} \frac{2(\sigma_{ij,lk}^{(1)} - \sigma_{ij,lk}^{(0)})}{\big(\pi - 2\arcsin(\delta_{ij,lk}^h)\big)\sqrt{1-(\delta_{ij,lk}^h)^2}}\frac{1}{H\big((u_i+u_l)/2\big)} \qquad (54)
$$

and by (22)

$$
\frac{\partial Q(\mathcal{Z};\Gamma^h)}{\partial h} = \sum_{\substack{1\le i<l\le d\\ 1\le j,k\le n}} (\sigma_{ij,lk}^{(1)} - \sigma_{ij,lk}^{(0)})E_{il}(j,k). \qquad (55)
$$

Therefore, by the assumption that $0 < \sigma_{ij,lk}^{(0)} \le \sigma_{ij,lk}^{(1)} < 1$ for $1 \le i < l \le d, 1 \le j,k \le n$, it is sufficient to show that

$$
E_{il}(j,k) \le \frac{2Q(\mathcal{Z};\Gamma^h)}{\big(\pi - 2\arcsin(\delta_{ij,lk}^h)\big)\sqrt{1-(\delta_{ij,lk}^h)^2}}\frac{1}{H\big((u_i+u_l)/2\big)}, \quad 1 \le i < l \le d, 1 \le j,k \le n. \qquad (56)
$$

From (34) we have (recall $\boldsymbol{u} = -\boldsymbol{\lambda}$)

$$\frac{E_{il}(j,k)}{\varphi(u_i,u_l;\delta^h_{ij,lk})} \le \mathbb{P}\left\{\cap_{s=1,s\neq i,l}^d \{Z_{s(n-1)} > \lambda_s\} \cap \{\boldsymbol{Z}_i' \in \{w_{i1}' \le n\}\} \cap \{\boldsymbol{Z}_l' \in \{w_{l1}' \le n\}\} \Big| \{Z_{ij} = \lambda_i, Z_{lk} = \lambda_l\}\right\}$$

$$= \mathbb{P}\left\{\cap_{s=1,s\neq i,l}^d \{Z_{s(2)} < u_s\} \cap \{\boldsymbol{Z}_i' \in \{v_{i1}' \le n\}\} \cap \{\boldsymbol{Z}_l' \in \{v_{l1}' \le n\}\} \Big| \{Z_{ij} = u_i, Z_{lk} = u_l\}\right\}, \tag{57}$$

where v_{i1}', v_{l1}' are defined by

$$v_{i1}' = \inf\{t : 1 \le t(\neq j) \le n, z_{it} < u_i\}, \quad v_{l1}' = \inf\{t : 1 \le t(\neq k) \le n, z_{lt} < u_l\}.$$

Define next

$$T_{ij} = \frac{(Z_{ij}-u_i) - \delta^h_{ij,lk}(Z_{lk}-u_l)}{1-(\delta^h_{ij,lk})^2}, \quad T_{lk} = \frac{(Z_{lk}-u_l) - \delta^h_{ij,lk}(Z_{ij}-u_i)}{1-(\delta^h_{ij,lk})^2}.$$

Since (Z_{ij}, Z_{lk}) is a bivariate Gaussian random vector with $N(0,1)$ marginals and correlation $\delta^h_{ij,lk}$, we have

$$\mathbb{E}\{T_{ij}Z_{ij}\} = \mathbb{E}\{T_{lk}Z_{lk}\} = 1, \quad \mathbb{E}\{T_{ij}Z_{lk}\} = \mathbb{E}\{T_{ij}Z_{lk}\} = 0.$$

Then it follows that the random vectors

$$\boldsymbol{Z}_v^* = (Z_{vw} - \delta^h_{vw,ij}T_{ij} - \delta^h_{vw,lk}T_{lk},\ 1 \le w \le n), \quad 1 \le v(\neq i,l) \le d,$$

$$\boldsymbol{Z}_i'^* = (Z_{it} - \delta^h_{it,ij}T_{ij} - \delta^h_{it,lk}T_{lk},\ 1 \le t(\neq j) \le n)$$

and

$$\boldsymbol{Z}_l'^* = (Z_{lt} - \delta^h_{lt,ij}T_{ij} - \delta^h_{lt,lk}T_{lk}, \quad 1 \le t(\neq k) \le n)$$

are independent of (Z_{ij}, Z_{lk}) and further independent of (T_{ij}, T_{lk}). Thus, by (57) and the fact that $0 \le \delta^h_{ij,lk} < 1$, $1 \le i < l \le d, 1 \le j,k \le n, h \in [0,1]$, we have as in Lemma 2.1 in [36]

