ON THE DISTRIBUTION OF DIVIDEND PAYMENTS IN A SPARRE ANDERSEN MODEL WITH GENERALIZED ERLANG(N) INTERCLAIM TIMES*

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Abstract

In this paper we present some results on the distribution of dividend payments until ruin under a Sparre Andersen risk model with generalized Erlang(n)-distributed inter-claim times and a constant dividend barrier. An integro-differential equation for the moment-generating function of the discounted sum of dividend payments until ruin is derived. Moreover, explicit solutions for arbitrary moments of the present value of dividend payments are obtained, when the individual claim amounts have a distribution with rational Laplace transform. Numerical illustrations of the results are given for an Erlang(2) risk model and Erlang(2)-distributed claim amounts.

Keywords: renewal risk model, dividend barrier, present value of dividend payments, moment-generating function

1 Introduction

In recent years there has been considerable interest in extending results from classical risk theory, where the surplus process of a non-life insurance portfolio is modeled by a compound Poisson process, to more flexible models. One such extension deals with replacing the Poisson claim number process by a more general renewal process leading to the so-called Sparre Andersen risk model. In this context, one particularly tractable class of distributions for the inter-claim times T_i ($i = 1, 2, \ldots$) is the class of generalized Erlang(n)-distributions, mainly because a generalized Erlang(n) random variable can be expressed as an independent sum of n exponential random variables and thus the lack-of-memory property of exponential random variables can be used for the analysis of the corresponding risk process. Various aspects of ruin in an Erlangian risk model are for instance studied in Dickson (1998), Dickson and Hipp (1998,2001), Cheng and Tang (2003), Sun and Yang (2004), Li and Garrido

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(2004a), Tsai and Sun (2004) and Gerber and Shiu (2005).

Another extension of the classical risk model is to allow for dividend payments to shareholders. The problem of finding optimal dividend payment strategies has a long history, see e.g. De Finetti (1957), Bühlmann (1970), Borch (1974), Gerber (1972,1979), Asmussen and Taskar (1997), Hubalek and Schachermayer (2004). Some recent papers on dividend barrier strategies are Paulsen and Gjessing (1997), Siegl and Tichy (1999), Albrecher and Kainhofer (2002), Albrecher et al. (2003) and Claramunt et al. (2003).

For the classical risk model and a constant dividend barrier, Lin et al. (2003) have studied the discounted penalty function at ruin, which is an important tool to quantify the riskiness of the barrier strategy. A second important quantity in assessing the quality of a dividend barrier strategy is the distribution of the discounted sum of dividend payments until ruin. Traditionally, only the first moment of this distribution was considered, but for the classical risk model and constant barrier strategy Dickson and Waters (2004) recently studied arbitrary moments (an extension of this analysis to linear dividend barriers can be found in Albrecher et al. (2005)). For a Brownian risk model, the distribution of dividend payments was investigated by Gerber and Shiu (2004a).
The analysis of the discounted penalty function in Lin et al. (2003) was recently generalized to a Sparre Andersen risk model with generalized Erlang(n)-distributed interclaim times by Li and Garrido (2004b) and it is natural to ask for a corresponding generalization of the results on the distribution of dividend payments.

In the present paper, we investigate the distribution of the discounted sum of dividend payments $D_{u,b}$ until ruin according to a constant dividend barrier b in a Sparre Andersen model with generalized Erlang(n)-distributed interclaim times (where u denotes the initial capital), in that way complementing the results of Li and Garrido (2004b). Using a differential approach, in Section 3 we derive an integro-differential equation for the moment-generating function of the dividend payments, which simplifies and generalizes some results of Dickson and Waters (2004). We then derive integro-differential equations for arbitrary moments of $D_{u,b}$ and solve them for claim size distributions with rational Laplace transform (Section 4). Finally, in Section 5 we illustrate the results for an Erlang(2)-model with Erlang(2)-distributed claim sizes.

