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Divisible homology classes in the special linear group of a number field

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Abstract

The integral homology groups of the infinite special linear group $SL(F)$ over a number field F are in general not finitely generated but they have the following property: for any integer $i \geq 0$, $H_i(SL(F); \mathbb{Z})$ is the direct sum of a free abelian group of finite rank and a torsion group. The purpose of this paper is to investigate partially the structure of that torsion subgroup. The main theorem asserts that, if $\bar{D}(i)$ denotes the subgroup of divisible elements in $H_i(SL(F); \mathbb{Z})$, then $\bar{D}(i)$ is an abelian group of finite exponent for any $i \geq 0$ (and $\bar{D}(i)$ is in general non-trivial). The following vanishing result is also proved: if N is a positive integer and ℓ a prime number $> N$ with the property that $K_{2n}F$ contains no ℓ-torsion divisible elements for all $n \leq N$, then the ℓ-torsion subgroup of $\bar{D}(i)$ is trivial for all $i \leq 2N$.

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0. Introduction

Let F be a number field and $SL(F)$ denote the infinite special linear group over F. The integral homology groups of $SL(F)$ are in general not finitely generated, but it was shown by the first author in Section 2 of [1] that, for all integers $i \geq 0$, $H_i(SL(F); \mathbb{Z})$ is the direct sum of a free abelian group of finite rank and a torsion group. The next interesting problem consists in understanding the structure of this torsion subgroup.

Recently, Banaszak looked at the corresponding question for the algebraic K-theory of number fields. The localization exact sequence in algebraic K-theory (see [11, Section 5; 13, Theorem 8; 14, Théorème 1])

$$\cdots \to K_iO \xrightarrow{r_*} K_iF \twoheadrightarrow \bigoplus_m K_{i-1}(O/m) \to \cdots,$$

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where O is the ring of algebraic integers in F, r_* the homomorphism induced by the inclusion $r: O \hookrightarrow F$, and where m runs over the set of maximal ideals of O, shows immediately that K_iF is a finitely generated group if i is odd and a torsion group if i is even, because K_iO is finitely generated for all positive i's and moreover finite for even i's (see [8, 12]), and because $K_{i-1}(O/m)$ is trivial for odd i's and finite cyclic for even i's by [10].

In [4, Chapter VIII], and [5, Section II], Banaszak investigated the subgroup $D(i)$ of divisible elements in K_iF (notice that a subgroup of divisible elements in an ambiant abelian group is not necessarily a divisible group; here for instance, $D(i)$ is finite). It follows from the above information on the structure of K_iF that $D(i)$ is trivial if i is odd and a torsion group if i is even. Moreover, since for i even, $\bigoplus_m K_{i-1}(O/m)$ is a direct sum of cyclic groups, hence, it contains no non-trivial divisible elements. Consequently, $D(i)$ is a subgroup of the image of r_*, hence, it is isomorphic to a subgroup of K_iO. Thus,

$D(i) = 0$ if i is odd and $D(i)$ is a finite group if i is even.

For any prime number ℓ, let $D(i)_\ell$ denote the ℓ-torsion subgroup of $D(i)$ (in other words, the subgroup of ℓ-divisible ℓ-torsion elements in K_iF). For $i = 2n$, n odd, Banaszak deduced that $D(2n)_\ell$ is in general non-trivial. Subsequently, together with Kolster, he obtained the following description (see [5, Theorem 3]): if F is totally real, n an odd positive integer and ℓ an odd prime, the order of $D(2n)_\ell$ is exactly given by the ℓ-adic absolute value of

$$\frac{w_{n+1}(F)\zeta_F(-n)}{\prod_{v|\ell} w_n(F_v)},$$

where $\zeta_F(-)$ is the Dedekind zeta function of F, $w_n(k)$ the biggest integer s such that the exponent of the Galois group $\mathrm{Gal}(k(\mu_s)/k)$ divides n for a field k (here μ_s is an sth primitive root of unity), and F_v the completion of F at v. For instance, if F is the field of rationals $\mathbb{Q}$, n an odd integer and ℓ an odd prime, the order of $D(2n)_\ell$ is equal to the ℓ-adic absolute value of the numerator of $B_{n+1}/(n+1)$, where B_{n+1} is the $(n+1)$th Bernoulli number. Notice that the knowledge of $D(2n)$ is of particular interest since it is related to the Lichtenbaum–Quillen conjecture (see [5, Section II.2]) and to étale K-theory (see [7]).

