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METRIC DISTRIBUTION RESULTS FOR SEQUENCES $(\{q_n\vec{\alpha}\})$

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Abstract

In this paper a recent result of PHILIPP and TICHY (2000) on the well-distribution measure of certain binary pseudorandom sequences in the unit interval is generalised. Furthermore the average value of the L^2-discrepancy of sequences $(\{q_n\vec{\alpha}\})_{n\geq 1}$ is calculated, where $(q_n)_{n\geq 1}$ is a given sequence of positive integers and $\vec{\alpha} \in [0,1]^d$.

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1 Introduction

Let $\{x\} := x - [x]$ denote the fractional part of a real number x and for any set M let c_M be the characteristic function of M. In this paper we study sequences of the type $\omega = (\{q_n\vec{\alpha}\})_{n\geq 1}$, where $\vec{\alpha} = (\alpha_1, \dots, \alpha_d)$ is a vector in the d-dimensional unit cube $U^d = [0,1]^d$ and $(q_n)_{n\geq 1}$ is a sequence of positive integers. Here $\{q_n\vec{\alpha}\}$ stands for the vector $(\{q_n\alpha_1\}, \{q_n\alpha_2\}, \dots, \{q_n\alpha_d\})$. In the special case $q_n = n$ we have the so-called Kronecker sequence $(n\vec{\alpha})$, which is uniformly distributed mod 1 if and only if $1, \alpha_1, \dots, \alpha_d$ are linearly independent over $\mathbb{Z}$ (cf. [1]).
For such a sequence $\omega = (\{q_n\vec{\alpha}\})_{n\geq 1}$, the standard discrepancy with arbitrary weights $k_i \geq 0$, $(i = 1, .., N)$, where $\sum_{i=1}^N k_i = 1$, is defined by

$$D_N(\omega) = \sup_{[\vec{x},\vec{y})\in J^d} \left| \sum_{n=1}^N k_n\, c_{[\vec{x},\vec{y})}(\{q_n\vec{\alpha}\}) - \lambda_d([\vec{x},\vec{y}))\right|, \tag{1.1}$$

where J^d is the set of all intervals of the form $[\vec{x},\vec{y}) := [x_1,y_1) \times [x_2,y_2) \times \dots \times [x_d,y_d)$ with $0 \leq x_i \leq y_i \leq 1, i = 1, \dots, d$ and λ_d denotes the d-dimensional Lebesgue measure. Furthermore the L^p-discrepancy of ω is defined by

$$D_N^{(p)}(\omega) := \left(\int_{J^d} \left| \sum_{n=1}^N k_n\, c_{[\vec{x},\vec{y})}(\{q_n\vec{\alpha}\}) - \lambda_d([\vec{x},\vec{y}))\right|^p d\vec{x}d\vec{y} \right)^{\frac{1}{p}}. \tag{1.2}$$

The L^p-discrepancy $D_N^{(p)}(\omega)$ for $p = 2$ and $d = 1$ is known as a diaphony which has been introduced by ZINTERHOF [12], see also STRAUCH [11].

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For a survey on discrepancies and other important concepts in the theory of uniform distribution, we refer to the textbooks of DRMOTA AND TICHY [1] and KUIPERS AND NIEDERREITER [5].

Let now $E_N = E_N(\omega) = \{e_1, \dots, e_N\}$ with

$$e_n := \begin{cases} +1 & \text{for } \{q_n \vec{\alpha}\} \in \left[0, \frac{1}{2^{1/d}}\right)^d \\ -1 & \text{for } \{q_n \vec{\alpha}\} \notin \left[0, \frac{1}{2^{1/d}}\right)^d \end{cases}, \quad 1 \le n \le N. \tag{1.3}$$

An important measure of the pseudorandomness of such a binary sequence E_N is its well-distribution measure defined as

$$W(E_N) := \max_{\substack{a \in \mathbb{Z},\, b,t \in \mathbb{N} \\ 1 \le a+bt \le N}} \left| \sum_{j \le t} e_{a+bj} \right|, \quad N \ge 1.$$