2 The model

In the Sparre Andersen model, the surplus process $R(t)$ of an insurance portfolio is given by

$$R(t) = u + c\,t - \sum_{i=1}^{N(t)} X_i, \tag{1}$$

where the claims X_i are positive i.i.d. random variables with distribution function F and mean $\mu < \infty$, u is the initial capital and c is the constant premium density. Here the claim number process $N(t) = \min\{k : T_1 + \ldots + T_{k+1} > t\}$ is an ordinary renewal process and the inter-occurrence times T_i $(i = 1, 2, \ldots)$ are a sequence of positive i.i.d. random variables (T_1 denotes the time until the first claim and, in general, T_i denotes the time between the $(i-1)$-*th* and i-*th* claim). The net profit condition is given by $c > \mu/E(T_i)$.

In this paper we will assume that the random variables T_i $(i = 1, 2, \ldots)$ are generalized Erlang(n)-distributed, i.e. each T_i is a convolution of n independent exponential random variables with possibly different parameters $\lambda_1, \ldots, \lambda_n$. This will allow to use Markovian arguments due to the lack-of-memory property of the exponential distribution.

The risk process (1) is now modified by introducing a constant dividend barrier b $(b \geq 0)$, i.e. whenever the surplus process reaches the level b, the premium income is paid out as dividends to shareholders and the modified surplus process remains at level b until the occurrence of the next claim. Let the random variable $D_{u,b}$ denote the present value of the discounted sum of dividend payments until ruin (with discount factor $\delta \geq 0$). In the sequel we will be interested in the moment generating function

$$M(u, y, b) = \mathbb{E}\left[e^{y\, D_{u,b}} \middle| R_0 = u\right]$$

(for those values of y where it exists) and the m-th moment

$$W_m(u, b) = \mathbb{E}\left[D_{u,b}^m \middle| R_0 = u\right] \qquad (m \in \mathbb{N}).$$

Note that $W_0(u, b) \equiv 1$. We will always assume that $0 \leq u \leq b$ (otherwise the overshoot $b - u$ is immediately paid out as dividends).

3 An integro-differential equation for $M(u, y, b)$

Let $\frac{\partial \cdot}{\partial y}$ denote the differentiation operator with respect to y and correspondingly $\frac{\partial \cdot}{\partial u}$ the differentiation operator with respect to u. Moreover, define $\prod_{j=2}^{1} \cdot = 1$.

Theorem 1. *The moment-generating function $M(u, y, b)$ is the solution of the partial integro-differential equation*

$$\left(\prod_{j=1}^{n}\left(\delta\, y\, \frac{\partial \cdot}{\partial y} - c\, \frac{\partial \cdot}{\partial u} + \lambda_j\right)\right) M(u, y, b) - \prod_{j=1}^{n} \lambda_j \int_0^u M(u - v, y, b)\, dF(v)$$
$$- \prod_{j=1}^{n} \lambda_j\, (1 - F(u)) = 0 \quad (2)$$

with boundary conditions

$$\left(\prod_{j=2}^{k}\left(\delta\, y\,\frac{\partial\cdot}{\partial y}-c\,\frac{\partial\cdot}{\partial u}+\lambda_{j-1}\right)\right)\,\frac{\partial M(u,y,b)}{\partial u}\Bigg|_{u=b} = y\,\left(\prod_{j=2}^{k}\left(\delta\, y\,\frac{\partial\cdot}{\partial y}-c\,\frac{\partial\cdot}{\partial u}+\lambda_{j-1}\right)\right)\,M(u,y,b)\Bigg|_{u=b},\quad k=1,\dots,n, \tag{3}$$

and

$$\lim_{b\to\infty} M(u,y,b)=1. \tag{4}$$

Proof. Let us decompose every inter-occurrence time with generalized Erlang(n)-distribution into the independent sum of n exponential random variables with parameters $\lambda_1,\dots,\lambda_n$, each causing a "sub-claim" of size 0 and at the time of the n-th sub-claim an actual claim with distribution function F occurs. This can be realized by considering n states of the risk process. Starting at time 0 in state 1, every sub-claim causes a transition to the next state and the time of occurrence of the n-th sub-claim, an actual claim with distribution F occurs and the risk process jumps into state 1 again. Let $M^{(j)}(u,y,b)$ denote the moment-generating function of $D_{u,b}$, if the risk process is in state j ($j=1,\dots,n$). Eventually we are then interested in $M(u,y,b):=M^{(1)}(u,y,b)$.
In this way, one has transformed the risk process into one with exponential inter-occurrence times and can use the lack-of-memory property of the exponential distribution together with a differential argument. Indeed, conditioning on the occurrence of a (sub-)claim, we obtain for $0\le u<b$ and $j=1,\dots,n-1$

