ON IDEALS OF A_p WITH BOUNDED APPROXIMATE UNITS AND CERTAIN CONDITIONAL EXPECTATIONS

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1. *Introduction*

Let G be an abelian locally compact group and H a closed subgroup. Hauenschild and Ludwig [**11**, Theorem 2.3, p. 170] obtained an explicit bijective correspondence between the set of all closed ideals of $L^1(H)$ and the set of all closed ideals of $L^1(G)$ invariant under the pointwise action of $L^\infty(G/H)$.

In this work, we obtain similar results for the Figà–Talamanca Herz algebra $A_p(G)$ of an amenable locally compact group G. For $1 < p < \infty$, $A_p(G)$ is the Banach algebra of all functions $\sum_{n=1}^{\infty} \bar{k}_n * \check{l}_n$ such that $\sum_{n=1}^{\infty} \|k_n\|_p \|l_n\|_{p'} < \infty$ with the usual norm [**12**].

Given a closed normal subgroup H, we prove (Theorem 5) the existence of a bijection e between the set of all closed ideals of $A_p(G/H)$ and the set of all closed ideals of $A_p(G)$ invariant under translations by elements of H. We show moreover (Corollary 12) that a closed ideal I of $A_p(G/H)$ has a bounded approximate unit if and only if $e(I)$ has a bounded approximate unit. The converse part of this assertion is delicate: it requires, in the L^1 case (as shown by Bekka [**1**]), different tools of integration theory on the dual of G which are missing for G non-abelian and even for G abelian if $p \neq 2$. We avoid this by considering $PM_p(G)$, the set of all p-pseudomeasures on G and the Banach algebra $\mathrm{Hom}_{A_p(G)}(PM_p(G))$ of all linear norm continuous maps Φ from $PM_p(G)$ into itself such that $\Phi(uT) = u\Phi(T)$ for $u \in A_p(G)$ and $T \in PM_p(G)$. We recall that $PM_p(G)$ is the ultraweak closure of the linear span of the set of right translations by elements of G in the space of all bounded operators of $L^p(G)$; for G abelian, $PM_2(G)$ is isomorphic to $L^\infty(\hat{G})$. There is a natural inclusion of $\mathrm{Hom}_{A_p(G/H)}(PM_p(G/H))$ into $\mathrm{Hom}_{A_p(G)}(PM_p(G))$. We prove (Theorem 10) the existence of a conditional expectation of $\mathrm{Hom}_{A_p(G)}(PM_p(G))$ onto $\mathrm{Hom}_{A_p(G/H)}(PM_p(G/H))$. This result seems to be new even for G abelian and $p = 2$. As an application we obtain the above mentioned result concerning the existence of approximate units in ideals of $A_p(G/H)$ and $A_p(G)$.

2. *Main definitions and notation*

We use notation and results of [**4, 5**]. We recall here the most important ones. The Banach space $PM_p(G)$ is the norm dual of $A_p(G)$, the duality being given by

$$\langle \bar{k} * \check{l}, T \rangle_{A_p(G), PM_p(G)} = \overline{\langle T\tau_p k, \tau_{p'} l \rangle}_{L^p(G), L^{p'}(G)}$$

where $\tau_p k = \check{k}\Delta_G^{-1/p}$ and $\langle f, g \rangle_{L^p(G), L^{p'}(G)} = \int_G f(x)\overline{g(x)}\,dx$. For $u \in A_p(G)$ and $T \in PM_p(G)$, uT is the operator defined by $\langle v, uT \rangle_{A_p(G), PM_p(G)} = \langle uv, T \rangle_{A_p(G), PM_p(G)}$ for all $v \in A_p(G)$; with the mapping $(u, T) \mapsto uT$, $PM_p(G)$ is a left Banach $A_p(G)$-module.

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Let ω be the canonical map from G onto G/H and β a Bruhat function for the pair (G,H) (that is, β is a non-negative continuous function on G, $\operatorname{supp}\beta\cap\omega^{-1}(K)$ is compact for every compact K of G/H and $\int_H \beta(xh)\,dh = 1$ for every $x\in G$ [16, p. 163]).

For $r,s\in C_{00}(G)$ (the space of continuous functions with compact support),

$$\langle f_{r,s}(T)m,n\rangle_{L^p(G/H),L^{p'}(G/H)} = \int_{G/H}\langle T[((_{\dot y}m\Delta_{G/H}^{1/p})\circ\omega)r],((_{\dot y}n\Delta_{G/H}^{1/p'})\circ\omega)s\rangle_{L^p(G),L^{p'}(G)}\,d\dot y$$

defines a bounded linear map $f_{r,s}$ of $PM_p(G)$ into $PM_p(G/H)$. Conversely, for $k,l\in C_{00}(G)$,

$$\langle \Omega_{k,l}(T)\phi,\psi\rangle_{L^p(G),L^{p'}(G)} = \int_G \langle TT_H[_{t^{-1}}(\check k\beta^{1/p'})\phi],T_H[_{t^{-1}}(\check l\beta^{1/p})\psi])\rangle_{L^p(G/H),L^{p'}(G/H)}\,dt,$$

where $T_H\alpha(\dot x) = \int_H \alpha(xh)\,dh$, defines a bounded linear map of $PM_p(G/H)$ into $PM_p(G)$. A compactness argument involving the maps $f_{r,s}$ allows one to define, in a canonical way, a linear contraction R of $cv_p(G)$ (the norm closure in $PM_p(G)$ of $\{T\in PM_p(G)\mid \operatorname{supp}T \text{ compact}\}$) onto $cv_p(G/H)$ which coincides on $L^1(G)$ with T_H. See [4] for details.