Clearly, $W(E_N)$ can be bounded by the discrepancy of the defining sequence $(\{q_n \vec{\alpha}\})_{n \ge 1}$ in the form

$$W(E_N) = \max_{\substack{a \in \mathbb{Z},\, b,t \in \mathbb{N} \\ 1 \le a+bt \le N}} \left| 2 \sum_{n=1}^{t} c_{[0, \frac{1}{2^{1/d}})^d}(\{q_n \vec{\alpha}\}) - t \right| \le 2 \max_{\substack{a \in \mathbb{Z},\, b,t \in \mathbb{N} \\ 1 \le a+bt \le N}} t D_t\Big(\{q_{a+bj} \vec{\alpha}\}, j \le t\Big), \tag{1.4}$$

where D_t is the discrepancy defined in (1.1) with equal weights $k_n = 1/N$, $n = 1, \dots, N$.

For $q_n = n^k$ ($n = 1, \dots, N$, $k \in \mathbb{N}$) and $d = 1$, sequences of type (1.3) were considered by MAUDUIT and SÁRKŐZY [6], who, among other things, proved metric results on asymptotic upper bounds for the right hand side of (1.4). Recently these bounds were improved by PHILIPP and TICHY [8] and at the same time generalised to arbitrary increasing sequences of positive integers $(q_n)_{n \ge 1}$.

In this paper we will derive a metric result on the asymptotic upper bound of (1.4) for arbitrary sequences of distinct positive integers $(q_n)_{n \ge 1}$ and arbitrary dimension $d \ge 1$. Finally, in Section 3 we calculate the L^2-norm of $D_N^{(2)}(\{q_n \vec{\alpha}\})$ and $D_N^{*(2)}(\{q_n \vec{\alpha}\})$ for arbitrary weights $k_n \ge 0, n = 1, \dots, N$, $\sum_{n=1}^{N} k_n = 1$ and arbitrary dimension $d \ge 1$.

2 A metric theorem for bounding $W(E_N)$

Theorem 2.1 *Let $(q_n,\ n \ge 1)$ be a sequence of distinct positive integers and let $\vec{\alpha} = (\alpha_1, \dots, \alpha_d) \in [0,1]^d$ for arbitrary $d \ge 1$. Then for almost all $\vec{\alpha}$ and arbitrary $\epsilon > 0$ we have*

$$\max \Big(t D_t(\{q_{a+bj} \vec{\alpha}\}, j \le t) : a \in \mathbb{Z}, b, t \in \mathbb{N}, 1 \le a + bt \le N \Big) \ll N^{2/3} (\log N)^{1+2d/3+\epsilon}. \tag{2.1}$$

Remark: This result is an extension of Theorem 1 of PHILIPP AND TICHY [8], where (2.1) has been established for increasing sequences (q_n) for the one-dimensional case $d = 1$ with the sharper estimate $N^{2/3}(\log N)^{1+\epsilon}$ on the right hand side.

Proof: The proof is based on a technique developed in [8]. Since the discrepancy $D_t \leq 1$, we only have to consider

$$N^{2/3}(\log N)^{1+2d/3+\epsilon} \leq t \leq N \tag{2.2}$$

and for the number b in the maximum we thus have without loss of generality

$$b \leq N/(t-1) \leq N^{1/3}(\log N)^{-1-2d/3-\epsilon}. \tag{2.3}$$

Furthermore, by application of the triangle equality, it is easy to see that we can assume

$$|a| \leq b \tag{2.4}$$

without loss of generality.
Now, for $\vec{h} = (h_1, \dots, h_d) \in \mathbb{Z}^d$ set

$$r(\vec{h}) = \prod_{j \leq d} \max(1, |h_j|) \quad \text{and} \quad \| \vec{h} \|_\infty = \max_{j=1,\dots,d} |h_j|.$$

