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On Extremal Index of Max-Stable Random Fields

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Abstract. For a given stationary max-stable random field $X(t), t \in \mathbb{Z}^d$ the corresponding generalised Pickands constant coincides with the classical extremal index $\theta_X \in [0,1]$. In this contribution we discuss necessary and sufficient conditions for θ_X to be 0, positive or equal to 1 and also show that θ_X is equal to the so-called block extremal index. Further, we consider some general functional indices of X and prove that for a large class of functionals they coincide with θ_X.

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1 Introduction

The connection between Pickands constant and extremal index of stationary max-stable Brown-Resnick random fields (rf's) has been initially pointed out in [16]. Calculation of Pickands constants for a general stationary max-stable rf $X(t), t \in \mathbb{Z}^d$ has been later dealt with in [25]. Previous investigations concerned with the calculation of extremal index in the context of max-stable processes are [8, 9, 21, 47]. Recent research in [2, 26, 45, 51] has shown, contrary to the prevailing intuitions, that there are certain subtilities (if $d > 1$) when dealing with stationary multivariate regularly varying rf's (see e.g., [48] for the definition) and the calculation of their extremal indices. Influenced by the findings of [7], several formulas for extremal indices of stationary regularly varying time series have appeared in the literature, see e.g., [35] and the references therein. Various (less well-known) formulas have been discovered also for Pickands constants in contributions unrelated to time series modelling. For instance in sequential analysis and statistical applications [42, 43] and extremes of random fields [29, 52] just to mention a few. For large classes of Gaussian rf's extremal indices have been discussed in [11, 24, 44], see also [4, 49] for non-Gaussian cases and related results.

Without loss of generality, we shall focus on the class of max-stable rf's with Fréchet marginals. Since these are limiting rf's, see e.g., [18], our formulas for their extremal indices are valid (with obvious modifications) also for the candidate extremal index of more general stationary regularly varying rf's (see [35] for recent findings). Studying max-stable rf's, instead of these more general rf's is also justified by Lemma 2 stated in Section 2 and Remark 2 iii).
In view of the well-known de Haan characterisation given in [13], the rf X with non-degenerated marginal distributions corresponds to some non-negative spectral rf $Z(t), t \in \mathbb{Z}^d$ having the following representation (in distribution)

$$X(t) = \max_{i \geq 1} \Gamma_i^{-1/\alpha} Z_i(t), \quad t \in \mathbb{Z}^d, \tag{1.1}$$

where $\Gamma_i = \sum_{k=1}^{i} Q_k$ with $Q_k, k \ge 1$ unit exponential random variables (rv's) independent of Z_i's which are independent copies of Z.

Clearly, Z is not unique since also $\tilde{Z}(t) = RZ(t), t \in \mathbb{Z}^d$ is a spectral rf for X, provided that R is a non-negative rv independent of Z such that $\mathbb{E}\{R^\alpha\} = 1$. Note that if for some $h \in \mathbb{Z}^d$ we have $Z(h) = 1$ almost surely, then in view of Balkema's lemma (stated in [14][Lem 4.1]) any spectral rf $\tilde{Z}$ of X has the same law as Z. We shall assume without loss of generality that for some $\alpha \in (0,\infty)$

$$\mathbb{P}\left\{\max_{t\in\mathbb{Z}^d} Z(t) > 0\right\} = 1, \quad \mathbb{E}\left\{Z^\alpha(t)\right\} = 1, \quad t \in \mathbb{Z}^d. \tag{1.2}$$

Lemma 8 in Appendix shows how to construct a spectral rf Z such that the first assumption in (1.2) holds. Note that $\mathbb{E}\left\{Z^\alpha(t)\right\} = 1$ implies that $X(t)$ has α-Fréchet distribution function $e^{-x^{-\alpha}}, x > 0$. This is no restriction since we are interested in stationary max-stable rf's. As in [25] define the Pickands constant (when the limit exists) with respect to the spectral rf Z by

$$\mathcal{H} = \lim_{n\to\infty} \frac{1}{n^d}\mathbb{E}\left\{\max_{t\in[0,n]^d\cap\mathbb{Z}^d} Z^\alpha(t)\right\} \le \lim_{n\to\infty} \frac{1}{n^d} \sum_{t\in[0,n]^d\cap\mathbb{Z}^d} \mathbb{E}\left\{Z^\alpha(t)\right\} \le 1. \tag{1.3}$$

Since the finite dimensional distributions (fidi's) of X can be calculated explicitly (see (6.1) below), if $\mathcal{H}$ exists, then

$$\mathbb{P}\left\{\max_{t\in[0,n]^d\cap\mathbb{Z}^d} X(t) \le n^d x\right\} = e^{-\frac{1}{n^d}\mathbb{E}\left\{\max_{t\in[0,n]^d\cap\mathbb{Z}^d} Z^\alpha(t)/x^\alpha\right\}} \to e^{-\mathcal{H}/x^\alpha} \tag{1.4}$$

as $n \to \infty$ is valid for all $x > 0$.

As argued in [16] and [10, 25] the sub-additivity of maximum functional implies that $\mathcal{H}$ is well-defined and finite, provided that X is stationary. Consequently, in view of (1.4) the extremal index (or using the terminology of [51], the classical extremal index) of the stationary max-stable rf X (denoted below by θ_X) always exists, does not depend on the particular spectral rf Z but on the law of the rf X and is given by

$$\theta_X = \mathcal{H} \in [0,1]. \tag{1.5}$$

In the special case

$$X(t) = V_t, \quad t \in \mathbb{Z}^d, \tag{1.6}$$

where V_t's are independent α-Fréchet rv's we have $\theta_X = 1$. We shall show that this is the only max-stable rf with unit Fréchet marginals satisfying $\theta_X = 1$. Using this fact and Lemma 2 we can construct a spectral rf Z for X, see Remark, 4 iii).

Hereafter we shall assume for simplicity that the max-stable rf X has unit Fréchet marginal distributions, i.e., below we shall consider the case

$$\alpha = 1.$$

If the spectral rf Z is not easy to determine or $X(t), t \in \mathbb{Z}^d$ is stationary but not max-stable, commonly the block extremal index (denoted below by $\widetilde{\theta_X}$) is utilised in various applications related to extreme value analysis. Assuming for simplicity that X has unit Fréchet marginals, it is defined by (see [23, 51])

$$\widetilde{\theta_X} := \lim_{n\to\infty} \frac{\mathbb{P}\{\max_{0\le i\le r_n, i\in\mathbb{Z}^d} X(i) > n\tau\}}{\prod_{j=1}^{d} r_{nj}\mathbb{P}\{X(0) > n\tau\}} \tag{1.7}$$

for any $\tau > 0$ and any sequence $r_n \in \mathbb{Z}^d, n \geq 1$ with non-decreasing integer-valued components $r_{nj}, j \leq d$ such that $\lim_{n\to\infty} r_{nj} = \lim_{n\to\infty} n/r_{nj}^d = \infty$ for any $j \leq d$. In (1.7) $i \leq r_n$ is interpreted component-wise, i.e., $i_j \leq r_{n_j}$ for all $j \leq d$ components of i and r_n, respectively.

Next, we define functional indices $\theta_{X,F}$ of X by

$$\theta_{X,F} = \mathbb{E}\left\{Z(0)F(Z)\right\} \in [0,1],$$

where $F : E \mapsto [0,1]$ is a measurable functional with respect to the product σ-field $\mathcal{E}$ on $E := [0,\infty)^{\mathbb{Z}^d}$. As mentioned above different choices of Z for X are possible. In order to make the definition of $\theta_{X,F}$ independent of the choice of Z and thus only dependent on the law of X, we shall also require that F is 0-homogeneous, i.e., $F(cf) = F(f)$ for any $c > 0, f \in E$. Indeed, under this assumption we have that

$$\theta_{X,F} = \mathbb{E}\left\{Z(0)F(Z/Z(0))\right\} = \mathbb{E}\left\{F(\Theta_0)\right\},$$

where the rf Θ_h is defined by (hereafter $\mathbb{I}(\cdot)$ denotes the indicator function)

$$\mathbb{P}\{\Theta_h \in A\} \;=\; \mathbb{E}\left\{Z(h)\mathbb{I}(Z/Z(h) \in A)\right\}, \quad \forall A \in \mathcal{E}. \tag{1.8}$$

It is known that for any $h \in \mathbb{Z}^d$ the law of Θ_h does not depend on the particular choice of the spectral rf Z and can be directly determined by X. In the case that for a spectral rf Z of X we have that $Z(h) > 0$ almost surely, this fact follows from Balkema's lemma. The proof for the general case follows from [25][Lem A.1], or from [50][Thm 1.1] and [31][Thm 2]. Consequently, the functional index $\theta_{X,F}$ depends only on the law of X. Note that for the definition of $\theta_{X,F}$ no stationarity of X is assumed.

It is well-known that a max-stable rf X with Fréchet marginals is a multivariate regularly varying rf. For general multivariate regularly varying rf's which are not max-stable, there is no spectral process Z as in our case of max-stable X and therefore the rf's $\Theta_h, h \in \mathbb{Z}^d$ are defined via a conditional limit, see e.g., [18, 40] and (2.1) below. The key advantage in the framework of max-stable rf's is that Θ_h is directly obtained by tilting a given spectral rf Z.

At this point two natural questions for a given stationary max-stable rf X arise:

Question 1: What is the relation between θ_X and $\widetilde{\theta_X}$?

Question 2: For what F is the functional index $\theta_{X,F}$ equal to θ_X?

In this contribution we show that we simply have $\theta_X = \widetilde{\theta_X}$ and then describe a large class of functionals F such that $\theta_X = \theta_{F,X}$. Further, we consider in some detail the cases $\theta_X = 0$ and $\theta_X = 1$.

Brief organisation of the rest of the paper: In the next section we discuss some basic properties of the rf's $\Theta_h, h \in \mathbb{Z}^d$ and then show how to construct a stationary max-stable rf X from a given rf Θ^* which in turn is necessary equal in law with Θ_0. In Section 3 we claim that $\theta_X = \widetilde{\theta_X}$ for any stationary max-stable rf's X. Additionally, we give equivalent conditions that guarantee $\theta_X > 0$ or $\theta_X = 0$ and then present several formulas for θ_X. Section 4 is concerned with the anti-clustering condition whereas Section 5 displays some examples. All the proofs are relegated to Section 6 which is followed by an Appendix.

2 Preliminaries

Unless otherwise specified we shall consider below a max-stable rf $X(t), t \in \mathbb{Z}^d$ as in the Introduction with spectral rf Z such that $\mathbb{E}\left\{Z(t)\right\} = 1, t \in \mathbb{Z}^d$. Hence $X(t)$ has unit Fréchet distribution $e^{-1/x}, x > 0$. We shall discuss first the case that X is non-stationary.