We also need several important subspaces of $PM_p(G)$: $PF_p(G)$ is the norm closure of $L^1(G)$ in $PM_p(G)$ and $C_p(G)$ the set of all $T\in PM_p(G)$ such that $TS\in PF_p(G)$ for every $S\in PF_p(G)$. We recall that $PF_p(G)\subset cv_p(G)\subset C_p(G)\subset PM_p(G)$.

If we assume G abelian and $p=2$, the preceding spaces have a very concrete and very simple description. Via the Fourier transform, $PF_2(G)$ is isomorphic to the space of all continuous functions on $\hat G$ vanishing at infinity, $cv_2(G)$ to the space of all bounded uniformly continuous functions on $\hat G$ and $C_2(G)$ to the space $C^b(\hat G)$ of all bounded continuous functions on $\hat G$. Moreover, for $T\in cv_2(G)$, $\widehat{R(T)} = \operatorname{Res}_{H^\perp}\hat T$ where $H^\perp = \{\chi\in\hat G\mid \chi(h)=1 \text{ for every } h\in H\}$.

For G arbitrary, $A_2(G)$ is the Fourier algebra of G introduced and investigated by Eymard, $PF_2(G)$ is the reduced C^*-algebra of G and $PM_2(G)$ is the von Neumann algebra of G [6].

Let $W_p(G)$ be the norm dual of $PF_p(G)$; we identify it as a subalgebra of $C^b(G)$ in such a way that for all $f\in L^1(G)$ and $w\in W_p(G)$,

$$\langle\lambda_G^p(f),w\rangle_{PF_p(G),W_p(G)} = \int_G f(x)\overline{w(x^{-1})}\,dx.$$

For G abelian, $W_2(G)$ is isomorphic to the Banach space of all bounded Radon measures on $\hat G$. For G amenable, $W_2(G)$ is the Fourier–Stieltjes algebra of G [6]. In analogy with the abelian case, there is a duality between $C_p(G)$ and $W_p(G)$: we can define it by $\langle T,w\rangle_{C_p(G),W_p(G)} = \lim\langle T\lambda_G^p(f_\iota),w\rangle_{PF_p(G),W_p(G)}$ where $(f_\iota)_{\iota\in I}$ is any bounded approximate unit in $L^1(G)$ (see [2, 3]). In [3], the interested reader can find, among other results, a detailed investigation of the duality between $C_p(G)$ and $W_p(G)$. We first need a little complement to that work.

LEMMA 1. *Let $T\in cv_p(G)$, $u,v\in W_p(G)$ and $w\in W_p(G/H)$. Then*
(1) $\langle T,uv\rangle_{C_p(G),W_p(G)} = \langle uT,v\rangle_{C_p(G),W_p(G)}$,
(2) $R(w\circ\omega T) = wR(T)$.

Proof. (1) Let (u_α) be a bounded approximate unit of $A_p(G)$, then we have by [2, Théorème 2, p. 136]

$$\begin{aligned}\langle T,uv\rangle_{C_p(G),W_p(G)} &= \lim\langle u_\alpha T,uv\rangle_{C_p(G),W_p(G)} = \lim\overline{\langle u_\alpha uv,T\rangle}_{A_p(G),PM_p(G)}\\ &= \lim\langle u_\alpha uT,v\rangle_{C_p(G),W_p(G)} = \langle uT,v\rangle_{C_p(G),W_p(G)}.\end{aligned}$$

(2) From [2, Théorème 3, p. 137], we have for all $v \in W_p(G/H)$

$$\begin{aligned}\langle R(w \circ \omega T), v\rangle_{C_p(G/H), W_p(G/H)} &= \langle w \circ \omega T, v \circ \omega\rangle_{C_p(G), W_p(G)} = \langle T, w \circ \omega v \circ \omega\rangle_{C_p(G), W_p(G)} \\ &= \langle R(T), wv\rangle_{C_p(G/H), W_p(G/H)} = \langle wR(T), v\rangle_{C_p(G/H), W_p(G/H)}.\end{aligned}$$

3. *A bijective correspondence between ideals of $A_p(G/H)$ and invariant ideals of $A_p(G)$*

DEFINITIONS. Let I be a closed ideal of $A_p(G/H)$. The closure in $A_p(G)$ of the linear span of $\{u v \circ \omega \mid u \in A_p(G), v \in I\}$ is a closed ideal $e(I)$ of $A_p(G)$. Observe that $e(I)$ is invariant by right and left translations by elements of H. Let I be closed ideal of $A_p(G)$. The closure in $A_p(G/H)$ of $T_H(I \cap C_{00}(G))$ is a closed ideal $r(I)$ of $A_p(G/H)$.

In the L^1-case, for G abelian, the corresponding definitions are the following ones. For a closed ideal I of $L^1(H)$, $e(I)$ is the closure in $L^1(G)$ of the linear span of $C_{00}(G) *_H I$. For a closed ideal I of $L^1(G)$, $r(I)$ is the closure in $L^1(H)$ of the linear span of $\{\mathrm{Res}_H f | f \in I * C_{00}(G)\}$ [11, p. 168].

Let I be closed ideal of $A_p(G)$, we recall that $I^\perp$ is the set of all $T \in PM_p(G)$ such that $\langle u, T\rangle_{A_p(G), PM_p(G)} = 0$ for all $u \in I$. The amenability of G implies that

$$I^\perp = \{T \in PM_p(G) \mid uT = 0 \text{ for all } u \in I\}.$$

The main result of this paragraph requires the following lemmas.