The purpose of the present paper is to study the *divisible elements in homology* of the infinite special linear group of a number field. Denote by $\bar{D}(i)$ the subgroup of divisible elements in $H_i(SL(F); \mathbb{Z})$, and for a prime ℓ, by $\bar{D}(i)_\ell$ the ℓ-torsion subgroup of $\bar{D}(i)$ (observe that $\bar{D}(i)$ is a torsion group because of the result of [1] mentioned above). In the first section (see Theorem 1.1), we prove that

$\bar{D}(i)$ is an abelian group of finite exponent for any $i \geq 0$.

In Section 2, we use the fact that the group $SL(F)$ has the same homology as the simply connected infinite loop space $BSL(F)^+$ obtained by performing the plus construction on the classifying space of $SL(F)$ and consider the Hurewicz homomorphism

$$h_i : K_iF \cong \pi_i BSL(F)^+ \to H_i(BSL(F)^+; \mathbb{Z}) \cong H_i(SL(F); \mathbb{Z})$$

for $i \geq 2$. We concentrate our attention to its restriction $h_i : D(i) \to \bar{D}(i)$ for $i = 2n$ and show the following assertion (see Corollary 2.5):

For any $n \geq 1$, $h_{2n} : D(2n)_\ell \to \bar{D}(2n)_\ell$ is a split injection if $\ell > n$.

We also observe that $\bar{D}(i)_\ell$ may contain non-trivial elements which do not belong to the image of $h_i : K_iF \to H_i(SL(F); \mathbb{Z})$; for example, it is possible that $\bar{D}(i)_\ell$ is non-trivial even if i is odd or if $i = 2n$ with n even. The last section is devoted to the following vanishing result (see Theorem 3.1):

If N is a positive integer and ℓ a prime number $> N$ such that $D(2n)_\ell = 0$ for $1 \leq n \leq N$, then $\bar{D}(i)_\ell = 0$ for $1 \leq i \leq 2N$.

Let us finally mention that the structure of the integral homology groups of the infinite *general* linear group $GL(F)$ may be deduced from the knowledge of the integral homology of $SL(F)$ by the Künneth formula, because of the homotopy equivalence $BGL(F)^+ \simeq BSL(F)^+ \times BF^\times$ which follows from the fact that $BSL(F)^+$ is the universal cover of $BGL(F)^+$ (see [1, proof of Corollary 9]).

1. A finiteness theorem

The first result on the structure of the integral homology groups of $SL(F)$ is given by Theorem 7 of [1]: for any $i \geq 0$, $H_i(SL(F); \mathbb{Z})$ is the direct sum of a free abelian group of finite rank and a torsion group. This has the following consequence for the integral *cohomology* of $SL(F)$: for any $i \geq 0$, $H^i(SL(F); \mathbb{Z})$ contains *no divisible elements* except 0 (see [1, Corollary 8]). The purpose of this section is to investigate the subgroup $\bar{D}(i)$ of divisible elements in $H_i(SL(F); \mathbb{Z})$.

Theorem 1.1. *For any $i \geq 0$, $\bar{D}(i)$ is an abelian group of finite exponent.*

Proof. According to [13, Theorem 4] or [11, Section 7, Proposition 3.2], there is a fibration

$$\prod_m BQP(O/m) \to BQP(O) \to BQP(F),$$

where $BQP(A)$ denotes the classifying space of the Q-construction over the category of finitely generated projective A-modules for a ring A (recall from [13, Theorem 1] that

$\Omega BQP(A) \simeq BGL(A)^+ \times K_0A$), $\prod$ the weak product (i.e., the direct limit of cartesian products with finitely many factors), and where m runs over the set of maximal ideals of O. By looping its base space and its total space, and by taking the universal covers, we obtain the fibration

$$BSL(O)^+ \xrightarrow{f} BSL(F)^+ \xrightarrow{g} \prod_m \widetilde{BQP}(O/m).$$

