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Spacetime Symmetries in Quantum Mechanics

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14.1 Introduction

In the last decades, the philosophy of physics has begun to pay attention to the meaning and the role of symmetries, an issue that has, however, had a great relevance in physics since, at least, the middle of the twentieth century. Notwithstanding this fact, this increasing interest in symmetries has not yet been transferred to the field of the interpretation of quantum mechanics. Although it is usually accepted that the Galilean group is the group of invariance of the theory, discussions about interpretations of quantum mechanics, with very few exceptions, have not taken into account symmetry considerations. But the invariance of a theory under a group does not guarantee the invariance of its interpretations, as they usually add interpretive assumptions to the formal structure of the theory. Symmetry considerations should thus be seriously taken into account in the field of the interpretation of quantum mechanics.

For this reason, in this chapter we shall focus on the spacetime symmetries of quantum mechanics. After briefly introducing certain terminological clarifications, we shall focus on two aspects of spacetime transformations. First, we shall consider the behavior of nonrelativistic quantum mechanics under the Galilean group, aiming at assessing its Galilean invariance in relation to interpretive concerns. Second, we shall analyze the widely-accepted view about the invariance of the Schrödinger equation under time reversal, in order to unveil some implicit assumptions underlying such a claim.

14.2 The Concepts of Invariance and Covariance

The meaning of the term ‘symmetry’ is rooted in ordinary language: Symmetry is a geometrical property of a body whose parts are equal in a certain sense. In mathematics, the term acquires a precise meaning in terms of *invariance* – an object is symmetric with respect to a certain transformation when it is invariant under such transformation; that is, it remains unchanged under its application. The concept of group, originally proposed by Galois in the first half of the nineteenth century, comes to supplement the notion of symmetry – a group clusters different transformations into a specific structure.

Despite its mathematical precision, in physics, the concept of symmetry has given rise to some disagreements on the meaning and the scope of the concept of invariance and of the closely-related concept of covariance. Commonly, the property of invariance only applies to mathematical objects and, derivatively, to the physical items to which they refer, and the property of covariance is reserved for equations and, derivatively, for the physical laws they express. However, some authors claim that the difference between invariance and covariance not only makes sense but is also relevant when applied to laws (Ohanian and Ruffini 1994, Suppes 2000, Brading and Castellani 2007). Hans Ohanian and Remo Ruffini (1994), for instance, claim that an equation is said to be covariant when its form is left unaltered under a certain transformation, and it is said to be invariant when it is covariant and its content, that is, its absolute objects (constants and nondynamical quantities) are also left unchanged by the transformation. Although inspired in this idea, we will not follow it in every detail. Here we will consider that an equation is invariant under a certain transformation when it does not change under the application of such transformation, and it is covariant under that transformation when its form is left unchanged by it (Suppes 2000). From this perspective, the invariance of a law does not imply the invariance of the objects contained in its representing equation.

Once one accepts that the concept of invariance makes sense in its application to laws, the conceptual implications both of the invariance of the law and of the involved objects under a particular group of transformations deserve to be considered. Moreover, when a law is covariant under a transformation and all the objects it contains are also invariant under the same transformation, the law is invariant under the transformation as well. Nevertheless, this is not the only way for a law to be invariant – if a law is covariant under a certain transformation, it can turn out to be invariant under the transformation even in the case that some of the objects it contains are not invariant under the same transformation (we will come back to this point in the next section, when discussing the invariance of the Schrödinger equation).

On the basis of these conceptual clarifications, some formal definitions can now be introduced.

Def. 1 Let us consider a set $\mathcal{A}$ of objects $a_i \in \mathcal{A}$, and a group G of transformations $g_\alpha \in G$, where the $g_\alpha : \mathcal{A} \to \mathcal{A}$ act upon the a_i as $a_i \to \tilde{a}_i$. An *object* $a_i \in \mathcal{A}$ is *invariant under the transformation* g_α if, for that transformation, $\tilde{a}_i = a_i$. In turn, the *object* $a_i \in \mathcal{A}$ is *invariant under the group* G if it is invariant under all the transformations $g_\alpha \in G$.

In physics, the objects to which transformations apply are usually those representing states s, observables O, and differential operators D, and each transformation acts upon them in a particular way. In turn, those objects are combined in equations representing the laws of a theory. Then,

> ***Def. 2*** Let L be a law represented by an equation $E(s, O_i, D_j) = 0$, where s represents a state, the O_i represent observables, and the D_j represent differential operators, and let G be a group of transformations $g_\alpha \in G$ acting upon the objects involved in the equation as $s \to \tilde{s}$, $O_i \to \tilde{O}_i$, and $D_j \to \tilde{D}_j$. L is *covariant under the transformation* g_α if $E(\tilde{s}, \tilde{O}_i, \tilde{D}_j) = 0$, and L is *invariant under the transformation* g_α if $E(\tilde{s}, O_i, D_j) = 0$. Moreover, L is *covariant – invariant – under the group* G if it is covariant – invariant – under all the transformations $g_\alpha \in G$.

On this basis, it is usually said that a certain group is the symmetry group of a theory:

> ***Def. 3*** A group G of transformations is said to be the *symmetry group* of a theory if the laws of the theory are covariant under the group G.

This means that the laws preserve their validity even when the transformations of the group are applied to the involved objects.

Some authors prefer to talk about symmetry instead of covariance. This is the case of John Earman (2004), who defines symmetry in the language of model theory:

> ***Def. 4*** Let $\mathcal{M}$ be the set of the models of a certain mathematical structure, and let $\mathcal{M}_L \subset \mathcal{M}$ be the subset of the models satisfying the law L. A symmetry of the law L is a map $S : \mathcal{M} \to \mathcal{M}$ that preserves $\mathcal{M}_L$, that is, for any $m \in \mathcal{M}_L$, $S(m) \in \mathcal{M}_L$.

When L is represented by a differential equation $E(s, O_i, D_j) = 0$, each model $m \in \mathcal{M}_L$ is represented by a solution $s = F(O_i, s_0)$ of the equation, corresponding to a possible evolution of the system. Then, the covariance of L under a transformation g – that is, the fact that $E(\tilde{s}, \tilde{O}_i, \tilde{D}_j) = 0$ – implies that if $s = F(O_i, s_0)$ is a solution of the equation, $\tilde{s} = \tilde{F}(\tilde{O}_i, s_0)$ is also a solution and, as a consequence, it represents a model $S(m) \in \mathcal{M}_L$. This means that the definition of covariance given by Def. 2 and the definition of symmetry given by Def. 4 are equivalent.

In turn, the covariance of a dynamical law – represented by a differential equation – does not imply the invariance of the possible evolutions – represented by the solutions of the equation. In fact, the covariance of the law L, represented by the equation $E(s, O_i, D_j) = 0$, implies that $s = F(O_i, s_0)$ and $\tilde{s} = \tilde{F}(\tilde{O}_i, s_0)$ are both solutions of the equation, but does not imply that $s = \tilde{s}$. In the model-theory language, the symmetry of L does not imply that $S(m) = m$. By contrast, invariance is a stronger property of the law: The invariance of L means that $E(\tilde{s}, O_i, D_j) = 0$; in this case $s = \tilde{s} = F(O_i, s_0)$ or, in the model language, $S(m) = m$.