LEMMA 2. *Let I be a closed ideal of $A_p(G)$ and $T \in PM_p(G)$ such that $R(uT) \in r(I)^\perp$ for every $u \in A_p(G)$. Then $T \in I^\perp$.*

Proof. Let $u \in I$, $v, w \in A_p(G) \cap C_{00}(G)$ and $r, s \in C_{00}(G)$. Observe that for every $y \in G$,

$$R({}_{y^{-1}}w \overline{\tau_p \bar{r}} * (\tau_p s)\check{} T) = f_{r,s}({}_{y^{-1}}wT)$$

(see [5, Proposition 3(iii), p. 98]). From [5, Proposition 13a), p. 104], we deduce that

$$\int_H \overline{\check{v} * w(h^{-1})} f_{r,s}(u_h T) dh = \int_G T_H({}_{y^{-1}}vu) f_{r,s}({}_{y^{-1}}wT)\, dy = 0.$$

This implies $f_{r,s}(uT) = 0$ and therefore $uT = 0$.

LEMMA 3. *Let I be a closed ideal in $A_p(G)$ such that $u_h \in I$ for $h \in H$ and $u \in I$. Then $R(vT) \in r(I)^\perp$ for every $v \in A_p(G)$ and $T \in I^\perp$.*

Proof. It suffices to consider $v \in A_p(G) \cap C_{00}(G)$. Let $u \in I \cap C_{00}(G)$ and $r, s \in C_{00}(G)$. We have

$$\int_H f_{r,s}(vu_h T)\, dh = 0.$$

From [5, Proposition 13b), p. 104], it follows that $T_H(u) f_{r,s}(vT) = 0$. This implies

$$T_H(u)\, R(\overline{\tau_p \bar{r}} * (\tau_{p'} s)\check{} vT) = 0$$

and therefore $T_H(u)\, R(vT) = 0$. We finally obtain $R(vT) \in r(I)^\perp$.

LEMMA 4. *Let I be a closed ideal of $A_p(G/H)$ and $T \in PM_p(G/H)$. Then $T \in I^\perp$ if and only if $\Omega_{k,l}(T) \in e(I)^\perp$ for every $k, l \in C_{00}(G)$.*

Proof. Recall that for $u \in A_p(G)$ and $v \in A_p(G/H)$, we have [**5**, Proposition 19, p. 111]

$$uv \circ \omega \Omega_{k,l}(T) = u\Omega_{k,l}(vT).$$

THEOREM 5. *Let G be an amenable locally compact group and H a closed normal subgroup of G.*

(1) *We have $I = r(e(I))$ for every closed ideal I of $A_p(G/H)$.*

(2) *For every closed ideal I of $A_p(G)$ such that $u_h \in I$ for $u \in I$ and $h \in H$, we have also $I = e(r(I))$.*

Hence e is a bijection between the set of all closed ideals of $A_p(G/H)$ and the set of all closed ideals of $A_p(G)$ invariant under H.

Proof. (1) From the very definitions we get $r(e(I)) \subset I$. For the reverse inclusion, we shall show that $r(e(I))^\perp \subset I^\perp$. Let $T \in r(e(I))^\perp$, and consider $v \in r(e(I))$, $u \in A_p(G)$ and $k, l \in C_{00}(G)$. From $vR(u\Omega_{k,l}(T)) = R(u\Omega_{k,l}(vT))$, we get $R(u\Omega_{k,l}(T)) \in r(e(I))^\perp$. Lemma 2 implies $\Omega_{k,l}(T) \in e(I)^\perp$, finally from Lemma 4 we conclude that $T \in I^\perp$.

(2) Taking into account (1) we need only to prove that for two closed ideals I, J of $A_p(G)$ with $u_h \in I$, $v_h \in J$ for $h \in H$, $u \in I$, $v \in J$ and $r(I) = r(J)$, it follows that $I = J$. Let $T \in I^\perp$; by Lemma 3, $R(uT) \in r(J)^\perp$ for $u \in A_p(G)$. Lemma 2 implies $T \in J^\perp$.

REMARKS. (1) For $A \subset G$, define $I(A)$ to be the ideal of $A_p(G)$ consisting of those functions which vanish on A. Let F be a closed subset of G/H, then $e(I(F)) = I(\omega^{-1}(F))$ and $r(I(\omega^{-1}(F))) = I(F)$.

(2) We conjecture that Theorem 5 is true for non-amenable groups. However amenability is used several times in our proof (existence of R, characterization of $I^\perp, \ldots$).

(3) Let I be a closed ideal of $A_p(G)$ such that ${}_h u \in I$ for every $u \in I$ and $h \in H$, then we have $u_h \in I$ for every $u \in I$ and $h \in H$. It suffices to apply the preceding theorem to the ideal $\{\tilde{u} \mid u \in I\}$ of $A_{p'}(G)$.

It is possible to extend partially the second part of Theorem 5 to arbitrary closed ideals of $A_p(G)$.

PROPOSITION 6. *Let I be a closed ideal of $A_p(G)$, then $e(r(I))$ coincides with I_H, the closure in $A_p(G)$ of the linear span of $\{u_h \mid u \in I, h \in H\}$. (Remark that I_H is the smallest H-invariant closed ideal of $A_p(G)$ containing H.)*

Proof. Because of Theorem 5, it suffices to verify that $r(I_H) \subset r(I)$, the converse inclusion being a direct consequence of $I \subset I_H$. Take $v \in I_H \cap C_{00}(G)$ and $\varepsilon > 0$ and let K be a compact neighbourhood of $\operatorname{supp} v$. We recall the existence of $C > 0$ such that $\|T_H u\|_{A_p(G/H)} \leqslant C\|u\|_{A_p(G)}$ for every $u \in A_p(G)$ with $\operatorname{supp} u \subset K$ [**14**, lemme 1, p. 188]. Choose $w \in A_p(G)$ with $w = 1$ on $\operatorname{supp} v$, $\operatorname{supp} w \subset K$ and $\|w\|_{A_p(G)} \leqslant 2$. Now there exists $u_1, \ldots, u_n \in I$ and $h_1, \ldots, h_n \in H$ such that

$$\left\| v - \sum_{i=1}^{n} (u_j)_{h_j} \right\|_{A_p(G)} < \frac{\varepsilon}{2C}.$$