Let C_f be the cofibre of the map f, $\theta: BSL(F)^+ \to C_f$ the collapsing map, and $\xi: C_f \to \prod_m \widetilde{BQP}(O/m)$ the map induced by g. Since K_iO is finitely generated [12], one can show that $g_*: K_iF \to \pi_i(\prod_m \widetilde{BQP}(O/m))$ and $\theta_*: K_iF \to \pi_iC_f$ are $\mathscr{C}$-isomorphisms for $i \geq 2$, where $\mathscr{C}$ is the Serre class of all finitely generated abelian groups. It then follows from $g = \xi\theta$ that $\xi_*: \pi_iC_f \to \pi_i(\prod_m \widetilde{BQP}(O/m))$ is a $\mathscr{C}$-isomorphism, and thus, since both groups are torsion groups (see [8]), a $\mathscr{D}$-isomorphism, where $\mathscr{D}$ denotes the Serre class of all abelian groups of finite exponent. By the mod $\mathscr{D}$ Whitehead theorem, the induced homomorphism $\xi_\#: H_i(C_f; \mathbb{Z}) \to H_i(\prod_m \widetilde{BQP}(O/m); \mathbb{Z})$ is also a $\mathscr{D}$-isomorphism for all $i \geq 2$. But the Künneth formula tells us that $H_i(\prod_m \widetilde{BQP}(O/m); \mathbb{Z})$ is a direct sum of finitely generated groups, since the integral homology groups of $\widetilde{BQP}(O/m)$ are finitely generated for all m, and therefore that it has no divisible elements except 0. This implies that the group of divisible elements in $H_i(C_f; \mathbb{Z})$ is contained in the kernel of $\xi_\#$, hence is of finite exponent. Note that $\bar{D}(i)$ is a torsion group, because $H_i(SL(F); \mathbb{Z})$ is the direct sum of a free abelian group of finite rank and a torsion group (see Section 2 of [1]). Consequently, the homology sequence of the cofibration $BSL(O)^+ \xrightarrow{f} BSL(F)^+ \xrightarrow{\theta} C_f$ enables us to conclude that $\bar{D}(i)$ belongs to $\mathscr{D}$ since $H_i(BSL(O)^+; \mathbb{Z})$ is finitely generated.

We shall check that $\bar{D}(i)$ is in general *non-trivial* (see Corollary 2.5).

Remark 1.2. The same argument proves that the subgroup of divisible elements in $H_i(\Omega^s BSL(F)^+; \mathbb{Z})$ is also of finite exponent for all $i \geq 0$ and $s \geq 0$.

2. The Hurewicz homomorphism

Denote by X_F a 1-connected Ω-spectrum whose 0th space is the infinite loop space $BSL(F)^+$: the homotopy groups of X_F are the K-groups of F in dimensions ≥ 2. This spectrum is of interest for algebraic K-theory because of the following result.

Theorem 2.1. *For $i \geq 2$, the Hurewicz homomorphism with coefficients in $\mathbb{Z}_{(\ell)}$, the integers localized at ℓ, $\bar{h}_i: K_i(F; \mathbb{Z}_{(\ell)}) \to H_i(X_F; \mathbb{Z}_{(\ell)})$ is an isomorphism if ℓ is a prime number $> \frac{1}{2}(i+1)$.*

Proof. Since the spectrum X_F is 1-connected, its Postnikov k-invariants $k^{i+1}(X_F)$ are cohomology classes of finite order ρ_i for $i \geq 3$, and ρ_i is only divisible by primes

$p \le \frac{1}{2}(i+1)$ (see [3, Theorem 1.5]). Now, let us write $X_F[i]$ for the ith Postnikov section of X_F (i.e., $X_F[i]$ is a spectrum with $\pi_j X_F[i] = 0$ for $j > i$, $\pi_j X_F \cong \pi_j X_F[i]$ for $j \le i$), and for any prime number ℓ, $(X_F[i])_{(\ell)}$ for its localization at ℓ, which has the property that $\pi_j(X_F[i])_{(\ell)} \cong (K_j F)_{(\ell)} \cong K_j(F; \mathbb{Z}_{(\ell)})$ for $j \le i$. If $\ell > \frac{1}{2}(i+1)$, all k-invariants of $(X_F[i])_{(\ell)}$ are trivial and $(X_F[i])_{(\ell)}$ is a wedge of Eilenberg–MacLane spectra:

$$(X_F[i])_{(\ell)} \simeq \bigvee_{j=2}^{i} \Sigma^j H(K_j(F; \mathbb{Z}_{(\ell)}))$$