2.1 General max-stable X

The importance of the rf's $\Theta_h, h \in \mathbb{Z}^d$ defined in (1.8) relates to the following conditional convergence results. Namely, in view of [25][Lem 2.1, A.1 & Rem 6.4] or by [18][Lem 3.5] we have that the convergence in distribution

$$X(t)/X(h)\Big|(X(h) > u) \xrightarrow{d} \Theta_h(t), \quad t \in \mathbb{Z}^d, \tag{2.1}$$

$$u^{-1}X(t)\Big|(X(h) > u) \xrightarrow{d} Y_h(t), \quad t \in \mathbb{Z}^d \tag{2.2}$$

hold as $u \to \infty$ in the product topology of $E = [0,\infty)^{\mathbb{Z}^d}$, where Θ_h is defined in (1.8) and

$$Y_h(t) = R\Theta_h(t), \quad t \in \mathbb{Z}^d,$$

with R an α-Pareto rv with survival function $x^{-\alpha}, x \ge 1$ independent of any other random element (recall that we consider $\alpha = 1$ for simplicity).
If for a given max-stable rf X if a spectral rf Z is known, it is often simpler to determine the law of Θ_h directly via (1.8) than deriving it from (2.1). In particular, if $\mathbb{P}\{Z(h) = 1\} = 1$, then the following equality in law

$$\Theta_h \stackrel{d}{=} Z \tag{2.3}$$

is valid. Below we determine the fidi's of Y_h in terms of Z and Θ_h.

Lemma 1. *For any $h, t_i \in \mathbb{Z}^d, x_i \in (0,\infty), i \le n$ we have*

$$\begin{aligned}\mathbb{P}\{Y_h(t_1) \le x_1, \ldots, Y_h(t_n) \le x_n\} &= \mathbb{E}\left\{\max\Big(1, \max_{1\le i\le n} \frac{\Theta_h(t_i)}{x_i}\Big) - \max_{1\le i\le n} \frac{\Theta_h(t_i)}{x_i}\right\} \\ &= \mathbb{E}\left\{\max\Big(Z(h), \max_{1\le i\le n} \frac{Z(t_i)}{x_i}\Big) - \max_{1\le i\le n} \frac{Z(t_i)}{x_i}\right\}.\end{aligned} \tag{2.4}$$

Remark 1. For the case of the stationary Brown-Resnick model (2.4) is stated in [51][Prop 6.1] for $h = 0$.

2.2 Stationary max-stable X

In view of [25][Thm 6.9] the max-stable rf $X(t), t \in \mathbb{Z}^d$ with unit Fréchet marginals is stationary, if and only if

$$\mathbb{E}\{Z(h)F(Z)\} = \mathbb{E}\Big\{Z(0)F(B^hZ)\Big\}, \quad \forall h \in \mathbb{Z}^d \tag{2.5}$$

is valid for any measurable function $F : E \mapsto [0,\infty]$ which is 0-homogeneous. Here B is the shift-operator so that $B^hZ(\cdot) = Z(\cdot - h), h \in \mathbb{Z}^d$. Note that for the stationary Brown-Resnick model the claim in (2.5) is first formulated in [16][Lem 5.2].
For notational simplicity we shall omit the subscript 0 and write simply Θ and Y instead of Θ_0 and Y_0, respectively; in our notation the origin of $\mathbb{R}^k, k \in \mathbb{N}$ is denoted by 0.
In view of [25][Thm 4.3] condition (2.5) is equivalent with the following equality in law

$$\Theta_h \stackrel{d}{=} B^h\Theta$$

valid for any $h \in \mathbb{Z}^d$.
Yet another equivalent formulation of condition (2.5) stated for the rf Θ is

$$\mathbb{E}\{\Theta(h)F(\Theta)\} = \mathbb{E}\left\{F(B^h\Theta)\mathbb{I}(B^h\Theta(0) \neq 0)\right\}, \quad \forall h \in \mathbb{Z}^d \tag{2.6}$$

valid again for all measurable functionals F as above, see e.g., [2, 18].
We note in passing that with the same arguments as in [18] it can be shown that (2.6) is equivalent to the so-called time-change formula derived in [2] for multivariate regularly varying rf's.

Next, since for stationary X we have that (2.2) holds, then in view of [2, 18] X is a multivariate regularly varying rf and Y is the so-called tail rf of X, whereas Θ is the so-called spectral tail rf. Therefore for a stationary max-stable rf X the rf Θ defined in (1.8) is simply the spectral tail rf of X.

Adopting the terminology of [28] for stationary max-stable rf's X, we shall refer to their spectral rf's Z as Brown-Resnick stationary (abbreviated as BRs) rf's.

From Z we can easily define the spectral tail rf Θ. Moreover, as mentioned in (2.3) we simply have $\Theta \stackrel{d}{=} Z$ if $Z(0) = 1$ almost surely. The key properties of BRs rf's Z and spectral tail rf's Θ are the TSF (2.5) and the identity (2.6), respectively. This is revealed by our next result, which shows how to construct a BRs rf Z from a given rf Θ^* that satisfies (2.6) and $\Theta^*(0) = 1$ almost surely, extending thus [27][Thm 4.2] to rf's.

Let in the following

$$\mathcal{I}_{fm}(p \cdot Y) = \min(i \in \mathbb{Z}^d : \max_{j \in \mathbb{Z}^d} |p_j Y(j)| = |p_i Y(i)|),$$

where $p_j's$ are non-negative numbers such that $\sum_{j\in\mathbb{Z}^d} p_j^\alpha = 1$ (recall $\alpha = 1$ in our case).
Hereafter N is a rv independent of any other random element such that $\mathbb{P}\{N = j\} = p_j > 0, j \in \mathbb{Z}^d$. Further, both $\min$ and $\max$ are defined with respect to a translation-invariant order on $\mathbb{Z}^d$, see [2] for the definition.

Lemma 2. *If $Y(t) = R\Theta^*(t), t \in \mathbb{Z}^d$ with R a unit Pareto rv independent of Θ^* which satisfies* (2.6) *and $\Theta^*(0) = 1$ almost surely, then Z_N given by*

$$Z_N(t) = \frac{B^N Y(t)}{\max_{i\in\mathbb{Z}^d} p_i B^N Y(i)} \mathbb{I}(\mathcal{I}_{fm}(p \cdot B^N Y) = N), \quad t \in \mathbb{Z}^d \tag{2.7}$$

is a spectral rf of some stationary max-stable rf $X(t), t \in \mathbb{Z}^d$ with unit Fréchet marginals. Moreover, the spectral tail rf Θ of X has the same law as Θ^*.

Remark 2. i) When $\alpha \neq 1$ the above construction is still valid if the denominator therein is substituting by $(\max_{i\in\mathbb{Z}^d} {p_i}^\alpha B^N Y(i))^{1/\alpha}$. In fact, (2.7) is a minor modification of the construction given in [18][Prop 2.12]. The other known constructions in [18, 27, 35] can be easily extended for the case $d > 1$, we omit the details.
ii) A $\mathbb{R}^q$-valued rf $\Theta(t), t \in \mathbb{Z}^d$ is called a spectral tail rf if it satisfies (2.6) where $\Theta(h), \Theta(-h)$ are substituted by $\|\Theta(h)\|, \|\Theta(-h)\|$ with $\|\cdot\|$ a norm on $\mathbb{R}^q$ and F is redefined accordingly and further $\mathbb{P}\{\|\Theta(0)\| = 1\} = 1$, see e.g., [2, 3, 35]. For such a rf, a BRs rf Z_N can be determined as in (2.7) by changing $\sum_{t\in\mathbb{Z}^d} p_t B^N Y(t)$ to $\sum_{t\in\mathbb{Z}^d} p_t B^N \|Y(t)\|$ and instead of $\max_{t\in\mathbb{Z}^d} p_t B^N Y(t)$ and $p \cdot B^N Y$ putting $\max_{t\in\mathbb{Z}^d} p_t B^N \|Y(t)\|, p \cdot B^N \|Y(t)\|$, respectively (with $Y(t) = R\Theta(t)$ and R a unit Pareto rv independent of Θ).

3 Classical, block & functional indices

As mentioned in the Introduction the classical extremal index θ_X of a stationary max-stable rf X always exists. We show first that it is equal to the block extremal index $\widetilde{\theta_X}$ defined in (1.7) and then answer the question when $\theta_X = 0$. This is already known for $d = 1$, see [10]. Our main findings in Theorem 2 gives several formulas for θ_X. The next result is a minor generalisation of the case $d = 1$ stated in [20].

Lemma 3. *If $X(t), t \in \mathbb{Z}^d$ is a stationary max-stable rf, then $\theta_X = \widetilde{\theta_X}$.*

Below we slightly modify the definition of anchoring maps introduced in [2]. Write next $\bar{\mathbb{Z}}^d$ for $\mathbb{Z}^d \cup \{\infty\}$ and recall that $E = [0, \infty)^{\mathbb{Z}^d}$ is equipped with the product σ-field $\mathcal{E}$.

Definition 1 *We call a measurable map $\mathcal{I} : E \mapsto \bar{\mathbb{Z}}^d$ anchoring if for $O = \{f \in E : \mathcal{I}(f) \in \mathbb{Z}^d\}$ the following conditions are satisfied for all $f \in O, i \in \mathbb{Z}^d$:*
$i)$ $\mathcal{I}(f) = i$ implies $f(i) \ge \min(f(0), 1)$;
$ii)$ $\mathcal{I}(f) = \mathcal{I}(B^i f) - i$.

As in [2] we define two important anchoring maps which are specified with respect to a translation-invariant order on $\mathbb{Z}^d$. In particular the minimum and maximum below are with respect to such an order. An instance of a translation-invariant order is the lexicographical one. Hereafter $\mathcal{S}(f) = \sum_{t \in \mathbb{Z}^d} f^\alpha(t)$ for any $f \in E$. Note that apart from Section 5.2 we have considered for simplicity only the case $\alpha = 1$.

Example 1. Let the non-empty set $O \in \mathcal{E}$ be given by

$$O = \Big\{ f \in E : \mathcal{S}(f) < \infty, \quad \max_{i \in \mathbb{Z}^d} f(i) > 0 \Big\}$$

and define the first maximum functional

$$\mathcal{I}_{fm}(f) = \min\Big(j \in \mathbb{Z}^d : f(j) = \max_{i \in \mathbb{Z}^d} f(i) \Big), \quad f \in O,$$

where $\mathcal{I}_{fm}(f) = \infty$ if $f \notin O$. Clearly, $\mathcal{I}_{fm}(f)$ is finite for $f \in O$ and condition $i)$ holds by the definition, whereas condition $ii)$ follows by the invariance (in the sense of [51]) of the translation-invariant order.