Def. 5 Let $\mathcal{M}$ be the set of the models of a certain mathematical structure, and let $\mathcal{M}_L \subset \mathcal{M}$ be the subset of the models satisfying the law L. Let a transformation be a map $S : \mathcal{M} \to \mathcal{M}$ that preserves $\mathcal{M}_L$. The law L is invariant under the transformation S if, for any $m \in \mathcal{M}_L$, $S(m) = m$.

The general definitions just described can be applied to the Schrödinger equation so as to explicitly state the conditions of covariance and invariance for quantum mechanics. Here we will focus on the evolution equation of the theory, leaving aside the collapse postulate, since it is an interpretive postulate in orthodox quantum mechanics. Given a transformation g acting as $|\varphi\rangle \to |\tilde{\varphi}\rangle$, $O \to \tilde{O}$, $d/dt \to \tilde{d}/dt$, and $i \to \tilde{i}$ (considering i as the shorthand for the operator iI), by making $\hbar = 1$ the Schrödinger equation is *covariant* under g when

$$\frac{\tilde{d}|\tilde{\varphi}\rangle}{dt} = -\tilde{i}\tilde{H}|\tilde{\varphi}\rangle \tag{14.1}$$

and it is *invariant* under g when

$$\frac{d|\tilde{\varphi}\rangle}{dt} = -iH|\tilde{\varphi}\rangle. \tag{14.2}$$

14.3 Quantum Mechanics and the Galilean Group

14.3.1 The Galilean Group

As time is represented by the variable $t \in \mathbb{R}$ and position is represented by the variable $\boldsymbol{r} = (x, y, z) \in \mathbb{R}^3$, the Galilean group $\mathcal{G} = \{T_\alpha\}$, with $\alpha = 1$ to 10, is a group of continuous spacetime transformations $T_\alpha : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3 \times \mathbb{R}$ such that

- $t \to t' = t + \tau$ (time-displacement)
- $\boldsymbol{r} \to \boldsymbol{r}' = \boldsymbol{r} + \boldsymbol{\rho}$ (space-displacement)
- $\boldsymbol{r} \to \boldsymbol{r}' = R_\theta \boldsymbol{r}$ (space-rotation)
- $\boldsymbol{r} \to \boldsymbol{r}' = \boldsymbol{r} + \boldsymbol{u}t$ (velocity-boost)

where $\tau \in \mathbb{R}$ is a real number representing a time interval, $\boldsymbol{\rho} = (\rho_x, \rho_y, \rho_z) \in \mathbb{R}^3$ is a triple of real numbers representing a space interval, $R_\theta \in \mathcal{M}^{3\times3}$ is a 3×3 matrix representing a space rotation by an angle θ, and $\boldsymbol{u} = (u_x, u_y, u_z) \in \mathbb{R}^3$ is a triple of real numbers representing a constant velocity.

For the Galilean group, $\mathcal{G}$ is a Lie group, the Galilean transformations T_α can be represented by unitary operators U_α over the Hilbert space, with the exponential parametrization $U_\alpha = e^{iK_\alpha s_\alpha}$, where s_α is a continuous parameter and K_α is a hermitian operator independent of s_α, the *generator* of the transformation T_α. Then, $\mathcal{G}$ is defined by 10 group generators K_α: one time-displacement K_τ, three space-displacements K_{ρ_i}, three space-rotations K_{θ_i}, and three velocity-boosts K_{u_i}, with $i = x, y, z$. Therefore, by taking $\hbar = 1$ as usual, the Galilean group is defined by the commutation relations between its generators:

$$
\begin{array}{ll}
\text{(a)}\ \left[K_{\rho_i}, K_{\rho_j}\right] = 0 & \text{(f)}\ \left[K_{u_i}, K_{\rho_j}\right] = i\delta_{ij}M \\
\text{(b)}\ \left[K_{u_i}, K_{u_j}\right] = 0 & \text{(g)}\ \left[K_{\rho_i}, K_\tau\right] = 0 \\
\text{(c)}\ \left[K_{\theta_i}, K_{\theta_j}\right] = i\varepsilon_{ijk}K_{\theta_j} & \text{(h)}\ \left[K_{\theta_i}, K_\tau\right] = 0 \\
\text{(d)}\ \left[K_{\theta_i}, K_{\rho_j}\right] = i\varepsilon_{ijk}K_{\rho_k} & \text{(i)}\ \left[K_{u_i}, K_\tau\right] = iK_{\rho_i} \\
\text{(e)}\ \left[K_{\theta_i}, K_{u_j}\right] = i\varepsilon_{ijk}K_{u_k} &
\end{array}
\tag{14.3}
$$

where ε_{ijk} is the Levi-Civita tensor. Strictly speaking, in the case of quantum mechanics the symmetry group is the group corresponding to the central extension of the Galilean algebra, obtained as a semi-direct product between the Galilean algebra and the algebra generated by a central charge, which in this case is the mass operator $M = mI$, where I is the identity operator and m is the mass. The mass operator as a central charge is a consequence of the projective representation of the Galilean group (see Bose 1995, Weinberg 1995). However, in order to simplify the presentation, we will use the expression "Galilean group" from now on to refer to the corresponding central extension.

In a closed, constant-energy system free from external fields, the generators K_α are given by the basic magnitudes of the theory: the energy $H = \hbar K_\tau$, the three momentum components $P_i = \hbar K_{\rho_i}$, the three angular momentum components $J_i = \hbar K_{\theta_i}$, and the three boost components $G_i = \hbar K_{u_i}$. Then, in this case the commutation relations turn out to be

$$
\begin{array}{ll}
\text{(a)}\ \left[P_i, P_j\right] = 0 & \text{(f)}\ \left[G_i, P_j\right] = i\delta_{ij}M \\
\text{(b)}\ \left[G_i, G_j\right] = 0 & \text{(g)}\ \left[P_i, H\right] = 0 \\
\text{(c)}\ \left[J_i, J_j\right] = i\varepsilon_{ijk}J_k & \text{(h)}\ \left[J_i, H\right] = 0 \\
\text{(d)}\ \left[J_i, P_j\right] = i\varepsilon_{ijk}P_k & \text{(i)}\ \left[G_i, H\right] = iP_i \\
\text{(e)}\ \left[J_i, G_j\right] = i\varepsilon_{ijk}G_k &
\end{array}
\tag{14.4}
$$

The rest of the physical magnitudes can be defined in terms of these basic ones. For instance, the three position components are $Q_i = G_i/m$, the three orbital angular momentum components are $L_i = \varepsilon_{ijk}Q_jP_k$, and the three spin components are

$S_i = J_i - L_i$. In the Hilbert formulation of quantum mechanics, each Galilean transformation $g_\alpha \in \mathcal{G}$ acts upon states and upon observables as

$$|\varphi\rangle \to |\tilde{\varphi}\rangle = U_{s_\alpha}|\varphi\rangle = e^{iK_\alpha s_\alpha}|\varphi\rangle \tag{14.5}$$

$$O \to \tilde{O} = U_{s_\alpha} O U_{s_\alpha}^{-1} = e^{iK_\alpha s_\alpha} O e^{-iK_\alpha s_\alpha} \tag{14.6}$$

The invariance of an observable O under a Galilean transformation T_α amounts to the commutation between O and the corresponding generator K_α:

$$\tilde{O} = e^{iK_\alpha s_\alpha} O e^{-iK_\alpha s_\alpha} = O \quad \Leftrightarrow \quad [O, K_\alpha] = 0 \tag{14.7}$$

It is worth bearing in mind that there are operators that are invariant under all the transformations of the group, and thereby, commute with all the generators of the group – the Casimir operators of a group. In the case of the Galilean group, the Casimir operators are the internal energy $W = H - P^2/2M$, the square of total spin $S^2 = \left(J - \frac{1}{M} G \times P\right)^2$, and the mass M, which are multiples of the identity in any irreducible representation.