$$M^{(j)}(u,y,b)=(1-\lambda_j dt)M^{(j)}(u+cdt,ye^{-\delta dt},b) + \lambda_j dt M^{(j+1)}(u+cdt,ye^{-\delta dt},b)+o(dt),$$

from which we obtain by Taylor expansion and collecting all terms of order dt

$$c\,\frac{\partial M^{(j)}}{\partial u}(u,y,b)-\lambda_j\,M^{(j)}(u,y,b)-\delta y\,\frac{\partial M^{(j)}}{\partial y}(u,y,b)+\lambda_j M^{(j+1)}(u,y,b)=0. \tag{5}$$

For $j=n$ we have

$$M^{(n)}(u,y,b)=(1-\lambda_n dt)M^{(n)}(u+cdt,ye^{-\delta dt},b) + \lambda_n dt\int_0^{u+c\,dt} M^{(1)}(u+cdt-v,ye^{-\delta dt},b)\,\mathrm{d}F(v)+\lambda_n dt\int_{u+c\,dt}^{\infty}\mathrm{d}F(v)+o(dt),$$

which leads to

$$c\,\frac{\partial M^{(n)}}{\partial u}(u,y,b)-\lambda_n\,M^{(n)}(u,y,b)-\delta y\,\frac{\partial M^{(n)}}{\partial y}(u,y,b) + \lambda_n\int_0^u M^{(1)}(u-v,y,b)\,\mathrm{d}F(v)+\lambda_n(1-F(u))=0. \tag{6}$$

From (5) it follows that

$$M^{(j+1)}(u,y,b) = \frac{\delta\, y\, \frac{\partial \cdot}{\partial y} - c\, \frac{\partial \cdot}{\partial u} + \lambda_j}{\lambda_j}\, M^{(j)}(u,y,b), \quad (j = 1,\ldots,n-1) \tag{7}$$

and subsequently

$$M^{(n)}(u,y,b) = \left(\prod_{j=1}^{n-1} \frac{\delta\, y\, \frac{\partial \cdot}{\partial y} - c\, \frac{\partial \cdot}{\partial u} + \lambda_j}{\lambda_j} \right) M^{(1)}(u,y,b),$$

which together with (6) yields (2).
For $u = b$ we obtain analogously for $j = 1,\ldots,n-1$

$$M^{(j)}(b,y,b) = (1-\lambda_j dt) e^{y\, c\, dt} M^{(j)}(b, y e^{-\delta dt}, b) + \lambda_j dt\, e^{y\, c\, dt}\, M^{(j+1)}(b, y e^{-\delta dt}, b) + o(dt),$$

which by Taylor expansion leads to

$$-\delta\, y\, \frac{\partial M^{(j)}}{\partial y}(b,y,b) + (y\, c - \lambda_j)\, M^{(j)}(b,y,b) + \lambda_j M^{(j+1)}(b,y,b) = 0. \tag{8}$$

Comparing these equations with the corresponding equations in (5), continuity of $M^{(j)}(u,y,b)$ at $u = b$ thus implies

$$c\, \frac{\partial M^{(j)}(u,y,b)}{\partial u}\bigg|_{u=b} = c\, y\, M^{(j)}(b,y,b), \quad (j = 1,\ldots,n-1). \tag{9}$$

An analogous continuity argument shows that (9) also holds for $j = n$. For $j = 1$, (9) is equivalent to (3) for $k = 1$. Now it just remains to express equations (9) for $j = 2,\ldots,n$ in terms of $M^{(1)} = M$, which is done by virtue of (7). Finally, condition (4) is obvious. □

Example 3.1. For $n = 1$ and $\lambda_1 := \lambda$ we retain the classical compound Poisson risk process and indeed (2) simplifies in this case to

$$\delta\, y\, \frac{\partial M}{\partial y}(u,y,b) - c\, \frac{\partial M}{\partial u}(u,y,b) + \lambda\, M(u,y,b) - \lambda \int_0^u M(u-v,y,b)\, dF(v) \\ - \lambda\, (1 - F(u)) = 0,$$

which is formula (1) of Albrecher (2004). Correspondingly, the boundary condition (3) simplifies to $\frac{\partial M(u,y,b)}{\partial u}\Big|_{u=b} = y\, M(b,y,b)$, which is equation (2) of Albrecher (2004).