The first and last maximum functionals are important since they are both anchoring and 0-homogeneous. Moreover, for a stationary max-stable rf $X(t), t \in \mathbb{Z}^d$ with spectral rf Θ and Fréchet marginals $\Phi(x) = e^{-1/x^\alpha}, x > 0$ we have that the law of X is specified by $\mathcal{I}_{fm}$ and Θ as follows

$$- \ln \mathbb{P}\{X(i) \le x_i, i \in \mathbb{Z}^d\} = \sum_{i \in \mathbb{Z}^d} \frac{1}{x_i^\alpha} \mathbb{P}\{\mathcal{I}_{fm}(\Theta/(B^{-i}x)) = 0\} \tag{3.1}$$

for any $x = (x_i)_{i \in \mathbb{Z}^d}$ with finitely many positive components and the rest equal to ∞; here $\Theta/(B^{-i}x) = (\Theta(j)/x_{j+i})_{j \in \mathbb{Z}^d}$. The proof of (3.1) is displayed in Appendix, see also [25][Eq. (6.10)]. Note in passing that (3.1) shows that the law of X is uniquely determined by Θ.

Example 2. Define the first exceedance functional by

$$\mathcal{I}_{fe}(f) = \min\Big(j \in \mathbb{Z}^d : f(j) > 1 \Big), \quad f \in O$$

and set $\mathcal{I}_{fe}(f) = \infty$ if $f \notin O$, where

$$O = \Big\{ f \in E : \mathcal{S}(f) < \infty, \quad \max_{t \in \mathbb{Z}^d} f(t) > 1 \Big\} \in \mathcal{E}.$$

Clearly, $\mathcal{I}_{fe}(f)$ for $f \in O$ is finite and $i)$ holds. Moreover since $\mathcal{I}_{fe}(f), f \in O$ is determined by a finite number of points in a neighbourhood of 0, then $\mathcal{I}_{fe}$ is measurable. Again condition $ii)$ is implied by the translation-invariance of the chosen order on $\mathbb{Z}^d$.

We call a measurable map $F : E \mapsto [0, \infty]$ shift-invariant if $F(B^h f) = F(f), h \in \mathbb{Z}^d, f \in E$.

Lemma 4. *Let $\Theta(t), t \in \mathbb{R}^d$ be a real-valued rf satisfying* (2.6) *with $\Theta(0) = 1$ almost surely. If R is a unit Pareto rv independent of Θ, then for any two anchoring maps $\mathcal{I}, \mathcal{I}'$ and any shift-invariant map F we have (set $Y(t) = R\Theta(t), t \in \mathbb{Z}^d$)*

$$\mathbb{P}\{\mathcal{I}(Y) = 0, \mathcal{I}'(Y) \in \mathbb{Z}^d, F(Y) < \infty\} = \mathbb{P}\{\mathcal{I}'(Y) = 0, \mathcal{I}(Y) \in \mathbb{Z}^d, F(Y) < \infty\}. \tag{3.2}$$

Moreover, $\mathbb{P}\{\mathcal{I}(Y) = 0, F(Y) < \infty\} = 0$ is equivalent with $\mathbb{P}\{\mathcal{I}(Y) \in \mathbb{Z}^d, F(Y) < \infty\} = 0$.

Remark 3. If $\mathcal{I}(Y), \mathcal{I}'(Y)$ are almost surely in $\mathbb{Z}^d$, then (3.2) boils down to $\mathbb{P}\{\mathcal{I}'(Y) = 0\} = \mathbb{P}\{\mathcal{I}(Y) = 0\}$, which is already shown in [2][Lem 3.5]. In general, $\mathcal{I}(Y)$ might not be finite almost surely.

Hereafter we consider anchoring maps $\mathcal{I} : E \mapsto \bar{\mathbb{Z}}^d$ such that

$$\mathbb{P}\{\mathcal{I}(Y) \in \mathbb{Z}^d, \mathcal{S}(Y) < \infty\} = \mathbb{P}\{\mathcal{S}(Y) < \infty\}, \tag{3.3}$$

which is in particular valid for both first (last) maximum and first (last) exceedance functionals.

Lemma 5. *If $X(t), t \in \mathbb{Z}^d$ is a stationary max-stable rf with some spectral rf Z and spectral tail rf Θ, then $\theta_X = 0$ if and only if $\mathbb{P}\{\mathcal{S}(\Theta) = \infty\} = \mathbb{P}\{\mathcal{S}(Z) = \infty\} = 1$. If further the anchoring map $\mathcal{I}$ satisfies* (3.3)*, then $\theta_X = 0$ is equivalent with*

$$\mathbb{P}\{\mathcal{I}(Y) = 0, \mathcal{S}(Y) < \infty\} = 0. \tag{3.4}$$

Since the first and last maximum functionals are 0-homogeneous and finite on the set $O = \{f \in E : \mathcal{S}(f) < \infty, \max_{i \in \mathbb{Z}^d} f(i) > 0\}$ we have that $\mathbb{P}\{\mathcal{S}(Z) = \infty\} = 1$ is equivalent with

$$\mathbb{P}\{\mathcal{I}_{fm}(Z) \notin \mathbb{Z}^d\} = 1$$

and the same also holds for the last maximum functional.
In view of Lemma 5, Lemma 9 and [19] $\theta_X = 0$ is equivalent with $\mathbb{P}\{\mathcal{S}(Z) = \infty\} = 1$. Further we have the following equivalent statements (below $\|\cdot\|$ is a norm on $\mathbb{R}^d$):
A1: $Z(t) \to 0$ almost surely as $\|t\| \to \infty$;
A2: $\Theta(t) \to 0$ almost surely as $\|t\| \to \infty$;
A3: $\mathcal{S}(Z) < \infty$ almost surely;
A4: $\mathcal{S}(\Theta) < \infty$ almost surely.
The equivalence of **A1** and **A3** is shown in [19], whereas the equivalence of **A1** and **A2** is a direct consequence of Lemma 9 and similarly for the equivalence of **A3** and **A4**. The equivalence **A2** and **A4** follows from [27] and [51]. Note further that $Y(t) = R\Theta(t) \to 0$ almost surely as $\|t\| \to \infty$ is equivalent with **A2** and $\mathcal{S}(Y) = R\mathcal{S}(\Theta) < \infty$ almost surely is equivalent with **A4**.

We state next the main result of this section; define in the following $\mathcal{B}(Y) = \sum_{t \in \mathbb{Z}^d} \mathbb{I}(Y(t) > 1)$ and interpret $0 : 0$ and $\infty : \infty$ as 0.

Theorem 2. *Let $\mathcal{I}, X$ be as in Lemma 5. If $\mathcal{I}$ satisfies* (3.3) *and $\mathbb{P}\{\mathcal{S}(\Theta) < \infty\} > 0$, then*

$$\begin{aligned}
\theta_X &= \mathbb{P}\{\mathcal{I}(Y) = 0, \mathcal{S}(Y) < \infty\} && (3.5)\\
&= \mathbb{P}\{\mathcal{I}_{fe}(Y) = 0\} && (3.6)\\
&= \mathbb{P}\{\mathcal{I}_{fm}(\Theta) = 0\} && (3.7)\\
&= \mathbb{P}\{\mathcal{I}(\Theta) = 0, \mathcal{S}(\Theta) < \infty\} && (3.8)\\
&= \mathbb{E}\left\{\frac{\max_{t \in \mathbb{Z}^d} \Theta(t)}{\sum_{t \in \mathbb{Z}^d} \Theta(t)}\right\} && (3.9)
\end{aligned}$$

$$= \mathbb{E}\left\{\frac{1}{\mathcal{B}(Y)}\right\}, \tag{3.10}$$

where (3.8) *holds if further* $\mathcal{I}$ *is 0-homogeneous. Moreover* $\{\mathcal{B}(Y) < \infty\} = \{\mathcal{S}(Y) < \infty\}$ *almost surely and in particular* $\theta_X = 1$ *if and only if* $\Theta(i) = 0$ *almost surely for all* $i \in \mathbb{Z}^d, i \neq 0$.

Remark 4. i) $\Theta(t) = \Theta_1(t_1)\Theta_2(t_2), t_1 \in \mathbb{Z}^k, t_2 \in \mathbb{Z}^m, t = (t_1, t_2) \in \mathbb{Z}^d$ with Θ_1, Θ_2 independent rf's satisfying (2.6) and $\mathbb{P}\{\Theta_i(0) = 1\} = 1, i = 1, 2$, then (3.9) implies that $\theta_X = \theta_{X_1}\theta_{X_2}$ where $X, X_i, i = 1, 2$ are stationary max-stable rf's with spectral rf Θ and $\Theta_i, i = 1, 2$, respectively.
ii) For $d = 1$ and $\theta_X = 1$ the claim that $\Theta(i) = 0, i \neq 0$ in Theorem 2 follows also from [30][Prop 2.2 (ii)].
iii) Since Θ uniquely defines X, then Theorem 2 implies that the only stationary max-stable rf X such that $\theta_X = 1$ is that given by (1.6). In view of (2.1) $\Theta(i) = 0, i \neq 0$ and hence by (2.7)

$$Z_N(t) = \frac{1}{p_t}\mathbb{I}(N = t), \quad t \in \mathbb{Z}^d$$

is a spectral rf for X specified in (1.6), where N is a discrete rv with positive probability mass function $p_t > 0, t \in \mathbb{Z}^d$.
iv) Taking $F(f) = \mathbb{I}(\mathcal{I}(f) = 0, \mathcal{S}(f) < \infty)$, then (3.8) implies $\theta_X = \theta_{X,F}$ under the further assumption that $\mathcal{I}$ is a 0-homogeneous functional satisfying (3.3).
v) It follows from the proof of Theorem 2 that (3.10) holds without the assumption that $\mathbb{P}\{\mathcal{S}(\Theta) < \infty\} > 0$. Hence $\theta_X = 0$ if and only if $\mathcal{B}(Y) = \infty$ almost surely. Further, from Theorem 2 we have that **A1, A2, A3** and **A4** are equivalent with **A5**: $\mathcal{B}(Y) < \infty$ almost surely.
iv) Formula (3.9) appears initially as extremal index in [38, 39] and in [17] as Pickands constant.

4 The anti-clustering condition

Since stationary max-stable rf's with Fréchet marginals are multivariate regularly varying (see for more details [2]) the classical extremal index of those rf's can be calculated using the findings of [2] and [51]. In the framework of stationary multivariate regularly varying rf's the anti-clustering condition of [7] plays a crucial role for the calculation of extremal index. Considering the stationary max-stable rf $X(t), t \in \mathbb{Z}^d$ with unit Fréchet marginals, in view of [2] the aforementioned condition reads as follows:

Condition C: Suppose that there exists a positive sequence of non-decreasing integers $r_n \to \infty$ as $n \to \infty$ and $\lim_{n\to\infty} r_n^d/n = 0$ such that for any $s > 0$

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mathbb{P}\left\{\max_{m<\|t\|<r_n, t\in\mathbb{Z}^d} X(t) > ns | X(0) > ns\right\} = 0.$$

The equivalence of Condition C and $\mathbb{P}\{\mathcal{S}(\Theta) < \infty\} = 1$ for the case $d = 1$ is known, see [18]. The case $d \geq 1$ of Brown-Resnick model is dealt with in [51][Prop 6.2]. Next we show that this equivalence holds for a general stationary max-stable rf X with spectral tail rf Θ and spectral rf Z.