14.3.2 The Covariance of the Schrödinger Equation

Given the Schrödinger equation, let us begin by (i) premultiplying its two members by $U = e^{iKs}$, (ii) adding and subtracting $(dU/dt)|\varphi\rangle$ to its first member, and (iii) using the property $U^{-1}U = I$:

$$U\frac{d|\varphi\rangle}{dt} + \frac{dU}{dt}|\varphi\rangle - \frac{dU}{dt}U^{-1}U|\varphi\rangle = -UiU^{-1}UHU^{-1}U|\varphi\rangle \tag{14.8}$$

Then, by recalling the transformations of states and observables of Eq. (14.5) and Eq. (14.6), we obtain

$$\frac{d|\tilde{\varphi}\rangle}{dt} - \frac{dU}{dt}U^{-1}|\tilde{\varphi}\rangle = -\tilde{i}\tilde{H}|\tilde{\varphi}\rangle \tag{14.9}$$

This shows that covariance obtains when the time-derivative operator transforms as

$$\frac{d}{dt} \to \frac{\tilde{d}}{dt} = \frac{D}{Dt} = \frac{d}{dt} - \frac{dU}{dt}U^{-1} \Rightarrow \frac{\tilde{d}|\tilde{\varphi}\rangle}{dt} = -\tilde{i}\tilde{H}|\tilde{\varphi}\rangle \tag{14.10}$$

This means that the transformed differential operator $\tilde{d}/dt$ is a covariant time-derivative D/Dt, which makes the Schrödinger equation to be Galilean-covariant in the sense of Eq. (14.1).

In a closed, constant-energy system free from external fields, H is time-independent and the P_i and the J_i are constants of motion (see Eq. (14.4g) and

Eq. (14.4h)). Then, for time-translations, space-translations and space-rotations, $dU/dt = de^{iKs}/dt = 0$, where K and s stand for H and τ, P_i and ρ_i, and J_i and θ_i, respectively. As a consequence, the time-derivative is invariant under time-displacements, space-displacements, and space-rotations (see Eq. (14.10)): $d/dt \rightarrow \tilde{d}/dt = d/dt$. But for boost-transformations this is not the case: The covariance of the Schrödinger equation implies the transformation of the differential operator as $d/dt \rightarrow D/Dt$. This means that covariance under boosts amounts to a sort of "nonhomogeneity" of time, which requires the covariant adjustment of the time-derivative. This conclusion should not be surprising since, when the system is described in a reference frame $\tilde{F}$ at uniform motion corresponding to a velocity u_x with respect to the original frame F, the boost-transformed state depends on a generator that is a linear function of time:

$$G_x = mQ_x = m(Q_{x0} + V_x t) = mQ_{x0} + P_x t \tag{14.11}$$

Then, if the Schrödinger equation is to be valid in $\tilde{F}$, where the state is $|\tilde{\varphi}\rangle$, the transformed time-derivative has to be adjusted to compensate the time-depending transformation of the state.

14.3.3 The Invariance of the Schrödinger Equation

As we have seen in the previous section, in a closed, constant-energy system free from external fields, H is time-independent and the P_i and the J_i are constants of motion. Then, for time-translations, space-translations, and space-rotations, it follows that $dU/dt = de^{iKs}/dt = 0$ and $d/dt \rightarrow \tilde{d}/dt = d/dt$. Moreover, for those transformations, $\tilde{i} = i$ follows trivially, and $\tilde{H} = H$ because (see Eq. (14.7)) (i) $[H, H] = 0$, (ii) $[P_i, H] = 0$ (Eq. (14.4g)), and (iii) $[J_i, H] = 0$ (Eq. (14.4h)). When these results apply to Eq. (14.9), it is easy to see that the Schrödinger equation is invariant under time-displacements, space-displacements, and space-rotations in the sense of Eq. (14.2).

The case of boost-transformations is different from the previous cases because, although $\tilde{i} = i$ still holds, the Hamiltonian is not boost-invariant even when the system is free from external fields (the same happens in classical mechanics, see Butterfield 2007: 6). In fact, under a boost-transformation corresponding to a velocity u_x, H changes as (see Eq. (14.4i): $[G_x, H] = iP_x \neq 0$)

$$\tilde{H} = e^{iG_x u_x} H e^{-iG_x u_x} \neq H \tag{14.12}$$

Since G_x is not time-independent, $dU/dt = de^{iG_x u_x}/dt \neq 0$, and Eq. (14.9) yields

$$\frac{d|\tilde{\varphi}\rangle}{dt} = -i\left[\tilde{H} + i\frac{de^{iG_x u_x}}{dt}e^{-iG_x u_x}\right] |\tilde{\varphi}\rangle \tag{14.13}$$

In order to know the value of the bracket in the right-hand side (r.h.s.) of Eq. (14.13), the two terms in the bracket must be computed. When the task is performed, it can be proved that the terms added to H in $\tilde{H}$ cancel out with those coming from the term containing the time-derivative (see Lombardi, Castagnino, and Ardenghi 2010: appendices). Therefore, Eq. (14.2) is again obtained and the invariance of the Schrödinger equation is proved to hold also for boost-transformations.

The case of boost-transformations illustrates a claim previously mentioned in Section 14.2: Even though a law is invariant under a transformation when it is covariant and all the involved objects are invariant, this is not the only way to obtain invariance. When the quantum system is free from external fields, the Schrödinger equation is invariant under boost-transformations, in spite of the fact that the Hamiltonian and the differential operator d/dt are not boost-invariant objects.

14.3.4 Galilean Group and External Fields

As explained in the previous subsection, when there are no external fields acting on the system, the Hamiltonian is invariant under time-displacements, space-displacements, and space-rotations, but not under boost-transformations. Despite this fact, the Schrödinger equation is completely invariant under the Galilean group, and this conceptually means that the state vector $|\varphi\rangle$ does not "see" the effect of the transformations – the evolutions of $|\varphi\rangle$ and $|\tilde{\varphi}\rangle$ are identical. In other words, the time-behavior of the system is independent of the reference frame used for the description.