Remark 3.1. The boundary condition (3) also extends the corresponding condition for a Brownian risk model (equation (4.5) of Gerber and Shiu (2004a)). The structure of this condition for $k = 1$ is discussed in Gerber and Shiu (2004b) in a more general framework.

4 Moments of $D_{u,b}$

4.1 An integro-differential equation

Recall that $W_m(u,b) = \mathbb{E}\left[D_{u,b}^m \middle| R_0 = u\right]$. Using the representation

$$M(u,y,b) = 1 + \sum_{m=1}^{\infty} \frac{y^m}{m!}\, W_m(u,b)$$

and equating the coefficients of $y^m\,(m = 0,1,2,..)$ in (2) leads to the following ordinary integro-differential equation for $W_m(u,b)\;(m = 1,2,\ldots)$:

$$\left(\prod_{j=1}^{n}\left(\delta\, m - c\,\frac{\partial\cdot}{\partial u} + \lambda_j\right)\right) W_m(u,b) - \prod_{j=1}^{n}\lambda_j \int_0^u W_m(u-v,b)\,dF(v) = 0. \qquad (10)$$

From (3) we correspondingly obtain for $k = 1,\ldots,n$

$$\left(\prod_{j=2}^{k}\left(\delta\, y\,\frac{\partial\cdot}{\partial y} - c\,\frac{\partial\cdot}{\partial u} + \lambda_{j-1}\right)\right) \sum_{m=1}^{\infty} \frac{y^m}{m!}\, \frac{\partial W_m(u,b))}{\partial u}\Bigg|_{u=b}$$
$$= y\left(\prod_{j=2}^{k}\left(\delta\, y\,\frac{\partial\cdot}{\partial y} - c\,\frac{\partial\cdot}{\partial u} + \lambda_{j-1}\right)\right) (1 + \sum_{m=1}^{\infty} \frac{y^m}{m!}\, W_m(u,b))\Bigg|_{u=b}.$$

Now, equating coefficients of $y^m\,(m = 1,2,..)$ leads to

$$\left(\prod_{j=2}^{k}\left(\delta\, m + \lambda_{j-1} - c\,\frac{\partial\cdot}{\partial u}\right)\right) \frac{\partial W_m(u,b))}{\partial u}\Bigg|_{u=b}$$
$$= m\left(\prod_{j=2}^{k}\left(\delta\,(m-1) + \lambda_{j-1} - c\,\frac{\partial\cdot}{\partial u}\right)\right) W_{m-1}(u,b))\Bigg|_{u=b}, \qquad (11)$$

which holds for $k = 1,\ldots,n$ and arbitrary $m \in \mathbb{N}$.
Furthermore, boundary condition (4) directly translates into

$$\lim_{b\to\infty} W_m(u,b) = 0 \quad (m = 1,2,..). \qquad (12)$$

Thus, the m-th moment $W_m(u,b)$ is the solution of the integro-differential equation (10) together with (11) and (12).

Remark 4.1. One could derive this result also by using a differential approach for $W_m(u,b)$ directly and then applying the binomial formula, a Taylor series expansion and collecting significant terms (see Albrecher (2004) for a corresponding procedure for the compound Poisson model ($n = 1$) with constant dividend barrier). Yet another possibility to derive the equations for $W_m(u,b)$ is to use a renewal equation, generalizing the derivation of Dickson and Waters (2004) for the compound Poisson case. However, the differential approach used above is considerably simpler.