Lemma 6. *The anti-clustering Condition C for* X *is equivalent with* **Ai**, $\mathbf{i} = 1, ..., 5$.

If $\mathbb{P}\{\mathcal{S}(\Theta) < \infty\} = 1$ or equivalently Condition C holds, then by [2] Lemma 3, Lemma 6 and [2][Prop 5.2] for any anchoring map $\mathcal{I}$

$$\theta_X = \mathbb{P}\{\mathcal{I}(Y) = 0\} = \mathbb{P}\{\mathcal{I}_{fm}(Y) = 0\} = \mathbb{P}\{\mathcal{I}_{fm}(\Theta) = 0\} \in (0, 1], \tag{4.1}$$

provided that $\mathbb{P}\{\mathcal{I}(Y) \in \mathbb{Z}^d\} = 1$. In the special case $\mathcal{I} = \mathcal{I}_{fe}$ (as shown already in [2])

$$\theta_X = \mathbb{P}\{\max_{0\prec t} Y(t) \leq 1\}. \tag{4.2}$$

Here $\prec$ denotes a translation-invariant order on $\mathbb{Z}^d$.

Remark 5. The expression in (4.2) is a well-known formula in the Gaussian setup and has appeared in numerous papers inspired by [1]. This special formula for the Gaussian setup is also referred to as Albin's constant, see [17]. In the context of stationary regularly varying time series the same formula has appeared in [3].

Next, consider the case that Condition C does not hold, i.e., $p = \mathbb{P}\{\mathcal{S}(\Theta) < \infty\} \in (0,1)$ and define the rf's $\Theta_1 = \Theta|(\mathcal{S}(\Theta) < \infty)$ and $\Theta_2 = \Theta|(\mathcal{S}(\Theta) = \infty)$. In view of [19][Thm 9, Prop 10], for two independent stationary max-stable rf's $\eta_i(t), t \in \mathbb{Z}^d, i = 1,2$ with unit Fréchet marginals and corresponding spectral tail rf's equal in law to $\Theta_i, i = 1,2$ we have that X has the same law as

$$\max(p\eta_1(t), (1-p)\eta_2(t)), \quad t \in \mathbb{Z}^d. \tag{4.3}$$

Since η_1 satisfies Condition C, then by [51][Prop 5.2], Lemma 3, (4.1) and Theorem 2

$$\theta_X = p\mathbb{P}\{\mathcal{I}_{fm}(\Theta_1) = 0\} = p\theta_{\eta_1} \in (0,1]. \tag{4.4}$$

Alternatively, since by the stationarity of X we have that θ_X exists and moreover $\theta_{\eta_2} = 0$, then Lemma 12 implies that $\theta_X = p\theta_{\eta_1}$. Consequently, we conclude that Condition C, Lemma 12, representation (4.3) together with the findings of [2] establish the validity of the first four expressions in Theorem 2.
We remark that from the above arguments, by (4.2) and Lemma 1 we obtain

$$\begin{aligned}\theta_X &= \mathbb{E}\left\{\max_{0 \preceq t} \Theta(t) - \max_{0 \prec t} \Theta(t); S(\Theta) < \infty\right\}\\ &= \mathbb{E}\left\{\max_{0 \preceq t} Z(t) - \max_{0 \prec t} Z(t); \mathcal{S}(Z) < \infty\right\}.\end{aligned}$$

The first formula above is already obtained for the Brown-Resnick model (see Section 5) in [51][Corr 6.3] and for the case $d = 1$ in [20][Thm 2.1].

5 Examples

We present below some examples starting first with the Brown-Resnick model. The second example and Lemma 2 show in particular how to construct stationary max-stable rf's starting from any α-summable deterministic sequence. We then discuss how to construct from some given rf a stationary max-stable rf X such that θ_X equals a given constant.

5.1 Brown-Resnick model

Consider $Z(t) = e^{W(t) - \sigma^2(t)/2}, t \in \mathbb{Z}^d$ with $W(t), t \in \mathbb{Z}^d$ a centered Gaussian rf with variance function σ^2 which is not identical to 0 and $\sigma(0) = 0$. Let $X(t), t \in \mathbb{Z}^d$ denote a max-stable rf with spectral rf Z. The case W is a standard Brownian motion and $d = 1$ is investigated in [6] and therefore this construction is referred to as the Brown-Resnick model.
For any fixed $h \in \mathbb{Z}^d$ the Gaussian rf (set $\gamma(s,t) = Var(W(t) - W(s)), s, t \in \mathbb{Z}^d$)

$$S_h(t) = W(t) - W(h) - \gamma(h,t)/2, \quad \forall t \in \mathbb{Z}^d$$

is such that $S_h(h) = 0$ almost surely and has variance function $\sigma_h^2(t) = \gamma(h,t)$.
With the same arguments as in [25], it follows that $Z_h(t) = e^{S_h(t)}, t \in \mathbb{Z}^d$ is also a spectral rf for X for any $h \in \mathbb{Z}^d$. Since $S_h(t), t \in \mathbb{Z}^d$ is a Gaussian rf with variance $Var(W(t) - W(h)) = \gamma(t,h)$, then the law of X

depends only on $\gamma(h,t)$ and not on σ^2. If we assume that W has stationary increments, then (2.5) implies that X is a stationary max-stable rf. The fact that $Z_h(h)=1$ for any $h\in\mathbb{Z}^d$ almost surely implies that $\Theta:=\Theta_0$ defined in (1.8) is simply given by $\Theta(t)=Z(t), t\in\mathbb{Z}^d$ and hence (recall $Y=R\Theta$)

$$Y(t)=e^{\widetilde{W}(t)+Q},\quad \widetilde{W}(t)=W(t)-\sigma^2(t)/2,\quad t\in\mathbb{Z}^d,$$

where $Q=\ln R$ is a unit exponential rv independent of W.
For an $N(0,1)$ rv V with distribution Φ being independent of Q and all $c>0, x\in\mathbb{R}$ (set $\bar{\Phi}=1-\Phi, V_c=cV-c^2/2$)

$$\begin{aligned}\mathbb{P}\{V_c+Q>x\} &= \mathbb{P}\{V_c+Q>x, V_c>x\}+\mathbb{P}\{V_c+Q>x, V_c\le x\}\\ &= \mathbb{P}\{V_c>x\}+e^{-x}\mathbb{E}\{e^{V_c}\mathbb{I}(V_c\le x)\}\\ &= \mathbb{P}\{V_c>x\}+e^{-x}\mathbb{P}\{cV\le x-c^2/2\},\end{aligned}\tag{5.1}$$

where we used that the exponentially tilted rv U defined by $\mathbb{P}\{U\le x\}=\mathbb{E}\{e^{V_c}\mathbb{I}(V_c\le x)\}, x\in\mathbb{R}$ has $N(c^2/2,c^2)$ distribution, see e.g., [25][Lem 7.1]. Consequently, for all $t\in\mathbb{Z}^d$ such that $c:=\sigma(t)>0$ and all $y>0$

$$\mathbb{P}\{Y(t)\le y\} = \Phi(c^{-1}\ln y+c/2)-e^{-1/y}\Phi(c^{-1}\ln y-c/2),\tag{5.2}$$

which agrees with the claim of [51][Prop 6.1] where the stationary case is considered.

Next, under the assumption that W has stationary increments, in view of (3.9) and (3.10)

$$\theta_X = \mathbb{E}\left\{\frac{1}{\sum_{t\in\mathbb{Z}^d}\mathbb{I}(\widetilde{W}(t)+Q>0)}\right\}=\mathbb{E}\left\{\frac{\max_{t\in\mathbb{Z}^d}e^{\widetilde{W}(t)}}{\sum_{t\in\mathbb{Z}^d}e^{\widetilde{W}(t)}}\right\},\tag{5.3}$$

which yields the following lower bound

$$\begin{aligned}\theta_X &= \mathbb{E}\left\{\frac{1}{\sum_{t\in\mathbb{Z}^d}\mathbb{I}(\widetilde{W}(t)+Q>0)}\right\}\ge\frac{1}{\mathbb{E}\{\sum_{t\in\mathbb{Z}^d}\mathbb{I}(\widetilde{W}(t)+Q>0)\}}\\ &= \frac{1}{\sum_{t\in\mathbb{Z}^d}\mathbb{P}\{\widetilde{W}(t)+Q>0\}}\\ &= \frac{1}{\sum_{t\in\mathbb{Z}^d}\bar{\Phi}(\sigma^2(t)/2)},\end{aligned}\tag{5.4}$$

where we used Fubini theorem for the first equality and (5.1) implies (5.4). The lower bound above is strictly positive under some growth conditions on σ, see [12] for similar calculations in the continuous case. Derivation of a tight positive lower bound is of general interest since in most of the cases direct evaluation of θ_X is not feasible.

It is of some interest to compare two different extremal indices of stationary max-stable Brown-Resnick rf's for different variance functions. With similar arguments as in [10][Thm 3.1] we can prove the following result:

Lemma 7. *Let $X_1(t), t\in\mathbb{Z}^d$ and $X_2(t), t\in\mathbb{Z}^d$ be two stationary max-stable Brown-Resnick rf's corresponding to two centered Gaussian processes W_1, W_2 with stationary increments, continuous trajectories and variance functions σ_1^2 and σ_2^2 which vanish at the origin. If $\sigma_1(t)\ge\sigma_2(t)$ holds for all $t\in\mathbb{Z}^d$, then $\theta_{X_1}\ge\theta_{X_2}$.*

Remark 6. i) Under the conditions of Lemma 7

$$\mathbb{E}\left\{\frac{1}{\sum_{t\in\mathbb{Z}^d}\mathbb{I}(\widetilde{W}_1(t)+Q>0)}\right\}\geq\mathbb{E}\left\{\frac{1}{\sum_{t\in\mathbb{Z}^d}\mathbb{I}(\widetilde{W}_2(t)+Q>0)}\right\}.$$

ii) The calculation of θ_X and different expressions for it have appeared in the literature in various contexts: the most prominent one concerns extremes of Gaussian rf's where in fact $\widetilde{\theta_X}$ has been originally calculated, see e.g., [15, 29, 34]. The first expression in (5.3) for the continuous setup, $d=1$ and the fractional Brownian motion case is obtained in [5][Thm 10.5.1]. Applications to sequential analysis and statistics have given rise to various forms of formula (5.3), see e.g., [32, 41]. As already shown in [17] (5.3) is useful for simulations of θ_X.