Then for fixed a, b, t and $\vec{\alpha}$ the ERDŐS-TURÁN inequality yields

$$tD_t(\{q_{a+bj}\vec{\alpha}\}, j \leq t) \leq \left(\frac{3}{2}\right)^d \left(\frac{2\,t}{H+1} + \sum_{0 < \| \vec{h} \|_\infty \leq H} \frac{1}{r(\vec{h})} \left| \sum_{j \leq t} e(\langle \vec{h}, \{q_{a+bj}\vec{\alpha}\}\rangle) \right| \right), \tag{2.5}$$

where $e(x) = \exp(2\pi i x)$, $\langle \cdot, \cdot \rangle$ denotes the dot product for d-dimensional vectors and H is an arbitrary positive integer (see for example DRMOTA AND TICHY [1]). From (2.5) we obtain

$$t^2 D_t^2(\{q_{a+bj}\vec{\alpha}\}, j \leq t) \leq \left(\frac{3}{2}\right)^{2d} \left(8(t/H)^2 + 2 \left(\sum_{\|\vec{h}\|_\infty \leq H} \frac{1}{r(\vec{h})} \left| \sum_{j \leq t} e(\langle \vec{h}, \{q_{a+bj}\vec{\alpha}\}\rangle) \right| \right)^2 \right). \tag{2.6}$$

For each $\vec{h}$ with $\|\vec{h}\|_\infty \leq H$ we have for all $1 \leq j_1 \leq j_2 \leq N$

$$\mathbb{E} \left| \sum_{j_1 \leq j \leq j_2} e(\langle \vec{h}, \{q_{a+bj}\vec{\alpha}\}\rangle) \right|^2 = \mathbb{E} \left| \sum_{j_1 \leq j \leq j_2} e^{2\pi i (h_1\alpha_1 + \dots + h_d\alpha_d) q_{a+bj}} \right|^2 =$$

$$= \mathbb{E} \left(\sum_{j_1 \leq l_1, l_2 \leq j_2} e^{2\pi i (h_1\alpha_1 + \dots + h_d\alpha_d)(q_{a+bl_1} - q_{a+bl_2})} \right) = j_2 - j_1 + 1,$$

since $(q_n,\ n \geq 1)$ is a sequence of distinct positive integers.
But now we can apply Lemma A.1 (see appendix) with $\gamma = 2$ and the superadditive function $g(i,j) := j - i + 1$ (that $g(i,j)$ is indeed superadditive, can be checked easily). Thereby we obtain

$$\mathbb{E} \max_{t \leq N/b} \left| \sum_{j \leq t} e(\langle \vec{h}, \{q_{a+bj}\vec{\alpha}\}\rangle) \right|^2 \leq C_1 \frac{N}{b} (\log N)^2 \tag{2.7}$$

for some constant $C_1 \geq 1$. By choosing $H = \lfloor (N/b)^{\frac{1}{2}} \rfloor + 1$ we obtain for fixed a and b from (2.6), (2.7) and MINKOWSKI'S inequality

$$\mathbb{E} \max_{t \leq N/b} t^2 D_t^2(\{q_{a+bj}\vec{\alpha}\}, j \leq t) \leq \left(\frac{3}{2}\right)^{2d} \left(C_2 N/b + C_3 N/b \cdot (\log N)^2 (\log N)^{2d}\right) \tag{2.8}$$

for constants $C_2, C_3 \geq 1$ and thus

$$\mathbb{E} \max_{t \leq N/b} t^2 D_t^2(\{q_{a+bj}\vec{\alpha}\}, j \leq t) \ll N/b \cdot (\log N)^{2d+2} \tag{2.9}$$

Now we can apply MARKOV'S inequality and together with (2.3) and (2.4) we obtain

$$\mathbb{P}\left(\max_{\substack{t \leq N/b \\ |a| \leq b \leq N^{1/3} (\log N)^{-1-2d/3-\epsilon}}} tD_t(\{q_{a+bj}\vec{\alpha}\}, j \leq t) \geq N^{2/3} (\log N)^{1+2d/3+\epsilon} \right) \ll$$
$$\ll N^{-\frac{4}{3}} (\log N)^{-4d/3-2-2\epsilon} \max_{b \leq N^{1/3}(\log N)^{-1-2d/3-\epsilon}} N/b \cdot (\log N)^{2d+2} \cdot b^2 \ll (\log N)^{-1-3\epsilon}.$$