Let $w_j = u_j w_{h_j^{-1}}$, so

$$\left\| T_H v - \sum_{i=1}^{n} T_H((w_j)_{h_j}) \right\|_{A_p(G/H)} < \varepsilon.$$

From $T_H((w_j)_{h_j}) \in r(I)$, we deduce finally $T_H v \in r(I)$ and the conclusion.

4. *A natural inclusion of* $\mathrm{Hom}_{A_p(G/H)}(PM_p(G/H))$ *into* $\mathrm{Hom}_{A_p(G)}(PM_p(G))$

DEFINITION. For $T \in cv_p(G)$ and $\phi \in cv_p(G)^*$, there is a unique $\phi \cdot T \in cv_p(G)$ with

$$\phi(uT) = \overline{\langle u, \phi \cdot T \rangle}_{A_p(G), PM_p(G)} \quad \text{for every} \quad u \in A_p(G).$$

For $\phi, \psi \in cv_p(G)^*$, we define $(\phi \cdot \psi)(T) = \phi(\psi \cdot T)$ for $T \in cv_p(G)$.

With this operation $cv_p(G)^*$ becomes a Banach algebra. We also have $\phi(vT) = \langle \phi \cdot T, v \rangle_{C_p(G), W_p(G)}$ for every $\phi \in cv_p(G)^*$, $v \in W_p(G)$ and $T \in cv_p(G)$.

PROPOSITION 7. *For $\phi \in cv_p(G)^*$ and $T \in PM_p(G)$, there is a unique*

$$\sigma_G(\phi)(T) \in PM_p(G)$$

such that

$$\phi(uT) = \overline{\langle u, \sigma_G(\phi)(T) \rangle}_{A_p(G), PM_p(G)}$$

for every $u \in A_p(G)$; σ_G is a Banach algebra isometry from $cv_p(G)^*$ onto $\mathrm{Hom}_{A_p(G)}(PM_p(G))$. *For every $\Phi \in \mathrm{Hom}_{A_p(G)}(PM_p(G))$ and $T \in cv_p(G)$, we have*

$$\sigma_G^{-1}(\Phi)(T) = \langle \Phi(T), 1_G \rangle_{C_p(G), W_p(G)}.$$

REMARKS. (1) This proposition is not new! All these assertions can be deduced, for example, from [9, p. 140]. The fact that σ_G is a bijective linear isometry is also due to Granirer [8, Proposition 2.1, p. 160]. We present here a very short self-contained approach.

(2) σ_G is onto if and only if G is amenable [8, Theorem 2.1, p. 160].

(3) Note that $\sigma_G(\phi)(S) = \phi \cdot S$ for every $S \in cv_p(G)$.

Proof of Proposition 7. (I) σ_G is a linear isometry.

It is clear that $\|\sigma_G(\phi)\| \leqslant \|\phi\|$. Let $\Phi \in \mathrm{Hom}_{A_p(G)}(PM_p(G))$. Define $\phi \in cv_p(G)^*$ by $\phi(T) = \langle \Phi(T), 1_G \rangle_{C_p(G), W_p(G)}$. We have $\|\phi\| \leqslant \|\Phi\|$ and, for $u \in A_p(G)$ and $T \in PM_p(G)$,

$$\overline{\langle u, \sigma_G(\phi)(T) \rangle}_{A_p(G), PM_p(G)} = \phi(uT) = \langle u\Phi(T), 1_G \rangle_{C_p(G), W_p(G)} = \overline{\langle u, \Phi(T) \rangle}_{A_p(G), PM_p(G)},$$

hence $\sigma_G(\phi) = \Phi$ and $\|\phi\| = \|\Phi\|$.

(II) $\sigma_G(\phi \cdot \psi) = \sigma_G(\phi)\sigma_G(\psi)$.

For $T \in PM_p(G)$ and $u \in A_p(G)$, we have

$$\begin{aligned}\overline{\langle u, \sigma_G(\phi)\sigma_G(\psi)(T) \rangle}_{A_p(G), PM_p(G)} &= \phi(u(\sigma_G(\psi)(T))) = \phi(\psi \cdot (uT)) \\ &= (\phi \cdot \psi)(uT) = \overline{\langle u, \sigma_G(\phi \cdot \psi)(T) \rangle}_{A_p(G), PM_p(G)}.\end{aligned}$$

THEOREM 8. *Let j be the map $\sigma_G \circ R^* \circ \sigma_{G/H}^{-1}$.*

(1) *j is a Banach algebra isometry from* $\mathrm{Hom}_{A_p(G/H)}(PM_p(G/H))$ *into* $\mathrm{Hom}_{A_p(G)}(PM_p(G))$.