5.2 Θ generated by summable sequences

Let $c_i, i\in\mathbb{Z}^d$ be non-negative constants satisfying $\sum_{i\in\mathbb{Z}^d}c_i^\alpha=C\in(0,\infty)$ for some $\alpha>0$ and define

$$\Theta(i)=\frac{c_{i+S}}{c_S},\quad i\in\mathbb{Z}^d$$

for a given rv S with values in $\mathbb{Z}^d$ satisfying

$$\mathbb{P}\{S=i\}=c_i^\alpha/C,\quad i\in\mathbb{Z}^d.$$

Clearly, $\Theta(0)=1$ almost surely and moreover Θ satisfies (2.6) stated for the case $\alpha>0$ as below, namely for any $h\in\mathbb{Z}^d$

$$\begin{aligned}\mathbb{E}\left\{\Theta^\alpha(h)F(\Theta)\right\} &= \mathbb{E}\{c_{h+S}^\alpha/c_S^\alpha\mathbb{I}(c_S\neq0)F(c_{\cdot+S})\}\\ &= \frac{1}{C}\sum_{i\in\mathbb{Z}^d}c_{h+i}^\alpha\mathbb{I}(c_i\neq0)F(c_{\cdot+i})\\ &= \mathbb{E}\left\{F(B^h\Theta)\mathbb{I}(\Theta(-h)\neq0)\right\}\end{aligned}$$

is valid for any 0-homogeneous measurable functional $F:E\mapsto[0,\infty]$.
Clearly, $\mathcal{S}(\Theta)=\sum_{t\in\mathbb{Z}^d}\Theta^\alpha(t)$ is finite almost surely, hence

$$\theta_X=\mathbb{E}\left\{\frac{\max_{t\in\mathbb{Z}^d}c_{t+S}^\alpha}{\sum_{t\in\mathbb{Z}^d}c_{t+S}^\alpha}\right\}=\frac{1}{C}\max_{t\in\mathbb{Z}^d}c_t^\alpha\in(0,1].\tag{5.5}$$

We note that θ_X given in (5.5) is the extremal index of a large class of stationary rf's, see e.g., [4, 45].

5.3 Constructions of X with given extremal index

From the previous example we conclude that for any $a\in(0,1]$ we can construct a stationary max-stable rf X such that $\theta_X=a$. We present below examples of rf X satisfying $\theta_X=0$ and then we construct stationary max-stable rf's $X^{(p)}$ indexed by $p\in(0,1)$ and calculate their extremal indices.
Consider next independent, non-negative rf's $\Theta_k(t), t\in\mathbb{Z}, k\leq d$ that satisfy (2.6) such that $\mathbb{P}\{\Theta_k(0)=1\}=1, k\leq d$. It follows that the rf $\Theta(t)=\prod_{1\leq k\leq d}\Theta_k(t_k), t=(t_1,\ldots,t_k)\in\mathbb{Z}^d$ also satisfies (2.6). In view of Lemma 2 we can construct stationary max-stable rf's $X, X_k, k\leq d$ corresponding to $\Theta,\Theta_k,k\leq d$. As already mentioned in Remark 4 $ii)$ we have $\theta_X=\prod_{k\leq d}\theta_{X_k}$ and therefore $\theta_X=0$ if some θ_{X_k} equals zero.

If we define $\Theta_k(j) = 1$ for all even integers j and $\Theta_k(j) = 0$ for all odd integers j, then Θ_k satisfies (2.6). Since $\mathcal{S}(\Theta_k) = \infty$ almost surely, then $\theta_{X_k} = 0$ follows and hence also $\theta_X = 0$.
In view of our examples, we can construct two independent stationary max-stable rf's $\eta_1(t), \eta_2(t), t \in \mathbb{Z}^d$ with unit Fréchet marginals and spectral tail rf's Z_1 and Z_2, respectively satisfying $\mathbb{P}\{\mathcal{S}(Z_1) < \infty\} = \mathbb{P}\{\mathcal{S}(Z_2) = \infty\} = 1$. The rf $X^{(p)}(t) = \max(p\eta_1(t), (1-p)\eta_2(t)), t \in \mathbb{Z}^d$ for any given $p \in (0,1)$ is stationary and further max-stable with unit Fréchet marginals. As already shown in the previous section, we have $\theta_{X^{(p)}} = p\theta_{\eta_1}$.

6 Proofs

PROOF OF LEMMA 1: For a given non-negative spectral rf Z of a max-stable rf X with unit Fréchet marginals by the de Haan representation of X for any $t_i \in \mathbb{Z}^d, x_i \in (0,\infty), i \le n$

$$-\ln \mathbb{P}\{X(t_1) \le x_1, \ldots, X(t_n) \le x_n\} = \mathbb{E}\left\{\max_{1\le i\le n} \frac{Z(t_i)}{x_i}\right\}. \tag{6.1}$$

Consequently, with $t_0 = h \in \mathbb{Z}^d$ and $x_0 = 1$ we obtain as $u \to \infty$

$$\begin{aligned}
&\mathbb{P}\{u^{-1}X(t_i) \le x_i, i = 1, \ldots, n | X(t_0) > u\}\\
&\sim \quad u\mathbb{P}\{u^{-1}X(t_i) \le x_i, i = 1, \ldots, n, u^{-1}X(t_0) > x_0\}\\
&= \quad u[\mathbb{P}\{u^{-1}X(t_i) \le x_i, i = 1, \ldots, n\} - \mathbb{P}\{u^{-1}X(t_i) \le x_i, i = 0, \ldots, n\}]\\
&\to \quad \mathbb{E}\left\{\max_{i=0,\ldots,n} \frac{Z(t_i)}{x_i} - \max_{i=1,\ldots,n} \frac{Z(t_i)}{x_i}\right\}, \quad u \to \infty\\
&= \quad \mathbb{E}\left\{\mathbb{I}(Z(t_0) > 0)\Big[\max_{i=0,\ldots,n} \frac{Z(t_i)}{x_i} - \max_{i=1,\ldots,n} \frac{Z(t_i)}{x_i}\Big]\right\}\\
&= \quad \mathbb{E}\left\{Z(t_0)\mathbb{I}(Z(t_0) > 0)\Big[\max_{i=0,\ldots,n} \frac{Z(t_i)}{Z(t_0)x_i} - \max_{i=1,\ldots,n} \frac{Z(t_i)}{Z(t_0)x_i}\Big]\right\}\\
&= \quad \mathbb{E}\left\{\max_{i=0,\ldots,n} \frac{\Theta_h(t_i)}{x_i} - \max_{i=1,\ldots,n} \frac{\Theta_h(t_i)}{x_i}\right\},
\end{aligned}$$

where the last line follows by the definition of Θ_h in (1.8). Hence in view of (2.2) and the fact that $\Theta_h(h) = 1$ almost surely, the proof is complete. □

PROOF OF LEMMA 2: Since by the assumptions $\sum_{j\in\mathbb{Z}^d} p_j = 1$ and Θ^* is non-negative we have for any $j \in \mathbb{Z}^d$

$$\mathbb{E}\left\{\sum_{i\in\mathbb{Z}^d} p_i\Theta^*(i-j)\right\} = \sum_{i\in\mathbb{Z}^d} p_i\mathbb{E}\{\Theta^*(i-j)\} = \sum_{i\in\mathbb{Z}^d} p_i\mathbb{P}\{\Theta^*(j-i) > 0\} \le 1,$$

which together with the non-negativity of Θ^* implies for some norm $\|\cdot\|$ on $\mathbb{R}^d$

$$\lim_{\|t\|\to\infty, t\in\mathbb{Z}^d} p_t\Theta^*(t-j) = \lim_{\|t\|\to\infty, t\in\mathbb{Z}^d} p_tY(t-j) = 0 \tag{6.2}$$

almost surely. Consequently, since further

$$\mathbb{P}\{p_N > 0\} = \mathbb{P}\{Y(0) > 1\} = 1,$$

then $\max_{t\in\mathbb{Z}^d} p_t B^N Y(t) \in (0,\infty)$ almost surely and thus Z_N in (2.7) is well-defined. Next, for any $a, h \in \mathbb{Z}^d$ and any 0-homogeneous measurable functional $F : E \mapsto [0,\infty]$, by the independence of N and Y applying Fubini theorem we obtain

$$\begin{aligned}
&\mathbb{E}\{Z_N(h)F(B^a Z_N)\}\\
&= \mathbb{E}\left\{\frac{B^N Y(h)}{\max_{s\in\mathbb{Z}^d} p_s B^N Y(s)} \mathbb{I}(\mathcal{I}_{fm}(p\cdot B^N Y) = N) F(B^{a+N}Y)\right\}\\
&= \sum_{j\in\mathbb{Z}^d} \mathbb{E}\left\{p_j \frac{B^j \Theta^*(h)}{\max_{s\in\mathbb{Z}^d} p_s \Theta^*(s-j)} \mathbb{I}(\mathcal{I}_{fm}(p\cdot B^j\Theta^*) = j) F(B^{a+j}\Theta^*)\right\}\\
&= \sum_{j\in\mathbb{Z}^d} \mathbb{E}\{B^j\Theta^*(h)\mathbb{I}(\mathcal{I}_{fm}(p\cdot B^j\Theta^*) = j)F(B^{a+j}\Theta^*)\}\\
&= \sum_{j\in\mathbb{Z}^d} \mathbb{E}\{\mathbb{I}(\mathcal{I}_{fm}(p\cdot B^h\Theta^*) = j, \Theta^*(j-h) > 0)F(B^{a+h}\Theta^*)\}\\
&= \mathbb{E}\left\{F(B^{a+h}\Theta^*)\sum_{j\in\mathbb{Z}^d} \mathbb{I}(\mathcal{I}_{fm}(p\cdot B^h\Theta^*) = j, \Theta^*(j-h) > 0)\right\}\\
&= \mathbb{E}\{F(B^{a+h}\Theta^*)\}\\
&= \mathbb{E}\{Z_N(a)F(B^h Z_N)\},
\end{aligned}$$

where the third equality follows since $\mathcal{I}_{fm}(p\cdot B^j\Theta^*) = j$ implies

$$\max_{s\in\mathbb{Z}^d} p_s\Theta^*(s-j) = p_j B^j\Theta^*(j) = p_j\Theta^*(0) = p_j > 0$$

almost surely, the fourth equality follows from (2.6) and the assumption that $\mathbb{P}\{\Theta^*(0) = 1\} = 1$, the sixth one is consequence of the following (which follows from (6.2))

$$\sum_{j\in\mathbb{Z}^d} \mathbb{I}(\mathcal{I}_{fm}(p\cdot B^h\Theta^*) = j) = \mathbb{I}(\mathcal{I}_{fm}(p\cdot B^h\Theta^*) \in \mathbb{Z}^d) = 1$$

almost surely and the fact that $\mathcal{I}_{fm}(p\cdot B^h\Theta^*) = j$ implies for any $h \in \mathbb{Z}^d$

$$p_j\Theta^*(j-h) \ge p_h\Theta^*(0) \ge p_h > 0$$

almost surely and consequently $\Theta^*(j-h) > 0$ almost surely. Finally, the last claimed equality is established by repeating the calculations for $\mathbb{E}\{Z_N(a)F(B^h Z_N)\}$. Hence the proof follows by (2.5) and the definition of the spectral tail rf Θ via the spectral rf Z. □