When the system is under the action of external fields, the fields modify the evolution of the system. But, in nonrelativistic quantum mechanics, fields are not quantized: They do not play the role of quantum systems that interact with other systems. For this reason, the effect of the fields on a system must be included in its Hamiltonian, because it is the only observable involved in the time-evolution law. It can be proved that the most general form of the Hamiltonian in the presence of external fields is (see, e.g., Ballentine 1998)

$$H = \frac{(P - A(Q))^2}{2M} + V(Q) \tag{14.14}$$

where $A(Q)$ is a vector potential and $V(Q)$ is a scalar potential. The covariance of the Schrödinger equation, as expressed in Eq. (14.9), fixes the way in which the potentials $A(Q)$ and $V(Q)$ must transform under the Galilean group. The electromagnetic field may be derived from a vector potential and a scalar potential; thus, a fully Galilean-covariant quantum theory of the Schrödinger field interacting with an external electromagnetic field is possible. However, the electric and magnetic fields that transform as required to preserve Galilean covariance, although related to the scalar and vector potentials in the usual way, are ruled by one of two sets of electromagnetic "field" equations. Those equations can be considered the nonrelativistic limits of Maxwell's equations in cases where either (i) magnetic effects predominate over electric ones ("magnetic limit": $c|B| >> |E|$), or (ii) electric effects predominate over magnetic ones ("electric limit": $c|B| << |E|$) (see Brown and Holland 1999, Colussi and Wickramasekara 2008). Nevertheless, $A(Q)$ and $V(Q)$ should not necessarily be identified with the electromagnetic potentials, because they are arbitrary functions that need not satisfy Maxwell's equations; for example, the Newtonian gravitational potential can also be included in the scalar $V(Q)$ (Ballentine 1998).

At this point, a relevant issue must be stressed. Space-displacements and space-rotations are purely geometric operations of displacing and rotating the system self-congruently to another place and to another direction, respectively. Analogously, time-displacements are purely geometric operations of displacing the system self-congruently to another time, and they may agree or not with dynamical evolutions. The commutation of two transformation generators means that the corresponding geometric operations can be performed in either order with the same result; for instance, the commutation $[K_{\rho_i}, K_{\rho_j}] = 0$ (see Eq. (14.3a)) means that the order in which space-displacements in different directions are performed does not modify the result. In particular, the validity of the Galilean group implies that time-displacements commute both with space-displacements and with space-rotations (see Eq. (14.3g) and Eq. (14.3h)). When there are no external fields acting on the system, this feature is given by the commutation relations involving the Hamiltonian, the three momentum components P_i, and the three angular momentum components J_i, $[P_i, H] = 0$ and $[J_i, H] = 0$ (Eq. (14.4g) and Eq. (14.4h)), because here the Hamiltonian *is* the time-displacement generator. But in the presence of external fields, since the action of the fields is incorporated into the Hamiltonian of the system, *the Hamiltonian is no longer the generator of time-displacements*: It only retains its role as the generator of the dynamical evolution (see Laue 1996, Ballentine 1998). For this reason, the commutation with the momentum components and with the angular momentum components gets broken: $[P_i, H] \neq 0$ and $[J_i, H] \neq 0$. However, to the extent that the covariance of the Schrödinger equation is retained, the commutation of time-displacements with both space-displacements and space-rotations still holds, and is still represented by $\left[K_{\rho_i}, K_\tau\right] = 0$ and $[K_{\theta_i}, K_\tau] = 0$, respectively (see Eq. (14.3g) and Eq. (14.3h)), where the momentum components are still the generators of space-displacements, $P_i = \hbar K_{\rho_i}$, and the angular momentum components are still the generators of space-rotations, $J_i = \hbar K_{\theta_i}$, but the Hamiltonian is no longer the generator of time-displacements, $H \neq \hbar K_\tau$ (we will come back to this point in the next section, about time-reversal invariance).

14.3.5 The Relevance to Interpretation

In principle, there are two possible interpretations of a transformation: active and passive. Under the active interpretation, the transformation corresponds to a change from a system to the transformed system; for instance, a system translated in space with respect to the original one. Under the passive interpretation, the transformation consists in a change of the viewpoint – the reference frame – from which the system is described; for instance, the space-translation of the observer that describes the system. In the case of continuous spacetime transformations, both active and passive interpretation are equally allowed; but such a situation is not so clear in the case of discrete transformations. In general, it is accepted that only the active interpretation makes sense in the case of discrete transformations (Sklar 1974: 363). Nevertheless, no matter which interpretation is adopted, the covariance of the fundamental law of a theory under its continuous symmetry group implies that the law still holds when the transformations are applied. In the active interpretation language, the original and the transformed systems are equivalent; in the passive interpretation language, the original and the transformed reference frames are equivalent.

As is typically accepted, the Galilean group is the symmetry group of continuous spacetime transformations of classical and quantum mechanics. In the language of the passive interpretation, the covariance of the dynamical laws amounts to the equivalence among inertial reference frames (time-translated, space-translated, space-rotated, or uniformly moving with respect to each other). In other words, Galilean transformations do not introduce any modification in the physical situation, but only express *a change in the perspective from which the system is described*.

These remarks are related to the fact that certain quantities are *physically irrelevant* in the light of theory's symmetries. For instance, the space-translation symmetry of a dynamical law means that the specific place where the system is located in space is irrelevant to its evolution governed by such law: "A global symmetry reflects the irrelevance of absolute values of a certain quantity: only relative values are relevant" (see Brading and Castellani 2007: 1360). In classical mechanics, for example, space-translation invariance implies that absolute position is irrelevant to the system's behavior – the equations of motion do not depend on absolute positions, only relative positions matter. The physical irrelevance of certain quantities is strongly linked with the issue of *objectivity*.

The intuition about a strong link between invariance and objectivity is rooted in a natural idea: What is objective should not depend on the particular perspective used for the description. When this intuition is translated into the group-theoretical language, it can be said that what is objective according to a theory is what is invariant under the symmetry group of the theory. This idea appeared in the domain of formal sciences in Felix Klein's "Erlangen Program" of 1872, with the attempt to characterize all known geometries by their invariants (see Kramer 1970). This idea passed to physics with the advent of relativity, regarding the ontological status of space and time (Minkowski 1923). The claim that objectivity means invariance becomes a main thesis of Hemann Weyl's book *Symmetry* (1952). Max Born also very clearly expressed his conviction about the strong link between invariance and objectivity: "I think the idea of invariance is the clue to a rational concept of reality" (Born 1953: 144). In recent times, the idea has strongly reappeared in several works. For instance, in her deep analysis of quantum field theory, Sunny Auyang (1995) makes her general concept of "object" to be founded on its invariance under transformations among all representations. The assumption of invariance as the root of objectivity is also the central theme of Robert Nozick's book *Invariances: The Structure of the Objective World* (2001). In the same vein, David Baker (2010) has argued that symmetries are a guide to finding out which quantities represent fundamental natural properties in a physical theory.

If the ontological meaning of symmetries is accepted, it is easy to see that symmetries must play an active role in the understanding of a physical theory. In the particular case of quantum mechanics, the consideration of its Galilean covariance cannot be overlooked in the discussions about interpretation.

