SOME MATHEMATICAL ASPECTS OF PRICE OPTIMISATION

ENKELEJD HASHORVA, GILDAS RATOVOMIRIJA, MAISSA TAMRAZ, AND YIZHOU BAI

Abstract: Calculation of an optimal tariff is a principal challenge for pricing actuaries. In this contribution we are concerned with the renewal insurance business discussing various mathematical aspects of calculation of an optimal renewal tariff. Our motivation comes from two important actuarial tasks, namely a) construction of an optimal renewal tariff subject to business and technical constraints, and b) determination of an optimal allocation of certain premium loadings. We consider both continuous and discrete optimisation and then present several algorithmic suboptimal solutions. Additionally, we explore some simulation techniques. Several illustrative examples show both the complexity and the importance of the optimisation approach.

Key Words: market tariff; optimal tariff; price optimisation; renewal business; sequential quadratic programming

1. INTRODUCTION

Commonly, insurance contracts are priced based on a tariff, here referred to as the *market tariff*. In mathematical terms such a market tariff is a function say $f : \mathbb{R}^d \to [m, M]$ where m, M are the minimal and the maximal premiums. For instance, a motor third party liability (MTPL) *market tariff* of key insurance market players in Switzerland has $d > 15$. Typically, the function f is neither linear nor a product of simple functions.

In non-life insurance, many insurance companies use different f for new business and renewal business. There are statistical and marketing reasons behind this practice. In this paper we are primarily concerned with non-life renewal business. Yet, some findings are of importance for pricing of insurance and other non-insurance products. We shall first discuss three important actuarial tasks and then present various mathematical aspects of relevance for pricing actuaries.

Practical actuarial task T1: *Given that a portfolio of N policyholders is priced under a given market tariff f, determine an optimal market tariff f^* that will be applied in the next portfolio renewal.*

Typically, actuarial textbooks are concerned with the calculation of the pure premium, which is determined by applying different statistical and actuarial methods to historical portfolio data, see e.g., [1–4]. The tariff that determines the pure premium of a given insurance contract will be here referred to as the *pure risk tariff*. In mathematical terms this is a function say $g : \mathbb{R}^{d_1} \to [m_1, M_1]$ with $d_1 \geq 1$.

In the actuarial practice, pure premiums are loaded, for instance for large claims, provisions, direct expenses and other costs (overheads, profit, etc.).

Actuarial mathematics explains various approaches for loading insurance premiums; in practice very commonly a linear loading is applied. We shall refer to the function that is utilised for the calculation of the premium of

an insurance coverage based on the costs related to that coverage as *actuarial tariff*; write $g_A : \mathbb{R}^{d_2} \to [m_2, M_2]$ for that function.

Despite the importance of task **T1**, the current actuarial literature has not dealt with its mathematical aspects. On the other side, practising actuaries are constantly confronted with various black-box type solutions available from external services or in few cases have developed their own internal models.

Practical actuarial task T2: *Given a pure risk tariff g, construct an optimal actuarial tariff g_A that includes various premium loadings.*

Since by definition there is no unique optimal *actuarial tariff*, the calculations leading to it can be performed depending on the resources of pricing and implementation team.

To this end, let us briefly mention an instance which motivates **T2**: Suppose for simplicity that the insurance portfolio in question consists of two groups of policyholders A and B with n_A and n_B policyholders, respectively. All the contracts are to be renewed, say at the next 1st of January. The pricing actuary calculates the *actuarial tariff* which shows that for group A, the yearly premium to be paid from each policyholder is 2'000 CHF and for group B, say 500 CHF. For this portfolio, overhead expenses (or expenses not directly allocated to an insurance policy) are calculated (estimated) to be X CHF for the next insurance period. The amount X can be distributed to $N = n_A + n_B$ policyholders in different ways, for instance each policyholder will have to pay $X/(n_A + n_B)$ of those expenses. Another alternative approach could be to calculate it as a fix percentage of the pure premiums. The principal challenge for pricing actuaries is that the policyholders are already in the portfolio and might be very sensitive to any change of their premiums, especially when the insurance risk does not change.