Hence we have for fixed $r \geq 1$

$$\mathbb{P}\left(\max_{\substack{t \leq 2^r/b \\ |a| \leq b \leq 2^{r/3} r^{-1-2d/3-\epsilon}}} tD_t(\{q_{a+bj}\vec{\alpha}\}, j \leq t) \geq 2^{2r/3} r^{1+2d/3+\epsilon} \right) \ll r^{-1-3\epsilon}$$

from which it finally follows by the Borel-Cantelli lemma that with probability 1

$$\max_{\substack{t \leq 2^r/b \\ |a| \leq b \leq 2^{r/3} r^{-1-2d/3-\epsilon}}} (tD_t(\{q_{a+bj}\vec{\alpha}\}, j \leq t)) \ll 2^{2r/3} r^{1+2d/3+\epsilon},$$

which completes the proof of (2.1). □

3 The mean of the L^2-discrepancy of $(\{q_n\vec{\alpha}\})$

We now allow for arbitrary sequences of positive integers $(q_n)_{n \geq 1}$ and define for $[\vec{x}, \vec{y}) \in J^d$ the remainder function

$$R_N(\vec{x}, \vec{y}, \vec{\alpha}) = \sum_{n=1}^{N} k_n c_{[\vec{x},\vec{y})}(\{q_n\vec{\alpha}\}) - \prod_{i=1}^{d} (y_i - x_i), \tag{3.1}$$

where $k_n \geq 0$, $n = 1, \ldots, N$ and $\sum_{n=1}^{N} k_n = 1$. This can be considered as a weighted local discrepancy function of the sequence $(\{q_n\vec{\alpha}\})$.
KOKSMA [4] was the first to investigate the integral $\int_{[0,1]^d} R_N^2(\vec{x}, \vec{y}, \vec{\alpha})\, d\vec{\alpha}$ for $d = 1$ and equal weights $k_n = \frac{1}{N}$, $(n = 1, \ldots, N)$, and STRAUCH [10] obtained an explicit expression for this case. The following Proposition generalises Theorem 1 of [10] in that it allows for arbitrary weights k_n and arbitrary dimension $d \geq 1$.

Proposition 3.1 *Let* (q_m, q_n) *denote the greatest common divisor of* q_m *and* q_n*. Then*

$$\int_{U^d} R_N^2(\vec{x},\vec{y},\vec{\alpha})\,d\vec{\alpha} =$$
$$= \sum_{m,n=1}^{N} k_n k_m \prod_{i=1}^{d}\left[(y_i-x_i)^2 + \frac{(q_m,q_n)^2}{q_m\cdot q_n} T\left(x_i, y_i, \frac{q_m}{(q_m,q_n)}, \frac{q_n}{(q_m,q_n)}\right)\right] - \prod_{i=1}^{d}(y_i-x_i)^2, \tag{3.2}$$

where

$$\begin{aligned} T(x_i,y_i,a,b) \;=\;& \Big(\{y_i a\}-\{x_i a\}\Big)\Big(\{x_i b\}-\{y_i b\}\Big) - \{y_i a\} + \{x_i a\} + \\ &+\max\Big(\{y_i a\},\{x_i b\}\Big) - \max\Big(\{x_i a\},\{x_i b\}\Big) + \\ &+\min\Big(\{y_i a\},\{y_i b\}\Big) - \min\Big(\{x_i a\},\{y_i b\}\Big). \end{aligned}$$

Proof: Since for every pair of real numbers x and y with $0 \le x \le y \le 1$ and $a \in \mathbb{N}$ we have

$$\int_0^1 c_{[x,y)}(\{a\alpha\})\,d\alpha = y - x$$

it follows from definition (3.1) that

$$\begin{aligned} \int_{U^d} R_N^2(\vec{x},\vec{y},\vec{\alpha})\,d\vec{\alpha} &= \int_{U^d}\left(\sum_{n=1}^{N} k_n c_{[\vec{x},\vec{y})}(\{q_n\vec{\alpha}\})\right)^2 d\vec{\alpha} - \\ &\quad - 2\prod_{i=1}^{d}(y_i-x_i)\sum_{n=1}^{N}\int_{U^d} k_n \prod_{i=1}^{d} c_{[x_i,y_i)}(\{q_n\alpha_i\})\,d\vec{\alpha} + \prod_{i=1}^{d}(y_i-x_i)^2 = \\ &= \int_{U^d}\left(\sum_{n=1}^{N} k_n \prod_{i=1}^{d} c_{[x_i,y_i)}(\{q_n\alpha_i\})\right)^2 d\vec{\alpha} - 2\prod_{i=1}^{d}(y_i-x_i)\sum_{n=1}^{N} k_n\prod_{i=1}^{d}(y_i-x_i) + \prod_{i=1}^{d}(y_i-x_i)^2 = \\ &= \int_{\alpha_1=0}^{1}\cdots\int_{\alpha_d=0}^{1}\left(\sum_{n=1}^{N} k_n \prod_{i=1}^{d} c_{[x_i,y_i)}(\{q_n\alpha_i\})\right)^2 d\alpha_1\cdots d\alpha_d - \prod_{i=1}^{d}(y_i-x_i)^2. \end{aligned} \tag{3.3}$$