PROOF OF LEMMA 3: Let $r_n \in \mathbb{Z}^d, n \ge 1$ be non-negative integers with components $r_{nj}, j \le d$ such that $\lim_{n\to\infty} n/r_{nj} = \lim_{n\to\infty} r_{nj} = \infty$. The stationarity of X yields further

$$C(A) = \mathbb{E}\left\{\max_{i\in A} Z(i)\right\} = C(A')$$

for any finite set of indices $A \subset \mathbb{Z}^d$ and any $A' \subset \mathbb{Z}^d$ which is a shift/translation of A. Moreover, by the sub-additivity of the maximum

$$C(A\cup B) \le C(A) + C(B).$$

Hence the growth of $C(A)$ is as that of the counting measure of A, see [16] for this argument and [33]. Consequently,

$$\lim_{n\to\infty} \frac{\mathbb{E}\left\{\max_{0\le i\le r_n, i\in\mathbb{Z}^d} Z(i)\right\}}{\prod_{j=1}^d r_{nj}} = \lim_{n\to\infty} n^{-d}\mathbb{E}\left\{\max_{i\in[0,n]^d, i\in\mathbb{Z}^d} Z(i)\right\} = \mathcal{H}.$$

The assumption on r_n and (6.1) imply that

$$\widetilde{\theta_X} \sim \frac{\mathbb{P}\{\max_{0\le i\le r_n, i\in\mathbb{Z}^d} X(i) > n\}}{\prod_{j=1}^d r_{nj}\mathbb{P}\{X(0)>n\}} \sim \frac{\mathbb{E}\left\{\max_{0\le i\le r_n, i\in\mathbb{Z}^d} Z(i)\right\}}{\prod_{j=1}^d r_{nj}}, \quad n\to\infty.$$

Hence $\mathcal{H} = \theta_X$ establishes the proof. □

PROOF OF LEMMA 4: We give first a key characterisation of tail rf's proved initially in [35] and also stated for rf's in [2]. Namely, for any measurable map $F : E \mapsto [0,\infty]$

$$\mathbb{E}\left\{F(Y)\mathbb{I}(Y(i) > 1/t)\right\} = t\mathbb{E}\left\{F(B^iY)\mathbb{I}(Y(-i) > t)\right\} \tag{6.3}$$

holds for all $i\in\mathbb{Z}^d, t>0$. If $\mathcal{I},\mathcal{I}'$ are two anchoring maps, since $Y(0)=R>1$ almost surely and $\mathcal{I}(Y)=i$ implies $Y(i)>1$ almost surely, by (6.3)

$$\begin{aligned}
&\mathbb{P}\{\mathcal{I}(Y)\in\mathbb{Z}^d, \mathcal{I}'(Y)=0, F(Y)<\infty\}\\
&= \sum_{i\in\mathbb{Z}^d}\mathbb{P}\{\mathcal{I}(Y)=i, \mathcal{I}'(Y)=0, F(Y)<\infty\}\\
&= \sum_{i\in\mathbb{Z}^d}\mathbb{P}\{\mathcal{I}(Y)=i, Y(i)>1, \mathcal{I}'(Y)=0, F(Y)<\infty\}\\
&= \sum_{i\in\mathbb{Z}^d}\mathbb{P}\{\mathcal{I}(B^iY)=i, Y(-i)>1, \mathcal{I}'(B^iY)=0, F(Y)<\infty\}\\
&= \sum_{i\in\mathbb{Z}^d}\mathbb{P}\{\mathcal{I}(Y)=0, F(Y)<\infty, Y(-i)>1, \mathcal{I}'(Y)=-i\}\\
&= \sum_{i\in\mathbb{Z}^d}\mathbb{P}\{\mathcal{I}(Y)=0, F(Y)<\infty, \mathcal{I}'(Y)=-i\}\\
&= \mathbb{P}\{\mathcal{I}'(Y)\in\mathbb{Z}^d, \mathcal{I}(Y)=0, F(Y)<\infty\}.
\end{aligned}$$

With similar arguments we obtain

$$\mathbb{P}\{\mathcal{I}(Y)\in\mathbb{Z}^d, F(Y)<\infty\} = \sum_{i\in\mathbb{Z}^d}\mathbb{P}\{\mathcal{I}(Y)=0, F(Y)<\infty, Y(-i)>1\}.$$

Consequently, $\mathbb{P}\{\mathcal{I}(Y)=0, F(Y)<\infty\}=0$ is equivalent with

$$\mathbb{P}\{\mathcal{I}(Y)\in\mathbb{Z}^d, F(Y)<\infty\}=0$$

establishing the proof. □

PROOF OF LEMMA 5: As shown in [19] condition $\mathbb{P}\{\mathcal{S}(Z)=\infty\}=1$ is equivalent with X being generated by a non-singular conservative flow. The latter is equivalent with $\theta_X=0$, see [21] (which follows by [38] if

$d = 1$ and by [37] for $d > 1$). In view of Lemma 4 and (3.3) $\mathbb{P}\{\mathcal{I}(Y) = 0, \mathcal{S}(Y) < \infty\} = 0$ is equivalent with $\mathbb{P}\{\mathcal{S}(Y) < \infty\} = 0$. Applying Lemma 9 in Appendix the latter is equivalent with $\mathbb{P}\{\mathcal{S}(Z) < \infty\} = 0$. This establishes the proof since the latter is equivalent with $\theta_X = 0$. □

PROOF OF THEOREM 2: We have that $\mathbb{P}\{\mathcal{S}(Z) < \infty\} = 0$ is equivalent with X is generated by a non-singular conservative flow, which in view of [36, 37, 38] is equivalent with $\theta_X = 0$. Applying Lemma 10 in Appendix to BRs spectral rf Z we have that $ZF(Z)$ is also a BRs spectral rf for any measurable functional $F : E \mapsto [0, \infty]$, which is 0-homogeneous and shift-invariant. Since both $\mathbb{I}(\mathcal{S}(f) = \infty), \mathbb{I}(\mathcal{S}(f) < \infty), f \in E$ are measurable 0-homogeneous and shift-invariant functionals and by the above

$$\lim_{n\to\infty} \frac{1}{n^d} \mathbb{E}\left\{ \max_{t\in[0,n]^d\cap\mathbb{Z}^d} Z(t)\mathbb{I}(\mathcal{S}(Z) = \infty) \right\} = 0$$

we have using further (1.5)

$$\begin{aligned} \theta_X = \mathcal{H} &= \lim_{n\to\infty} \frac{1}{n^d} \mathbb{E}\left\{ \max_{t\in[0,n]^d\cap\mathbb{Z}^d} Z(t) \right\} \\ &= \lim_{n\to\infty} \frac{1}{n^d} \mathbb{E}\left\{ \max_{t\in[0,n]^d\cap\mathbb{Z}^d} Z(t)\mathbb{I}(\mathcal{S}(Z) < \infty) \right\}. \end{aligned} \tag{6.4}$$

Next, assuming that $\mathbb{P}\{\mathcal{S}(Z) < \infty\} > 0$ by Lemma 9 $\mathbb{P}\{\mathcal{S}(\Theta) < \infty\} > 0$ and the converse also holds. Setting $Z_*(t) = Z(t)\mathbb{I}(\mathcal{S}(Z) < \infty)$ by Lemma 10 it is BRs and further $\mathcal{S}(Z_*) < \infty$ almost surely. In view of Lemma 8 we can assume that $\mathcal{S}(Z_*) > 0$ almost surely. Applying (2.5) and using the equivalence of **A1** and **A3** we obtain further

$$\begin{aligned} \theta_X &= \lim_{n\to\infty} \frac{1}{n^d} \sum_{h\in[0,n]^d\cap\mathbb{Z}^d} \mathbb{E}\left\{ Z_*(h) \frac{\max_{t\in[0,n]^d\cap\mathbb{Z}^d} Z_*(t)}{\sum_{t\in[0,n]^d\cap\mathbb{Z}^d} Z_*(t)} \right\} \\ &= \lim_{n\to\infty} \frac{1}{n^d} \sum_{h\in[0,n]^d\cap\mathbb{Z}^d} \mathbb{E}\left\{ Z_*(0) \frac{\max_{t\in[0,n]^d\cap\mathbb{Z}^d} B^h Z_*(t)}{\sum_{t\in[0,n]^d\cap\mathbb{Z}^d} B^h Z_*(t)} \right\} \\ &= \lim_{\varepsilon\downarrow 0} \lim_{n\to\infty} \frac{1}{n^d} \sum_{h\in[\varepsilon n,(1-\varepsilon)n]^d\cap\mathbb{Z}^d} \mathbb{E}\left\{ Z_*(0) \frac{\max_{t\in[0,n]^d\cap\mathbb{Z}^d} B^h Z_*(t)}{\sum_{t\in[0,n]^d\cap\mathbb{Z}^d} B^h Z_*(t)} \right\} \\ &= \mathbb{E}\left\{ Z_*(0) \frac{\max_{t\in\mathbb{Z}^d} Z_*(t)}{\sum_{t\in\mathbb{Z}^d} Z_*(t)} \right\} \\ &= \mathbb{E}\left\{ \frac{\max_{t\in\mathbb{Z}^d} \Theta(t)}{\mathcal{S}(\Theta)} \mathbb{I}(\mathcal{S}(\Theta) < \infty) \right\}. \end{aligned}$$

Since by definition the events $\{\mathcal{I}_{fm}(\Theta) \in \mathbb{Z}^d\}$ and $\{\mathcal{S}(\Theta) < \infty\}$ are almost surely the same, the 0-homogeneity of $\mathcal{I}_{fm}(\cdot)$ implies (recall $\Theta(0) = 1$ almost surely)

$$\begin{aligned} \theta_X &= \mathbb{E}\left\{ \frac{\max_{t\in\mathbb{Z}^d} \Theta(t)}{\mathcal{S}(\Theta)} \mathbb{I}(\mathcal{I}_{fm}(\Theta) \in \mathbb{Z}^d) \right\} \\ &= \sum_{j\in\mathbb{Z}^d} \mathbb{E}\left\{ \frac{\max_{t\in\mathbb{Z}^d} \Theta(t)}{\mathcal{S}(\Theta)} \mathbb{I}(\mathcal{I}_{fm}(\Theta) = j) \right\} \\ &= \sum_{j\in\mathbb{Z}^d} \mathbb{E}\left\{ \Theta(j) \frac{\Theta(0)}{\mathcal{S}(\Theta)} \mathbb{I}(\mathcal{I}_{fm}(\Theta) = j) \right\} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j\in\mathbb{Z}^d} \mathbb{E}\left\{\frac{\Theta(-j)}{\mathcal{S}(\Theta)}\mathbb{I}(\mathcal{I}_{fm}(B^j\Theta)=j)\right\}\\
&= \mathbb{E}\left\{\sum_{j\in\mathbb{Z}^d}\frac{\Theta(-j)}{\mathcal{S}(\Theta)}\mathbb{I}(\mathcal{I}_{fm}(\Theta)=0)\right\}\\
&= \mathbb{P}\{\mathcal{I}_{fm}(\Theta)=0\}\\
&= \mathbb{P}\{\mathcal{I}_{fm}(\Theta)=0, \mathcal{S}(\Theta)<\infty\},
\end{aligned}
$$