At renewal (abbreviated as @$\mathcal{R}$ in the following) given that the insured risk does not change, if the new offered premium is different from the current one, the policyholder can cancel the contract. Clearly, another common reason for cancelling the insurance contract is also the competition in the insurance market. Consequently, the solutions of **T1-T2** need to take into account the probability of renewal of the policies at the point of renewal.

As illustrated above for **T2**, the percentage of premium increase δ_i for the ith policyholder @$\mathcal{R}$ can be fixed, i.e., $\delta_i \in \Delta$ where say $\Delta = \{0\%, 5\%, 10\%\}$. Such increases are often used in practice especially if the distribution channel is primarily dominated by the tied agents. A clear advantage of such type of tariff modification is that it can be straightforwardly implemented with minimal implementation costs. Therefore, instead of **T2** a simpler task which is very often encountered in the insurance practice (but surprisingly not in actuarial literature) is as follows:

Practical actuarial task T3: *Modify for any $i \le N$ the premium P_i of the ith policyholder @$\mathcal{R}$ by a fixed percentage, say $\delta_i \in \Delta_i$ with Δ_i a discrete set (for instance $\Delta_i = \{0\%, 5\%, 10\%\}$) so that the new set of premiums*

$$P_i^* = P_i(1 + \delta_i), \quad 1 \le i \le N$$

are optimal under several business constraints. Moreover, determine the new market tariff f^* which yields P_i^*'s.

There are several difficulties related to the solutions of the actuarial tasks **T1-T3**. In practice the *market tariff* is very complex for key insurance coverages such as motor or household insurance. A typical f utilised in insurance practice is as follows (consider only two arguments for simplicity)

(1.1) $$f(x,y) = \min\Big(M_0, \max(e^{ax+by}, m_0 + m_1 x + m_2 y)\Big).$$

The exponential term in f is very common in practice since both claim frequency and average claim sizes are modelled using generalised linear models (GLM's) with log-link function. The reason for the choice of log-link functions is the ease of IT implementation. Both min and max functions in (1.1) prevent the premiums from being extremes. These are often decided by empirical findings and insurance market constraints.

Even if we know the optimal P_i^*'s that solve **T3**, when the structure of f (and also of f^*) is fixed say as in (1.1), then the existence of an optimal f^* that gives exactly P_i^*'s is in general not guaranteed. Note that due to technical reasons, the actuaries can change the coefficients that determine f, say a, b and so on, but the structure of the tariff, i.e., the form of f in (1.1) is in general fixed when preparing a new renewal tariff due to implementation costs.

The main goal of this contribution is to discuss various mathematical aspects that lead to optimal solutions of both actuarial tasks **T1** and **T3**. Further we analyse eventual implementations of our optimisation problems for renewal business. Optimisation problems related to new business are more involved and will therefore be treated in a forthcoming contribution.

To this end, we note that in the last 12 years many insurance companies in Europe have already implemented price optimisation techniques. Very recent contributions focus on the issues of price optimisation, mainly from the ethical and regulation points of view, see [5–7]. It is important to note that optimality issues in insurance and reinsurance business, not directly related to the problems treated in this contribution, have been discussed in various contexts, see e.g., [8–14] and the references therein.

Brief organisation of the rest of the paper: Section 2 describes the different optimisation settings from the insurer's point of view. In Section 3, we provide partial solutions for problem **T3**. Section 4 describes the different algorithms used to solve the optimisation problems followed by some insurance applications to the motor line of business presented in Section 5.

2. Objective functions and Business Constraints

2.1. **Theoretical Settings.** For simplicity, and without loss of generality, we shall assume that the renewal time is fixed for all $i = 1, \ldots, N$ policyholders already insured in the portfolio with the ith policyholder paying the insurance premium P_i for the current insurance period. Each policyholder can be insured for different insurance periods. Without loss of generality, we shall suppose that @$\mathcal{R}$ each insurance contract has the option to be renewed for say one year, with a renewal premium $P_i^* := P_i(1 + \delta_i)$.

Suppose that the renewal probability for the ith contract is a function of P_i and some parameters describing the risk characteristics of the policyholder. At renewal, by changing the premium, this probability will depend on the premium change δ_i, the previous premium P_i and other risks characteristics. Therefore we shall assume that this probability is

(2.1) $$\Psi_i(P_i, \delta_i),$$

where Ψ_i is a strictly positive function depending eventually on i (when the risk characteristics of the ith policyholder are tractable). This is a common assumption in logistic regression, where Ψ_i is the inverse of the logit function (called also expit), or Ψ_i is a univariate distribution function.