It has been shown in [10] that for every pair of real numbers x and y with $0 \le x \le y \le 1$ and positive integers q_m, q_n

$$\int_{\alpha=0}^{1} c_{[x,y)}(\{q_m\alpha\})c_{[x,y)}(\{q_n\alpha\})d\alpha \;=\; (y-x)^2 + \frac{(q_m,q_n)^2}{q_m\cdot q_n}T\Big(x, y, \frac{q_m}{(q_m,q_n)}, \frac{q_n}{(q_m,q_n)}\Big),$$

which can now be used to calculate the integrals in (3.3) by using

$$\int_{\alpha_1=0}^{1}\cdots\int_{\alpha_d=0}^{1}\prod_{i=1}^{d}c_{[x_i,y_i)}(\{q_m\alpha_i\})\prod_{i=1}^{d}c_{[x_i,y_i)}(\{q_n\alpha_i\})\,d\alpha_1\cdots d\alpha_d =$$

$$=\prod_{i=1}^{d}\int_{\alpha_i=0}^{1}c_{[x_i,y_i)}(\{q_m\alpha_i\})c_{[x_i,y_i)}(\{q_n\alpha_i\})d\alpha_i,$$

which completes the proof of (3.2). □

Proposition 3.1 can now be used to calculate the L^2-norm of the L^2-discrepancy $D_N^{(2)}(\{q_n\vec{\alpha}\})$ with arbitrary weights k_i:

Theorem 3.1

$$\left\|D_N^{(2)}(\{q_n\vec{\alpha}\})\right\|_2=\sqrt{\left(\frac{1}{6^d}-\frac{1}{12^d}\right)\sum_{\substack{m,n=1\\ q_n=q_m}}^{N}k_nk_m}. \tag{3.4}$$

Proof: By changing the order of integration we get

$$\int_{U^d}\left(D_N^{(2)}(\{q_n\vec{\alpha}\})\right)^2 d\vec{\alpha}=\iint_{J^d}\int_{U^d}R_N^2(\vec{x},\vec{y},\vec{\alpha})\,d\vec{\alpha}\,d\vec{x}d\vec{y},$$

which by (3.2) yields

$$\int_{U^d}\left(D_N^{(2)}(\{q_n\vec{\alpha}\})\right)^2 d\vec{\alpha}=\sum_{m,n=1}^{N}k_nk_m\iint_{J^d}\prod_{i=1}^{d}\Bigg[(y_i-x_i)^2+$$

$$+\frac{(q_m,q_n)^2}{q_m\cdot q_n}T\left(x_i,y_i,\frac{q_m}{(q_m,q_n)},\frac{q_n}{(q_m,q_n)}\right)\Bigg]d\vec{x}d\vec{y}-\iint_{J^d}\prod_{i=1}^{d}(y_i-x_i)^2d\vec{x}d\vec{y}=$$

$$=\sum_{m,n=1}^{N}k_nk_m\prod_{i=1}^{d}\iint_{0\le x_i\le y_i\le 1}\left[(y_i-x_i)^2+\frac{(q_m,q_n)^2}{q_m\cdot q_n}T\left(x_i,y_i,\frac{q_m}{(q_m,q_n)},\frac{q_n}{(q_m,q_n)}\right)\right]dx_i\,dy_i-$$