(2) *For $T \in PM_p(G)$, $u \in A_p(G)$, $v \in W_p(G/H)$ and $\Phi \in \mathrm{Hom}_{A_p(G/H)}(PM_p(G/H))$, we have*

$$\langle uv \circ \omega, j(\Phi)(T) \rangle_{A_p(G), PM_p(G)} = \overline{\langle \Phi(R(uT)), v \rangle}_{C_p(G/H), W_p(G/H)}.$$

(3) *Let I be a closed ideal of $A_p(G/H)$. Assume the existence of a projection P of $PM_p(G/H)$ onto $I^\perp$* with *$P \in \mathrm{Hom}_{A_p(G/H)}(PM_p(G/H))$. Then $j(P)$ is a projection of $PM_p(G)$ onto $e(I)^\perp$.*

Proof. (1) From [4, Theorem 6, p. 21], we know that R^* is an isometry from $cv_p(G/H)^*$ into $cv_p(G)^*$; so it suffices to show that R^* is an algebra homomorphism. For $u \in A_p(G)$, $T \in cv_p(G)$ and $\phi \in cv_p(G/H)^*$, we have

$$\langle u, R(R^*(\phi)\cdot T)\rangle_{A_p(G/H), PM_p(G/H)} = \overline{\langle R^*(\phi)\cdot T, u\circ\omega\rangle}_{C_p(G), W_p(G)}$$

[2, théorème 2, p. 136].

But $\overline{\langle R^*(\phi)\cdot T, u\circ\omega\rangle}_{C_p(G), W_p(G)} = \langle u, \phi\cdot R(T)\rangle_{A_p(G/H), PM_p(G/H)}$, so we get $R(R^*(\phi)\cdot T) = \phi\cdot R(T)$. Letting $\phi, \psi \in cv_p(G/H)^*$ and $T \in cv_p(G)$, we have

$$(R^*(\phi)\cdot R^*(\psi))(T) = \phi(R(R^*(\psi)\cdot T)) = \phi(\psi\cdot R(T)) = (R^*(\phi\cdot\psi))(T).$$

(2)
$$\begin{aligned}\langle v\circ\omega u, j(\Phi)(T)\rangle_{A_p(G), PM_p(G)} &= R^*(\sigma_{G/H}^{-1}(\Phi))(v\circ\omega uT)\\ &= \overline{\langle \Phi(R(v\circ\omega uT)), 1_{G/H}\rangle}_{C_p(G/H), W_p(G/H)}\\ &= \overline{\langle \Phi(vR(uT)), 1_{G/H}\rangle}_{C_p(G/H), W_p(G/H)}\\ &= \overline{\langle \Phi(R(uT)), v\rangle}_{C_p(G/H), W_p(G/H)}.\end{aligned}$$

(3) Letting $u \in A_p(G)$, $v \in I$ and $T \in PM_p(G)$, we have

$$\langle v\circ\omega u, j(P)(T)\rangle_{A_p(G), PM_p(G)} = \overline{\langle P(R(uT)), v\rangle}_{C_p(G/H), W_p(G/H)}$$

and therefore $j(P)(T) \in e(I)^\perp$. Let $T \in e(I)^\perp$. For $u \in A_p(G)$ and $v \in A_p(G/H)$, we have $R(uT) \in I^\perp$ and using (2)

$$\langle v\circ\omega u, j(P)(T)\rangle_{A_p(G), PM_p(G)} = \langle v, R(uT)\rangle_{A_p(G/H), PM_p(G/H)} = \langle uv\circ\omega, T\rangle_{A_p(G), PM_p(G)};$$

this implies $j(P)(T) = T$.

REMARK. Putting $v = 1_{G/H}$ in (2), we get

$$\langle u, j(\Phi)(T)\rangle_{A_p(G), PM_p(G)} = \overline{\langle \Phi(R(uT)), 1_{G/H}\rangle}_{C_p(G/H), W_p(G/H)}.$$

5. *A conditional expectation of* $\mathrm{Hom}_{A_p(G)}(PM_p(G))$ *onto* $\mathrm{Hom}_{A_p(G/H)}(PM_p(G/H))$. *An application*

We first need some complements to [14, lemme 1] and [4, p. 13 and 17].

THEOREM 9. *Let* $u \in A_p(G)$, $T \in PM_p(G/H)$, $k, l \in C_{00}(G)$ *and* $k' = \check{k}\beta^{1/p'}$, $l' = \check{l}\beta^{1/p}$. *Then,*

(1)
$$\|\Delta_{G/H}^{1/p'} T_H(u\tilde{k}' * l'\Delta_G^{-1/p'})\|_{A_p(G/H)} \leqslant \|k\|_p \|l\|_{p'} \|u\|_{A_p(G)}$$

and

$$\langle u, \Omega_{k,l}(T)\rangle_{A_p(G), PM_p(G)} = \langle \Delta_{G/H}^{1/p'} T_H(\Delta_G^{-1/p'} u\tilde{k}' * l'), T\rangle_{A_p(G/H), PM_p(G/H)}.$$

(2)
$$R(u\Omega_{k,l}(T)) = (\Delta_{G/H}^{1/p'} T_H(\Delta_G^{-1/p'}\tilde{k}' * l'u))\, T.$$

Proof. (1) We only need to consider $u = \bar{r} * \check{s}$ with $r, s \in C_{00}(G)$. We have

$$\langle \bar{r} * \check{s}, \Omega_{k,l}(T)\rangle_{A_p(G), PM_p(G)} = \int_G \langle w(t), T\rangle_{A_p(G/H), PM_p(G/H)}\, dt,$$

where

$$w(t) = \overline{\tau_p(T_H({}_{t^{-1}}k'\tau_p r))} * (\tau_{p'}(T_H({}_{t^{-1}}l'\tau_{p'} s)))^\vee.$$

