Exact Asymptotics and Limit Theorems for Supremum of Stationary χ-processes Over a Random Interval

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Abstract: Let $\{\chi_k(t), t \geq 0\}$ be a stationary χ-process with k degrees of freedom being independent of some non-negative random variable $\mathcal{T}$. In this paper we derive the exact asymptotics of $\mathbb{P}\left\{\sup_{t\in[0,\mathcal{T}]} \chi_k(t) > u\right\}$ as $u \to \infty$ when $\mathcal{T}$ has a regularly varying tail with index $\lambda \in [0, 1)$. Three other novel results of this contribution are the mixed Gumbel limit law of the normalised maximum over an increasing random interval, the Piterbarg inequality and Seleznjev pth-mean theorem for stationary χ-processes.

Key Words: χ-process; limit theorems; Piterbarg inequality; Piterbarg theorem for χ-processes; Seleznjev pth-mean approximation theorem.

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1 Introduction

Let $\{X(t), t \geq 0\}$ be a Gaussian process with continuous sample paths, and let $\mathcal{T}$ be a non-negative random variable independent of this process. In several important contributions Dębicki and his co-authors (see e.g., Dębicki (2002), Dębicki et al. (2004), Zwart et al. (2005), Dębicki and van Uitert (2006), Arendarczyk and Dębicki (2011,2012)) have derived exact tail asymptotic behaviour of the supremum $M(\mathcal{T}) = \sup_{t\in[0,\mathcal{T}]} X(t)$ of this process over the random interval $[0, \mathcal{T}]$, i.e., there is a known function h such that

$$\mathbb{P}\{M(\mathcal{T}) > u\} = h(u)(1 + o(1)), \quad u \to \infty. \tag{1}$$

The function $h(\cdot)$ is determined therein assuming that $\{X(t), t \geq 0\}$ is either a standard (with mean zero and unit variance) stationary Gaussian process or it has stationary increments, and supposing further that $\mathcal{T}$ has either regularly varying tail behaviour at ∞ or it is a Weibullian random variable. As pointed out in Zwart et al. (2005), Palmowski and Zwart (2007) several important applications in queuing theory, insurance and hydrodynamics are related to the tail asymptotics of the supremum of random processes over some random intervals.

If $\mathcal{T}$ is not random, say it is a deterministic constant and $\{X(t), t \geq 0\}$ is a standard stationary Gaussian process, then $h(\cdot)$ in (1) is given by the classical Pickands result (see Pickands (1969), Berman (1992), Leadbetter et al. (1983) or Piterbarg (1996))

$$\mathbb{P}\{M(\mathcal{T}) > u\} = \mathcal{T} H_\alpha u^{2/\alpha} \Psi(u)(1 + o(1)), \quad u \to \infty, \tag{2}$$

where $\Psi(\cdot)$ is the survival function of a $N(0, 1)$ random variable, provided that the correlation function $r(t)$ of X satisfies

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(A1). $r(t) = 1 - |t|^\alpha + o(|t|^\alpha)$ as $t \to 0$, with $\alpha \in (0, 2]$

and further $r(t) < 1$ for all $t > 0$. We note in passing that a deep contribution which gives the first rigorous proof of Pickands theorem presented in Pickands (1969) is Piterbarg (1972). Further we remark that Pickands constant H_α is defined as

$$H_\alpha = \lim_{S \to \infty} S^{-1} \mathbb{E}\left\{ \exp\Big(\sup_{t \in [0,S]} Z(t) \Big) \right\} \in (0, \infty),$$

with $\{Z(t), t \geq 0\}$ a fractional Brownian motion with continuous sample paths, mean function $\mathbb{E}\{Z(t)\} = -t^\alpha$ and covariance function

$$\operatorname{cov}(Z(s), Z(t)) = |t|^\alpha + |s|^\alpha - |t - s|^\alpha.$$

In this paper we are interested in the tail asymptotics of supremum $M(\mathcal{T})$ of a stationary χ-process when $\mathcal{T}$ has a regularly varying tail. The impetus for this investigation comes from Arendarczyk and Dębicki (2012) where a standard stationary Gaussian process with correlation function $r(t)$ is considered. If the non-negative random variable $\mathcal{T}$ has a finite expectation, then $\mathcal{T}$ in (2) can be substituted by $\mathbb{E}\{\mathcal{T}\}$. Another tractable case is when $\mathbb{E}\{\mathcal{T}\} = \infty$ and $\mathcal{T}$ has a regularly varying tail with index $\lambda \in [0, 1)$, i.e.,

$$\lim_{t \to \infty} \frac{\mathbb{P}\{\mathcal{T} > xt\}}{\mathbb{P}\{\mathcal{T} > t\}} = x^{-\lambda}, \quad \forall x > 0. \tag{3}$$

Since $\mathcal{T}$ can be large with large probability, as shown in Arendarczyk and Dębicki (2012) the Berman condition

(A2). $\lim_{t \to \infty} r(t) \ln t = 0$

is crucial for the derivation of the exact tail asymptotics of $M(\mathcal{T})$. The result derived in the aforementioned paper (which we formulate below) is important for our deviations.

Theorem AD. *Let $\{X(t), t \geq 0\}$ be a standard stationary Gaussian process with continuous sample paths and correlation function $r(t)$ satisfying (A1).*

i) If the non-negative random variable $\mathcal{T}$ independent of this process is such that $\mathbb{E}\{\mathcal{T}\} < \infty$, then

$$\mathbb{P}\{M(\mathcal{T}) > u\} = \mathbb{E}\{\mathcal{T}\}\mu(u)(1 + o(1)), \quad u \to \infty,$$

where $\mu(u) = H_\alpha u^{2/\alpha} \Psi(u)$.

ii) If both (A2) and (3) hold, then

$$\mathbb{P}\{M(\mathcal{T}) > u\} = \Gamma(1 - \lambda)\mathbb{P}\{\mu(u)\mathcal{T} > 1\}(1 + o(1)), \quad u \to \infty, \tag{4}$$

where $\Gamma(\cdot)$ is the Euler gamma function.

The recent paper Tan and Hashorva (2013a) discusses extensions of Arendarczyk-Dębicki Theorem AD for strongly dependent Gaussian processes. In this paper we are concerned with the tail asymptotics of the supremum over random intervals of χ-processes. The major difficulty when dealing with this class of non-Gaussian processes is that several important results like Berman's Normal Comparison Lemma and other well-established techniques presented in Piterbarg (1996) are not directly available.

Guided by the findings of Arendarczyk and Dębicki (2012) it is reasonable to conjecture that both cases $\mathbb{E}\{\mathcal{T}\} < \infty$ and $\mathbb{E}\{\mathcal{T}\} = \infty$ should be dealt with separately leading to two different results. Clearly, instead of Pickands result (2) we need to rely on Piterbarg theorem for χ-processes, see (6) below.

Our main results show that Arendarczyk-Dębicki theorem can be extended to χ-processes by choosing the appropriate substitute of the function $\mu(\cdot)$ appearing in Piterbarg theorem on supremum of χ-processes.

In this paper we also present limit theorems for $T \to \infty$. Since for approximation purposes Seleznjev pth-mean convergence theorem is of certain important, we conclude this paper with an extension of the aforementioned theorem for χ-processes.

Organisation of the paper: In the next section, we present the Arendarczyk-Dębicki theorems in the settings of this paper considering weakly and strongly dependent stationary χ-processes. Section 3 then contains two results, namely the limit theorem when $T \to \infty$ and Seleznjev pth-mean convergence theorem. All the proofs are relegated to Section 4.