$$\begin{aligned}
&E_{il}(j,k)\frac{\mathbb{P}\{T_{ij}<0, T_{lk}<0\}}{\varphi(u_i,u_l;\delta^h_{ij,lk})} \\
&\le \mathbb{P}\left\{\cap_{s=1,s\neq i,l}^d \{Z^*_{s(2)} < u_s\} \cap \{\boldsymbol{Z}_i'^* \in \{v_{i1}' \le n\}\} \cap \{\boldsymbol{Z}_l'^* \in \{v_{l1}' \le n\}\} \cap \{T_{ij}<0\} \cap \{T_{lk}<0\}\right\} \\
&\le \mathbb{P}\left\{\cap_{s=1,s\neq i,l}^d \{Z_{s(2)} < u_s\} \cap \{\boldsymbol{Z}_i' \in \{v_{i1}' \le n\}\} \cap \{\boldsymbol{Z}_l' \in \{v_{l1}' \le n\}\} \cap \{Z_{ij}<u_i\} \cap \{Z_{lk}<u_l\}\right\}
\end{aligned}$$

$$= Q(\mathcal{Z};\Gamma^h). \tag{58}$$

Moreover, by Lemma 2.2 in [36]

$$\frac{\mathbb{P}\{T_{ij}<0, T_{lk}<0\}}{\varphi(u_i,u_l;\delta^h_{ij,lk})} \ge \frac{\pi - 2\arcsin(\delta^h_{ij,lk})}{2}\sqrt{1-(\delta^h_{ij,lk})^2}H\left(\frac{u_i+u_l}{2}\right),$$

which together with (58) implies (56), hence the proof for $r=2$ is complete.
For $2 < r \le n$, we need to show that (56) holds for $\widetilde{E}_{il}(j,k)$. This follows by similar arguments as for $r=2$, using the inequality (43) instead of (34). □

5. Appendix

In the appendix we give detailed proofs of Theorem 3.1 and Proposition 3.4. Before proceeding to the proofs we present two lemmas which will be used in the proof of Theorem 3.1. For notational simplicity, we set $q = q(u) = u^{-2/\alpha}, u > 0$ and write $[x]$ for the integer part of $x \in \mathbb{R}$.

Lemma 5.1. *Under the assumptions of Theorem 3.1 with $\gamma = 0$, for any $a, T > 0$ we have*

$$\limsup_{u\to\infty} \sum_{j=[T/(aq)]}^{[\varepsilon/\mathbb{P}\{X_{n-r+1:n}(0)>u\}]} \mathbb{P}\left\{X_{n-r+1:n}(aqj) > u \Big| X_{n-r+1:n}(0) > u\right\} \to 0, \quad \varepsilon \downarrow 0. \tag{59}$$

Proof of Lemma 5.1: By Lemma 2 in [11] (see the proof of (3.5) therein), for sufficiently large u

$$a_u(t) := \mathbb{P}\left\{X_{n-r+1:n}(t) > u \Big| X_{n-r+1:n}(0) > u\right\} \le 2\mathbb{P}\left\{X_{1:r}(t) > u \Big| X_{1:r}(0) > u\right\}.$$

Since further $X(t) - \rho(t)X(0), t > 0$ is independent of $X(0)$, we have for some constant $K > 0$ (below the value of K might change from line to line)

$$\begin{aligned} a_u(t) &\le 2^{r+1}\left(\mathbb{P}\left\{X(t) > X(0) > u \Big| X(0) > u\right\}\right)^r \\ &\le 2^{r+1}\left(\mathbb{P}\left\{X(t) - \rho(t)X(0) > u(1-\rho(t)) \Big| X(0) > u\right\}\right)^r \\ &= 2^{r+1}\left(1 - \Phi\left(u\sqrt{\frac{1-\rho(t)}{1+\rho(t)}}\right)\right)^r \\ &\le Ku^{-r}\left(\frac{1-|\rho(t)|}{1+|\rho(t)|}\right)^{-r/2} \exp\left(-\frac{ru^2}{2}\frac{1-|\rho(t)|}{1+|\rho(t)|}\right), \end{aligned} \tag{60}$$

where the last inequality follows by (50).