As it is well known, the Kochen-Specker theorem (Kochen and Specker 1967) establishes a barrier to any realist classical-like interpretation of quantum mechanics: It proves the impossibility of ascribing definite values to *all* the physical quantities (observables) of a quantum system simultaneously, while preserving the functional relations between commuting observables. This result is a manifestation of the *contextuality* of quantum mechanics – the ascription of definite values to the observables of a quantum system is always contextual. As a consequence of the Kochen-Specker theorem, any realist interpretation of quantum mechanics is committed to selecting a subset of definite-valued observables from the set of all the observables of the system (or a preferred basis from all the formally equivalent bases of the Hilbert space). The observables of that subset will be those that acquire definite values without violating quantum contextuality. It is at this point that the symmetry group of the theory becomes a leading character. As noticed by Harvey Brown, Mauricio Suárez, and Guido Bacciagaluppi (1998), any interpretation that selects the set of the definite-valued observables of a quantum system in a given state is committed to considering how that set is transformed under the Galilean group.

However, now the link between invariance and objectivity comes into play. The study of the role of symmetries is particularly pressing in the case of realist interpretations of quantum mechanics, which conceive a definite-valued observable as a physical magnitude that *objectively* acquires an actual definite value among all its possible values: The fact that a certain observable acquires a definite value should be an objective fact that should not depend on the descriptive perspective. Therefore, the set of the definite-valued observables of a system picked out by the interpretation should be left invariant by the Galilean transformations. From a realist viewpoint, it would be unacceptable that such a set changed as result of a mere change in the perspective from which the system is described.

In his article "Aspects of objectivity in quantum mechanics," Harvey Brown (1999) explicitly tackles the problem in discussing the objectivity of "sharp values." In particular, he focuses on interpretations that specify state-dependent rules for assigning sharp values to some of the self-adjoint operators representing quantum magnitudes, such as the interpretations whose value-assignment rules coincide with the eigenstate-eigenvalue link, or the modal interpretations that make the set of definite-valued observables to depend on the instantaneous state of the system. Brown clearly explains the difference between the classical and the quantum case. In classical mechanics, Galilean noninvariant magnitudes modify their values with the change of reference frame; for this reason, if their objectivity is to be retained, they must be regarded not as intrinsic properties but as relational properties. For instance, the values of classical position and momentum can be conceived as relational properties that link the system and the reference frame. In quantum mechanics, by contrast, the relational nature acquires a further degree; whereas, in the classical case the sharp value of a magnitude depends on the reference frame, in the quantum case the *very sharpness* of an observable's value must be relational in order to preserve its objectivity. For instance, the fact that the position of a system has a sharp (definite) value in a certain reference frame, and, as a consequence, that the system can be conceived as a localized particle, is itself relational. In a different reference frame, the system may have an unsharp value of position and, then, may behave as a delocalized particle. Brown also correctly stresses that it is not just boosts that produce this kind of situation; passive spatial translations can cause that some sharp-valued observables to become unsharp. On the basis of this analysis, he concludes that

> If, in the hope of providing an ontological interpretation of quantum mechanics, we introduce *state-dependent* rules for assigning sharp values to magnitudes associated with a specific quantum system, we should recognise that the objective status of such sharp values is relational, not absolute.
>
> (Brown 1999: 66–67)

In the same article, Brown considers an interpretation in which the rule of definite-value ascription is not state-dependent; he analyzes the problem of covariance in the de Broglie-Bohm pilot-wave interpretation of quantum mechanics, but he discards it because, although no privileged frame is picked out by the hidden dynamics of the corpuscles, the forces acting upon the corpuscles and generated by the guiding wave are Aristotelian, not Newtonian, because they produce velocities, not accelerations. This seems to lead him to interpretations that introduce state-dependent rules of definite-value ascription as the only alternative. This maneuver is opposed to that which led Jeffrey Bub (1997) to advocate for Bohmian mechanics, conceived as a modal interpretation whose rule of definite-value ascription picks out the position observable: The difficulties of the original modal interpretations to deal with nonideal measurements due to their state-dependent rules turns Bohmian mechanics into a natural alternative. But what both Bub and Brown seem to overlook is that there are other interpretive strategies beyond Bohmian mechanics and traditional modal interpretations that make the definite-valued observables to depend on the state of the system. One of them is even more natural than the Bohmian proposal when the aim is to preserve the objectivity of definite-valuedness (or of sharpness, in Brown's terms) in the light of Galilean symmetry. In fact, the natural way to reach this goal, without making the objective status of definite-valuedness relational, is to appeal to the Casimir operators of the Galilean group: If the interpretation has to select a Galilean-invariant set of definite-valued observables, such a set must depend on those Casimir operators, insofar as they are invariant under all the transformations of the Galilean group. An interpretation that has adopted this interpretive strategy is the modal-Hamiltonian interpretation in its Galilean invariant version (Ardenghi, Castagnino, and Lombardi 2009, Lombardi, Castagnino, and Ardenghi 2010), which has been successfully applied to many well-known physical situations and has proved to be effective for solving the measurement problem, both in its ideal and its nonideal versions (Lombardi and Castagnino 2008).

Considering that the Casimir operators of the Galilean group represent the definite-valued observables of a quantum system has the advantage of being very general. When the system is free from external fields and the Galilean group is defined by the commutation relations Eq. (14.4), the Casimir operators correspond to the observables mass M, squared-spin S^2, and internal energy W. Yet the Casimir operators of the Galilean group can always be defined, even when Eq. (14.4) does not hold. The group must thus be defined in the completely general way as expressed by Eq. (14.3). Therefore, when there are external fields applied to the system, and such fields do not break the covariance of quantum mechanics under the Galilean group, the strategy of defining the definite-valued observables in terms of the Casimir operators remains valid, whatever they represent.

Furthermore, the strategy admits a further generalization in its application to relativistic quantum theories, such as relativistic quantum mechanics and quantum field theory: The interpretive postulate of endowing the Casimir operators with definite-valuedness and objectivity is retained, but the relevant group structure is replaced by changing the Galilean group by the Poincaré group.

14.4 Quantum Mechanics and Time Reversal

14.4.1 A General Notion of Time Reversal

What was said in the previous section seems not to straightforwardly apply to the question about time-reversal symmetry. To begin with, time reversal is a transformation that does not belong to the Galilean group. Furthermore, while all Galilean transformations are continuous, time reversal is a discrete transformation that, at first glance, simply performs the transformation $t \to -t$. Notwithstanding these facts, time reversal encloses even more subtle features as it is somehow related to the nature of time and its unavoidable differences with respect to space; whereas one can move freely in all directions of space, it seems that one "moves" in just one direction of time, from past to future and never the other way around. From early twentieth century, the very notion of time reversal was quite relevant not only for many physicists working on the foundations of physics, but also for many philosophers aiming at getting a grasp of the nature of time.

Unfortunately, time is not the kind of thing one can experiment on: Unlike electrons, pendulums or electromagnetic fields, one cannot directly find out time's properties by running an experiment. However, physicists and philosophers have managed this setback by devising a formal way to dig into time's properties – the notions of time reversal and time-reversal invariance have been keystones for the famous *problem of the arrow of time* in physics (problem lying on the borderline between the physics and the metaphysics of time). In fact, the arrow of time has largely been introduced in terms of time-reversal symmetry: If physical laws somehow fail to be time-reversal invariant, then one might come to the conclusion that time is headed according to such a physical law. To put it differently, time-reversal symmetry is supposed to shed light on the structure of time according to a given theory. Quoting Jill North:

> If the fundamental laws cannot be formulated without reference to a particular kind of structure, then this structure must exist in order to support the laws – "support" in the sense that the laws could not be formulated without making reference to that structure.
>
> (North 2009: 203)

The idea is that, by knowing how dynamical equations (standing for physical laws) behave under time reversal, one can learn about the nature of time according to a theory. This kind of principle is well-seated in the literature (see also Earman 1974, Sklar 1974, Arntzenius 1997), and it is the key element that links time's properties, physical laws, and the time-reversal transformation.