where we applied (2.6) in the last third line combined with condition $ii)$ in the definition of anchoring maps and also used that $\mathcal{S}(f), f\in E$ is a shift-invariant functional. Clearly, the last two formulas hold also for the last maximum functional. Since (3.3) implies

$$
\mathbb{P}\{\mathcal{I}(Y)\notin\mathbb{Z}^d, \mathcal{S}(Y)<\infty\}=0, \tag{6.5}
$$

then using Lemma 4 to obtain the second equality below we have

$$
\begin{aligned}
\mathbb{P}\{\mathcal{I}_{fm}(\Theta)=0,\mathcal{S}(\Theta)<\infty\} &= \mathbb{P}\left\{\mathcal{I}_{fm}(Y)=0,\mathcal{S}(Y)<\infty,\mathcal{I}(Y)\in\mathbb{Z}^d\right\}\\
&\quad +\mathbb{P}\left\{\mathcal{I}_{fm}(Y)=0,\mathcal{S}(Y)<\infty,\mathcal{I}(Y)\notin\mathbb{Z}^d\right\}\\
&= \mathbb{P}\left\{\mathcal{I}_{fm}(Y)\in\mathbb{Z}^d,\mathcal{S}(Y)<\infty,\mathcal{I}(Y)=0\right\}\\
&= \mathbb{P}\left\{\mathcal{I}(Y)=0,\mathcal{S}(Y)<\infty\right\}
\end{aligned}
$$

and hence $\theta_X=\mathbb{P}\{\mathcal{I}_{fe}(Y)=0\}$ follows and the same is true also for the last exeedance functional. In view of the equivalence **A2** and **A4** we have

$$
\{\mathcal{S}(Y)<\infty\}\subset\{\mathcal{B}(Y)<\infty\}, \tag{6.6}
$$

with $\mathcal{B}(Y):=\sum_{t\in\mathbb{Z}^d}\mathbb{I}(Y(t)>1)$. Hence since $Y(0)=R\Theta(0)=R>1$ almost surely implies $\mathcal{B}(Y)\ge 1$ almost surely

$$
\begin{aligned}
&\mathbb{E}\left\{\frac{\mathcal{B}(Y)}{\mathcal{B}(Y)}\mathbb{I}(\mathcal{I}(Y)=0,\mathcal{S}(Y)<\infty)\right\}\\
&= \sum_{t\in\mathbb{Z}^d}\mathbb{E}\left\{\frac{1}{\mathcal{B}(Y)}\mathbb{I}(\mathcal{I}(Y)=0,Y(t)>1,\mathcal{S}(Y)<\infty)\right\}\\
&= \mathbb{E}\left\{\frac{1}{\mathcal{B}(Y)}\sum_{t\in\mathbb{Z}^d}\mathbb{I}(\mathcal{I}(Y)=-t,Y(-t)>1,\mathcal{S}(Y)<\infty)\right\}\\
&= \mathbb{E}\left\{\frac{1}{\mathcal{B}(Y)}\mathbb{I}(\mathcal{I}(Y)\in\mathbb{Z}^d,\mathcal{S}(Y)<\infty)\right\}\\
&= \mathbb{E}\left\{\frac{1}{\mathcal{B}(Y)}\mathbb{I}(\mathcal{S}(Y)<\infty)\right\}\\
&= \mathbb{E}\left\{\frac{1}{\mathcal{B}(Y)}\mathbb{I}(\mathcal{S}(\Theta)<\infty)\right\},
\end{aligned}
$$

where we used (6.3) to derive the last fourth line and the last second equality follows from (6.5). With the same arguments as in the proof of [46][Lem 2.5] considering the discrete setup as in [18] for any $n > 0$

$$\mathbb{E}\left\{\max_{t\in[0,n]^d\cap\mathbb{Z}^d} Z(t)\right\} = \sum_{t\in[0,n]^d\cap\mathbb{Z}^d} \mathbb{E}\left\{\frac{1}{\sum_{s\in[0,n]^d\cap\mathbb{Z}^d}\mathbb{I}(Y(s-t)>1)}\right\}. \tag{6.7}$$

Since $Y(0) > 1$ almost surely and thus the denominator in the expectation above is greater equal 1 and converges as $n\to\infty$ almost surely to $\mathcal{B}(Y)$, it follows by the dominated convergence theorem that

$$\theta_X = \lim_{n\to\infty} n^{-d}\mathbb{E}\left\{\max_{t\in[0,n]^d\cap\mathbb{Z}^d} Z(t)\right\} = \mathbb{E}\left\{\frac{1}{\mathcal{B}(Y)}\right\} \le 1,$$

hence (3.10) holds. From the last two expressions of θ_X we conclude that $\mathbb{E}\left\{\frac{1}{\mathcal{B}(Y)}\mathbb{I}(\mathcal{S}(Y)=\infty)\right\} = 0$. Consequently, almost surely $\{\mathcal{B}(Y)<\infty\}\subset\{\mathcal{S}(Y)<\infty\}$, which together with (6.6) implies that almost surely

$$\{\mathcal{B}(Y)<\infty\} = \{\mathcal{S}(Y)<\infty\}.$$

Next, if $\mathbb{P}\{\Theta(i)=0\}=1$ for all $i\neq 0, i\in\mathbb{Z}^d$, then

$$\theta_X = \mathbb{E}\left\{\frac{\max_{t\in\mathbb{Z}^d}\Theta(t)}{\sum_{t\in\mathbb{Z}^d}\Theta(t)}\mathbb{I}(\mathcal{S}(\Theta)<\infty)\right\} = 1.$$

Conversely, if $\theta_X = 1$, then necessarily $\mathbb{P}\{\mathcal{S}(\Theta)<\infty\}=1$ and thus

$$\theta_X = 1 = \mathbb{E}\left\{\frac{\max_{t\in\mathbb{Z}^d}\Theta(t)}{\sum_{t\in\mathbb{Z}^d}\Theta(t)}\right\}$$

implying that $\max_{t\in\mathbb{Z}^d}\Theta(t) = \sum_{t\in\mathbb{Z}^d}\Theta(t)$ almost surely. Taking $\mathcal{I}(f) = \mathcal{I}_{fm}(f)$ we have that $\theta_X = \mathbb{P}\{\mathcal{I}(\Theta)=0\}=1$ implies that $\max_{t\in\mathbb{Z}^d}\Theta(t)=\Theta(0)=1$ almost surely and therefore

$$\sum_{t\in\mathbb{Z}^d}\Theta(t) = 1 + \sum_{t\in\mathbb{Z}^d, t\neq 0}\Theta(t) = 1$$

almost surely. Consequently, (recall $\Theta(i)$'s are non-negative) $\mathbb{P}\{\Theta(i)=0\}=1$ for all $i\neq 0, i\in\mathbb{Z}^d$ establishing the proof. □

PROOF OF LEMMA 6: For any $s>0$ and any non-decreasing sequence of integers $r_n, n\in\mathbb{N}$ tending to infinity such that $\lim_{n\to\infty} r_n^d/n = 0$ we have for any positive integer m (recall $\mathbb{E}\{Z(t)\}=1$ for any $t\in\mathbb{Z}^d$)

$$n^{-1}\mathbb{E}\left\{\max_{m<\|t\|<r_n, t\in\mathbb{Z}^d} Z(t)\right\} \le n^{-1}\sum_{m<\|t\|<r_n, t\in\mathbb{Z}^d}\mathbb{E}\{Z(t)\} \to 0, \quad n\to\infty,$$

hence by (6.1) and the dominated convergence theorem

$$\begin{aligned} 1 - &\lim_{n\to\infty}\mathbb{P}\{\max_{m<\|t\|<r_n, t\in\mathbb{Z}^d} X(t) > ns | X(0)>ns\} \\ &= s\lim_{n\to\infty} n\mathbb{P}\{\max_{m<\|t\|<r_n, t\in\mathbb{Z}^d} X(t)\le ns, X(0)>ns\} \\ &= \mathbb{E}\left\{\max_{m<\|t\|<\infty, t\in\mathbb{Z}^d, t=0} Z(t) - \max_{m<\|t\|<\infty, t\in\mathbb{Z}^d} Z(t)\right\} \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\left\{\mathbb{I}(Z(0)>0)\Big[\max_{m<\|t\|<\infty, t\in\mathbb{Z}^d, t=0} Z(t) - \max_{m<\|t\|<\infty, t\in\mathbb{Z}^d} Z(t)\Big]\right\}\\
&= \mathbb{E}\left\{Z(0)\mathbb{I}(Z(0)>0)\Big[\max_{m<\|t\|<\infty, t\in\mathbb{Z}^d, t=0} \frac{Z(t)}{Z(0)} - \max_{m<\|t\|<\infty, t\in\mathbb{Z}^d} \frac{Z(t)}{Z(0)}\Big]\right\}\\
&= \mathbb{E}\left\{\Big(1-\max_{m<\|t\|<\infty, t\in\mathbb{Z}^d} \Theta(t)\Big)_+\right\}
\end{aligned}
$$

for any positive integer m (recall $\Theta(0)=1$ almost surely). If **A1** holds, then by the dominated convergence theorem

$$
\lim_{m\to\infty} \mathbb{E}\left\{\max_{m<\|t\|<\infty, t\in\mathbb{Z}^d, t=0} Z(t) - \max_{m<\|t\|<\infty, t\in\mathbb{Z}^d} Z(t)\right\} = \mathbb{E}\left\{Z(0)\right\} = 1,
$$

hence Condition C is satisfied.
Conversely, if Condition C is satisfied for some sequence $r_n, n\ge 1$ of non-negative increasing integers, then by the above calculations

$$
\begin{aligned}
1 - \lim_{m\to\infty}\lim_{n\to\infty} \mathbb{P}\{\max_{m<\|t\|<r_n, t\in\mathbb{Z}^d} X(t) > ns | X(0) > ns\}&\\
= \lim_{m\to\infty} \mathbb{E}\left\{(1-\max_{m<\|t\|<\infty, t\in\mathbb{Z}^d} \Theta(t))_+\right\} &= 1
\end{aligned}
$$

and thus almost surely as $m\to\infty$

$$
\max_{m<\|t\|<\infty, t\in\mathbb{Z}^d} \Theta(t) \to 0.
$$

Consequently, by Lemma 11 in Appendix condition **A2** holds, hence the proof follows from Remark 4. □

7 Appendix

For notational simplicity we consider the case $\alpha=1$ in the following. The results for $\alpha>0$ can be formulated with obvious modifications.