4.2 Explicit solutions for claim sizes with rational Laplace transform

Let us ignore for a moment that $W_m(u,b)$ is only defined for $0 \le u \le b$ and define the Laplace transform $\tilde{W}_m(s,b) := \int_0^\infty e^{-su} W_m(u,b)\, du$ and the Laplace-Stieltjes transform $\tilde{f}(s) := \int_0^\infty e^{-sv}\, \mathrm{d}F(v)$. Equation (10) does not depend on b and thus the idea is to use Laplace transforms to obtain the structure of $W_m(u,b)$ up to constants and then to use the boundary conditions to determine these constants.
Define the n-th degree polynomial

$$\gamma(s) := \prod_{j=1}^{n} (\delta\, m - c\, s + \lambda_j)\,.$$

Then taking the Laplace transform of (10) yields

$$\gamma(s)\tilde{W}_m(s,b) + G_{n-1}(s) - \prod_{j=1}^{n} \lambda_j\, \tilde{W}_m(s,b)\, \tilde{f}(s) = 0,$$

where $G_{n-1}(s)$ is an $(n-1)$-th degree polynomial in s, whose coefficients involve the quantities $\frac{\partial^j W_m}{\partial u^j}(0,b)$ $(j = 0, \dots, n-1)$. Thus we obtain

$$\tilde{W}_m(s,b) = \frac{G_{n-1}(s)}{\gamma(s) - (\prod_{j=1}^{n} \lambda_j)\, \tilde{f}(s)}. \tag{13}$$

Setting the denominator equal to zero, we obtain a variant of the so-called Lundberg fundamental equation, which already appeared in Gerber and Shiu (2005) when studying the discounted penalty function. There it was shown by Rouché-type arguments that this equation has exactly n zeroes in the positive halfplane.

Let us now restrict the further analysis to the case of claim size distributions with rational Laplace transform, i.e.

$$\tilde{f}(s) = \frac{Q_{r-1}(s)}{P_r(s)},$$

where $Q_{r-1}(s)$ and $P_r(s)$ denote polynomials of degree (at most) $r-1$ and r, respectively $(r \ge 1)$. Note that this class of distributions contains the phase-type distributions and in particular the Erlang distributions as a special case.
For this choice of claim size distribution, the denominator of (13) has exactly $n+r$ zeroes, which are denoted by $R_1, \dots, R_{n+r}$ (and which do not depend on b). From the above we conclude that r of these zeroes are located in the negative halfplane. For simplicity of notation, let us assume first that $R_1, \dots, R_{n+r}$ are real and distinct. Then, using partial fractions in (13), one obtains

$$W_m(u,b) = \sum_{i=1}^{n+r} \alpha_i(b) e^{R_i\, u}. \tag{14}$$

In order to determine the coefficients $\alpha_i(b)$, $n+r$ equations are needed. The first n equations directly follow from (11). For the remaining r equations, one has to substitute (14) into (10). After performing the integration in the last summand of the left-hand side of (10), it is not difficult to see that equating coefficients of the resulting exponential terms leads to exactly r additional equations (for an illustration, see the example in Section 5). Note that $\lim_{b\to\infty}\alpha_i(b)=0$ has to hold for all $i=1,\dots,n+r$ in order to satisfy (12).

If the zeroes $R_1,\dots,R_{n+r}$ are not all distinct and/or real, then a little more care is needed, since then the coefficients α_i are functions of u also, but one can obtain the solution in a completely analogous way.

5 Numerical illustrations for an Erlang(2) model

As an illustration of the results of the previous section, consider the case of a Sparre Andersen model with Erlang$(2,\lambda)$ interclaim time, i.e. $\mathbb{P}(T_i\le t)=1-(\lambda t+1)\,e^{-\lambda t}$ $(t\ge 0)$. This corresponds to the case $n=2$ and $\lambda_1=\lambda_2:=\lambda$. It thus follows from (10) that the m-th moment of discounted dividend payments $W_m(u,b)$ (for arbitrary $m\in\mathbb{N}$) is the solution of

$$c^2\,\frac{\partial^2 W_m(u,b)}{\partial u^2}-2\,c\,(\delta\,m+\lambda)\,\frac{\partial W_m(u,b)}{\partial u}+(\delta\,m+\lambda)^2\,W_m(u,b)\\ -\lambda^2\int_0^u W_m(u-v,b)\,dF(v)=0 \quad (15)$$

and the boundary condition (11) simplifies to

$$\left.\frac{\partial W_m(u,b))}{\partial u}\right|_{u=b}=m\,W_{m-1}(b,b) \quad (16)$$