However, here nothing has yet been said about the time-reversal transformation in itself, other than that it minimally performs the transformation $t \rightarrow -t$. As typically noted in the literature, the topic is somewhat tricky as there is no shared understanding of what time reversal is exactly supposed to do nor of what properties it should instantiate (see Savitt 1996 for a careful analysis of varied time-reversal operators; see Peterson 2015 for an updated approach). Furthermore, time reversal's properties seem to change from theory to theory, to the extent that dynamics also changes. This remark already assumes a strong premise – that the structure of time, in particular, regarding its time-reversal symmetry property, is closely tied up to the theory's dynamics.

14.4.2 Time Reversal in Quantum Mechanics

Independently of the considerations discussed previously, the community of physicists has reached a wide consensus about the appropriate time-reversal transformation for standard quantum mechanics. The traditional procedure starts out by arguing that time reversal can no longer be considered as in Hamiltonian classical mechanics. As it is well known, the time-reversal operator in classical contexts is typically defined as the operator that changes the sign of the variable t. But in quantum mechanics the rationale seems to be different; in fact, Ballentine (1998) warns us:

> One might suppose that time reversal would be closely analogous to space inversion, with the operation $t \rightarrow -t$ replacing $x \rightarrow -x$. In fact, this simple analogy proves to be misleading at almost every step.
>
> (Ballentine 1998: 377)

Quantum mechanics textbooks rarely offer a thorough justification for such a claim and commonly go on by formally introducing the "proper" way to reverse time in quantum mechanics. In some cases, the only justification is based on a classical analogy: The transformation $t \rightarrow -t$ does not lead to the transformation of momentum as $P \rightarrow -P$, which is expected because this is the way in which momentum transforms under time reversal in classical mechanics. But mere analogy does not seem to be a sufficiently good argument; for this reason, Bryan Roberts (2017) has very recently brought up an updated and purely quantum-mechanic-based reasoning for defending the standard procedure. Here we will not analyze in detail those arguments; rather, we will consider the problem in the light of the symmetries of the Schrödinger equation related to the reversal of time.

Let us use θ to call a generic time-reversal operator, which performs at least the transformation $t \to -t$ but whose precise form is not defined yet. As explained in Section 14.2, this operator acts as $|\varphi\rangle \to |\tilde{\varphi}\rangle$, $O \to \tilde{O}$, $d/dt \to \tilde{d}/dt$, and $i \to \tilde{i}$; in particular, $|\tilde{\varphi}\rangle = \theta|\varphi\rangle$, $\tilde{O} = \theta O \theta^{-1}$, $\tilde{d}/dt = \theta d/dt\, \theta^{-1}$, and $\tilde{i} = \theta i \theta^{-1}$. The Schrödinger equation is *covariant/invariant* under θ when Eq. (14.2)/(14.3) holds, respectively. So, let us begin by premultiplying the two members of the Schrödinger equation by θ and using the property $\theta^{-1}\theta = I$:

$$\theta\left(\frac{d}{dt}\right)\theta^{-1}\theta|\varphi\rangle = -\theta i\theta^{-1}\theta H\theta^{-1}\theta|\varphi\rangle \tag{14.15}$$

As long as θ is not a function of t, it is easy to prove that the Schrödinger equation is covariant under the application of θ:

$$\frac{\tilde{d}|\tilde{\varphi}\rangle}{dt} = -\tilde{i}\tilde{H}|\tilde{\varphi}\rangle \tag{14.16}$$

Now, in order to know if the Schrödinger equation is also invariant under the application of θ, it is necessary to define the precise form of θ to see how it acts upon d/dt, i, and H. As in the case of the Galilean group, the situation of a closed, constant-energy system will be considered.

Case (*i*): If $\theta = T$ only performs the transformation $t \to -t$, then

$$\frac{\tilde{d}}{dt} = T\left(\frac{d}{dt}\right)T^{-1} = -\frac{d}{dt} \qquad \tilde{i} = TiT^{-1} = i \qquad \tilde{H} = THT^{-1} = H \tag{14.17}$$

Introducing these equations into Eq. (14.16) leads to the conclusion that the Schrödinger equation is not T-invariant, since

$$\frac{d|\tilde{\varphi}\rangle}{dt} = iH|\tilde{\varphi}\rangle \tag{14.18}$$

Case (*ii*): If $\theta = T^*$ performs the transformation $t \to -t$ and the complex conjugation $i \to -i$, then

$$\frac{\tilde{d}}{dt} = T^*\left(\frac{d}{dt}\right)T^{*-1} = -\frac{d}{dt} \qquad \tilde{i} = T^*iT^{*-1} = -i \qquad \tilde{H} = T^*HT^{*-1} = H \tag{14.19}$$

Introducing these equations into Eq. (14.16) leads to the conclusion the Schrödinger equation is T^*-invariant, since

$$\frac{d|\tilde{\varphi}\rangle}{dt} = -iH|\tilde{\varphi}\rangle \tag{14.20}$$

Summing up, independently of any interpretation, the results given by Eq. (14.18) and Eq. (14.20) show that the Schrödinger equation is not invariant under the unitary operator T, and it is invariant under the antiunitary operator T^*. The conceptual question is which of the two operators, T or T^*, represents the operation of time reversal.

14.4.3 *Time-Reversal Invariance: Between Petitio Principii* and *a Priori* Truth

A common answer to that conceptual question is saying that the unitary operator T is unacceptable as the time-reversal operator because it breaks the requirement that the energy of the system must be bounded from below. In Jun John Sakurai's words:

> Consider an energy eigenket $|n\rangle$ with energy eigenvalue E_n. The corresponding time-reversed state would be $\Theta|n\rangle$ [where Θ stands for our T], and we would have, because of (4.4.27) [$-H\Theta = \Theta H$]
>
> $$H\Theta|n\rangle = -\Theta H|n\rangle = (-E_n)\Theta|n\rangle \tag{4.4.28}$$
>
> This equation says that $\Theta|n\rangle$ is an eigenket of the Hamiltonian with energy eigenvalues $-E_n$. But this is nonsensical even in the very elementary case of a free particle. We know that the energy spectrum of the free particle is positive semidefinite – from 0 to +∞. There is no state lower than a particle at rest (momentum eigenstate with momentum eigenvalue zero); the energy spectrum ranging from $-\infty$ to 0 would be completely unacceptable.
>
> (Sakurai 1994: 272–273)

Why does the unitary operator T not meet that requirement? It is not unusual to read that the reason is that the unitary operator T transforms the Hamiltonian as $THT^{-1} = -H$. This sounds very strange because, by performing only the transformation $t \to -t$, the operator T should leave the time-independent Hamiltonian invariant. So, now the question is: Why does the Hamiltonian transform as $H \to -H$? Although not always explicit, a typical answer is that offered by Stephen Gasiorowicz:

> we find that [the equation of motion] *can be invariant* only if
>
> $$THT^{-1} = -H$$
>
> This, however, is an unacceptable condition, because time reversal cannot change the spectrum of H, which consists of positive energies only. If T is taken to be anti-unitary [our T^*], the * operator changes the i to $-i$ [in the equation of motion] and the trouble does not occur.
>
> (Gasiorowicz 1966: 27; italics added)

In a similar vein, Robert Sachs clearly explains that

> we require that the [time-reversal] transformations leave the equations of motion *invariant* when all forces or interactions vanish.
>
> (Sachs 1987: 7; italics added)

In other words, under time reversal, the Hamiltonian should transform as $H \to -H$ in order to preserve the time-reversal invariance of the Schrödinger equation; for this reason, the unitary T is unacceptable as time-reversal operator, and the right one is the antiunitary T^*.