It is easy to verify that w is a continuous map from G into the Banach space $A_p(G/H)$ and that

$$\int_G \|w(t)\|_{A_p(G/H)}\,dt \leqslant \|k\|_p\|l\|_{p'}\|r\|_p\|s\|_{p'}.$$

There is a unique $a\in A_p(G/H)$ with

$$\langle a,S\rangle_{A_p(G/H),\,PM_p(G/H)} = \int_G \langle w(t),S\rangle_{A_p(G/H),\,PM_p(G/H)}\,dt$$

for every $S\in PM_p(G/H)$. For $x\in G$ we obtain

$$\begin{aligned}
a(\dot{x}) &= \int_G (\Delta_{G/H}^{1/p'}\,T_H({}_{t^{-1}}k'\tau_p r)^*) * (\Delta_{G/H}^{1/p'}\,T_H({}_{t^{-1}}l'\tau_{p'}s))(\dot{x})\,dt\\
&= \Delta_{G/H}^{1/p'}(\dot{x})\int_G\int_H\int_G \Delta_G^{-1/p'}(xh)\,\overline{k'(t^{-1}y^{-1})}\,\overline{r(y)}\,l'(t^{-1}y^{-1}xh)\,\check{s}(y^{-1}xh)\,dy\,dh\,dt\\
&= \Delta_{G/H}^{1/p'}(\dot{x})\,T_H(\Delta_G^{-1/p'}\tilde{k}' * l'\bar{r} * \check{s})(\dot{x}),
\end{aligned}$$

this implies $\langle \bar{r}*\check{s},\Omega_{k,l}(T)\rangle_{A_p(G),\,PM_p(G)} = \langle a,T\rangle_{A_p(G/H),\,PM_p(G/H)}$ and

$$\|\Delta_{G/H}^{1/p'}\,T_H(\Delta_G^{-1/p'}\tilde{k}' * l'\bar{r}*\check{s})\|_{A_p(G/H)} \leqslant \|k\|_p\|l\|_{p'}\|r\|_p\|s\|_{p'}.$$

(2) Using again [2, Théorème 2, p. 136], we have for $w\in W_p(G/H)$

$$\begin{aligned}
\langle R(u\Omega_{k,l}(T)),w\rangle_{C_p(G/H),\,W_p(G/H)} &= \langle u\Omega_{k,l}(T),w\circ\omega\rangle_{C_p(G),\,W_p(G)}\\
&= \overline{\langle u\,w\circ\omega,\Omega_{k,l}(T)\rangle_{A_p(G),\,PM_p(G)}}\\
&= \overline{\langle \Delta_{G/H}^{1/p'}\,T_H(\Delta_G^{-1/p'}u\,w\circ\omega\,\tilde{k}' * l'),T\rangle}_{A_p(G/H),\,PM_p(G/H)}\\
&= \langle(\Delta_{G/H}^{1/p'}\,T_H(\Delta_G^{-1/p}u\tilde{k}' * l'))\,T,w\rangle_{C_p(G/H),\,W_p(G/H)}.
\end{aligned}$$

REMARK. The amenability of G is not needed for (1).

THEOREM 10. *There is a linear contraction E of* $\mathrm{Hom}_{A_p(G)}(PM_p(G))$ *onto* $\mathrm{Hom}_{A_p(G/H)}(PM_p(G/H))$ *such that*

(1) $E(j(\Psi)) = \Psi$ *for* $\Psi\in\mathrm{Hom}_{A_p(G/H)}(PM_p(G/H))$,

(2) $E(\Phi j(\Psi)) = E(\Phi)\Psi$ *for*

$$\Phi\in\mathrm{Hom}_{A_p(G)}(PM_p(G)) \quad \textit{and} \quad \Psi\in\mathrm{Hom}_{A_p(G/H)}(PM_p(G/H)).$$

(3) *Let I be a closed ideal of $A_p(G/H)$. Assume the existence of a projection $P\in\mathrm{Hom}_{A_p(G)}(PM_p(G))$ of $PM_p(G)$ onto $e(I)^\perp$. Then $E(P)$ is a projection of $PM_p(G/H)$ onto $I^\perp$.*

Proof. (I) Existence of E with (1) and (2).

Let X be the Banach space of all continuous bilinear functionals on $cv_p(G/H)\times cv_p(G)^*$. For a subset C of X, $\bar{C}$ denotes the closure of C in X for the topology $\sigma(X, cv_p(G/H)\times cv_p(G)^*)$. For $k,l\in C_{00}(G/H)$, we define $\alpha_{k,l}\in X$ by $\alpha_{k,l}(T,\phi) = \phi(\Omega_{\nu_p(k),\,\nu_{p'}(l)}(T))$ where $\nu_p(k) = \check{\Delta}_G^{1/p}\check{\beta}^{1/p}(k\Delta_{G/H}^{1/p})\circ\omega$. For $\varepsilon>0$ and F an arbitrary finite subset of $A_p(G/H)$, let $C_{F,\varepsilon}$ be the set of all $\alpha_{k,l}$ such that $k,l\in C_{00}(G/H)$, $\|k\|_p\|l\|_{p'}=1$ and $\|u-u\tilde{k}*\check{l}\|_{A_p(G/H)}<\varepsilon$ for every $u\in F$. There is $\alpha\in\bigcap\{\bar{C}_{F,\varepsilon}\,|\,F$ finite subset of $A_p(G/H),\varepsilon>0\}$. For every $\phi\in cv_p(G)^*$ there is a unique $Q(\phi)\in cv_p(G/H)^*$ such that $\alpha(T,\phi)=Q(\phi)(T)$ for every $T\in cv_p(G/H)$. The map Q is a linear contraction of $cv_p(G)^*$ into $cv_p(G/H)^*$.

We show that $Q(R^*(\phi)) = \phi$ for every $\phi \in cv_p(G/H)^*$.