Lemma 8. *If* $X(t), t\in\mathbb{Z}^d$ *is a max-stable rf with de Haan representation* (1.1) *and some spectral rf* Z *satisfying* $\mathbb{E}\{Z(t)\}\in(0,\infty)$ *for all* $t\in\mathbb{Z}^d$, *then we can find a spectral rf* Z_* *for* X *such that* $\max_{t\in\mathbb{Z}^d} Z_*(t) > 0$ *almost surely.*

PROOF OF LEMMA 8: Let $w_i, i\in\mathbb{Z}^d$ be positive constants such that

$$
\mathbb{E}\left\{\sum_{i\in\mathbb{Z}^d} w_i Z(i)\right\} \in (0,\infty).
$$

w_i's exist since $\mathbb{E}\{Z(i)\}\in(0,\infty)$ for any $i\in\mathbb{Z}^d$. By the choice of w_i's we have that

$$
M = \max_{i\in\mathbb{Z}^d} w_i Z(i)
$$

is a non-negative rv and $a=\mathbb{E}\{M\}\in(0,\infty)$. Let $Z_*(t), t\in\mathbb{Z}^d$ be a rf defined by

$$
\mathbb{P}\{Z_*\in A\} = \mathbb{E}\{M\mathbb{I}(aZ/M\in A)/a\}
$$

for any measurable set $A \subset E$. Since by the above definition

$$\mathbb{P}\{\max_{i\in\mathbb{Z}^d} w_i Z_*(i) = 0\} = \mathbb{E}\{M\mathbb{I}(\max_{i\in\mathbb{Z}^d} w_i Z(i)/M = 0)/a\} = 0$$

it follows that $\mathbb{P}\{\max_{i\in\mathbb{Z}^d} Z_*(i) = 0\} = 0$. Moreover, for any $x_i \in (0,\infty), t_i \in \mathbb{Z}^d, i \le n$

$$\begin{aligned}
&-\ln \mathbb{P}\{X(t_1) \le x_1, \ldots, X(t_n) \le x_n\}\\
&= \mathbb{E}\{\max_{1\le i\le n} Z(t_i)/x_i\}\\
&= \mathbb{E}\{\mathbb{I}(\max_{1\le i\le n} Z(t_i) > 0) \max_{1\le i\le n} Z(t_i)/x_i\}\\
&= \mathbb{E}\{M/a\mathbb{I}(M>0)\mathbb{I}(\max_{1\le i\le n} Z(t_i) > 0) \max_{1\le i\le n} aZ(t_i)/(Mx_i)\}\\
&= \mathbb{E}\{\mathbb{I}(\max_{1\le i\le n} Z_*(t_i) > 0) \max_{1\le i\le n} Z_*(t_i)/x_i\}\\
&= \mathbb{E}\{\max_{1\le i\le n} Z_*(t_i)/x_i\},
\end{aligned}$$

where the third equality is a simple consequence of $\max_{1\le i\le n} Z(t_i) > 0$ implies $M > 0$. Hence Z_* is a spectral rf for X. The calculations above show that we can define alternatively $Z_*(t) = \mathbb{P}\{\max_{s\in\mathbb{Z}^d} Z(s) > 0\}Z(t)$ conditioned on $\max_{s\in\mathbb{Z}^d} Z(s) > 0$, which was suggested by the reviewer. □

Proof of (3.1): As in the proof of Lemma 8, we can assume without loss of generality that Z is such that $\max_{t\in\mathbb{Z}^d}(Z(t)/x_t) > 0$ almost surely for any $x = (x_j)_{j\in\mathbb{Z}^d}$ a positive sequence. Suppose for simplicity that $\alpha = 1$ and let next x be a sequence with finite number of positive elements and the rest equal to ∞ (we interpret a/∞ as 0). Since further Z/x consists of zeros and finitely many positive numbers, then $\mathcal{I}_{fm}(Z/x) \in \mathbb{Z}^d$ almost surely. Consequently, by (6.1), Fubini theorem and the fact that $\mathcal{I}_{fm}(Z/x) = j$ implies $\max_{i\in\mathbb{Z}^d}(Z(t_i)/x_i) = Z(j)/x_j$ almost surely

$$\begin{aligned}
-\ln \mathbb{P}\{X(i) \le x_i, i \in \mathbb{Z}^d\} &= \mathbb{E}\{\max_{i\in\mathbb{Z}^d} Z(t_i)/x_i\mathbb{I}(\mathcal{I}_{fm}(Z/x) \in \mathbb{Z}^d)\}\\
&= \sum_{j\in\mathbb{Z}^d} \mathbb{E}\{\max_{i\in\mathbb{Z}^d} Z(t_i)/x_i\mathbb{I}(\mathcal{I}_{fm}(Z/x) = j)\}\\
&= \sum_{j\in\mathbb{Z}^d} \frac{1}{x_j}\mathbb{E}\{Z(j)\mathcal{I}_{fm}(Z/x) = j\}\\
&= \sum_{j\in\mathbb{Z}^d} \frac{1}{x_j}\mathbb{E}\{Z(0)\mathcal{I}_{fm}(B^j Z/x) = j\}\\
&= \sum_{j\in\mathbb{Z}^d} \frac{1}{x_j}\mathbb{E}\{Z(0)\mathbb{I}(\mathcal{I}_{fm}((B^j Z/x)/Z(0)) = j)\}\\
&= \sum_{j\in\mathbb{Z}^d} \frac{1}{x_j}\mathbb{P}\{\mathcal{I}_{fm}(B^j(\Theta/(B^{-j}x))) = j\}\\
&= \sum_{j\in\mathbb{Z}^d} \frac{1}{x_j}\mathbb{P}\{\mathcal{I}_{fm}(\Theta/(B^{-j}x)) = 0\},
\end{aligned}$$

where the fourth first equality follows from (2.5) and the last equality follows since $\mathcal{I}_{fm}$ is an anchoring map. □

Lemma 9. *Let $Z(t), t \in \mathbb{Z}^d$ be a BRs rf satisfying* (1.2). *If $F : E \mapsto [0,\infty]$ is a shift-invariant and 0-homogeneous measurable map, then $\mathbb{E}\{F(Z)\} = 0$ is equivalent with $\mathbb{E}\{F(\Theta)\} = 0$. If further F is bounded by 1, then $\mathbb{E}\{F(Z)\} = 1$ is equivalent with $\mathbb{E}\{F(\Theta)\} = 1$.*

PROOF OF LEMMA 9: By the shift-invariance of F and (2.5) we have

$$\begin{aligned} 0 &= \mathbb{E}\{F(\Theta)\} = \mathbb{E}\{Z(0)F(Z/Z(0))\} = \sum_{i\in\mathbb{Z}^d} \mathbb{E}\{Z(0)F(B^{-i}Z)\} \\ &= \sum_{i\in\mathbb{Z}^d} \mathbb{E}\{Z(i)F(Z)\} \ge \mathbb{E}\left\{\Big(\max_{i\in\mathbb{Z}^d} Z(i)\Big)F(Z)\right\}, \end{aligned}$$

hence since Z is chosen such that $\max_{i\in\mathbb{Z}^d} Z(i) > 0$ almost surely, then $\mathbb{E}\{F(Z)\} = 0$ follows. If $\mathbb{E}\{F(Z)\} = 0$, then $F(Z) = 0$ almost surely and thus

$$0 = \mathbb{E}\{Z(0)F(Z)\} = \mathbb{E}\{F(\Theta)\} = 0$$

follows. Next, $\mathbb{E}\{F(\Theta)\} = 1$ is the same as $\mathbb{E}\{1 - F(\Theta)\} = 0$, which is equivalent with $\mathbb{E}\{1 - F(Z)\} = 0$ as shown above, establishing the proof. □

Lemma 10. *If $F : E \mapsto [0,\infty]$ is a 0-homogeneous measurable functional and $Z(t), t \in \mathbb{Z}^d$ is a BRs rf, then $Z_* = ZF(Z)$ is also a BRs rf, provided that $\mathbb{E}\{Z_*(t_0)\} \in (0,\infty)$ for some $t_0 \in \mathbb{Z}^d$.*

PROOF OF LEMMA 10: Using (2.5) we have that $\mathbb{E}\{Z_*(t)\} = \mathbb{E}\{Z_*(t_0)\} \in (0,\infty)$ for any $t \in \mathbb{Z}^d$ and in particular $\mathbb{P}\{F(Z) = 0\} < 1$ and $\mathbb{P}\{F(Z) = \infty\} = 0$. Since F is 0-homogeneous, we have that Z_* satisfies (2.5), which is an equivalent condition for a spectral rf to be a BRs rf, see [25]. □

Lemma 11. *If $V(t), t \in \mathbb{Z}^d$ is a non-negative rf, then $\mathbb{P}\{\lim_{\|t\|\to\infty} V(t) = 0\} = 1$ is equivalent with there exists a non-decreasing sequence of integers $r_n, n \ge 1$ that converge to infinite as $n \to \infty$ such that*

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mathbb{P}\{\max_{m\le\|t\|\le r_n} V(t) > \delta\} = 0 \tag{7.1}$$

is valid for any $\delta > 0$.

PROOF OF LEMMA 11: It is well-known that (see e.g., [22][A1.3])

$$\mathbb{P}\{\lim_{\|t\|\to\infty} V(t) = 0\} = 1$$

if and only if for all large m and any δ, ε positive

$$\mathbb{P}\{\max_{\|t\|\ge m} V(t) > \delta\} < \varepsilon,$$

which clearly implies (7.1). Assuming that the latter condition holds, then for given δ, ε positive there exists N such that for all m, n larger than N we have $\mathbb{P}\{\max_{m\le\|t\|\le r_n} V(t) > \delta\} < \varepsilon$. Since $\lim_{n\to\infty} r_n = \infty$, then $\mathbb{P}\{\max_{m\le\|t\|} V(t) > \delta\} \le \varepsilon$, hence the claim follows. □

Lemma 12. *Let $\eta_i(t), i = 1,2, t \in \mathbb{Z}^d$ be two independent stationary rf's with unit Fréchet marginal distributions. If the extremal indices of both η_1 and η_2 exist, then the rf $X(t) = \max(p\eta_1(t), (1-p)\eta_2(t)), t \in \mathbb{Z}^d$ has for any $p \in (0,1)$ extremal index $\theta_X = p\theta_{\eta_1} + (1-p)\,\theta_{\eta_2} \in [0,1]$.*

PROOF OF LEMMA 12: By the independence of η_1 and η_2 we have that X is stationary with unit Fréchet marginal distributions. In order to show the claim it suffices to prove that $\max_{t\in[0,n]^d} X(t)/n^d$ converges in distribution as $n \to \infty$ to $(p\theta_{\eta_1} + (1-p)\theta_{\eta_2})\xi$, where ξ is a unit Fréchet rv. As $n \to \infty$, by the assumptions $\max_{t\in[0,n]^d} \eta_i(t)/n^d$ converge for $i = 1,2$ in distribution to $p_i\theta_{\eta_i}\xi_i$ with ξ_1, ξ_2 two independent unit Fréchet rv's and $p_1 = 1 - p_2 = p$. Since $\max(p_1\theta_{\eta_1}\xi_1, p_2\theta_{\eta_2}\xi_2)$ has the same df as $(p_1\theta_{\eta_1} + p_2\theta_{\eta_2})\xi$, the claim follows by the independence of η_i's and Slutsky's lemma. $\square$

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