2 Exact Tail Asymptotics

Define a stationary χ-process with k degrees of freedom as

$$\chi_k(t) = ||\mathbf{X}(t)|| = \left(X_1^2(t) + \cdots + X_k^2(t)\right)^{1/2}, \quad t \geq 0, \tag{5}$$

where $\mathbf{X}(t) = (X_1(t), \cdots, X_k(t))$ is a Gaussian vector process whose components are independent copies of a standard stationary Gaussian process $\{X(t), t \geq 0\}$ with correlation function $r(t)$. If $r(t)$ satisfies condition (A1) and $r(t) < 1$ for all $t > 0$, then by Piterbarg (1994) (see also Corollary 7.3 in Piterbarg (1996)) for any fixed $T > 0$ and $M_k(T) := \sup_{t\in[0,T]} \chi_k(t)$ we have

$$\mathbb{P}\left\{M_k(T) > u\right\} = T\mu_k(u)(1 + o(1)), \quad u \to \infty, \tag{6}$$

where

$$\mu_k(u) = \frac{2^{1-k/2}H_\alpha}{\Gamma(k/2)} u^{2/\alpha+k-2} \exp(-u^2/2).$$

The asymptotic properties of $M_k(T)$ have been studied by many authors, see e.g., Adler (1990), Albin (1990), Piterbarg (1994, 1996), Albin and Jarušková (2003), Konstantinides et al. (2004), Piterbarg and Stamatovic (2004), Jarušková (2010), Stamatovic and Stamatovic (2010), Jarušková and Piterbarg (2011) and Tan and Hashorva (2013b, 2013c) for various results.

We know from Tan and Hashorva (2013a) that Arendarczyk-Dębicki theorem can be extended to strongly dependent stationary Gaussian processes, which are naturally introduced replacing (A2) by

(A3). $\lim_{t\to\infty} r(t) \ln t = r \in [0, \infty)$.

When the correlation function of the standard Gaussian process $\{X(t), t \geq 0\}$ satisfies (A3) with $r > 0$ we refer to $\{\chi_k(t), t \geq 0\}$ as a strongly dependent stationary χ-process.

Clearly, the properties of χ-process $\{\chi_k(t), t \geq 0\}$ are determined by those of the standard Gaussian process $\{X(t), t \geq 0\}$. Next we present the analogous result of Theorem AD for weakly and strongly dependent χ-processes. The claim for $k = 1$ i.e., for stationary Gaussian processes follows immediately from Tan and Hashorva (2013a), therefore the proof of Theorem 2.1 will be given for $k \geq 2$.

Theorem 2.1. *Let $\{X(t), t \geq 0\}$ be a standard stationary Gaussian process with continuous sample paths and correlation function $r(t)$ satisfying (A1). Define $\{\chi_k(t), t \geq 0\}$ as in* (5) *and suppose that it is independent of*

$\mathcal{T} \geq 0$.

i) If $\mathbb{E}\{\mathcal{T}\} \in (0, \infty)$, *then*

$$\mathbb{P}\{M_k(\mathcal{T}) > u\} = \mathbb{E}\{\mathcal{T}\}\mu_k(u)(1 + o(1)), \quad u \to \infty. \tag{7}$$

ii) If both (A3) and (3) *hold, then*

$$\mathbb{P}\{M_k(\mathcal{T}) > u\} = g_{r,k}(\lambda)\mathbb{P}\{\mu_k(u)\mathcal{T} > 1\}(1 + o(1)), \quad u \to \infty, \tag{8}$$

where

$$g_{r,k}(\lambda) = \int_0^\infty \mathbb{E}\left\{\exp\left(-xe^{-r+\sqrt{2r}\chi_k(1)}\right)\right\} x^{-\lambda} dx.$$

Note in passing that when $r = 0$, then $g_{0,k}(\lambda) = \Gamma(1 - \lambda)$.

3 Limit Theorems

The Gumbel limit theorem for $a_T(M_1(T) - b_T)$ with $T \to \infty$ has been discussed in many important contributions, see e.g., the classical manuscripts Leadbetter et al. (1983), Adler (1990), Berman (1992), Piterbarg (1996) and Azaïs and Wschbor (2009). Typically, under the Berman condition the limit law is the Gumbel distribution $\Lambda(x) = \exp(-\exp(-x))$, and

$$a_T = b_T(1 + o(1)) = \sqrt{2 \ln T}, \quad T \to \infty.$$

When the Berman condition is substituted by the strong dependence assumption (A3) with $r > 0$, then the limit theorems still hold (see e.g., Mittal and Ylvisaker (1975), Piterbarg (1996), Kudrov and Piterbarg (2007), or Tan et al. (2012)). The limiting distribution is not Gumbel but a mixed Gumbel distribution. A direct consequence of a mixed Gumbel limit law is the convergence in probability

$$\frac{M_k(T)}{\sqrt{2 \ln T}} \overset{p}{\to} 1, \quad T \to \infty. \tag{9}$$

In general, (9) does not imply the mean convergence $\lim_{T\to\infty} \mathbb{E}\{M_k(T)/b_T\} = 1$.

A key contribution in the approximation of the distribution function of maxima of Gaussian random fields is Seleznjev (2006) which shows that the above convergence holds also in the pth mean, for any $p > 0$.

The aforementioned paper shows for the case $r = 0$ and $k = 1$ under a global condition on the Gaussian processes that in fact not only the mean convergence above is true, but also the pth-mean convergence; we shall refer to such a result as Seleznjev pth-mean convergence theorem, see Theorem 3.2 below.

It is intuitive that when $\mathcal{T}$ is a non-negative random variable, then there is a certain connection of the result in (8) and the limit law for the normalised maximum.

Theorem 3.1. *Let* $\{X(t), t \geq 0\}$ *and* $\{\chi_k(t), t \geq 0\}$ *be as in Theorem 2.1, and let* $\mathcal{T}_t$ *be a non-negative random variable such that* $\mathcal{T}_T/T \overset{p}{\to} \mathcal{T}$ *in probability, as* $T \to \infty$. *If further* $\{\chi_k(t), t \geq 0\}$ *and* $\{\mathcal{T}_t, t \geq 0\}$ *are independent and the correlation function* $r(t)$ *of* X *satisfies (A1) and (A3), then*

$$\lim_{T\to\infty} \sup_{x\in\mathbb{R}} \left| \mathbb{P}\left\{a_T\left(M_k(\mathcal{T}_T) - b_T\right) \leq x\right\} - \mathbb{E}\left\{\left(\Lambda_{r,k}(x)\right)^{\mathcal{T}}\right\} \right| = 0, \tag{10}$$

where

$$a_T = \sqrt{2\ln T}, \quad b_T = a_T + a_T^{-1}\ln\Big(K a_T^{2/\alpha-2+k}\Big), \quad K = \frac{2^{1-k/2}H_\alpha}{\Gamma(k/2)}$$

and for any $x \in \mathbb{R}$

$$\Lambda_{r,k}(x) \quad = \quad \mathbb{E}\left\{\exp\left(-e^{-x-r+\sqrt{2r}\chi_k(1)}\right)\right\}. \tag{11}$$

In view of Theorem 3.1

$$\lim_{T\to\infty}\sup_{x\in\mathbb{R}}\Big|\mathbb{P}\left\{a_T\left(M_k(T)-b_T\right)\le x\right\}-\Lambda_{r,k}(x)\Big| = 0, \tag{12}$$

which yields further (9).

In order to state Seleznjev pth-mean convergence theorem for χ-processes we show first Piterbarg inequality for χ-processes which is given for multiparameter Gaussian processes in Theorem 8.1 of Piterbarg (1996), see alternatively Theorem 8.1 in the seminal contribution Piterbarg (2001).