At this point, one seems to be caught in the dilemma between *petitio principii* and *a priori* truth. If our original question was whether the Schrödinger equation is time-reversal invariant, an argument that selects the right operator describing time reversal by taking the time-reversal invariance of the equation as one of its premises clearly begs the question. However, some authors do not fall in *petitio principii* by considering that certain symmetries of the physical laws have an *a priori* status:

> A symmetry can be a priori, i.e., the physical law is built in such a way that it respects that particular symmetry by construction. This is exemplified by spacetime symmetries, because spacetime is the theater in which the physical law acts … and must therefore respect the rules of the theater.
>
> (Dürr and Teufel 2009: 43–44)

From this perspective, the invariance under the Galilean group must be built into the Schrödinger equation due to the homogeneity and the isotropy of space and the homogeneity of time. This view may sound reasonable to the extent that those are features of space and time that we, in a certain sense, can experience. But, why should we impose time-reversal invariance? We have no experience of the isotropy of time since we cannot travel backwards in time. Despite this, time-reversal invariance must be introduced as a postulate:

> One should ask why we are so keen to have this feature in the fundamental laws when we experience the contrary. Indeed, we typically experience thermodynamic changes which are irreversible, i.e., which are not time reversible. The simple answer is that our platonic idea (or mathematical idea) of time and space is that they are without preferred direction, and that the "directed" experience we have is to be explained from the underlying time symmetric law.
>
> (Dürr and Teufel 2009: 47)

Challenging the most widely-held position about time reversal in the field of quantum mechanics, a few authors have raised their voices against it by appealing to philosophical reasons: The antiunitary operator T^* would fail to offer a conceptually sound and clear-cut representation of time reversal. On the one hand, a far-reaching tradition, which tracks back to the work of Giulio Racah (1937) and Satosi Watanabe (1955), pleads for a unitary time-reversal operator in quantum theories. Oliver Costa de Beauregard (1980) has argued for such a view by claiming that a unitary time-reversal operator that merely reverses the direction of time by flipping the sign of the variable t goes more naturally along with relativistic contexts and is more naturally akin to the Feynmann zig-zag philosophy. On the other hand, philosophers such as Craig Callender (2000) and David Albert (2000) have claimed that the Schrödinger equation should actually be considered as nontime-reversal invariant, since it is not invariant under T, which more fairly represents what one means by "reversing the direction of time." Without any further ado, Jill North claims:

> What is a time-reversal transformation? Just a flipping of the direction of time! That is all there is to a transformation that changes how things are with respect to time: change the direction of time itself.
>
> (North 2009: 212)

For these philosophers, if the time-reversal invariance of a theory will tell us something about the structure of time, time reversal should only reverse the direction of time without extra additions. In particular, if the question at issue is the problem of the arrow or time and the time-reversal invariance of the theory is considered relevant to this problem, imposing the time-reversal invariance of the Schrödinger equation as a requirement that the theory must satisfy is a circular strategy.

14.4.4 Wigner's Definition

The line of argumentation sketched in the previous section seems to be not completely convincing for adopting the antiunitary operator T^* as the adequate representation of time reversal. Yet a thorough argument can be introduced by appealing to the authority of Eugene Wigner, who defines time reversal as a transformation such that:

> The following four operations, carried out in succession on an arbitrary state, will result in the system returning to its original state. The first operation is time inversion, the second time displacement by *t*, the third again time inversion, and the last on again time displacement by *t*.
>
> (Wigner 1931/1959: 326)

In other words,

time reversal $\times$ *displacement by* Δt $\times$ *time reversal* $\times$ *displacement by* Δt = *identity*

This requirement is commonly interpreted in formal terms as follows:

$$U_{\Delta t}\theta^{-1}U_{\Delta t}\theta s = s \tag{14.21}$$

where s is the arbitrary state and $U_{\Delta t}$ is the evolution operator for Δt. This requirement is precisely Sakurai's starting point in the argument that led him to the conclusion stated previously. Under the assumption that the evolution operators form a group (an assumption that is not always satisfied, see Bohm and Gadella [1989], where the time evolution is represented by a semigroup), a $U_{-\Delta t}$ exists such that $U_{-\Delta t} U_{\Delta t} = I$. In this case, Eq. (14.21) becomes

$$\theta^{-1} U_{\Delta t} \theta s = U_{-\Delta t} s \tag{14.22}$$

In quantum mechanics, the argument continues, $U_{\Delta t} = e^{-iH\Delta t}$:

$$\theta^{-1} e^{-iH\Delta t} \theta |\varphi\rangle = e^{iH\Delta t} |\varphi\rangle \tag{14.23}$$

Therefore,

$$\theta^{-1}(-iH)\theta = iH \tag{14.24}$$

So, when θ is the unitary operator T, this leads to

$$-H = THT^{-1} \tag{14.25}$$

which is unacceptable because it leads to values of energy that are unbounded from below. Since Wigner (1931/1959) also proved that any symmetry transformation is represented by a unitary or an antiunitary operator, the argument concludes that the right time-reversal operator is the antiunitary operator T^*.

This argument in favor of T^* is certainly much better than the previous one, which takes the time-reversal invariance of the Schrödinger equation as one of the premises. Nevertheless, as we will see, this second argument imposes the time-reversal invariance of the dynamical equation beforehand as well, even if in a more subtle way.