$$-\prod_{i=1}^{d}\iint_{0\le x_i\le y_i\le 1}(y_i-x_i)^2dx_i\,dy_i.$$

But from Corollary 2 in [10] it follows that for all $a,b\in\mathbb{N}$

$$\iint_{0\le x_i\le y_i\le 1}T(x_i,y_i,a,b)\,dx_i\,dy_i=\begin{cases}0, & \text{for}\quad a\neq b,\\ \frac{1}{12}, & \text{for}\quad a=b,\end{cases}$$

so that

$$\begin{aligned}
\int_{U^d}\left(D_N^{(2)}(\{q_n\vec{\alpha}\})\right)^2 d\vec{\alpha} &= \sum_{\substack{m,n=1\\ q_n=q_m}}^{N} k_n k_m \prod_{i=1}^{d}\left(\frac{1}{12}+\frac{1}{12}\frac{(q_m,q_n)^2}{q_m q_n}\right)+\sum_{\substack{m,n=1\\ q_n\neq q_m}}^{N} k_n k_m \prod_{i=1}^{d}\frac{1}{12}-\frac{1}{12^d}\\
&= \sum_{m,n=1}^{N} k_n k_m \frac{1}{12^d}-\sum_{\substack{m,n=1\\ q_n=q_m}}^{N} k_n k_m \frac{1}{12^d}+\sum_{\substack{m,n=1\\ q_n=q_m}}^{N} k_n k_m \frac{1}{6^d}-\frac{1}{12^d}\\
&= \left(\frac{1}{6^d}-\frac{1}{12^d}\right)\sum_{\substack{m,n=1\\ q_n=q_m}}^{N} k_n k_m.
\end{aligned}$$

□

Example 1: For equal weights $k_n = \frac{1}{N}$, $(n = 1, \dots, N)$ and $d = 1$ in (3.4), we obtain

$$\int_0^1 \left(D_N^{(2)}(\{q_n\alpha\})\right)^2 d\alpha = \frac{1}{12N^2}\sum_{\substack{m,n=1\\ q_n=q_m}}^{N} 1,$$

which is given in Theorem 2 in [10].

Example 2: If $(q_n)_{n\geq 1}$ is a sequence of distinct positive integers, equation (3.4) gives

$$\int_{U^d}\left(D_N^{(2)}(\{q_n\vec{\alpha}\})\right)^2 d\vec{\alpha} = \left(\frac{1}{6^d}-\frac{1}{12^d}\right)\sum_{n=1}^{N} k_n^2. \tag{3.5}$$

Remark: In [9] SCHOISSENGEIER pointed out that for $q_n = n, (n = 1, \dots, N)$, equal weights $k_n = \frac{1}{N}$ and $d = 1$ the asymptotic order of the L^2-norm of D_N is $1/\sqrt{N}$. Equation (3.5) shows that in this case $1/\sqrt{N}$ is also the right order of magnitude of the L^2-norm of $D_N^{(2)}$ and, more generally, that this asymptotic result also holds for arbitrary, but fixed dimension d and arbitrary sequences of distinct positive integers $(q_n)_{n\geq 1}$.

In the theory of uniform distribution it is of particular interest to consider the discrepancy of sequences with the underlying set system J_0^d consisting of intervals of the form $[0,\vec{y}) = [0,y_1)\times[0,y_2)\times\ldots\times[0,y_d)$ with $0 \leq y_i \leq 1$, $i = 1,\dots,d$, which is called the star discrepancy D_N^* of $\omega = (\{q_n\vec{\alpha}\})_{n\geq 1}$, so

$$D_N^*(\omega) = \sup_{[0,\vec{y})\in J_0^d}\left|\sum_{n=1}^{N} k_n\, c_{[0,\vec{y})}(\{q_n\vec{\alpha}\}) - \lambda_d([0,\vec{y}))\right|,$$

and correspondingly

$$D_N^{*\,(p)}(\omega) := \left(\int_{J_0^d}\left|\sum_{n=1}^{N} k_n\, c_{[0,\vec{y})}(\{q_n\vec{\alpha}\}) - \lambda_d([0,\vec{y}))\right|^p d\vec{y}\right)^{\frac{1}{p}}.$$

By ROTH's theorem (see e.g. [1]), for any dimension d there exists an absolute constant $c_d > 0$ such that $(D_N^{*\,(2)})^2 \geq c_d\frac{(\log N)^{d-1}}{N^2}$ for any N points $\vec{x}_1,\dots,\vec{x}_N \in [0,1]^d$.