Next, let $g(u)$ be a function such that $\lim_{u\to\infty} g(u) = \infty, |\rho(g(u))| = u^{-2}$. It follows from $u^{-2}\ln g(u) = o(1)$ that $g(u) \le \exp(\epsilon' u^2)$ for some (recall that $|\rho(T)| < 1$; see [21], p. 86)

$$0 < \epsilon' < r/2(1 - |\rho(T)|)/(1 + |\rho(T)|)$$

and sufficiently large u. Next, we split the sum in (59) at $aqj = g(u)$. The first term is

$$\begin{aligned} &\sum_{j=[T/(aq)]}^{[g(u)/(aq)]} \mathbb{P}\left\{X_{n-r+1:n}(aqj) > u \Big| X_{n-r+1:n}(0) > u\right\} \\ &\quad \le K\frac{g(u)}{aq}u^{-r}\left(\frac{1-|\rho(T)|}{1+|\rho(T)|}\right)^{-r/2} \exp\left(-\frac{ru^2}{2}\frac{1-|\rho(T)|}{1+|\rho(T)|}\right) \\ &\quad \le Ku^{2/\alpha - r}\exp\left(\epsilon' u^2 - \frac{ru^2}{2}\frac{1-|\rho(T)|}{1+|\rho(T)|}\right) \to 0, \quad u \to \infty. \end{aligned}$$

For the remaining term, it follows by Lemma 1 in [11] and (60)

$$\begin{aligned} &\sum_{j=[g(u)/(aq)]}^{[\varepsilon/\mathbb{P}\{X_{n-r+1:n}(0)>u\}]} \mathbb{P}\left\{X_{n-r+1:n}(aqj) > u \Big| X_{n-r+1:n}(0) > u\right\} \\ &\le K\frac{\varepsilon}{\mathbb{P}\{X_{n-r+1:n}(0) > u\}}u^{-r}\left(\frac{1-u^{-2}}{1+u^{-2}}\right)^{-r/2} \exp\left(-\frac{ru^2}{2}\frac{1-u^{-2}}{1+u^{-2}}\right) \\ &\le K\varepsilon\exp\left(-\frac{ru^2}{2}\left(\frac{1-u^{-2}}{1+u^{-2}} - 1\right)\right) \\ &\le K\varepsilon, \quad u \to \infty. \end{aligned}$$

Therefore, the claim follows by letting $\varepsilon \downarrow 0$. □

Next, with the notation as in (14) we set

$$T = T(u) = \frac{1}{c_{n,r}\mathcal{A}_{r,\alpha}}(2\pi)^{\frac{r}{2}}u^{r-\frac{2}{\alpha}}\exp\left(\frac{ru^2}{2}\right), \quad u > 0. \tag{61}$$

Lemma 5.2. *Let $T = T(u)$ be defined as in (61) and $a > 0, 0 < \lambda < 1$ be given constants. Under the assumptions of Lemma 5.1 for any $0 \le s_1 < \cdots < s_p < t_1 < \cdots < t_{p'}$ in $\{aqj : j \in \mathbb{Z}, 0 \le aqj \le T\}$ with $t_1 - s_p \ge \lambda T$*

$$\begin{aligned} &\Big|\mathbb{P}\left\{\cap_{i=1}^{p}\{X_{n-r+1:n}(s_i) \le u\}, \cap_{j=1}^{p'}\{X_{n-r+1:n}(t_j) \le u\}\right\} \\ &\quad -\mathbb{P}\left\{\cap_{i=1}^{p}\{X_{n-r+1:n}(s_i) \le u\}\right\}\mathbb{P}\left\{\cap_{j=1}^{p'}\{X_{n-r+1:n}(t_j) \le u\}\right\}\Big| \to 0, \quad u \to \infty. \end{aligned} \tag{62}$$

Proof of Lemma 5.2: Denote

$$X_{ij} = X_j(s_i)\mathbb{I}\{i \le p\} + X_j(t_{i-p})\mathbb{I}\{p < i \le p+p'\}, \quad 1 \le i \le p+p',\ 1 \le j \le n$$

and $\{Y_{ij}, 1 \le i \le p, 1 \le j \le n\} \overset{d}{=} \{X_{ij}, 1 \le i \le p, 1 \le j \le n\}$, independent of $\{Y_{ij}, p+1 \le i \le p+p', 1 \le j \le n\} \overset{d}{=} \{X_{ij}, p+1 \le i \le p+p', 1 \le j \le n\}$. Applying Theorem 2.4 with

$$X_{i(n-r+1)} = X_{n-r+1:n}(s_i)\mathbb{I}\{i \le p\} + X_{n-r+1:n}(t_{i-p})\mathbb{I}\{p < i \le p+p'\}$$

and

$$Y_{i(n-r+1)} = Y_{n-r+1:n}(s_i)\mathbb{I}\{i \le p\} + Y_{n-r+1:n}(t_{i-p})\mathbb{I}\{p < i \le p+p'\},$$