In order to consider the renewal probabilities in the tariff and premium optimisation tasks, the actuary needs to know/determine $\Psi_i(P_i, \delta_i)$ for any $\delta_i \in \Delta_i$, where Δ_i is the range of possible changes of premium (commonly

$0 \in \Delta_i$). Estimation of Ψ_i's is non-trivial; it can be handled for instance using logistic regression, see Section 4.1.2 below for more details.

In practice, depending on the market position and the strategy of the insurance company, different objective functions can be used for the determination of an optimal *actuarial tariff* or *market tariff*. We discuss below two important objective functions:

O1) <u>Maximise the future expected premium volume @$\mathcal{R}$:</u>
In our model, the current premium volume for the portfolio in question is $V = \sum_{i=1}^{N} P_i$, whereas the premium volume in case of complete renewal is

$$V^* = \sum_{i=1}^{N} P_i^* = \sum_{i=1}^{N} P_i(1 + \delta_i).$$

Given the fact that not all policies might renew, let us denote by $N_{@\mathcal{R}}$ the random number of policies which will be renewed. Since we can treat each contract as an independent risk, then

$$N_{@\mathcal{R}} = \sum_{i=1}^{N} I_i,$$

with $I_1, \ldots, I_N$ independent Bernoulli random variables satisfying

$$\mathbb{P}\{I_i = 1\} = \Psi_i(P_i, \delta_i), \quad 1 \leq i \leq N.$$

Clearly, the expected percentage of the portfolio to renew is given by (set below $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_N)$)

(2.2)
$$\theta(\boldsymbol{\delta}) = \frac{\mathbb{E}\{N_{@\mathcal{R}}\}}{N} = \sum_{i=1}^{N} \frac{\mathbb{E}\{I_i\}}{N} = \frac{1}{N}\sum_{i=1}^{N}\Psi_i(P_i, \delta_i).$$

The premium volume @$\mathcal{R}$ (which is random) will be denoted by $\mathcal{V}_{@\mathcal{R}}$. It is simply given by

$$\mathcal{V}_{@\mathcal{R}} := \sum_{i=1}^{N} I_i P_i(1 + \delta_i).$$

Consequently, considering the interest in maximising the premium volume, then the objective function in this setting is given by

(2.3)
$$q_{vol}(\boldsymbol{\delta}) := \mathbb{E}\{\mathcal{V}_{@\mathcal{R}}\} = \sum_{i=1}^{N} P_i(1 + \delta_i)\mathbb{E}\{I_i\} = \sum_{i=1}^{N} P_i(1 + \delta_i)\Psi_i(P_i, \delta_i).$$

Note that $P_1, \ldots, P_N$ are known, therefore the optimisation will be performed with respect to δ_i's only.

O1') <u>Minimise the variance of $\mathcal{V}_{@\mathcal{R}}$:</u> If the variance of $\mathcal{V}_{@\mathcal{R}}$ is large, the whole renewal process can be ruined. Therefore along **O1** the minimisation of the variance of $\mathcal{V}_{@\mathcal{R}}$ is important. In this model we have

(2.4)
$$q_{var}(\boldsymbol{\delta}) := Var(\mathcal{V}_{@\mathcal{R}}) = \sum_{i=1}^{N}[P_i(1 + \delta_i)]^2\Psi_i(P_i, \delta_i)[1 - \Psi_i(P_i, \delta_i)].$$

O2) <u>Maximise the expected premium difference @$\mathcal{R}$:</u> Let $\tau_i = P_i\delta_i$ be the premium difference for the *ith* policyholder and set $\boldsymbol{\tau} := (\tau_1, \ldots, \tau_N)$. The total premium difference @$\mathcal{R}$ is $\sum_{i=1}^{N} I_i\tau_i$, with expectation

(2.5)
$$q_{dif}(\boldsymbol{\tau}) = \mathbb{E}\left\{\sum_{i=1}^{N} I_i\tau_i\right\} = \sum_{i=1}^{N}\tau_i\Psi_i(P_i, \delta_i).$$

It is not difficult to formulate other objective functions, for instance related to the classical ruin probability, Parisian ruin (see e.g., [15]), or future solvency and market position of the insurance company. Moreover, the objective functions can be formulated over multiple insurance periods.