Proposition 3.2. *Let* $\{X(\boldsymbol{t}), \boldsymbol{t}\in\mathbb{R}^n\}, n\in\mathbb{N}$ *be a centered Gaussian random field with continuous sample paths and set* $\sigma(\boldsymbol{t}) = \sqrt{Var\{X(\boldsymbol{t})\}} > 0, \boldsymbol{t}\in\mathbb{R}^n$*. Suppose that the global Hölder condition*

$$\mathbb{E}\{(X(\boldsymbol{t})-X(\boldsymbol{s}))^2\} \le G||\boldsymbol{t}-\boldsymbol{s}||^\gamma, \quad \forall \boldsymbol{s},\boldsymbol{t}\in(0,\infty)^n \tag{13}$$

holds for some $G>0, \gamma\in(0,\infty)$*, and define* $\chi_k(\boldsymbol{t}) = \sqrt{X_1^2(\boldsymbol{t})+\cdots+X_k^2(\boldsymbol{t})}$ *with* $X_1,\cdots,X_k$ *independent copies of* X*. Then for any* $u>0, T>0$ *and any closed set* $E\subset[0,T]^n$ *we have*

$$\mathbb{P}\Big\{\sup_{\boldsymbol{t}\in E}\chi_k(\boldsymbol{t}) > u\Big\} \le CT^n u^{2n/\gamma}\Psi\Big(\frac{u}{\sup_{\boldsymbol{s}\in E}\sigma(\boldsymbol{s})}\Big), \tag{14}$$

with $C>0$ *not depending on* u *and* T*.* We conclude this section with Seleznjev pth-mean convergence theorem.

Theorem 3.3. *Let* $\{X(t), t\ge 0\}$ *be a standard stationary Gaussian process with continuous sample paths and correlation function* $r(t)$ *and define* $\{\chi_k(t), t\ge 0\}$ *as in* (5)*. If both (A1) and (A3) hold, then for any* $p>0$ *we have*

$$\lim_{T\to\infty}\mathbb{E}\left\{\Big(\frac{M_k(T)}{\sqrt{2\ln T}}\Big)^p\right\} = 1. \tag{15}$$

Remarks: a) For $k=1$ a uniform version of (12) motivated by Seleznjev (1991) is shown in Tan et al. (2012). In the aforementioned paper $\Lambda_{r,1}$ is not given by (11) but from the following equivalent formula

$$\Lambda_{r,1}(x) \quad = \quad \mathbb{E}\left\{\exp\left(-e^{-x-r-\ln 2}\big(e^{\sqrt{2r}\mathcal{W}}-e^{-\sqrt{2r}\mathcal{W}}\big)\right)\right\}, \tag{16}$$

with $\mathcal{W}$ a $N(0,1)$ random variable. We note that a uniform version of the limit theorem presented above for $\chi-$processes is possible to derive.

b) Clearly, for any integer k we have that $\Lambda_{0,k}(x) = \exp(-\exp(-x))$ is the unit Gumbel distribution. Thus in the weak dependence case corresponding to $r=0$ (i.e., when the Berman condition holds) the limit law of the normalised maximum is Gumbel, which is a well-known result for Gaussian processes, see e.g., Lifshits (1995), Leadbetter and Rootzén (1988) and Piterbarg (1996). In case of χ-processes the Gumbel limit law is shown in Piterbarg and Stamatovic (2004) and Stamatovic and Stamatovic (2010).

4 Proofs

This section consists of five lemmas and the proofs of the claimed results in Section 2 and 3. We first present some notation and details which will be useful for the proofs below. Crucial in the following is the construction of a grid $\mathfrak{R}_{b,u,\varepsilon}$ of points originally designated by Piterbarg and Stamatovic (2004), see also Konstantinides et al. (2004). For simplicity we shall consider the case $k \geq 2$ partitioning the sphere S_{k-1} onto $N(\varepsilon)$ parts $A_1, \cdots, A_{N(\varepsilon)}$ as follows: consider polar coordinates on the sphere S_{k-1}, $(x_1, \cdots, x_k) = S(\varphi_1, \cdots, \varphi_{k-1})$, $\varphi_1, \cdots, \varphi_{k-2} \in [0, \pi)$, $\varphi_{k-1} \in [0, 2\pi)$. Divide the interval $[0, \pi]$ on intervals of length ε (or less for the last interval), and do the same for the interval $[0, 2\pi]$. This partition of $[0, \pi]^{k-2} \times [0, 2\pi]$ generates the partition $A_1, \cdots, A_{N(\varepsilon)}$ of the sphere. Set next $L_u = L\mu_k^{-1}(u)$, with $L > 0$.

In order to construct $\mathfrak{R}_{b,u,\varepsilon}$ we choose in any A_j an inner point and consider the tangent plane to $[0, L_u] \times S_{k-1}$ at the chosen point. Introduce in the tangent plane rectangular coordinates, with origin at the tangent point; the first coordinate is assigned to the direction t. Consider the grid of points

$$\mathfrak{R}_{b,u,\varepsilon}^{j,P} := \left(bi_1 u^{-\frac{2}{\alpha}}, bi_2 u^{-1}, \cdots, bi_k u^{-1} \right), \quad j = 1, \cdots, N(\varepsilon),$$

where $(i_1, \cdots, i_k) \in \mathbb{Z}^k$. Suppose that ε is so small that the orthogonal projection of all $[0, T] \times A_j$ onto the corresponding tangent planes are one-to-one. Denote by A_j^P the projection of A_j at the tangent plane, and by $\mathfrak{R}_{b,u,\varepsilon}^j$ the prototype of $\mathfrak{R}_{b,u,\varepsilon}^{j,P}$ under this projection. The grid

$$\mathfrak{R}_b := \mathfrak{R}_{b,u,\varepsilon} = \bigcup_{j=1}^{N(\varepsilon)} \mathfrak{R}_{b,u,\varepsilon}^j,$$

with an appropriate choice of its parameters, will be used in the proofs below.

For a given $\delta > 0$ we partition the interval $[0, L_u]$ onto intervals of length one intermittent with intervals of length δ. If $M_u = [\frac{L_u}{1+\delta}] = O(\mu_k^{-1}(u))$, then the number of all intervals with length one is M_u; such intervals are index as $K_1, \cdots, K_{M_u}$ and we set for their union

$$K_u^* = \bigcup_{j=1}^{M_u} K_j.$$

In view of Piterbarg (1996), see also Lifshits (2012) for any closed non-empty set $E \subset [0, T]$ we have

$$\sup_{t \in E} \chi_k(t) = \sup_{(t, \mathbf{v}) \in E \times S_{k-1}} Y(t, \mathbf{v}),$$

where the Gaussian field $\{Y(t, \mathbf{v}), (t, \mathbf{v}) \in [0, T] \times S_{k-1}\}$ is given by

$$Y(t, \mathbf{v}) = X_1(t)v_1 + \cdots + X_k(t)v_k, \quad t \in [0, T], \tag{17}$$

with

$$\mathbf{v} = (v_1, \cdots, v_k) \in S_{k-1} := \{(x_1, \cdots, x_k) : x_1^2 + \cdots + x_k^2 = 1\}.$$

Denote by $r_Y(t,s)$ the correlation function of the field $Y(t,\mathbf{v})$, we have $r_Y(t,s) = r(t,s)A(\mathbf{v},\mathbf{w})$, where

$$A(\mathbf{v},\mathbf{w}) = 1 - \frac{||\mathbf{v}-\mathbf{w}||^2}{2} \le 1, \quad \mathbf{v},\mathbf{w} \in S_{k-1}.$$