it follows that, the left-hand side of (62) is bounded from above by

$$\begin{aligned} &Ku^{-2(r-1)}\left(\frac{T}{q}\right)\sum_{\lambda T \le t_j - s_i \le T} \exp\left(-\frac{ru^2}{1+|\rho(t_j-s_i)|}\right)\int_0^{|\rho(t_j-s_i)|}\frac{(1+|h|)^{2(r-1)}}{(1-h^2)^{r/2}}\,dh \\ &\le Ku^{-2(r-1)}\left(\frac{T}{q}\right)\sum_{\lambda T \le aqj \le T}|\rho(aqj)|\exp\left(-\frac{ru^2}{1+|\rho(aqj)|}\right) \quad \text{for } u \text{ large}, \end{aligned} \tag{63}$$

where K is some constant. The rest of the proof consists of the similar arguments as that of Lemma 12.3.1 in [21]. Indeed, letting $\gamma(t) = \sup\{|\rho(s)|\ln s : s \ge t\}, t > 1$, we have that $|\rho(t)| \le \gamma(t)/\ln t$ and $\gamma(t) \le M$ for some positive constant M and all sufficiently large t. Recalling (61), we have

$$u^2 = \frac{2}{r}\ln T + \left(\frac{2}{r\alpha} - 1\right)\ln\ln T + \ln\left(\left(\frac{r}{2}\right)^{1-2/(r\alpha)}\frac{(c_{n,r}\mathcal{A}_{r,\alpha})^{2/r}}{2\pi}\right)(1+o(1)), \quad u \to \infty,$$

which implies that

$$\begin{aligned} \exp\left(-\frac{ru^2}{1+|\rho(aqj)|}\right) &\le \exp\left(-ru^2\left(1 - \frac{\gamma(\lambda T)}{\ln(\lambda T)}\right)\right) \\ &\le K\exp\left(-ru^2\right) \le KT^{-2}(\ln T)^{r-2/\alpha} \end{aligned}$$

for all T large. Consequently, the right-hand side of (63) is bounded from above by

$$\begin{aligned} &Ku^{-2(r-1)}\left(\frac{T}{q}\right)^2\left(\frac{1}{T/q}\sum_{\lambda T \le aqj \le T}|\rho(aqj)|\ln(aqj)\right)\frac{1}{\ln(\lambda T)}T^{-2}(\ln T)^{r-2/\alpha} \\ &\le K\frac{1}{T/q}\sum_{\lambda T \le aqj \le T}|\rho(aqj)|\ln(aqj), \end{aligned}$$

which tends to 0 as $T \to \infty$ since $\rho(t)\ln t = o(1)$. Hence the proof is complete. □

Below W denotes an $N(0,1)$ random variable which is independent of any other random element involved.

Proof of Theorem 3.1: a) Note that (14) and Lemmas 5.1 and 5.2 hold for the rth upper order statistics process $\{X_{n-r+1:n}(t), t \ge 0\}$. In view of Theorem 10 in [2] we have for $T = T(u)$ defined as in (61)

$$\lim_{u \to \infty}\mathbb{P}\left\{\sup_{t \in [0,T(u)]}X_{n-r+1:n}(t) \le u + \frac{x}{ru}\right\} = \exp\left(-e^{-x}\right), \quad x \in \mathbb{R}.$$

Expressing u in terms of T using (61) we obtain the required claim for any $x \in \mathbb{R}$, with $a_{r,T}, b_{r,T}$ given as in (15); the uniform convergence in x follows since all functions (with respect to x) are continuous, bounded and increasing.

b) The proof follows from the main arguments of Theorem 3.1 in [27] by showing that, for any $\varepsilon > 0$ and $x \in \mathbb{R}$

$$\begin{aligned} \Phi(x-\varepsilon) &\le \liminf_{T \to \infty}\mathbb{P}\left\{M_X(T) \le c_T b_{r,T} + \sqrt{\rho(T)}x\right\} \\ &\le \limsup_{T \to \infty}\mathbb{P}\left\{M_X(T) \le c_T b_{r,T} + \sqrt{\rho(T)}x\right\} \le \Phi(x+\varepsilon), \end{aligned} \tag{64}$$

where

$$M_X(T) := \sup_{t\in[0,T]} X_{n-r+1:n}(t), \quad c_T := \sqrt{1-\rho(T)}.$$

We start with the proof of the first inequality. Let $\rho^*(t), t \ge 0$ be a correlation function of a stationary Gaussian process such that $\rho^*(t) = 1 - 2\left|t\right|^\alpha + o(\left|t\right|^\alpha)$ as $t \to 0$. There exists some $t_0 > 0$ such that for T large

$$\rho^*(t)c_T^2 + \rho(T) \le \rho(t), \quad 0 \le t \le t_0. \tag{65}$$

Denote by $\{Y_k(t), t \ge 0\}, k \in \mathbb{N}$ independent centered stationary Gaussian processes with a.s. continuous sample paths and common covariance function $\rho^*(\cdot)$, and define $\{Y(t), t \ge 0\}$ by

$$Y(t) = \sum_{k=1}^{\infty} Y_k(t)\mathbb{I}\{t \in [(k-1)t_0, kt_0)\}, \quad t \ge 0. \tag{66}$$