Choose $T \in cv_p(G/H)$ and $\varepsilon > 0$. By the Cohen–Hewitt factorization theorem [**13**, 32.22], there is $w \in A_p(G/H)$ and $T' \in cv_p(G/H)$ such that $T = wT'$. There is also

$$\alpha_{k,l} \in C_{\{w\},\, \varepsilon/2(1+|||T'|||_p)(1+\|\phi\|)} \quad \text{such that} \quad |\alpha(T, R^*(\phi)) - \alpha_{k,l}(T, R^*(\phi))| < \frac{\varepsilon}{2}.$$

We have $\alpha(T, R^*(\phi)) = Q(R^*(\phi))(T)$, $\alpha_{k,l}(T, R^*(\phi)) = \phi(\bar{k} * \check{l}T)$ and

$$|\phi(\bar{k} * \check{l}T) - \phi(T)| < \varepsilon/2.$$

We conclude that $|Q(R^*(\phi))(T) - \phi(T)| < \varepsilon$ and therefore $Q(R^*(\phi)) = \phi$.

For $\psi \in cv_p(G)^*$ and $\phi \in cv_p(G/H)^*$, we have $Q(\psi \cdot R^*(\phi)) = Q(\psi) \cdot \phi$.

Let $\varepsilon > 0$, $S \in cv_p(G/H)$ and $w \in A_p(G/H)$. There is $\alpha_{k,l} \in C_{\{w\},\varepsilon}$ with

$$|\alpha(S, \psi \cdot R^*(\phi)) - \alpha_{k,l}(S, \psi \cdot R^*(\phi))| < \frac{\varepsilon}{2} \quad \text{and} \quad |\alpha(\phi \cdot S, \psi) - \alpha_{k,l}(\phi \cdot S, \psi)| < \frac{\varepsilon}{2}.$$

We have $\alpha(S, \psi \cdot R^*(\phi)) = Q(\psi \cdot R^*(\phi))(S)$, $\alpha(\phi \cdot S, \psi) = (Q(\psi) \cdot \phi)(S)$ and

$$\alpha_{k,l}(S, \psi \cdot R^*(\phi)) = \psi(R^*(\phi) \cdot \Omega_{v_p(k),\, v_{p'}(l)}(S)).$$

For $u \in A_p(G)$ we have

$$\begin{aligned}\overline{\langle u, R^*(\phi) \cdot \Omega_{v_p(k),\, v_{p'}(l)}(S) \rangle}_{A_p(G),\, PM_p(G)} &= \phi(R(u\Omega_{v_p(k),\, v_{p'}(l)}(S))) \\ &= \phi((\Delta_{G/H}^{1/p'} T_H(\Delta_G^{-1/p'} uv))\, S),\end{aligned}$$

where

$$v = ((v_p(k))\check{}\beta^{1/p})\tilde{} * (v_{p'}(l))\check{}\beta^{1/p}.$$

We obtain

$$\langle u, R^*(\phi) \cdot \Omega_{v_p(k),\, v_{p'}(l)}(S) \rangle_{A_p(G),\, PM_p(G)} = \langle u, \Omega_{v_p(k),\, v_{p'}(l)}(\phi \cdot S) \rangle_{A_p(G),\, PM_p(G)},$$

which implies

$$R^*(\phi) \cdot \Omega_{v_p(k),\, v_{p'}(l)}(S) = \Omega_{v_p(k),\, v_{p'}(l)}(\phi \cdot S).$$

We finally get

$$|Q(\psi \cdot R^*(\phi))(S) - (Q(\psi) \cdot \phi)(S)| < \varepsilon.$$

The map $E = \sigma_{G/H} \circ Q \circ \sigma_G^{-1}$ is therefore a linear contraction of $\mathrm{Hom}_{A_p(G)}(PM_p(G))$ onto $\mathrm{Hom}_{A_p(G/H)}(PM_p(G/H))$ which satisfies (1) and (2).

(II) Let Φ be an element of $\mathrm{Hom}_{A_p(G)}(PM_p(G))$. For $T \in PM_p(G/H)$, F a finite subset of $A_p(G/H)$, $\varepsilon, \varepsilon' > 0$, there is $\alpha_{k,l} \in C_{F,\varepsilon'}$ with

$$\left| \langle w, E(\Phi)(T) \rangle_{A_p(G/H),\, PM_p(G/H)} - \overline{\langle \Phi(\Omega_{v_p(k),\, v_{p'}(l)}(T)), w \circ \omega \rangle}_{C_p(G),\, W_p(G)} \right| < \varepsilon$$

for every $w \in F$.

For there is $\alpha_{k,l} \in C_{F,\varepsilon'}$ with $|\alpha(wT, \sigma_G^{-1}(\Phi)) - \alpha_{k,l}(wT, \sigma_G^{-1}(\Phi))| < \varepsilon$ for every $w \in F$. We have

$$\alpha(wT, \sigma_G^{-1}(\Phi)) = \overline{\langle w, E(\Phi)(T) \rangle}_{A_p(G/H),\, PM_p(G/H)}$$

and

$$\begin{aligned}\alpha_{k,l}(wT, \sigma_G^{-1}(\Phi)) &= \langle \Phi(\Omega_{v_p(k),\, v_{p'}(l)}(wT)), 1_G \rangle_{C_p(G),\, W_p(G)} \\ &= \langle w \circ \omega\, \Phi(\Omega_{v_p(k),\, v_{p'}(l)}(T)), 1_G \rangle_{C_p(G),\, W_p(G)} \\ &= \langle \Phi(\Omega_{v_p(k),\, v_{p'}(l)}(T)), w \circ \omega \rangle_{C_p(G),\, W_p(G)}.\end{aligned}$$

(III) We prove (3).

(a) For every $T \in PM_p(G/H)$ we have $E(P)(T) \in I^{\perp}$.