Denote by $Y_j(t,\mathbf{v}), (t,\mathbf{v}) \in K_j \times S_{k-1},\ j = 1, \cdots, M_u$ independent copies of the Gaussian field $Y(t,\mathbf{v}), (t,\mathbf{v}) \in K_j \times S_{k-1}$ and let $Z_1, \cdots, Z_k$ be standard Gaussian random variables so that the components of the random vector

$$\Big(Y(t,\mathbf{v}), Y_1(t,\mathbf{v}), \cdots, Y_{M_u}(t,\mathbf{v}), Z_1, \cdots, Z_k\Big)$$

are mutually independent and set

$$Z(\mathbf{v}) := Z_1 v_1 + \cdots + Z_k v_k, \quad \mathbf{v} \in S_{k-1}.$$

Further, set $\varrho(L_u) = r/\ln L_u$ and define

$$Y_0(t,\mathbf{v}) = \sqrt{1-\varrho(L_u)}Y_j(t,\mathbf{v}) + \sqrt{\varrho(L_u)}Z(\mathbf{v}), \quad (t,\mathbf{v}) \in K_u^* \times S_{k-1}$$

if $(t,\mathbf{v}) \in K_j \times S_{k-1}, j = 1, \cdots, M_u$. Denote by $r_{Y_0}((t,\mathbf{v}),(s,\mathbf{w}))$ the correlation function of the field $Y_0(t,\mathbf{v})$. We have

$$r_{Y_0}((t,\mathbf{v}),(s,\mathbf{w})) = r^*(t,s)A(\mathbf{v},\mathbf{w}),$$

where

$$r^*(t,s) = \begin{cases} \varrho(L_u) + (1-\varrho(L_u))r(t,s), & t \in K_i, s \in K_j, i = j, \\ \varrho(L_u), & t \in K_i, s \in K_j, i \ne j. \end{cases} \tag{18}$$

Both Lemma 4.1 and Lemma 4.2 below are taken from Piterbarg and Stamatovic (2004).

Lemma 4.1. *For given positive constants $\theta_1 < \theta_2$ there exits a grid $\mathfrak{R}_b := \mathfrak{R}_{b,u,\varepsilon}$ on $[0,L_u] \times S_{k-1}$, $L_u = L\mu_k^{-1}(u)$ such that*

$$\mathbb{P}\Big\{\max_{(t,\mathbf{v})\in([0,L_u]\times S_{k-1})\cap\mathfrak{R}_b} Y(t,\mathbf{v}) > u\Big\} - \mathbb{P}\Big\{\max_{(t,\mathbf{v})\in[0,L_u]\times S_{k-1}} Y(t,\mathbf{v}) > u\Big\} \to 0, \quad u \to \infty, \quad b \downarrow 0 \tag{19}$$

holds uniformly for $L \in [\theta_1, \theta_2]$. Further, for the gird $\mathfrak{R}_b := \mathfrak{R}_{b,u,\varepsilon}$ the convergence

$$\mathbb{P}\Big\{\max_{(t,\mathbf{v})\in(K_u^*\times S_{k-1})\cap\mathfrak{R}_b} Y(t,\mathbf{v}) > u\Big\} - \mathbb{P}\Big\{\max_{(t,\mathbf{v})\in([0,L_u]\times S_{k-1})\cap\mathfrak{R}_b} Y(t,\mathbf{v}) > u\Big\} \to 0, \quad b \downarrow 0 \tag{20}$$

holds uniformly for $L \in [\theta_1, \theta_2]$.

Lemma 4.2. *The claim in* (19) *holds with the same grid $\mathfrak{R}_b$ also for the field $Y_0(t,\mathbf{v})$.*

Lemma 4.3. *For given positive constants $\theta_1 < \theta_2$ and the grid $\mathfrak{R}_b := \mathfrak{R}_{b,u,\varepsilon}$*

$$\mathbb{P}\Big\{\max_{(t,\mathbf{v})\in(K_u^*\times S_{k-1})\cap\mathfrak{R}_b} Y(t,\mathbf{v}) > u\Big\} - \mathbb{P}\Big\{\max_{(t,\mathbf{v})\in(K_u^*\times S_{k-1})\cap\mathfrak{R}_b} Y_0(t,\mathbf{v}) > u\Big\} \to 0, \quad u \to \infty \tag{21}$$

holds uniformly for $L \in [\theta_1, \theta_2]$.

Proof. The proof uses similar arguments as that of Lemma 15.4 in Piterbarg (1996) and Lemma 5 in Stamatovic and Stamatovic (2010). Introduce next the Gaussian random field

$$Y_h(t,\mathbf{v}) = \sqrt{h}Y(t,\mathbf{v}) + \sqrt{1-h}Y_0(t,\mathbf{v}), \quad (t,\mathbf{v}) \in K_u^* \times S_{k-1},$$

with $h \in (0,1)$ and denote by $r_h((t,\mathbf{v}),(s,\mathbf{w}))$, $(t,\mathbf{v}),(s,\mathbf{w}) \in K_u^* \times S_{k-1}$ its covariance function. It is easy to calculate $r_h((t,\mathbf{v}),(s,\mathbf{w})) = r_h(t,s)A(\mathbf{v},\mathbf{w})$, where $r_h(t,s) = hr(t,s) + (1-h)r^*(t,s)$. By Berman's inequality (see Piterbarg (1996))

$$\begin{aligned}
&\left|\mathbb{P}\Big\{\max_{(t,\mathbf{v})\in(K_u^*\times S_{k-1})\cap\mathfrak{R}_b} Y(t,\mathbf{v}) > u\Big\} - \mathbb{P}\Big\{\max_{(t,\mathbf{v})\in(K_u^*\times S_{k-1})\cap\mathfrak{R}_b} Y_0(t,\mathbf{v}) > u\Big\}\right| \\
&\le \frac{1}{2\pi} \sum_{\substack{(t,\mathbf{v}),(s,\mathbf{w})\in[K_u^*\times S_{k-1}]\cap\mathfrak{R}_b,\\ (t,\mathbf{v})\neq(s,\mathbf{w})}} |r_Y((t,\mathbf{v}),(s,\mathbf{w})) - r_{Y_0}((t,\mathbf{v}),(s,\mathbf{w}))| \\
&\quad\times \int_0^1 \frac{1}{\sqrt{(1-r_h(t,s))}} \exp\left(-\frac{u^2}{1+r_h((t,\mathbf{v}),(s,\mathbf{w}))}\right) dh. \qquad (22)
\end{aligned}$$