(for any abelian group G, $H(G)$ denotes the Eilenberg–MacLane spectrum having all homotopy groups trivial except for G in dimension 0). Then, it is easy to compute

$$H_i(X_F; \mathbb{Z}_{(\ell)}) \cong H_i(X_F[i]_{(\ell)}; \mathbb{Z}) \cong \bigoplus_{j=2}^{i} H_i(\Sigma^j H(K_j(F; \mathbb{Z}_{(\ell)})); \mathbb{Z}).$$

But it follows from [9, Théorème 2] or [3, Proposition 1.3] that $H_i(\Sigma^j H(K_j(F; \mathbb{Z}_{(\ell)})); \mathbb{Z})$ is trivial if $j < i < j + 2\ell - 2$. Consequently, the condition $i < 2\ell - 1$ produces the desired assertion since $H_i(X_F; \mathbb{Z}_{(\ell)}) \simeq H_i(\Sigma^i H(K_i(F; \mathbb{Z}_{(\ell)}), i); \mathbb{Z}) \cong K_i(F; \mathbb{Z}_{(\ell)})$.

Remark 2.2. From the theorem, it is true that $K_i F$ and $H_i(X_F; \mathbb{Z})$ have isomorphic subgroups of ℓ-torsion divisible elements if $\ell > \frac{1}{2}(i+1)$. We shall prove in another paper that for any bounded below spectrum, the cokernel of the Hurewicz homomorphism is a group of finite exponent. Consequently, *all* divisible elements in $H_i(X_F; \mathbb{Z})$ belong to the image of the Hurewicz homomorphism $\tilde{h}_i : K_i F \to H_i(X_F; \mathbb{Z})$ (but, perhaps, they are images of elements which are not divisible in $K_i F$).

Remark 2.3. If we look at integers $i \ge 1$, we may also consider the Hurewicz homomorphism $K_i(F; \mathbb{Z}_{(\ell)}) \to H_i(Y_F; \mathbb{Z}_{(\ell)})$, where Y_F denotes a 0-connected Ω-spectrum whose 0th space is $BGL(F)^+$: then, the conclusion of Theorem 2.1 holds for primes $\ell > \frac{i}{2} + 1$.

It is also useful to consider the Hurewicz homomorphism on the space level

$$h_i : K_i(F; \mathbb{Z}_{(\ell)}) \to H_i(BSL(F)^+; \mathbb{Z}_{(\ell)}) \cong H_i(SL(F); \mathbb{Z}_{(\ell)})$$

for $i \ge 2$, and the commutative diagram

$$\begin{array}{ccc} K_i(F; \mathbb{Z}_{(\ell)}) \cong \pi_i(BSL(F)^+; \mathbb{Z}_{(\ell)}) & \xrightarrow{h_i} & H_i(BSL(F)^+; \mathbb{Z}_{(\ell)}) \cong H_i(SL(F); \mathbb{Z}_{(\ell)}) \\ \downarrow \cong & & \downarrow \sigma \\ K_i(F; \mathbb{Z}_{(\ell)}) \cong \pi_i(X_F; \mathbb{Z}_{(\ell)}) & \xrightarrow{\tilde{h}_i} & H_i(X_F; \mathbb{Z}_{(\ell)}), \end{array}$$

where σ denotes the iterated homology suspension. Thus, Theorem 2.1 has the following immediate consequence (see also [2, Section 2]).

Corollary 2.4. *If i is an integer ≥ 2 and ℓ a prime number $> \frac{1}{2}(i+1)$, then the Hurewicz homomorphism $h_i: K_i(F; \mathbb{Z}_{(\ell)}) \to H_i(SL(F); \mathbb{Z}_{(\ell)})$ is a split injection.*

Since we know that $D(i) = 0$ for odd i's, let us consider $i = 2n$ and obtain the following splitting result.