Let w be an element of the ideal I. For $\varepsilon > 0$, there is $\alpha_{k,l} \in C_{\{w\},\varepsilon}$ with

$$\left|\langle w, E(P)(T)\rangle_{A_p(G/H),\,PM_p(G/H)} - \overline{\langle P\Omega_{\nu_p(k),\,\nu_{p'}(l)}(T), w\circ\omega\rangle}_{C_p(G),\,W_p(G)}\right| < \varepsilon.$$

Choose $u \in A_p(G) \cap C_{00}(G)$ equal to 1 on a neighbourhood of the support of $P(\Omega_{\nu_p(k),\,\nu_{p'}(l)}(T))$. We have

$$\langle P\Omega_{\nu_p(k),\,\nu_{p'}(l)}(T), w\circ\omega\rangle_{C_p(G),\,W_p(G)} = \langle P\Omega_{\nu_p(k),\,\nu_{p'}(l)}(T), uw\circ\omega\rangle_{C_p(G),\,W_p(G)} = 0.$$

This implies $E(P)(T) \in I^{\perp}$.

(b) We prove that $E(P)(T) = T$ for every $T \in I^{\perp}$.

Let $\varepsilon > 0$ and $w \in A_p(G/H)$. There is $\alpha_{k,l} \in C_{\{w\},\,\varepsilon/2(1+|||T|||_p)}$ with

$$|\langle w, E(P)(T)\rangle_{A_p(G/H),\,PM_p(G/H)} - \overline{\langle P\Omega_{\nu_p(k),\,\nu_{p'}(l)}(T), w\circ\omega\rangle}_{C_p(G),\,W_p(G)}| < \frac{\varepsilon}{2}.$$

Taking into account Proposition 4 and Theorem 8, we obtain

$$\begin{aligned}\langle P\Omega_{\nu_p(k),\,\nu_{p'}(l)}(T), w\circ\omega\rangle_{C_p(G),\,W_p(G)} &= \langle \Omega_{\nu_p(k),\,\nu_{p'}(l)}(T), w\circ\omega\rangle_{C_p(G),\,W_p(G)}\\ &= \langle R(\Omega_{\nu_p(k),\,\nu_{p'}(l)}(T)), w\rangle_{C_p(G/H),\,W_p(G/H)}\\ &= \langle \bar{k} * \check{l}T, w\rangle_{C_p(G/H),\,W_p(G/H)}.\end{aligned}$$

This implies

$$|\langle w, E(P)(T)\rangle_{A_p(G/H),\,PM_p(G/H)} - \langle w, T\rangle_{A_p(G/H),\,PM_p(G/H)}| < \varepsilon.$$

We conclude that $E(P)(T) = T$.

REMARK. We are not able to show that $E(j(\Psi)\Phi) = \Psi E(\Phi)$ for $\Psi \in \mathrm{Hom}_{A_p(G/H)}(PM_p(G/H))$ *and* $\Phi \in \mathrm{Hom}_{A_p(G)}(PM_p(G))$.

PROPOSITION 11. *Let A be a commutative normed algebra with an approximate unit bounded by* 1. *Let I be a closed ideal of A and $C > 0$. The ideal I has an approximate unit bounded by C if and only if there is a projection P of A^* onto $I^{\perp}$ with $\|\mathrm{id} - P\| \leqslant C$ and such that $P(af) = aP(f)$ for $a \in A$ and $f \in A^*$.*

This proposition is due to Françoise Lust-Piquard [**15**, pp. 7 and 15]. The condition on C is not there but requires no new idea. See also [7, Proposition 6.4, p. 17].

COROLLARY 12. *Let I be a closed ideal of $A_p(G/H)$ and $C > 0$. The ideal I has an approximate unit bounded by C if and only if $e(I)$ has an approximate unit bounded by C.*

Proof. This corollary is a direct consequence of Proposition 11, Theorem 8(3) and Theorem 10(3).

REMARKS. (1) Assume that I has an approximate unit bounded by C. It is possible to prove **directly** that $e(I)$ has an approximate unit bounded by C. It suffices to adapt the proof of the L^1-case (see [**10**, Lemma 1, p. 170; **1**, Theorem, p. 392]).

Let $u \in e(I)$ and $\varepsilon > 0$. There is $u_1, \ldots, u_n \in A_p(G)$ and $v_1, \ldots, v_n \in I$ such that

$$\left\| u - \sum_{k=1}^{n} u_k v_k \circ \omega \right\|_{A_p(G)} < \frac{\varepsilon}{4(1+C)}.$$

There is $b \in A_p(G)$ such that $\|b\|_{A_p(G)} \leqslant 1$ and

$$\left\| \sum_{k=1}^{n} u_k v_k \circ \omega - \left(\sum_{k=1}^{n} u_k v_k \circ \omega \right) b \right\|_{A_p(G)} < \frac{\varepsilon}{4(1+C)}.$$

One can find $w \in I$ such that

$$\|w\|_{A_p(G/H)} \leqslant C \quad \text{and} \quad \|v_j - v_j w\|_{A_p(G/H)} < \frac{3}{4\left(1 + \sum_{k=1}^{n} \|u_k\|_{A_p(G)}\right)}$$

for every $1 \leqslant j \leqslant n$. If we choose $d = b w \circ \omega$, we conclude that $d \in e(I)$, $\|d\|_{A_p(G)} \leqslant C$ and $\|u - ud\|_{A_p(G)} < \varepsilon$.

The proof of the converse assertion seems to require the map E (Theorem 10) and Proposition 11.

(2) As far as we know, Theorem 10 seems to be new even for G abelian and $p = 2$.

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