As in Piterbarg (1996) the summands in the last sum above will be denoted by $\beta(t,s,\mathbf{v},\mathbf{w})$. Next, let $B_i, i \le 4, C_i, i \le 13$ be positive constants and consider first s,t that belong to the same interval from K_u^*. The condition (A1) implies that there exists a number $\tau \in (0, 2^{-1/\alpha})$ such that for all $|t-s| < \tau$,

$$\frac{1}{2}|t-s|^\alpha \le 1 - r(t,s) \le 2|t-s|^\alpha. \qquad (23)$$

By the assumptions $\varrho(L_u) < \frac{B_1}{u^2}$. Further, from the construction of $\mathfrak{R}_b$ the number of points from $(K_j \times S_{k-1}) \cap \mathfrak{R}_b, j = 1, \cdots, M_u$ does not exceed $B_2 u^{2/\alpha+k-1}$, and the number of points from $K_j \cap \mathfrak{R}_b, j = 1, \cdots, M_u$ does not exceed $B_3 u^{2/\alpha}$. Similarly, the number of points from $S_{k-1} \cap \mathfrak{R}_b$ is not greater than $B_4 u^{k-1}$. Next for some $x > 0$ define

$$\mathcal{A}_x := \sum_{\substack{(t,\mathbf{v}),(s,\mathbf{w})\in[K_u^*\times S_{k-1}]\cap\mathfrak{R}_b,\\ (t,\mathbf{v})\neq(s,\mathbf{w}),|t-s|\le x}} \beta(t,s,\mathbf{v},\mathbf{w})$$

and similarly $\mathcal{A}_x^c$ which is as above where we require further $|t-s| > x$. We have with $\overline{K_{u,j}} = K_j \cap \{biu^{-2/\alpha}, i = 1, \cdots\}, j = 1, 2, \cdots, M_u$

$$\begin{aligned}
\mathcal{A}_\tau &\le C_1 u^{-2} \sum_{\substack{\mathbf{w},\mathbf{v}\in S_{k-1}\cap\mathfrak{R}_b,|t-s|\in\overline{K_{u,j}},|t-s|\le\tau\\ j=1,2,\cdots,M_u}} \sqrt{1-r(t,s)} \exp\left(-\frac{u^2}{1+r(t,s)|A(\mathbf{v},\mathbf{w})|}\right) \\
&= C_1 u^{-2} \exp\left(-\frac{u^2}{2}\right) \sum_{\substack{\mathbf{w},\mathbf{v}\in S_{k-1}\cap\mathfrak{R}_b,|t-s|\in\overline{K_{u,j}},|t-s|\le\tau\\ j=1,2,\cdots,M_u}} \sqrt{1-r(t,s)} \\
&\quad \times \exp\left(-u^2 \frac{1-r(t,s)|A(\mathbf{v},\mathbf{w})|}{2(1+r(t,s)|A(\mathbf{v},\mathbf{w})|)}\right), \\
&= C_1 u^{-2} \exp\left(-\frac{u^2}{2}\right) \sum_{\substack{\mathbf{w},\mathbf{v}\in S_{k-1}\cap\mathfrak{R}_b,|t-s|\in\overline{K_{u,j}},|t-s|\le\tau\\ j=1,2,\cdots,M_u}} \sqrt{1-r(t,s)} \exp\left(-u^2 \frac{1-r(t,s)}{2(1+r(t,s))}\right) \\
&\quad \times \exp\left(-u^2 \frac{r(t,s)(1-|A(\mathbf{v},\mathbf{w})|)}{2(1+r(t,s))(1+r(t,s)|A(\mathbf{v},\mathbf{w})|)}\right) \\
&= C_1 u^{k-3} \exp\left(-\frac{u^2}{2}\right) \sum_{\substack{\mathbf{w}\in S_{k-1}\cap\mathfrak{R}_b,|t-s|\in\overline{K_{u,j}},|t-s|\le\tau\\ j=1,2,\cdots,M_u}} \sqrt{1-r(t,s)} \exp\left(-u^2 \frac{1-r(t,s)}{2(1+r(t,s))}\right)
\end{aligned}$$

$$\times \exp\left(-u^2 \frac{r(t,s)(1-|A(\mathbf{v}_0,\mathbf{w})|)}{2(1+r(t,s))(1+r(t,s)|A(\mathbf{v}_0,\mathbf{w})|)}\right)$$

where $\mathbf{v}_0$ is a fixed point on $S_{k-1}\cap\mathfrak{R}_b$. Since

$$\sum_{\substack{\mathbf{w}\in S_{k-1}\cap\mathfrak{R}_b, |t-s|\in\overline{K_{u,j}}, |t-s|\le\tau \\ j=1,2,\cdots,M_u}} \exp\left(-u^2 \frac{r(t,s)(1-|A(\mathbf{v}_0,\mathbf{w})|)}{2(1+r(t,s))(1+r(t,s)|A(\mathbf{v}_0,\mathbf{w})|)}\right)$$
$$\le \sum_{\mathbf{w}\in S_{k-1}\cap\mathfrak{R}_b} \exp\left(-C_2u^2||\mathbf{v}_0-\mathbf{w}||^2\right) \le C_3,$$

then by (23)

$$\begin{aligned}
\mathcal{A}_\tau &\le C_4 u^{k-3}\exp\left(-\frac{u^2}{2}\right) \sum_{\substack{|t-s|\in\overline{K_{u,j}}, |t-s|\le\tau \\ j=1,2,\cdots,M_u}} \sqrt{2|t-s|^\alpha}\exp\left(-\frac{u^2|t-s|^\alpha}{8}\right)\\
&\le C_4 M_u u^{2/\alpha+k-3}\exp\left(-\frac{u^2}{2}\right)\sum_{|z|\in\overline{K_{u,1}}, |z|\le\tau}\sqrt{2|z|^\alpha}\exp\left(-\frac{u^2|z|^\alpha}{8}\right)\\
&\le C_5u^{-2}\sum_{i=1}^{\infty} i^{\alpha/2}\exp\left(-\frac{(bi)^\alpha}{8}\right) = O(u^{-2}), \quad u\to\infty.
\end{aligned}$$

Denote by $\gamma = \frac{1}{4}\inf_{|t-s|>\tau}(1-|r(t,s)|)$. For any $|t-s|>\tau$

$$\exp\left(-u^2\frac{1-|r(t,s)|}{2(1+|r(t,s)|)}\right) \le \exp\left(-u^2\frac{4\gamma}{4}\right) = \exp(-\gamma u^2).$$

With similar arguments as for $\mathcal{A}_\tau$ and the above fact, we get

$$\begin{aligned}
\mathcal{A}_\tau^c &\le C_6 M_u u^{k-3}u^{-2}\sum_{|t-s|\in\overline{K_u,1}, |t-s|>\tau, \mathbf{w}\in S_{k-1}\cap\mathfrak{R}_b}\exp\left(-\frac{u^2}{1+|r(t,s)|}\right)\\
&\le C_7 M_u u^{k-3}u^{-2}\sum_{|t-s|\in\overline{K_u,1}, \mathbf{w}\in S_{k-1}\cap\mathfrak{R}_b}\exp\left(-\frac{u^2}{2}\right)\exp\left(-\gamma u^2\right)\\
&\le C_7 M_u u^{2/\alpha+k-3}u^{-2}\sum_{z\in\overline{K_u,1}, \mathbf{w}\in S_{k-1}\cap\mathfrak{R}_b}\exp\left(-\frac{u^2}{2}\right)\exp\left(-\gamma u^2\right)\\
&\le C_8 M_u u^{4/\alpha+2k-2}u^{-2}\exp\left(-u^2(\gamma+\frac{1}{2})\right) = o(1), \quad u\to\infty.
\end{aligned}$$

Next, we estimate the parts of the sum (22) where $t\in K_i, s\in K_j, i\neq j$. From (A3) it follows that there exists $\eta_2\in(0,1)$, such that $r(t,s)<1-\eta_2$, where $|t-s|>\delta$. Consider $t\in K_i$, $s\in K_j$ such that $\sup\{|t-s| : t\in K_i, s\in K_j\}\le L_u^{\eta_1}$, where $\eta_1\in(0,\frac{\eta_2}{2-\eta_2})$. It follows that as $u\to\infty$

$$\begin{aligned}
\mathcal{A}_{L_u^{\eta_1}} &= O\Bigg(\sum_{\substack{(t,\mathbf{v}),(s,\mathbf{w})\in[K_u^*\times S_{k-1}]\cap\mathfrak{R}_b, \\ (t,\mathbf{v})\neq(s,\mathbf{w}), |t-s|\le L_u^{\eta_1}}}\exp\left(-\frac{u^2}{2-\eta_2}\right)\Bigg)\\
&= O\Big(M_u u^{2/\alpha+k-1}L_u^{\eta_1}u^{2/\alpha+k-1}e^{-\frac{u^2}{2-\eta_2}}\Big) = o(1).
\end{aligned}$$