It follows from (65) that for T sufficiently large

$$\mathbb{E}\left\{X(s)X(t)\right\} \ge \mathbb{E}\left\{\left(c_TY(s) + \sqrt{\rho(T)}W\right)\left(c_TY(t) + \sqrt{\rho(T)}W\right)\right\}, \quad s,t \ge 0.$$

Therefore, by Proposition 3.2

$$\begin{aligned}\mathbb{P}\left\{M_X(T) \le c_Tb_{r,T} + \sqrt{\rho(T)}x\right\} &\ge \mathbb{P}\left\{c_TM_Y(T) + \sqrt{\rho(T)}W \le c_Tb_{r,T} + \sqrt{\rho(T)}x\right\}\\ &\ge \Phi(x-\varepsilon)\left(\mathbb{P}\left\{\sup_{t\in[0,t_0]} Y_{n-r+1:n}(t) \le b_{r,T} + \varepsilon\sqrt{\rho(T)}\right\}\right)^{[T/t_0]+1}.\end{aligned}$$

Noting that $a = \inf_{0<t\le t_0}(1-\rho^*(t))\left|t\right|^\alpha > 0$, we have by Theorem 1.1 in [11] (see also (14))

$$\lim_{T\to\infty} \frac{\mathbb{P}\left\{\sup_{t\in[0,t_0]} Y_{n-r+1:n}(t) > b_{r,T} + \varepsilon\sqrt{\rho(T)}\right\}}{t_0c_{n,r}b_{r,T}^{2/\alpha}\left(1-\Phi(b_{r,T}+\varepsilon\sqrt{\rho(T)})\right)^r} = 2^{1/\alpha}\mathcal{A}_{r,\alpha}.$$

Consequently, since $\gamma = \infty$ we have

$$\begin{aligned}&\lim_{T\to\infty}([T/t_0]+1)\ln\mathbb{P}\left\{\sup_{t\in[0,t_0]} Y_{n-r+1:n}(t) \le b_{r,T} + \varepsilon\sqrt{\rho(T)}\right\}\\ &= -\lim_{T\to\infty}\frac{T}{t_0}\mathbb{P}\left\{\sup_{t\in[0,t_0]} Y_{n-r+1:n}(t) > b_{r,T} + \varepsilon\sqrt{\rho(T)}\right\}\\ &= -\lim_{T\to\infty} Tc_{n,r}2^{1/\alpha}\mathcal{A}_{r,\alpha}b_{r,T}^{2/\alpha}\left(1-\Phi(b_{r,T}+\varepsilon\sqrt{\rho(T)})\right)^r\\ &= 0\end{aligned}$$

establishing the first inequality in (64).

Next, we consider the last inequality in (64). Note that, by the convexity of $\rho(\cdot)$, there exists a separable stationary Gaussian process $\{Y(t), t \in [0,T]\}$ with correlation function given by (using the well-known Polya criteria, see e.g., [15])

$$\widetilde{\rho}(t) = \frac{\rho(t)-\rho(T)}{1-\rho(T)}, \quad t \in [0,T]. \tag{67}$$

We have the equality in distribution

$$M_X(T) \stackrel{d}{=} c_TM_Y(T) + \sqrt{\rho(T)}W$$

implying

$$\begin{aligned}\mathbb{P}\left\{M_X(T) \le c_Tb_{r,T} + \sqrt{\rho(T)}x\right\} &= \int_{-\infty}^{\infty}\mathbb{P}\left\{M_Y(T) \le b_{r,T} + \frac{\sqrt{\rho(T)}}{c_T}(x-u)\right\}\varphi(u)\,du\\ &\le \Phi(x+\varepsilon) + \mathbb{P}\left\{M_Y(T) \le b_{r,T} - \varepsilon\frac{\sqrt{\rho(T)}}{c_T}\right\}.\end{aligned} \tag{68}$$