Due to the nature of insurance business, there are several constraints that should be taken into account for the renewal business optimisation, see [16] and the references therein. Typically, the most important business constraints relate to the strategy of the company and the concrete insurance market. We formulate few important constraints below:

C1) Expected retention level $@\mathcal{R}$ should not be less say than 70%. Although the profit and the volume of premiums at renewal are important, all insurance companies are interested in keeping most of the policyholders in their portfolios. Therefore there is commonly a lower bound imposed on the expected retention level ℓ at renewal. For instance $\ell \geq 90\%$ means that the expected percentage of customers that will not renew their contracts should not exceed 10%. In mathematical terms, this is formulated as

$$(2.6) \qquad \theta_{rlevel}(\boldsymbol{\delta}) = \frac{\mathbb{E}\{N^*\}}{N} \geq \ell.$$

C2) A simple constraint is to require that the renewal premiums P_i^*'s are not too different from the "old" ones, i.e.,

$$(2.7) \qquad \delta_i \in [a, b], \quad \tau_i := P_i \delta_i \in [A, B], \quad 1 \leq i \leq N$$

for instance $a = -5\%, b = 10\%$ and $A = -50, B = 300$.

Several other constraints including those related to reputational risk, decrease of provision level for tied-agents, and loss of loyal customers can be formulated similarly and will therefore not be treated in detail.

2.2. **Practical Settings.** In insurance practice the cost of optimisation itself (actuarial and other resources) needs to be also taken into account. Additionally, since the total volume of premiums at renewal is large, an optimal renewal tariff is of interest (business relevant) only if it produces a significant improvement to the current tariff. Therefore, for practical implementations, we need to redefine the objective functions. For a given positive constant c, say $c = 1'000$, we redefine (2.3) as

$$(2.8) \qquad q_{vol}^c(\delta_1, \ldots, \delta_N) := c \left\lfloor \mathbb{E}\{\mathcal{V}_{@\mathcal{R}}\}/c \right\rfloor = c \left\lfloor \sum_{i=1}^N P_i(1 + \delta_i)\Psi_i(P_i, \delta_i)/c \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the largest integer smaller than x. Similarly, we redefine (2.4) as

$$(2.9) \qquad c \left\lfloor Var(\mathcal{V}_{@\mathcal{R}})/c \right\rfloor = c \left\lfloor \sum_{i=1}^N \{P_i(1 + \delta_i)\}^2 \Psi_i(P_i, \delta_i)[1 - \Psi_i(P_i, \delta_i)]/c \right\rfloor.$$

Finally, (2.5) can be written as

$$(2.10) \qquad q_{dif}^c(\tau_1, \ldots, \tau_N) = c \left\lfloor \sum_{i=1}^N \tau_i \Psi_i(P_i, \delta_i)/c \right\rfloor.$$

For implementation purposes and due to business constraints, τ_i's can be assumed to be certain given positive integers. Therefore a modification of (2.7) can be formulated as

$$(2.11) \qquad \delta_i \in [a, b] \cap (c_1^{-1}\mathbb{Z}), \quad \tau_i := P_i \delta_i \in [A, B] \cap (c_1 \mathbb{Z}), \quad 1 \leq i \leq N,$$

where $c_1 > 0$, for instance $c_1 = 100$.

Such modifications of both objective functions and constraints show that for practical implementation, there is no unique optimal solution of the optimisation problem of interest.