Let us begin by noticing that, when Eq. (14.21) is used to formalize Wigner's requirement, the "*displacement by* Δt" is represented by the time evolution of the system by Δt according to the dynamical law of the theory – the Schrödinger equation, here expressed as $s = U_{\Delta t} s_0$. However, as stressed in Section 14.3.4, when considering the Galilean group, *time displacement is not time evolution*. Time evolution is ruled by the dynamical law of the theory, in this case quantum mechanics: According to the Schrödinger equation, the Hamiltonian is the generator of the dynamical evolution. By contrast, spacetime transformations are purely geometric operations of displacing or rotating the system self-congruently. In particular, time-displacement is a purely geometric operation that displaces the system self-congruently to another time, and may agree or not with time evolution. In fact, as Hans Laue (1996) and Leslie Ballentine (1998) stress, in a generic case, the Hamiltonian is not the generator of time displacements and only retains its role as the generator of the dynamical evolution. This clearly shows that time displacement and time evolution are different concepts. Hence, insofar as

Wigner's definition involves time displacement and not time evolution, it must be formally expressed as (recall how Galilean transformations act upon states, Eq. (14.5))

$$U_\tau \theta^{-1} U_\tau \theta s = e^{iK_\tau \Delta t} \theta^{-1} e^{iK_\tau \Delta t} \theta s = s \tag{14.26}$$

where K_τ is the generator of time-displacement (see Eq. (14.3)). Now, since $U_{-\tau}$ exists such that $U_{-\tau} U_\tau = I$, an analogue of Eq. (14.23) can be obtained:

$$\theta^{-1} e^{iK_\tau \Delta t} \theta |\varphi\rangle = e^{-iK_\tau \Delta t} |\varphi\rangle \tag{14.27}$$

where the difference in sign between Eq. (14.27) and Eq. (14.22) is due to the inverse relation between transformations on function space and transformations on coordinates (see Ballentine 1998: 67). Eq. (14.27) expresses what time reversal means: It should be a transformation such that

time reversal $\times$ *displacement by* Δt $\times$ *time reversal* = *displacement by* $-\Delta t$

More explicitly, take an arbitrary state, time-reverse it, time-displace it by Δt in a given time-displacement direction, and time-reverse it again; these three operations must be equivalent to time-displace the original state the same time interval Δt in the opposite time-displacement direction. Eq. (14.27) represents formally this condition, but no conclusion about how the Hamiltonian is transformed by time reversal can be drawn from it.

Even if accepting the conceptual difference between time displacement and time evolution in Wigner's definition of time reversal, somebody might retort by saying that there are cases in which time displacement amounts to time evolution: as discussed in Section 14.3, in those cases the generator K_τ of time displacement is equal to the generator H of time evolution. If K_τ is replaced by H in Eq. (14.27), when θ is the unitary operator T the relation $-H = THT^{-1}$ of Eq. (14.25) obtains again, and this is sufficient to discard the unitary operator T as the adequate representation of time reversal.

Although the argument just discussed seems to be conclusive, when considered in detail, the implicit assumptions come to light. In fact, the argument equates time displacement and time evolution, both in the case of Δt and in the case of $-\Delta t$. We have good empirical reasons to accept that, in certain cases, the time displacement toward the future by Δt is equivalent to the time evolution given by $U_{\Delta t} = e^{-iH\Delta t}$. But we do not know how the system would evolve in time toward the past, as we have no experience at all of such an evolution; this is the specific feature that makes time so different than space. If we impose that, when time displacement is time evolution toward the future, this is the case toward the past too, then we are introducing the time-reversal invariance of the dynamical law by hand.

In other words, the question about the time-reversal invariance of a law is precisely the question of whether the time displacement of the system toward the past is also ruled by the dynamical law, that is, whether it is also a time evolution. If the answer is positive, the law is time-reversal invariant, if the answer is negative, the law is not time-reversal invariant. Therefore, supposing from the very beginning that any time displacement toward the past is a dynamical evolution amounts to putting the cart before the horse.

But, then, what do Eq. (14.21) and Eq. (14.22) mean? Actually, those equations express the conditions that define what can be called *motion reversal*:

$$\textit{motion reversal} \times \textit{evolution by } \Delta t \times \textit{motion reversal} \times \textit{evolution by } \Delta t = \textit{identity}$$

$$\textit{motion reversal} \times \textit{evolution by } \Delta t \times \textit{motion reversal} = \textit{evolution by } -\Delta t$$

To put it precisely, the *motion-reversal operator* is the operator that reverses the direction of a lawful motion of the system so as to obtain another lawful motion. Then, the argument that, starting by Eq. (14.21), concludes with discarding the unitary operator T is a proof of the fact that the antiunitary operator T^* is the right *motion-reversal operator* for quantum mechanics.

Even though the difference between motion reversal and time reversal has not been sufficiently stressed, it is acknowledged by some authors. For example, Ballentine clearly states:

> In the first place, the term "time reversal" is misleading, and the operation that is the subject of this section would be more accurately described as *motion reversal*. We shall continue to use the traditional but less accurate expression "time reversal", because it is so firmly entrenched.
>
> (Ballentine 1998: 377; italics in original)

Sakurai also emphasizes the point just at the beginning of the section devoted to time reversal:

> In this section we study another discrete symmetry operator, called **time reversal**. This is a difficult topic for the novice, partly because the term *time reversal* is a misnomer; it reminds us of science fiction. Actually what we do in this section can be more appropriately characterized by the term *reversal of motion*. Indeed, that is the terminology used by E. Wigner, who formulated time reversal in a very fundamental paper written in 1932.
>
> (Sakurai 1994: 266; bold and italics in original)

Summing up, it is quite clear that the antiunitary operator T^* is the motion-reversal operator in quantum mechanics. But the initial question still remains: which is the right quantum time-reversal operator?

14.5 Conclusions

In this chapter we have focused on the spacetime symmetries of quantum mechanics under the assumption that exploring the meaning of those symmetries is relevant to the interpretation of the theory.

In the first part, we have considered the behavior of nonrelativistic quantum mechanics under the Galilean group. We have shown that the Schrödinger equation is always covariant under the Galilean group, but its Galilean invariance can only be guaranteed when it is applied to a closed system free from external fields. We have also discussed the relevance of symmetries to interpretation; in particular, any realist interpretation that intends to select a Galilean-invariant set of definite-valued observables should make that set to depend on the Casimir operators of the Galilean group, since they are invariant under all the transformations of the group. In future works, these conclusions can be extended in two senses. On the one hand, they can be transferred to quantum field theory by changing the symmetry group accordingly: The definite-valued observables of a system in quantum field theory would be those represented by the Casimir operators of the Poincaré group. Since the mass operator M and the squared-spin operator S^2 are the only Casimir operators of the Poincaré group, they would always represent definite-valued observables, a view that stands in agreement with a usual physical assumption in quantum field theory. On the other hand, if invariance is a mark of objectivity, there is no reason to focus only on spacetime global symmetries. Internal or gauge symmetries should also be considered as relevant in the definition of objectivity and, as a consequence, in the identification of the definite-valued observables of the system.

In the second part of the chapter, we have carefully disentangled the different notions involved in the issue of the time-reversal invariance of the Schrödinger equation. We have assessed the usual claim about the matter, according to which the Schrödinger equation is time-reversal invariant and the quantum time-reversal operator is antiunitary. We have argued that the antiunitary operator is actually a motion-reversal operator and that the question about the right time-reversal operator in quantum mechanics is still an open question. Those who think that time is ontologically independent of and prior to the processes in it will stress the difference between time reversal and motion reversal and, consequently, may tend to prefer a time-reversal operator that only flips the direction of time. Others, by contrast, may claim that the very concept of time as independent of motion has no meaning. From this relationalist-like view, distinguishing between time reversal and motion reversal as different operations makes no sense and, as a consequence, the right time-reversal operator is necessarily a motion-reversal operator. This shows that the question about the time-reversal invariance of quantum mechanics involves deep issues about the very nature of time. But the further development of this aspect of the problem will be the subject of future work.

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