Consquently, we only need to prove that

$$\lim_{T\to\infty} \mathbb{P}\left\{M_Y(T) \le b_{r,T} - \varepsilon\sqrt{\rho(T)}\right\} = 0.$$

To this end, using again the convexity of $\widetilde{\rho}(\cdot)$, we construct a separable stationary Gaussian process $\{Z(t), t \in [0,T]\}$ with the correlation function (recall $\widetilde{\rho}(\cdot)$ in (67))

$$\sigma(t) = \max\left(\widetilde{\rho}(t), \widetilde{\rho}\big(T\exp\big(-\sqrt{\ln T}\big)\big)\right), \quad t\in[0,T]. \tag{69}$$

Again by Proposition 3.2, we have

$$\mathbb{P}\left\{M_Y(T) \le b_{r,T} - \varepsilon\sqrt{\rho(T)}\right\} \le \mathbb{P}\left\{M_Z(T) \le b_{r,T} - \varepsilon\sqrt{\rho(T)}\right\}. \tag{70}$$

Next, we construct a grid of intervals as follows. Let $I_1, \ldots, I_{[T]}$ be $[T]$ consecutive unit intervals with an interval of length δ removed from the right-hand side of each one with $\delta\in(0,1)$ given, and

$$\mathcal{G}_T = \left\{k(2\ln T)^{-3/\alpha},\ k\in\mathbb{N}\right\} \cap \big(\cup_{i=1}^{[T]} I_i\big).$$

It follows from Theorem 10 in [2] and Theorem 1.1 in [11] that, $\sup_{t\in[0,T]} Z_{n-r+1:n}(t)$ and $\sup_{t\in\mathcal{G}_T} Z_{n-r+1:n}(t)$ have the same asymptotic distribution and thus we only need to show that

$$\lim_{T\to\infty} \mathbb{P}\left\{\sup_{t\in\mathcal{G}_T} Z_{n-r+1:n}(t) \le b_{r,T} - \varepsilon\sqrt{\rho(T)}\right\} = 0.$$

Let $\{Z'_{n-r+1:n}(t), t\ge 0\}$ be generated by $\{Z'(t), t\in[0,T]\}$ which is again a separable stationary process with the correlation function (recall $\sigma(\cdot)$ in (69))

$$\sigma^*(t) = \frac{\sigma(t)-\sigma(T)}{1-\sigma(T)}, \quad t\in[0,T].$$

Analogously to the derivation of (68) we obtain

$$\begin{aligned}
&\mathbb{P}\left\{\sup_{t\in\mathcal{G}_T} Z_{n-r+1:n}(t) \le b_{r,T} - \varepsilon\sqrt{\rho(T)}\right\}\\
&\quad = \mathbb{P}\left\{\sqrt{1-\sigma(T)}\max_{t\in\mathcal{G}_T} Z'_{n-r+1:n}(t) + \sqrt{\sigma(T)}W \le b_{r,T} - \varepsilon\sqrt{\rho(T)}\right\}\\
&\quad \le \Phi\left(-\frac{1}{2}\varepsilon\left(\frac{\rho(T)}{\sigma(T)}\right)^{1/2}\right) + \mathbb{P}\left\{\max_{t\in\mathcal{G}_T} Z'_{n-r+1:n}(t) \le b_{r,T} + \frac{b_{r,T}\sigma(T)}{\sqrt{1-\sigma(T)}(1+\sqrt{1-\sigma(T)})} - \frac{\varepsilon\sqrt{\rho(T)}}{2\sqrt{1-\sigma(T)}}\right\},
\end{aligned}$$

which tends to 0 as $T\to\infty$. The proof of it is the same as that of Theorem 3.1 in [27], by using instead Theorem 1.1 in [11] and our Theorem 2.4. Consequently, the last inequality in (64) follows by (68) and (70). We complete the proof for $\gamma=\infty$.

c) Given $\delta\in(0,1)$, take $I_1,\ldots,I_{[T]}$ as in b). For $\{Y_k(t), t\ge 0\}, k\in\mathbb{N}$ independent copies of X define

$$Y(t) := \sum_{k=1}^{\infty} Y_k(t)\mathbb{I}\{t\in[k-1,k)\}, \quad t\ge 0$$

and

$$X_*(t) := \sqrt{1-\rho_*(T)}Y(t) + \sqrt{\rho_*(T)}W, \quad t\in\cup_{k=1}^{[T]} I_k,$$

where $\rho_*(T) = \gamma/\ln T$. The rest of the proof is similar to that as of Theorem 2.1 in [34] by using our Theorem 2.4 instead of Berman's inequality. We omit the details.