Let $\varpi(t,s)=\max\{r(t,s), r^*(t,s)\}$ and $\vartheta(t)=\sup_{t<kq-lq\le T}\{\varpi(kq-lq)\}$, where $q=bu^{-2/\alpha}$. Using further (18) we get

$$\mathcal{A}_{L_u^{\eta_1}}^c \le \sum_{\substack{(t,\mathbf{v}),(s,\mathbf{w})\in[K_u^*\times S_{k-1}]\cap\mathfrak{R}_b, \\ (t,\mathbf{v})\neq(s,\mathbf{w}), |t-s|>L_u^{\eta_1}}} |r(t,s)-r^*(t,s)|\exp\left(-\frac{u^2}{1+\varpi(t,s)}\right)$$

$$\leq \quad C_9 M_u u^{k-1} u^{k-1} \sum_{t,s\in[K_u^*\cap\Re_b],|t-s|>L_u^{\eta_1}} |r(t,s) - r/\ln L_u| \exp\left(-\frac{u^2}{1+\varpi(t,s)}\right)$$

$$\leq \quad C_{10} M_u u^{2k-2} \sum_{L_u^{\eta_1}<|kq-lq|<L_u} |r(kq,lq) - r/\ln L_u| \exp\left(-\frac{u^2}{1+\varpi(kq,lq)}\right).$$

Moreover, by the assumption (A3), we have $\vartheta(t)\ln t \leq C_{11}$ for all sufficiently large t. Thus, $\varpi(kq,lq) \leq \vartheta(L_u^{\eta_1}) \leq C_{11}/\ln L_u^{\eta_1}$ for $|kq-lq| > L_u^{\eta_1}$. The following inequality holds

$$\sum_{L_u^{\eta_1}<|kq-lq|<L_u} |r(kq,lq) - r/\ln L_u| \exp\left(-\frac{u^2}{1+|\varpi(kq,lq)|}\right)$$

$$\leq \left(\frac{L_u}{q\ln L_u}\exp\left(-\frac{u^2}{1+\vartheta(L_u^{\eta_1})}\right)\right)\left(\frac{q\ln L_u}{L_u}\sum_{L_u^{\eta_1}<|kq-lq|<L_u,} |r(kq,lq) - r/\ln L_u|\right). \tag{24}$$

From (A3) we deduce that $\sup_{t>k}|r(t)|\ln k \leq \delta_k + r$, $k \geq k_0$, $\delta_k > 0$ and $\delta_k = o(1)$ as $k\to\infty$, and thus $\sup_{t>s}|r(t)|\ln s$ is bounded. Hence there exists C_{12} such that $\vartheta(L_u^{\eta_1}) < C_{12}/\ln L_u$ and

$$M_u u^{2/\alpha+2k-2}\frac{L_u}{q\ln L_u}\exp\left(-\frac{u^2}{1+\vartheta(L_u^{\eta_1})}\right) = O\left(\exp\left(u^2 - \frac{u^2}{1+C_{13}/u^2}\right)\right)$$

is bounded. Considering the second term on the right hand side of (24), if (A3) holds, we have

$$\frac{q\ln L_u}{u^{2/\alpha}L_u}\sum_{L_u^{\eta_1}<|kq-lq|<L_u}\left|r(kq,lq) - \frac{r}{\ln(|kq-lq|)}\right| \leq \frac{L_u - L_u^{\eta_1}}{\eta_1 L_u}\max_{L_u^{\eta_1}<kq<L_u}|r(kq)\ln(kq) - r| = o(1) \tag{25}$$

as $u\to\infty$ and by an estimate as in the proof of Lemma 6.4.1 of Leadbetter et al. (1983)

$$\frac{q\ln L_u}{u^{2/\alpha}L_u}\sum_{L_u^{\eta_1}<|kq-lq|<L_u}\left|\frac{r}{\ln(|kq-lq|)} - \frac{r}{\ln L_u}\right| = \frac{q\ln L_u}{L_u}\sum_{L_u^{\eta_1}<kq<L_u}\left|\frac{r}{\ln(kq)} - \frac{r}{\ln L_u}\right| = o(1), \quad u\to\infty \tag{26}$$

Since the first term on the right-hand side of (24) is bounded, we conclude from (25) and (26) that the left-hand side of (24) convergence to zero, and hence the proof is established. □

The following result plays a crucial role in the proof of Theorem 2.1; set below

$$m(u) := \mu_k^{-1}(u).$$

Lemma 4.4. *Let $\{X(t), t\geq 0\}$ be a standard stationary Gaussian process with correlation $r(t)$, and define χ_k as in (5). If $r(t)$ satisfies both (A1) and (A3), then for any $0<\theta_1<\theta_2<\infty$*

$$\lim_{u\to\infty}\sup_{x\in[\theta_1,\theta_2]}\left|\mathbb{P}\left\{\sup_{s\in[0,xm(u)]}\chi_k(s)\leq u\right\} - \mathbb{E}\left\{\exp(-xe^{-r+\sqrt{2r}\chi_k(1)})\right\}\right| = 0.$$

Proof. By the definition of the field $Y_0(t,\mathbf{v})$ with $L=x$ we obtain

$$\mathbb{P}\left\{\max_{(t,\mathbf{v})\in(K_u^*\times S_{k-1})\cap\Re_b} Y_0(t,\mathbf{v})\leq u\right\}$$
$$= \frac{1}{(2\pi)^{n/2}}\int_{\boldsymbol{z}\in\mathbb{R}^k} e^{-\frac{||\boldsymbol{z}||^2}{2}}\mathbb{P}\left\{\max_{(t,\mathbf{v})\in(K_u^*\times S_{k-1})\cap\Re_b} Y_0(t,\mathbf{v})\leq u|Z_1=z_1,\cdots,Z_k=z_k\right\}dz_1\cdots dz_k$$
$$= \frac{1}{(2\pi)^{n/2}}\int_{\boldsymbol{z}\in\mathbb{R}^k} e^{-\frac{||\boldsymbol{z}||^2}{2}}\mathbb{P}\left\{\max_{(t,\mathbf{v})\in[(0,1)\times S_{k-1}]} Y(t,\mathbf{v})\leq\frac{u-\sqrt{\varrho(L_u)}||\boldsymbol{z}||}{\sqrt{1-\varrho(L_u)}}\right\}^{M_u}dz_1\cdots dz_k, \tag{27}$$

where

$$M_u = \left[\frac{L_u}{1+\delta}\right] = \left[\frac{x\mu_k^{-1}(u)}{1+\delta}\right].$$

Since as $u \to \infty$

$$u_{\boldsymbol{z}} \quad := \quad \frac{u-\rho^{1/2}(L_u)||\boldsymbol{z}||}{(1-\varrho(L_u))^{1/2}} = u + \frac{-\sqrt{2r}||\boldsymbol{z}||+r}{u} + o(\frac{1}{u}),$$

then we have

$$M_u \mu_k(u_{\boldsymbol{z}}) = -xe^{-r+\sqrt{2r}||\boldsymbol{z}||}(1+o(1)), \quad u \to \infty.$$