for $k=1$. For $k=2$ in (11), one obtains

$$(\delta\,m+\lambda)\left.\frac{\partial W_m(u,b))}{\partial u}\right|_{u=b}-c\left.\frac{\partial^2 W_m(u,b)}{\partial u^2}\right|_{u=b}\\ =m\,(\delta\,(m-1)+\lambda)W_{m-1}(b,b)-m\,c\left.\frac{\partial W_{m-1}(u,b)}{\partial u}\right|_{u=b},$$

which simplifies to

$$\left.\frac{\partial^2 W_m(u,b)}{\partial u^2}\right|_{u=b}=m\left.\frac{\partial W_{m-1}(u,b)}{\partial u}\right|_{u=b}+\frac{m\,\delta}{c}W_{m-1}(b,b),$$

or equivalently

$$\left.\frac{\partial^2 W_m(u,b)}{\partial u^2}\right|_{u=b}=m(m-1)W_{m-2}(b,b)+\frac{m\,\delta}{c}W_{m-1}(b,b) \quad (17)$$

(with the understanding that $W_{-1}(u,b) \equiv 0$). Thus it remains to solve (15) together with (12), (16) and (17).

If in addition we assume that $X_i \sim$ Erlang $(2,\eta)$, then $\tilde{f}(s) = \frac{\eta^2}{(s+\eta)^2}$ and from Section 4.2 we obtain the representation

$$W_m(u,b) = \sum_{i=1}^{4} \alpha_i(b) e^{R_i u}, \tag{18}$$

where R_i ($i = 1, \dots, 4$) are the solutions in R of

$$(\delta\, m - c\, R + \lambda)^2 - \frac{\lambda^2 \eta^2}{(R+\eta)^2} = 0. \tag{19}$$

Substituting (18) into (15) yields, after rearranging terms,

$$\sum_{i=1}^{4} \alpha_i(b) \Big((c\, R_i - \lambda - m\delta)^2 - \frac{\eta^2 \lambda^2}{(R_i+\eta)^2} \Big) e^{R_i u} = \eta^2 \lambda^2 \sum_{i=1}^{4} \alpha_i(b) \left(\frac{u}{R_i+\eta} - \frac{1}{(R_i+\eta)^2} \right) e^{-\eta u},$$

from which we obtain the two conditions

$$\sum_{i=1}^{4} \frac{\alpha_i(b)}{(R_i+\eta)} = 0 \quad \text{and} \quad \sum_{i=1}^{4} \frac{\alpha_i(b)}{(R_i+\eta)^2} = 0. \tag{20}$$

Now, for $m = 1$ we have from (16) $\frac{\partial W_1(u,b))}{\partial u}\Big|_{u=b} = 1$ so that

$$\sum_{i=1}^{4} \alpha_i(b) R_i e^{R_i b} = 1 \tag{21}$$

and from (17) we have $\frac{\partial^2 W_1(u,b)}{\partial u^2}\Big|_{u=b} = \frac{\delta}{c}$ leading to

$$\sum_{i=1}^{4} \alpha_i(b) R_i^2 e^{R_i b} = \frac{\delta}{c}. \tag{22}$$

Hence the coefficients $\alpha_i(b)$ can be determined as the solution of the four linear equations (20), (21) and (22).

Example 4.1. Let $T_i \sim$ Erlang(2,2), $X_i \sim$ Erlang (2,2), $c = 1.1$ and $\delta = 0.03$. Let us first consider $m = 1$. In this case the solutions of (19) are $R_1 = -2.79$, $R_2 = -0.32$, $R_3 = 0.17$, $R_4 = 2.63$ (here and in the sequel, all numerical values rounded to their last digit) and we have from (18)

$$W_1(u,b) = \alpha_1(b) e^{-2.79\, u} + \alpha_2(b) e^{-0.32 u} + \alpha_3(b) e^{0.17\, u} + \alpha_4(b) e^{2.63\, u}.$$