Combining all the arguments for the three cases above, we complete the proof of Theorem 3.1. □

Proof of Proposition 3.4: First note that by (16) we have $\pi_{r,\alpha}(x), x\in\mathbb{R}$ defined by (with $Y_i(t) := X_i^*(t)$ for simplicity)

$$\pi_{r,\alpha}(x) = -\lim_{T\to\infty}\frac{1}{T}\mathbb{P}\left\{\sup_{0\le t\le T} Y_{r:n}(t) \le x\right\}, \quad x\in\mathbb{R}$$

exists and is left-continuous. Next, we show that $\pi_{r,\alpha}(x)$ is right-continuous, which will be crucial for our proof. As in Theorem 3.1 in [24] we first show that, for all $x \in \mathbb{R}, y > 0, m \geq 1, \theta \in (0,1)$ and $h \in (0, \infty)$

$$\mathbb{P}\left\{\sup_{0\le t\le mh} Y_{r:n}(t) \le x+y\right\} \le \Phi^{-m}\left(\frac{-y + x\left(\sqrt{1+a_{h,\theta}^2}-1\right)}{a_{h,\theta}}\right)\mathbb{P}\left\{\sup_{0\le t\le \theta mh} Y_{r:n}(t) \le x\right\}. \tag{71}$$

Let therefore $W_k, 1 \le k \le m$ be independent $N(0,1)$ random variables which are further independent of X_i^*, $1 \le i \le n$, and write, for simplicity, $a = a_{h,\theta}$. We have

$$\begin{aligned}
p_{h,\theta}(x,Y) &:= \mathbb{P}\left\{\max_{1\le k\le m}\sup_{(k-1)h\le t\le kh}\frac{Y_{r:n}(t,aW_k)}{\sqrt{1+a^2}} \le x\right\}\\
&\ge \mathbb{P}\left\{\max_{1\le k\le m}\sup_{(k-1)h\le t\le kh} Y_{r:n}(t) \le x+y\right\}\mathbb{P}\left\{\max_{1\le k\le m} aW_k \le -y + x\big(\sqrt{1+a^2}-1\big)\right\}\\
&= \mathbb{P}\left\{\max_{1\le k\le m}\sup_{(k-1)h\le t\le kh} Y_{r:n}(t) \le x+y\right\}\Phi^m\left(\frac{-y+x\big(\sqrt{1+a^2}-1\big)}{a}\right),
\end{aligned}$$

where $\{Y_{r:n}(t, aW_k), t \in [(k-1)h, kh]\}$ is the rth order statistics process generated by $\{Y_i(t) + aW_k, t \in [(k-1)h, kh)\}, 1 \le i \le n$. Furthermore, it follows by (17) and the monotonicity of $\rho(\cdot)$ that (set $I_k = [(k-1)h, kh)$)

$$\begin{aligned}
&\frac{\mathbb{E}\left\{(Y_1(t)+aW_{[t/h]+1})(Y_1(s)+aW_{[s/h]+1})\right\}}{1+a^2} - \mathbb{E}\left\{Y_1(\theta t)Y_1(\theta s)\right\}\\
&= \begin{cases} \frac{1-\rho(|t-s|)}{1+a^2}\left(a^2 - \frac{\rho(\theta|t-s|)-\rho(|t-s|)}{1-\rho(|t-s|)}\right), & t,s\in I_k,\\ \frac{\rho(|t-s|)}{1+a^2} - \rho(\theta\,|t-s|), & t\in I_k, s\in I_l, k\ne l\end{cases}\\
&\le 0,
\end{aligned}$$

which by Proposition 3.2 implies that

$$p_{h,\theta}(x,Y) \le \mathbb{P}\left\{\max_{1\le k\le m}\sup_{(k-1)h\le t\le kh} Y_{r:n}(\theta t) \le x\right\}$$

establishing (71). Now, let $f(s,x) := \ln \mathbb{P}\left\{\sup_{0\le t\le s} Y_{r:n}(t) \le x\right\}, s > 0, x \in \mathbb{R}$. We have from (71) that, for any $y > 0$

$$\frac{f(mh,x)}{mh} \le \frac{f(mh,x+y)}{mh} \le \theta\frac{f(\theta mh,x)}{\theta mh} - \frac{\ln\Phi\left(\frac{-y}{a_{h,\theta}} + A_{h,\theta}x\right)}{h},$$

with $A_{h,\theta} = \frac{a_{h,\theta}}{\sqrt{1+a_{h,\theta}^2}+1}$. Letting first $m \to \infty$ and then $y \downarrow 0$ in the above we have

$$\pi_{r,\alpha}(x) \ge \lim_{y\downarrow 0}\pi_{r,\alpha}(x+y) \ge \theta\pi_{r,\alpha}(x) + \frac{\ln\Phi(A_{h,\theta}x)}{h}. \tag{72}$$