Utilising thus (6) we may further write

$$\begin{aligned}
\mathbb{P}\Big\{\max_{(t,\mathbf{v})\in(0,1)\times S_{k-1}} Y(t,\mathbf{v}) \le u_{\boldsymbol{z}}\Big\}^{M_u} &= \left(1-\mathbb{P}\left\{\max_{t\in(0,1)}\chi_k(t) > u_{\boldsymbol{z}}\right\}\right)^{M_u} \\
&= \exp\left(M_u \ln\left(1-\mathbb{P}\left\{\max_{t\in(0,1)}\chi_k(t) > u_{\boldsymbol{z}}\right\}\right)\right) \\
&= \exp\left(-M_u\mu_k\left(u_{\boldsymbol{z}}\right)(1+o(1))\right) \\
&\to \exp\left(-xe^{-r+\sqrt{2r}||\boldsymbol{z}||}\right), \quad u \to \infty. \qquad (28)
\end{aligned}$$

Consequently, as $u \to \infty$

$$\lim_{u\to\infty} \mathbb{P}\Big\{\max_{(t,\mathbf{v})\in K_u^*\times S_{k-1}} Y_0(t,\mathbf{v}) > u\Big\} = \frac{1}{(2\pi)^{n/2}} \int_{\boldsymbol{z}\in\mathbb{R}^k} e^{-\frac{||\boldsymbol{z}||^2}{2}} \exp\left(-xe^{-r+\sqrt{2r}||\boldsymbol{z}||}\right) dz_1\cdots dz_k,$$

hence the proof follows by Lemma 4.1, 4.2 and 4.3 and dominated convergence theorem. □

Lemma 4.5. *Let $\{M(t), t\ge 0\}$ be non-negative random variables such that for constants $a_t > 0, b_t, t \ge 0$ we have the convergence in distribution*

$$a_t(M(t)-b_t) \overset{d}{\to} \mathcal{M}, \quad t \to \infty,$$

with $\mathcal{M}$ some non-degenerate random variable. If $\lim_{t\to\infty} a_t b_t = \infty$, then for any $p > 0$ we have

$$\mathbb{E}\{[M(t)]^p\} \ge b_t^p(1+o(1)), \quad t \to \infty$$

and if further for some positive constants α, C, λ, τ

$$\lambda b_t^\tau = \ln t + o(1) \quad \text{and} \quad \mathbb{P}\{M(t) > y\} \le Cty^\alpha \exp(-\lambda y^\tau) \qquad (29)$$

hold for any t large and all y large enough, uniformly in t, then

$$\mathbb{E}\{[M(t)]^p\} = b_t^p(1+o(1)), \quad t \to \infty. \qquad (30)$$

Proof. Borrowing the idea of the proof of Theorem 1 in Seleznjev (2006) the first claim follows by applying Chebyshev's inequality, using additionally the assumptions $\lim_{t\to\infty} a_t b_t = \infty$ and $\mathcal{M}$ has a non-degenerate distribution function. Proceeding as in the proof of the aforementioned theorem define for some $z > 0$

$$c_t = u_t + zu_t^{1-\tau/p}\ln u_t, \quad u_t = b_t^p.$$

By assumption (29) we have $\lim_{t\to\infty} b_t = \infty$, hence

$$\begin{aligned}\mathbb{E}\{[M(t)]^p\} &\le u_t\left(1+z\frac{\ln u_t}{b_t^\tau}\right)+\int_{c_t}^\infty \mathbb{P}\left\{M(t)>y^{1/p}\right\}dy\\ &\le b_t^p\left(1+zp\frac{\ln b_t}{b_t^\tau}\right)+(1+o(1))C^* t c_t^\kappa \exp(-\lambda c_t^{\tau/p})\end{aligned}$$

for some positive constants κ and C^*. Since

$$c_t^{\tau/p}=b_t^\tau+\tau z\ln b_t+o(1),\quad \ln c_t = p\ln b_t - zp\frac{\ln b_t}{b_t^\tau},\quad t\to\infty,$$

then by (29) as $t\to\infty$

$$\ln(C^* t c_t^\kappa \exp(-\lambda c_t^{\tau/p})) = O(1)-(l\tau z-\kappa p)\ln b_t,$$

hence choosing z large enough we obtain

$$\mathbb{E}\{[M(t)]^p\} \le b_t^p\left(1+zp\frac{\ln b_t}{b_t^\tau}\right)+o\Big(\frac{1}{b_t}\Big)=b_t^p(1+o(1)),\quad t\to\infty$$

and thus the claim follows. □

Proof of Theorem 2.1. The proof of the first assertion is the same as that of Theorem 3.1 of Arendarczyk and Dębicki (2012). Next we prove the second assertion; we define

$$\Upsilon_{r,k}(x):=\mathbb{E}\left\{\exp\left(-xe^{-r+\sqrt{2r}\chi_k(1)}\right)\right\},\quad \overline{\Upsilon}_{r,k}(x):=1-\Upsilon_{r,k}(x),\quad x\in\mathbb{R}.$$

Case $\lambda>0$: Following Arendarczyk and Dębicki (2012) we make the following decomposition with F the distribution function of $\mathcal{T}$:

$$\begin{aligned}\mathbb{P}\left\{M_k(\mathcal{T})>u\right\} &= \int_0^{\theta_1 m(u)}\mathbb{P}\left\{\sup_{s\in[0,t]}\chi_k(s)>u\right\}dF(t)+\int_{\theta_1 m(u)}^{\theta_2 m(u)}\mathbb{P}\left\{\sup_{s\in[0,t]}\chi_k(s)>u\right\}dF(t)\\ &\quad+\int_{\theta_2 m(u)}^\infty \mathbb{P}\left\{\sup_{s\in[0,t]}\chi_k(s)>u\right\}dF(t)=:I_1+I_2+I_3.\end{aligned}$$

From the proof of Theorem 3.2 of Arendarczyk and Dębicki (2012) as $u\to\infty$ we have

$$I_1\le\frac{\lambda}{1-\lambda}\theta_1^{1-\lambda}\mathbb{P}\left\{\mathcal{T}>m(u)\right\}(1+o(1))$$

and

$$I_3\le\mathbb{P}\left\{\mathcal{T}>\theta_2 m(u)\right\}=\theta_2^{-\lambda}\mathbb{P}\left\{\mathcal{T}>m(u)\right\}(1+o(1)).$$

Applying Lemma 4.4, for $\epsilon>0$ and sufficiently large u we obtain the following upper bound

$$\begin{aligned}\frac{I_2}{1+\epsilon} &= \frac{1}{1+\epsilon}\int_{\theta_1}^{\theta_2}\mathbb{P}\left\{\sup_{s\in[0,tm(u)]}\chi_k(s)>u\right\}dF(xm(u))\le\int_{\theta_1}^{\theta_2}\overline{\Upsilon}_{r,k}(x)dF(xm(u))\\ &= \int_{\theta_1}^{\theta_2}\Upsilon_{r,k}(x)\mathbb{P}\left\{\mathcal{T}>xm(u)\right\}dx-\overline{\Upsilon}_{r,k}(\theta_2)\mathbb{P}\left\{T>\theta_2 m(u)\right\}+\overline{\Upsilon}_{r,k}(\theta_1)\mathbb{P}\left\{\mathcal{T}>\theta_1 m(u)\right\}.\end{aligned}$$

Similarly, for $\epsilon\in(0,1)$ and sufficiently large u

$$\frac{I_2}{1-\epsilon}\ge\int_{\theta_1}^{\theta_2}\Upsilon_{r,k}(x)\mathbb{P}\left\{\mathcal{T}>xm(u)\right\}dx-\overline{\Upsilon}_{r,k}(\theta_2)\mathbb{P}\left\{\mathcal{T}>\theta_2 m(u)\right\}+\overline{\Upsilon}_{r,k}(\theta_1)\mathbb{P}\left\{\mathcal{T}>\theta_1 m(u)\right\}.$$