The coefficients $\alpha_i(b)$ can easily be determined by (20), (21) and (22) and involve exponentials in b. Table 1 gives some numerical values of $W_1(u,b)$ and Figure 1

depicts the behavior of $W_1(u,b)$ as a function of b for some given values of initial capital u.
For $m=2$, the solutions of (19) are $R_1=-2.78$, $R_2=-0.40$, $R_3=0.27$, $R_4=2.65$ and here the coefficients $\alpha_i(b)$ are the solutions of the system of the following four linear equations:

$$\begin{cases} \sum_{i=1}^{4} \alpha_i(b) R_i e^{R_i b} = 2\, W_1(b,b) \\ \sum_{i=1}^{4} \alpha_i(b) R_i^2 e^{R_i b} = 2 + \frac{2\delta}{c}\, W_1(b,b) \\ \sum_{i=1}^{4} \dfrac{\alpha_i(b)}{R_i+2} = 0 \\ \sum_{i=1}^{4} \dfrac{\alpha_i(b)}{(R_i+2)^2} = 0, \end{cases}$$

which can be evaluated in a straight-forward way. In Table 2, some numerical values of the standard deviaton $SD(u,b) := \sqrt{W_2(u,b) - W_1^2(u,b)}$ of the discounted sum of dividend payments are given and Figure 2 depicts the behavior of $SD(u,b)$ as a function of b for some given values of initial capital u. Figure 3 depicts the variation coefficient of $D_{u,b}$ defined by $\frac{SD(u,b)}{W_1(u,b)}$.

Moreover, one can observe that $W_1(b,b)$ and $SD(b,b)$ tend towards a finite value, namely $\lim_{b\to\infty} W_1(b,b) = 6.245$ and $\lim_{b\to\infty} SD(b,b) = 2.904$, which occurs because each of the terms $\alpha_i(b)e^{R_i b}$ either goes to zero or to a finite limit $(i = 1,\ldots,4)$, see also Figure 4.

The following final remark is in order: The appropriate choice of a dividend barrier height b strongly depends on the optimization criterion under consideration. If a safety criterion involving the probability and/or time of ruin is applied, then the results on the discounted penalty function of Li and Garrido (2004b) might be used. Figure 1 shows that if one instead wants to maximize expected dividend payments until ruin, then in this model b should be chosen as small as possible. However, one can see from Figure 2 that the standard deviation of these payments has a maximum for rather small values of b, indicating that one should choose the optimization criteria with great care and that consideration of the first moment is not sufficient to represent the profit-participation in practice.

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$b \setminus u$	0	1	2	3	4	5	6	7	8	9
0	1.076									
1	0.836	1.808								
2	0.856	1.847	2.846							
3	0.848	1.828	2.815	3.803						
4	0.801	1.728	2.661	3.597	4.574					
5	0.730	1.575	2.424	3.277	4.174	5.143				
6	0.648	1.397	2.151	2.908	3.705	4.575	5.538			
7	0.565	1.218	1.875	2.535	3.229	3.988	4.840	5.799		
8	0.486	1.049	1.615	2.184	2.782	3.436	4.170	5.010	5.967	
9	0.416	0.897	1.381	1.867	2.379	2.938	3.566	4.285	5.118	6.073

Table 1: Exact values for the expectation $W_1(u,b)$ of the discounted dividend payments.

$b \setminus u$	0	1	2	3	4	5	6	7	8	9
0	0.744									
1	1.240	1.399								
2	1.667	2.11	2.193							
3	1.864	2.456	2.695	2.742						
4	1.884	2.528	2.846	2.989	3.02					
5	1.797	2.436	2.783	2.981	3.085	3.111				
6	1.656	2.263	2.613	2.836	2.988	3.08	3.104			
7	1.496	2.058	2.396	2.629	2.807	2.945	3.035	3.06		
8	1.334	1.847	2.167	2.399	2.59	2.755	2.892	2.984	3.011	
9	1.181	1.644	1.942	2.167	2.362	2.54	2.705	2.845	2.942	2.969

Table 2: Exact values for the standard deviation $\sqrt{W_2(u,b) - W_1^2(u,b)}$ of the discounted dividend payments.

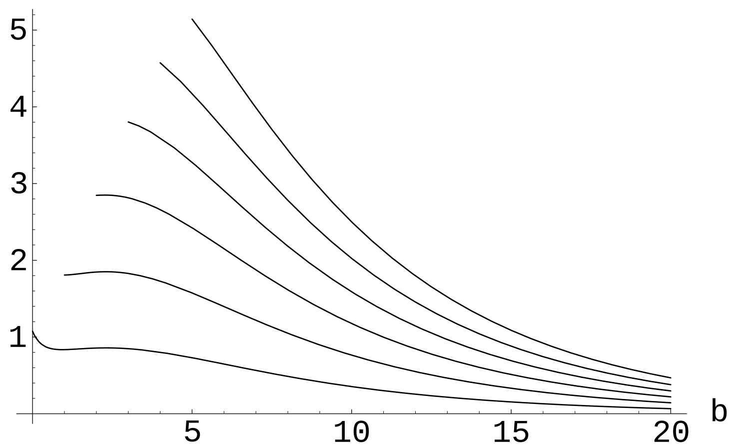

Figure 1: $W_1(u,b)$ as a function of b for $u = 0, 1, \ldots, 5$ (from bottom to top)