Remarks 2.1. *i) If two different insurance contracts are renewed through different distributional channels, then typically different constraints are to be applied to each of those policies. Additionally, the cancellation probabilities could be different, even in the case where both policyholders have the same risk profile. Therefore, in order to allow for different distributional channels, we only need to adjust the constraints and assume an appropriate cancellation pattern.*

ii) In practice, Ψ_i's can be estimated by using for instance logistic regression. At random, customers are offered @R higher/lower premiums than their P_i's i.e., δ_i's are chosen randomly with respect to some prescribed distribution function. An application of the logistic regression to the data obtained (renewal/non renewal) explains the cancellation (or renewal) probability in terms of risk factors as well as other predictors (social status, etc.). In an insurance market dominated by tied-agents this approach is quite difficult to apply.

iii) Different policyholders can renew their contracts for different periods. This case is included in our assumptions above.

iv) Most tariffs utilised in practice, for instance an MTPL one, consist of hundreds of coefficients (typically more than 300). Due to a dominating product structure, modern insurance tariffs consists of many individual cells, say 200'000 in average. However, most of these tariff cells are empty. In most cases less than 15% of the cells determine 80% of the total premium volume in the portfolio. For instance, it is quite rare that a Ferrari is insured for a TPL risk by a 90 years old lady, living in a very small village. With this in mind, the relevant number N in practical optimisation problems does not exceed 10'000. Our algorithms and simulation methods work fairly well for such N.

3. Solutions for **T3**

The main difficulty when dealing with the actuarial task **T3** lies on the complexity of Ψ_i's since these functions are:

a) in general not known,

b) difficult to estimate if past data are partially available,

c) even when these functions are known, the constraints **C1-C2** and the objective functions **O1,O1',O2** are in general not convex. We discuss next a partial solution for **T3**.

<u>Problem T3a</u>: Given $P_1, \ldots, P_N$ determine $\boldsymbol{\delta}^* = (\delta_1^*, \ldots, \delta_N^*)$ such that

$$(3.1) \qquad\qquad q_{vol}(\boldsymbol{\delta}^*) \text{ is maximal}, \quad q_{var}(\boldsymbol{\delta}^*) \text{ is minimal}$$

under the constraints

$$\theta_{rlevel}(\boldsymbol{\delta}) \geq \ell, \quad \boldsymbol{l} \leq \boldsymbol{\delta} \leq \boldsymbol{u},$$

where $\boldsymbol{l}$ and $\boldsymbol{u}$ are 2 vectors such that their components l_i, $u_i \in (-1, 1)$ for $i \leq N$.

<u>Problem T3b</u>: Determine the market tariff f^* from $P_1^*, \ldots, P_N^*$.

The solution (an approximate one) of **T3b** can be easily derived. Given $P_1^*, \ldots, P_N^*$, and since the structure of the *market tariff* is known, then f^* can be determined (approximately) by running a non-linear regression analysis with resoine variables P_i^*'s.

Below we shall focus on task **T3a** dealing with the determination of the optimal premiums P_i^*'s at renewal.

In insurance practice, the functions $\Psi_i, i \leq n$ can be assumed to be piece-wise linear and non-decreasing. This assumption is indeed reasonable, since for very small τ_i or δ_i the cancelation probability should not change. However, that assumption can be violated if for instance at renewal the competition modifies also their new business premiums. For simplicity, these cases will be excluded in our analysis, and thus we assume that the

decision for accepting the renewal offer is not influenced by the competition.

We list below some tractable choices for Ψ_i's:

Ma) Suppose that for given known constants π_i, a_i, b_i

$$\Psi_i(P_i, \delta_i) = \pi_i(1 + a_i\delta_i + b_i\delta_i^2), \quad 1 \leq i \leq N.$$

In practice, π_i, a_i, b_i need to be estimated. Clearly, the case that b_i's are equal to 0 is quite simple and tractable.

Note in passing that a simple extension of the above model is to allow a_i and b_i to differ depending on the sign of δ_i.

Mb) One choice motivated by the logistic regression model commonly used for estimation of renewal probabilities is the expit function, i.e.,

$$\Psi_i(P_i, \delta_i) = \frac{1}{1 + c_i^{-1}e^{-T_i\delta_i}}, \quad 1 \leq i \leq N,$$

where c_i, T_i's are known constants (to be estimated in applications), see e.g., [17] .

We note that Model **Ma)** can be seen as an approximation of Model **Mb**).

Mc) Finally, we consider the case where Ψ_i's are determined only for specific δ_i's. For instance, for the ith policyholder Ψ_i depends on P_i and δ_i as follows

$index$	1	2	3	4	5	6	7	8	9
$\delta_i($ in %)	-20 %	-15 %	-10%	-5%	0%	5%	10%	15%	20%
$P_i(1+\delta_i)$	80	85	90	95	100	105	110	115	120
$\Psi_i(P_i, \delta_i)$	0.999	0.995	0.990	0.975	0.950	0.925	0.900	0.875	0.825

TABLE 3.1. Renewal probabilities as a function of premiums of the ith policyholder.