Corollary 2.5. *If n is a positive integer and ℓ a prime number $> n$, then the Hurewicz homomorphism $h_{2n}: D(2n)_\ell \to \bar{D}(2n)_\ell$ is a split injection.*

Of course, if F is totally real, $i = 2n$ an even integer with n odd and ℓ a prime $> n$, then Banaszak's formula for the order of $D(2n)_\ell$ asserts that $\bar{D}(2n)_\ell$ is *non-trivial* for suitable n and ℓ. If $F = \mathbb{Q}$ for instance, $D(2n)_\ell$ is non-trivial if ℓ is an irregular prime and n an odd integer such that ℓ divides the numerator of $B_{n+1}/(n+1)$. Actually, it turns out that, in general, $\bar{D}(i)_\ell$ is bigger than $D(i)_\ell$ ($\ell > \frac{1}{2}(i+1)$).

Theorem 2.6. *If F is a totally real number field, there exist positive integers i and prime numbers $\ell > \frac{1}{2}(i+1)$ such that the group $H_i(SL(F); \mathbb{Z})$ contains non-trivial ℓ-torsion divisible elements which do not belong to the image of the Hurewicz homomorphism $h_i: K_iF \to H_i(SL(F); \mathbb{Z})$. In particular, $H_i(SL(F); \mathbb{Z})$ may contain non-trivial divisible elements even if i is odd or if $i = 2n$ with n even.*

Proof. If i is a positive integer and ℓ a prime $> \frac{1}{2}(i+1)$, then all k-invariants of the localized ith Postnikov section $(BSL(F)^+[i])_{(\ell)}$ of $BSL(F)^+$ are trivial since this is the case for the spectrum $(X_F[i])_{(\ell)}$ (see the proof of Theorem 2.1). Therefore, $(BSL(F)^+[i])_{(\ell)}$ is a product of Eilenberg–MacLane spaces:

$$(BSL(F)^+[i])_{(\ell)} \simeq \prod_{j=2}^{i} K(K_j(F; \mathbb{Z}_{(\ell)}), j).$$

This homotopy equivalence and the Künneth formula provide a calculation of

$$H_i(SL(F); \mathbb{Z}_{(\ell)}) \cong H_i((BSL(F)^+[i])_{(\ell)}; \mathbb{Z}) \cong H_i\left(\prod_{j=2}^{i} K(K_j(F; \mathbb{Z}_{(\ell)}), j); \mathbb{Z}\right).$$

This homology group has not only $H_i(K(K_i(F; \mathbb{Z}_{(\ell)}), i); \mathbb{Z}) \cong K_i(F; \mathbb{Z}_{(\ell)})$ as direct summand, but also mixed terms, for instance of the form

$$K_{2m}(F; \mathbb{Z}_{(\ell)}) \otimes (K_{j_1}(F; \mathbb{Z}_{(\ell)}) \otimes K_{j_2}(F; \mathbb{Z}_{(\ell)}) \otimes \cdots \otimes K_{j_s}(F; \mathbb{Z}_{(\ell)})),$$

where $2m + j_1 + j_2 + \cdots + j_s = i$; however, the right hand side of this tensor product includes a free $\mathbb{Z}_{(\ell)}$-module if $j_1, j_2, \ldots, j_s$ are $\equiv 1 \pmod 4$ and ≥ 5 (see [8]). If this occurs for m odd, then all elements of $D(2m)_\ell$ are divisible in the above mixed term. Consequently, $\bar{D}(i)_\ell$ contains not only $D(i)_\ell$, but also $D(2m)_\ell$ for suitable choices of $m \leq \frac{1}{2}(i-5)$. This may happen even if i is odd or if $i = 2n$ with n even.

Example 2.7. Take $F = \mathbb{Q}$ and $\ell = 691$. It is known that $D(22)_{691}$ in non-trivial (see [4, Section VIII.3]) and that $K_j\mathbb{Q}$/torsion is infinite cyclic if $j \equiv 1 \pmod 4$ and ≥ 5. The argument introduced in the previous proof exhibits, for instance, non-trivial elements in $\bar{D}(27)_{691}$, in $\bar{D}(36)_{691}/D(36)_{691}$, *and in* $\bar{D}(66)_{691}/D(66)_{691}$.

It is easy to deduce from Theorem 2.1 that the divisible elements detected by Theorem 2.6 vanish under σ.

Corollary 2.8. *If i is an integer* ≥ 2 *and* ℓ *a prime number* $> \frac{1}{2}(i+1)$, *then the iterated homology suspension* $\sigma : H_i(SL(F); \mathbb{Z}_{(\ell)}) \to H_i(X_F; \mathbb{Z}_{(\ell)})$ *satisfies* $\sigma(\bar{D}(i)_\ell / D(i)_\ell) = 0$.