The regularly varying tail of $\mathcal{T}$ combined with Theorem 1.5.2 in Bingham et al. (1987) imply

$$\int_{\theta_1}^{\theta_2} \Upsilon_{r,k}(x)\mathbb{P}\{\mathcal{T} > xm(u)\} dx = \mathbb{P}\{\mathcal{T} > m(u)\} \int_{\theta_1}^{\theta_2} \Upsilon_{r,k}(x)x^{-\lambda}dx(1+o(1))$$

as $u \to \infty$. Thus for each $\epsilon \in (0,1)$, and $\theta_2 > \theta_1 > 0$

$$\frac{1}{1+\epsilon} \limsup_{u\to\infty} \frac{I_2}{\mathbb{P}\{\mathcal{T} > m(u)\}} \leq \int_{\theta_1}^{\theta_2} \Upsilon_{r,k}(x)x^{-\lambda}dx - \overline{\Upsilon}_{r,k}(\theta_2)\theta_2^{-\lambda} + \overline{\Upsilon}_{r,k}(\theta_1)\theta_1^{-\lambda}$$

and

$$\frac{1}{1-\epsilon} \liminf_{u\to\infty} \frac{I_2}{\mathbb{P}\{T > m(u)\}} \geq \int_{\theta_1}^{\theta_2} \Upsilon_{r,k}(x)x^{-\lambda}dx - \overline{\Upsilon}_{r,k}(\theta_2)\theta_2^{-\lambda} + \overline{\Upsilon}_{r,k}(\theta_1)\theta_1^{-\lambda}.$$

Hence, letting $\theta_1 \to 0$, $\theta_2 \to \infty$ and $\epsilon \to 0$, we conclude that I_1 and I_3 are negligible ($u \to \infty$) compared with I_2, and moreover

$$I_2 = \int_0^\infty \Upsilon_{r,k}(x)x^{-\lambda}dx\mathbb{P}\{\mathcal{T} > m(u)\}(1+o(1)), \quad u \to \infty.$$

<u>Case $\lambda = 0$</u>. The proof is similar to that of Theorem 3.3 of Arendarczyk and Dębicki (2012). For given $\theta_2 > 0$ Lemma 4.4 implies

$$\begin{aligned}\mathbb{P}\{M_k(\mathcal{T}) > u\} &\geq \mathbb{P}\left\{\sup_{s\in[0,\theta_2 m(u)]} X_1(s) > u\right\}\mathbb{P}\{\mathcal{T} > \theta_2 m(u)\}(1+o(1)) \\ &= \overline{\Upsilon}_{r,k}(\theta_2)\mathbb{P}\{\mathcal{T} > m(u)\}(1+o(1)), \quad u \to \infty.\end{aligned}$$

Thus, letting $\theta_2 \to \infty$ we have

$$\mathbb{P}\{M_k(\mathcal{T}) > u\} \geq \mathbb{P}\{\mathcal{T} > m(u)\}(1+o(1)), \quad u \to \infty.$$

By Karamata's theorem (see e.g., Resnick (1987)) and (5) we obtain further

$$\begin{aligned}\mathbb{P}\{M_k(\mathcal{T}) > u\} &\leq \int_0^{\theta_1 m(u)} \mathbb{P}\left\{\sup_{s\in[0,t]} \chi_k(s) > u\right\} dF(t) + \mathbb{P}\{\mathcal{T} > m(u)\} \\ &\leq \mathbb{P}\left\{\sup_{s\in[0,1]} \chi_k(s) > u\right\}\left[\int_0^{\theta_1 m(u)} \mathbb{P}\{\mathcal{T} > t\} dt + 1\right] + \mathbb{P}\{\mathcal{T} > m(u)\} \\ &= (1+\theta_1)\mathbb{P}\{\mathcal{T} > m(u)\}(1+o(1)), \quad u \to \infty,\end{aligned}$$

which together with the fact that θ_1 can be arbitrary small establishes the proof. □

Proof of Theorem 3.1. For $u_T(x) = a_T^{-1}x + b_T$ we obtain

$$T\mu_k(u_T(x)) = e^{-x}(1+o(1)), \quad T \to \infty.$$

Hence, if we replace L_u by T and u by $u_T(x)$, then by checking the proofs of Lemmas 4.1-4.3 it follows that they hold uniformly for $x \in \mathbb{R}$. Thus, repeating the steps of the proof of Lemma 4.4 we obtain

$$\lim_{T\to\infty} \sup_{x\in\mathbb{R}} \left|\mathbb{P}\{a_T(M_k(T) - b_T) \leq x\} - \Lambda_{r,k}(x)\right| = 0, \tag{31}$$

where $\Lambda_{r,k}$ is defined in (11). Since further the following convergence

$$\frac{a_T}{a_{\mathcal{T}_T}} \to 1, \quad a_T(b_{\mathcal{T}_T} - b_T) \to \ln \mathcal{T}, \quad T \to \infty$$

holds in probability, then in view of the transfer theorem of Gnedenko and Fahim (1969) it follows that

$$\begin{aligned}\mathbb{P}\left\{a_T\left(M_k(\mathcal{T}_T)-b_T\right)\le x\right\} &= \mathbb{P}\left\{\frac{a_T}{a_{\mathcal{T}_T}}a_{\mathcal{T}_T}\left(M_k(\mathcal{T}_T)-b_{\mathcal{T}_T}\right)+a_T(b_{\mathcal{T}_T}-b_T)\le x\right\}\\ &\to \mathbb{P}\left\{W_{r,k}+\ln\mathcal{T}\le x\right\},\quad T\to\infty, \qquad (32)\end{aligned}$$

where $W_{r,k}$ is a random variable with distribution function $\Lambda_{r,k}$ being further independent of $\mathcal{T}$, and thus the proof is complete. □

Proof of Proposition 3.2. As in (14) For any closed subset E of $[0,T]^n$ we have

$$\max_{\boldsymbol{t}\in E}\chi_k(\boldsymbol{t})=\max_{(\boldsymbol{t},\mathbf{v})\in E\times S_{k-1}}Y(\boldsymbol{t},\mathbf{v}),$$

where the random field $Y(\boldsymbol{t},\mathbf{v})$ is defined as in (17). Consequently, since for any $u>0$

$$\mathbb{P}\Big\{\max_{\boldsymbol{t}\in E}\chi_k(\boldsymbol{t})>u\Big\}=\mathbb{P}\Big\{\max_{(\boldsymbol{t},\mathbf{v})\in E\times S_{k-1}}Y(\boldsymbol{t},\mathbf{v})>u\Big\}\le\mathbb{P}\Big\{\max_{(\boldsymbol{t},\mathbf{v})\in E\times S_{k-1}}|Y(\boldsymbol{t},\mathbf{v})|>u\Big\}$$

and further $Y(\boldsymbol{t},\mathbf{v})$ satisfies the global Hölder condition the proof follows by Piterbarg inequality shown in Theorem 8.1 of Piterbarg (1996). □

Proof of Theorem 3.3. It follows easily that $\{X(t),t\ge0\}$ satisfies the global Hölder condition with γ equal to α, hence by Proposition 3.2 Piterbarg inequality (14) holds for $M_k(T)$. Consequently, in view of (12) the proof follows applying Lemma 4.5 with $b_t=\sqrt{2\ln t}$ and $l=1/2,\tau=2$. □

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