The Model **Ma**) is simple and tractable and can be seen as an approximation of a more complex one. Moreover, it leads to some crucial simplification of the objective functions in question.

4. OPTIMISATION ALGORITHMS

4.1. **Maximise the expected premium volume** @$\mathcal{R}$**.** In this section, we consider the objective function **O1** subject to the constraint function **C1**. Our optimisation problem can be formulated as follows

$$\max_{\boldsymbol{\delta}} q_{vol}(\boldsymbol{\delta}), \quad \boldsymbol{\delta} := (\delta_1, \ldots, \delta_N)$$

(4.1)
$$\text{subject to } \theta_{rlevel}(\boldsymbol{\delta}) \geqslant \ell,$$

$$\boldsymbol{l} \leq \boldsymbol{\delta} \leq \boldsymbol{u},$$

where q_{vol} and θ_{rlevel} are defined repectively in (2.3) and (2.6). Further $\boldsymbol{l} := (l_1, \ldots, l_N)$, $\boldsymbol{u} := (u_1, \ldots, u_N)$ are such that $l_i, u_i \in (-1, 1)$ for $i \leq N$.

4.1.1. *Probability of renewal Ψ_i as in **Ma)**.* We consider the case where the probability of renewal Ψ_i is

$$\Psi_i := \Psi_i(P_i, \delta_i) = \pi_i(1 + a_i\delta_i + b_i\delta_i^2).$$

- Setting $b_i = 0$, we have

(4.2)
$$\Psi_i := \Psi_i(P_i, \delta_i) = \pi_i(1 + a_i\delta_i).$$

Since $\Psi_i \in (0, 1)$ should hold for all policyholders $i \leq N$, we require that

$$a_i \in (1 - \frac{1}{\pi_i}, \quad \frac{1}{\pi_i} - 1), \quad \delta_i \in (-1, 1), \quad \pi_i > 0$$

for all $i \leq N$.

The assumption $b_i = 0$ implies that (4.1) is a quadratic programming (QP) problem subject to linear constraints. It has a global maximum if and only if its objective function is concave, which is the case when $a_i < 0$. Hence we shall assume that $a_i \in (1 - \frac{1}{\pi_i}, 0)$ for any $i \leq N$.

Scenario 1: We consider the optimisation problem (4.2) without the upper and lower bounds constraints. In view of (4.2), the optimisation problem (4.1) can be reformulated as follows

$$
(4.3) \quad \min_{\boldsymbol{\delta}} f(\boldsymbol{\delta}) = \frac{1}{2}\boldsymbol{\delta}^{\top} Q \boldsymbol{\delta} + \boldsymbol{c}^{\top}\boldsymbol{\delta}, \quad \boldsymbol{\delta} = (\delta_1, \ldots, \delta_N)^{\top},
$$
$$
\text{subject to } g(\boldsymbol{\delta}) = \boldsymbol{a}^{\top}\boldsymbol{\delta} - b \leq 0,
$$

where

$$
\boldsymbol{c} = (-\pi_1 P_1 (1 + a_1), \ldots, -\pi_N P_N (1 + a_N))^{\top}
$$

describes the coefficient of the linear terms of f, Q is a diagonal and positive definite matrix describing the coefficients of the quadratic terms of f determined by

$$
Q = \begin{pmatrix} -2\pi_1 P_1 a_1 & 0 & 0 & \ldots & 0 \\ 0 & -2\pi_2 P_2 a_2 & 0 & \ldots & 0 \\ 0 & \ldots & -2\pi_i P_i a_i & \ldots & 0 \\ 0 & 0 & 0 & \ldots & -2\pi_N P_N a_N \end{pmatrix}.
$$

Since (4.3) has only one constraint, $\boldsymbol{a}$ is a vector related to the linear coefficients of g and is given by

$$
\boldsymbol{a} = -(\pi_1 a_1, \pi_2 a_2, \ldots, \pi_N a_N)^{\top}.
$$

Furthermore, we have that

$$
b = \sum_{i=1}^{N} \pi_i - N\ell.
$$

Note in passing that the constant term of the objective function f is not accounted for in the resolution of (4.3).