Remark 2.9. As we mentioned in the introduction, all divisible elements in K_iF belong to the image of the homomorphism $r_* : K_iO \to K_iF$ induced by the inclusion $r : O \hookrightarrow F$. If i is a positive integer and ℓ a prime $> \frac{1}{2}(i+1)$, it follows obviously from Theorem 2.1 that the ℓ-torsion divisible elements in $H_i(X_F; \mathbb{Z})$ are also elements of the image of the induced homomorphism $r_* : H_i(X_O; \mathbb{Z}) \to H_i(X_F; \mathbb{Z})$. We do not know the answer to the following *question*: is $\bar{D}(i)_\ell$ contained in the image of $r_* : H_i(SL(O); \mathbb{Z}) \to H_i(SL(F); \mathbb{Z})$?

3. A vanishing theorem

The study of the Serre spectral sequence of the fibration

$$\prod_m BQP(O/m) \to BQP(O) \to BQP(F)$$

(introduced in Section 1) shows that $H_i(SL(F); \mathbb{Z})$ contains in general a lot of ℓ-torsion elements for all primes ℓ. The goal of this section is to prove that for certain choices of the integer i and the prime ℓ, the group $H_i(SL(F); \mathbb{Z})$ has no non-trivial ℓ-torsion divisible elements.

Theorem 3.1. *If N is a positive integer and* ℓ *a prime number* $> N$ *with the property that* $D(2n)_\ell = 0$ *for all positive* $n \leq N$, *then* $\bar{D}(i)_\ell = 0$ *for all positive* $i \leq 2N$.

Proof. As in the proof of Theorem 2.6, the assumption $\ell > N$ provides a homotopy equivalence

$$(BSL(F)^+[2N])_{(\ell)} \simeq \prod_{j=2}^{2N} K(K_j(F; \mathbb{Z}_{(\ell)}), j).$$

According to [5, Section II.1, Corollary 1], the vanishing of $D(2n)_\ell$ implies the splitting

$$K_{2n}(F; \mathbb{Z}_{(\ell)}) \cong K_{2n}(O; \mathbb{Z}_{(\ell)}) \oplus \left(\bigoplus_m K_{2n-1}(O/m; \mathbb{Z}_{(\ell)}) \right).$$

Therefore, $K_{2n}(F;\mathbb{Z}_{(\ell)})$ is a direct sum of finitely generated $\mathbb{Z}_{(\ell)}$-modules and the same is true for $H_k(K(K_{2n}(F;\mathbb{Z}_{(\ell)}),2n);\mathbb{Z})$, for all $k \geq 1$ $(2 \leq 2n \leq 2N)$. On the other hand, $K_j(F;\mathbb{Z}_{(\ell)})$ is finitely generated if j is odd because of the localization exact sequence. We may finally conclude by the Künneth formula that, for $i \leq 2N$,

$$H_i(SL(F);\mathbb{Z}_{(\ell)}) \cong H_i((BSL(F)^+[2N])_{(\ell)};\mathbb{Z}) \cong H_i\left(\prod_{j=2}^{2N} K(K_j(F;\mathbb{Z}_{(\ell)}),j);\mathbb{Z}\right)$$

is again a direct sum of finitely generated $\mathbb{Z}_{(\ell)}$-modules, and hence has no non-trivial ℓ-torsion divisible elements, since the ℓ-torsion subgroup of any finitely generated $\mathbb{Z}_{(\ell)}$-module is finite. In other words, we get $\bar{D}(i)_\ell = 0$ for $i \leq 2N$.

Remark 3.2. It is shown in [6] that, for $F = \mathbb{Q}$, the Kummer–Vandiver conjecture [15, p. 159] holds if and only if $D(2n)_\ell = 0$ for n even, ℓ odd. It is known (loc.cit.) that this conjecture holds for $\ell < 125\,000$. Thus, the formula in the introduction for the order of $D(2n)_\ell$, for n odd, makes it easy to check the hypothesis of Theorem 3.1 for $\ell < 125\,000$ and $F = \mathbb{Q}$.

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Note added in proof

The problem mentioned in Remark 2.9 was partially solved in [16].

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