Furthermore, since

$$a_{h,\theta}^2 = \inf_{0<t\le h}\frac{\rho(\theta t)-\rho(t)}{1-\rho(t)} \le \inf_{0<t\le 1}\frac{\rho(\theta t)-\rho(t)}{1-\rho(t)}$$

holds for any $h > 1$, we conclude that $\lim_{\theta\uparrow 1} A_{h,\theta} = 0$. Thus, letting $\theta \uparrow 1$ and $h \to \infty$ in (72) we obtain the right-continuity of $\pi_{r,\alpha}(x)$.

To complete the proof, it suffices to show that

$$\begin{aligned}
-\frac{2}{\alpha}\pi_{r,\alpha} &\le \liminf_{x\downarrow 0}\frac{\ln\mathbb{P}\left\{\sup_{0\le t\le 1}X_{r:n}(t)\le x\right\}}{\ln(1/x)}\\
&\le \limsup_{x\downarrow 0}\frac{\ln\mathbb{P}\left\{\sup_{0\le t\le 1}X_{r:n}(t)\le x\right\}}{\ln(1/x)} \le -\frac{2}{\alpha}\pi_{r,\alpha}.
\end{aligned} \tag{73}$$

By the self-similarity of process $\{X_{r:n}(t), t \ge 0\}$, we have for any $x \in (0,1)$

$$\begin{aligned}
\mathbb{P}\left\{ \sup_{0\le t\le 2/\alpha \ln(1/x)} Y_{r:n}(t) \le 0 \right\} &= \mathbb{P}\left\{ \sup_{x^{2/\alpha}\le t\le 1} X_{r:n}(t) \le 0 \right\} \\
&\le \mathbb{P}\left\{ \sup_{x^{2/\alpha}\le t\le 1} X_{r:n}(t) \le x \right\} \\
&\le \frac{\mathbb{P}\left\{ \sup_{0<t\le 1} X_{r:n}(t) \le x \right\}}{\mathbb{P}\left\{ \sup_{0<t\le x^{2/\alpha}} X_{r:n}(t) \le x \right\}} \\
&= \frac{\mathbb{P}\left\{ \sup_{0<t\le 1} X_{r:n}(t) \le x \right\}}{\mathbb{P}\left\{ \sup_{0<t\le 1} X_{r:n}(t) \le 1 \right\}},
\end{aligned}$$

where the second inequality follows from Proposition 3.2 and the fact that $\mathbb{E}\{X(s)X(t)\} \ge 0$ for all $s,t \ge 0$. Consequently, the lower bound in (73) follows. Next, we establish the upper bound in (73). It follows that, for $y > 0$ sufficiently small

$$\begin{aligned}
&\frac{1}{(\alpha/2)h} \ln \mathbb{P}\left\{ \sup_{0\le t\le h} Y_{r:n}(t) \le y \right\} \\
&= \frac{1}{\alpha h/2} \ln \mathbb{P}\left\{ \sup_{e^{-h}\le t\le 1} (t^{-\alpha/2} X_{r:n}(t)) \le y \right\} \\
&\ge \frac{1}{\alpha h/2} \ln \mathbb{P}\left\{ \sup_{e^{-h}\le t\le 1} X_{r:n}(t) \le y e^{-\alpha h/2} \right\} \\
&\ge \frac{1}{\alpha h/2} \ln \mathbb{P}\left\{ \sup_{0\le t\le 1} X_{r:n}(t) \le y e^{-\alpha h/2} \right\} \\
&= \frac{\alpha h/2 - \ln y}{\alpha h/2} \frac{1}{\ln(1/(y e^{-\alpha h/2}))} \ln \mathbb{P}\left\{ \sup_{0\le t\le 1} X_{r:n}(t) \le y e^{-\alpha h/2} \right\}.
\end{aligned}$$

Letting $h \to \infty$ in the above we obtain that

$$\limsup_{x\downarrow 0} \frac{\ln \mathbb{P}\left\{ \sup_{0\le t\le 1} X_{r:n}(t) \le x \right\}}{\ln(1/x)} \le -\frac{2}{\alpha}\pi_{r,\alpha}(y) \to -\frac{2}{\alpha}\pi_{r,\alpha}, \quad y \downarrow 0,$$

where the last step follows by the right-continuity of $\pi_{r,\alpha}(x)$ at 0. Consequently, (73) holds and thus the proof is complete. ☐

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