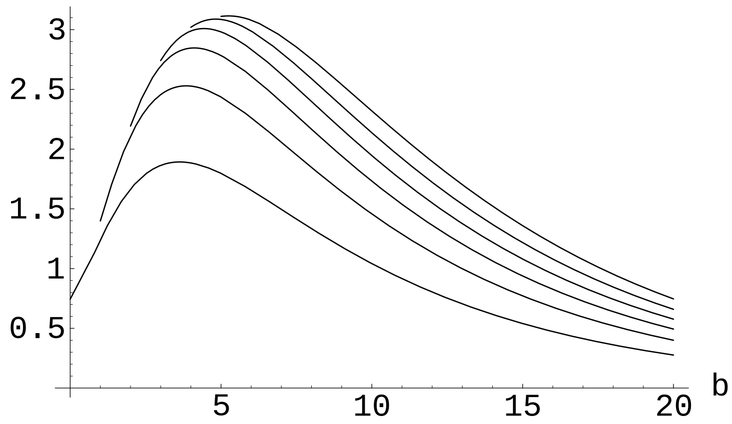

Figure 2: Standard deviation of $D_{u,b}$ as a function of b for $u = 0, 1, \ldots, 5$ (from bottom to top)

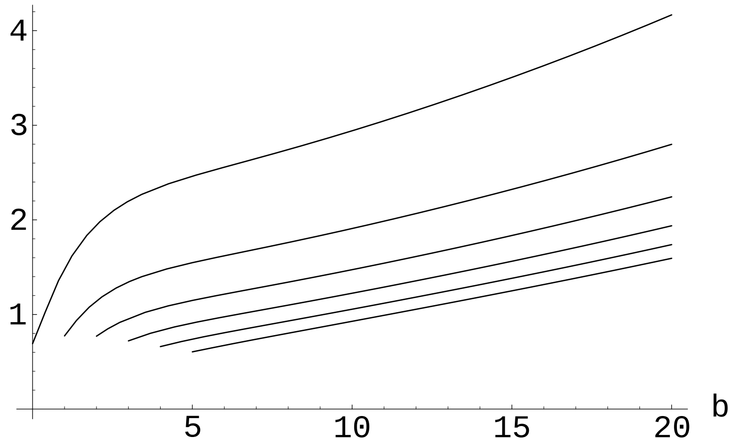

Figure 3: Variation coefficient of $D_{u,b}$ as a function of b for $u = 0, 1, \ldots, 5$ (from top to bottom)

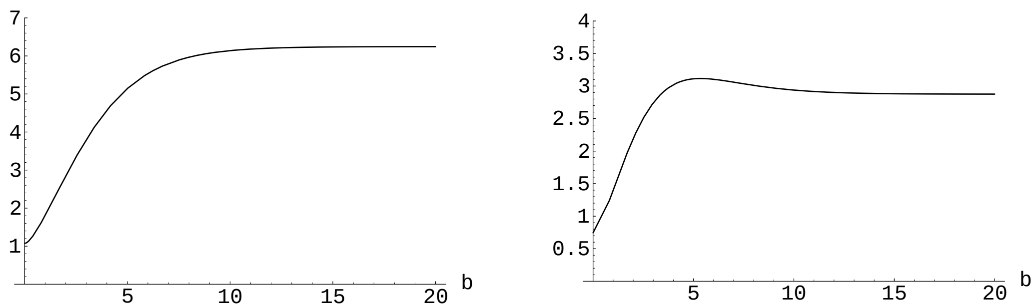

Figure 4: Limiting behavior of $W_1(b, b)$ (left) and the standard deviation $\sqrt{W_2(b,b) - W_1^2(b,b)}$ (right)