Proposition 3.1 allows us to investigate the average value of $D_N^{*\,(2)}(\omega)$ (with respect to the L^2-norm):

Theorem 3.2

$$\int_{U^d} \left(D_N^{*(2)}(\{q_n\vec{\alpha}\})\right)^2 d\vec{\alpha} = \sum_{m,n=1}^{N} k_m k_n \left(\frac{1}{3} + \frac{1}{12}\frac{(q_m,q_n)^2}{q_m \cdot q_n}\right)^d + + \sum_{\substack{m,n=1\\ q_n=q_m}}^{N} k_m k_n \left(\frac{1}{2^d} - \left(\frac{5}{12}\right)^d\right) - \frac{1}{3^d}. \qquad (3.6)$$

Proof: We proceed similarly to the proof of Theorem 3.1. As we now have $x_i = 0$, $(i = 1, \dots, d)$ we see from (3.2) that

$$\int_{U^d} \left(D_N^{*(2)}(\{q_n\vec{\alpha}\})\right)^2 d\vec{\alpha} = \int_{y_1=0}^{1} \cdots \int_{y_d=0}^{1} \int_{U^d} R_N^2(0,\vec{y},\vec{\alpha})\, d\vec{\alpha}\, dy_1 \cdots dy_d =$$

$$= \int_{y_1=0}^{1} \cdots \int_{y_d=0}^{1} \sum_{m,n=1}^{N} k_n k_m \prod_{i=1}^{d} \left[y_i^2 + \frac{(q_m,q_n)^2}{q_m q_n} \left(\min\left(\left\{ \frac{q_m y_i}{(q_m,q_n)} \right\}, \left\{ \frac{q_n y_i}{(q_m,q_n)} \right\} \right) - \right.\right.$$

$$\left.\left. - \left\{ \frac{q_m y_i}{(q_m,q_n)} \right\} \left\{ \frac{q_n y_i}{(q_m,q_n)} \right\} \right) \right] dy_1 \cdots dy_d - \int_{y_1=0}^{1} \cdots \int_{y_d=0}^{1} \prod_{i=1}^{d} y_i^2 dy_1 \cdots dy_d =$$

$$= \sum_{m,n=1}^{N} k_n k_m \prod_{i=1}^{d} \int_{y_i=0}^{1} \left[y_i^2 + \frac{(q_m,q_n)^2}{q_m q_n} \left(\min\left(\left\{ \frac{q_m y_i}{(q_m,q_n)} \right\}, \left\{ \frac{q_n y_i}{(q_m,q_n)} \right\} \right) - \right.\right.$$

$$\left.\left. - \left\{ \frac{q_m y_i}{(q_m,q_n)} \right\} \left\{ \frac{q_n y_i}{(q_m,q_n)} \right\} \right) \right] dy_i - \frac{1}{3^d}.$$

Since

$$\int_0^1 \left(\min(\{a y_i\}, \{b y_i\}) - \{a y_i\}\{b y_i\} \right) dy_i = \begin{cases} \frac{1}{12}, & \text{for } a \neq b, \\ \frac{1}{6}, & \text{for } a = b, \end{cases}$$

for arbitrary $a, b \in \mathbb{N}$ (see [10]), we conclude that

$$\int_{U^d} \left(D_N^{*(2)}(\{q_n\vec{\alpha}\})\right)^2 d\vec{\alpha} = \sum_{\substack{m,n=1\\ q_m \neq q_n}}^{N} k_n k_m \left(\frac{1}{3} + \frac{1}{12}\frac{(q_m,q_n)^2}{q_m q_n}\right)^d + \sum_{\substack{m,n=1\\ q_m=q_n}}^{N} k_n k_m \frac{1}{2^d} - \frac{1}{3^d} =$$

$$= \sum_{m,n=1}^{N} k_m k_n \left(\frac{1}{3} + \frac{1}{12}\frac{(q_m,q_n)^2}{q_m \cdot q_n}\right)^d + \sum_{\substack{m,n=1\\ q_n=q_m}}^{N} k_m k_n \left(\frac{1}{2^d} - \left(\frac{5}{12}\right)^d\right) - \frac{1}{3^d}.$$

□

Example 3: For $d = 1$ equation (3.6) gives

$$\int_{y=0}^{1} \int_{\alpha=0}^{1} R_N^2(0,y,\alpha)\, dy\, d\alpha = \frac{1}{12} \sum_{m,n=1}^{N} k_m k_n \frac{(q_m,q_n)^2}{q_m q_n} + \frac{1}{12} \sum_{\substack{m,n=1\\ q_m=q_n}}^{N} k_m k_n,$$

which for $k_n = \frac{1}{N}$, $(n = 1, \ldots, N)$ was already derived in [10].

Example 4: If $(q_n)_{n\geq 1}$ is a sequence of distinct positive integers, then it follows from (3.6) that

$$\int_{U^d} \left(D_N^{*(2)}(\{q_n\vec{\alpha}\})\right)^2 d\vec{\alpha} = \sum_{m,n=1}^{N} k_m k_n \left(\frac{1}{3} + \frac{1}{12}\frac{(q_m, q_n)^2}{q_m q_n}\right)^d + \left(\frac{1}{2^d} - \left(\frac{5}{12}\right)^d\right) \sum_{n=1}^{N} k_n^2 - \frac{1}{3^d}. \tag{3.7}$$

Remark: Gál [3] showed that

$$\sum_{m,n=1}^{N} \frac{(q_m, q_n)^2}{q_m q_n} \ll N(\log\log N)^2$$

for every finite sequence $(q_n)_{n\geq 1}$ of distinct positive integers[1]. This bound is also tight. Thus for equal weights $k_n = \frac{1}{N}$, $(n = 1, \ldots, N)$ and arbitrary, but fixed $d \geq 1$ we can determine the asymptotic behavior of (3.7):

$$\int_{U^d} \left(D_N^{*(2)}(\{q_n\vec{\alpha}\})\right)^2 d\vec{\alpha} = \frac{1}{3^d} + \frac{d}{12N^2 \cdot 3^{d-1}} \sum_{m,n=1}^{N} \frac{(q_m, q_n)^2}{q_m q_n} +$$
$$+ \frac{1}{N^2} \sum_{j=0}^{d-2} \binom{d}{j} \frac{1}{3^j} \sum_{m,n=1}^{N} \left(\frac{(q_m, q_n)^2}{q_m q_n}\right)^{d-j} + \left(\frac{1}{2^d} - \left(\frac{5}{12}\right)^d\right) \frac{1}{N} - \frac{1}{3^d} \ll \frac{(\log\log N)^2}{N},$$

so that the L^2-norm of $D_N^{*(2)}(\{q_n\vec{\alpha}\})$ is of asymptotic order $\frac{\log\log N}{\sqrt{N}}$.

Appendix

Let $g(i,j)$ be a superadditive function, i.e. a function satisfying

$$g(i,j) \geq 0 \quad \text{for all} \quad 1 \leq i \leq j \leq n$$
$$g(i,j) \leq g(i,j+1) \quad \text{for all} \quad 1 \leq i \leq j \leq n$$
$$g(i,j) + g(j+1,k) \leq g(i,k) \quad \text{for all} \quad 1 \leq i \leq j \leq n.$$

The following lemma is a special case of [7, Corollary 3.1]:

Lemma A.1. *Let $X_1, \ldots, X_n$ be arbitrary random variables and put $S(i,j) = X_i + \ldots + X_j$ and $M(i,j) = \max\{|S(i,i)|, |S(i,i+1)|, \ldots, |S(i,j)|\}$ for $1 \leq i \leq j \leq n$. Suppose that there exists a superadditive function $g(i,j)$ such that*

$$\mathbb{E}\left|S(i,j)\right|^\gamma \leq g(i,j) \quad \textit{for all} \quad 1 \leq i \leq j \leq n$$

for a given real $\gamma \geq 1$. Then

$$\mathbb{E}M^\gamma(1,n) \leq g(1,n)\left(\lfloor \log n \rfloor + 1\right)^\gamma.$$

[1] This result was extended for weighted sums in Dyer and Harman [2].

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