Next, we define the Lagrangian function

$$
\mathcal{L}(\boldsymbol{\delta}, \lambda) = f(\boldsymbol{\delta}) + \lambda g(\boldsymbol{\delta}),
$$

where λ is the Lagrangian multiplier.

Given that Q is a positive definite matrix, the well-known Karush-Kuhn-Tucker (KKT) conditions (see for details [18][p. 342])

$$
(4.4) \quad \begin{cases} \nabla \mathcal{L}(\boldsymbol{\delta}^*, \lambda^*) & = & 0, \\ \lambda^* g(\boldsymbol{\delta}^*) & = & 0, \\ g(\boldsymbol{\delta}^*) & \leq & 0, \\ \lambda^* & \geq & 0 \end{cases}
$$

are sufficient for a global minimum of (4.3) if they are satisfied for a given vector $(\boldsymbol{\delta}^*, \lambda^*)$. Thus, in the sequel, we provide an explicit solution for this type of optimisation problem. Typically, (4.3) can be reduced to the Markowitz mean-variance optimisation problem, see [19, 20].

Setting $\boldsymbol{\delta_1} = \boldsymbol{\delta} + Q^{-1}\boldsymbol{c}$, then (4.3) can be expressed as the following standard quadratic program

$$
(4.5) \quad \min_{\boldsymbol{\delta_1}} \frac{1}{2}\boldsymbol{\delta_1}^{\top} Q \boldsymbol{\delta_1},
$$
$$
\text{subject to } \boldsymbol{a_1}^{\top}\boldsymbol{\delta_1} \leq b_1,
$$

with $b_1 = b + \boldsymbol{a}^\top Q^{-1} \boldsymbol{c}$. It should be noted that the constant term (when replacing $\boldsymbol{\delta_1}$ by $\boldsymbol{\delta} + Q^{-1}\boldsymbol{c}$ in (4.5)) does not play any role in the resolution of (4.5).

Let $\boldsymbol{\delta^*}$ be the optimal solution of (4.5). The KKT conditions defined in (4.4) can be explicitly written as follows:

$$
\begin{cases}
Q\boldsymbol{\delta^*} + \lambda^* \boldsymbol{a} = \boldsymbol{0}, & \text{(4.6a)} \\
\lambda^*(\boldsymbol{a}^\top \boldsymbol{\delta^*} - b_1) = 0, & \text{(4.6b)} \\
\boldsymbol{a}^\top \boldsymbol{\delta^*} - b_1 \leq 0, & \text{(4.6c)} \\
\lambda^* \geq 0, & \text{(4.6d)}
\end{cases}
$$

where $\boldsymbol{0} = (0,\ldots,0)^\top \in \mathbb{R}^N$.

If $\lambda^* = 0$, then $\boldsymbol{\delta^*} = \boldsymbol{0}$ follows directly from (4.6a) implying

$$\boldsymbol{\delta^*} = -Q^{-1}\boldsymbol{c}.$$

In view of (4.6d) the other possibility is $\lambda^* > 0$, which in view of (4.6b) implies $\boldsymbol{a}^\top \boldsymbol{\delta^*} = b_1$. Further from (4.6a) $\boldsymbol{\delta^*} = -\lambda^* Q^{-1}\boldsymbol{a}$, hence

$$\boldsymbol{\delta^*} = -Q^{-1}(\lambda^* \boldsymbol{a} + \boldsymbol{c}),$$

with $\lambda^* = -(\boldsymbol{a}^\top Q^{-1}\boldsymbol{a})^{-1} b_1$.

Scenario 2: We consider that (4.3) has lower and upper bounds constraints. Thus, the optimisation problem at hand can be formulated as follows

$$
\begin{aligned}
&\min_{\boldsymbol{\delta}} \frac{1}{2}\boldsymbol{\delta}^\top Q \boldsymbol{\delta} + \boldsymbol{c}^\top \boldsymbol{\delta}, \\
&\text{subject to } \boldsymbol{a}^\top \boldsymbol{\delta} - b \leq 0, \\
&\boldsymbol{l} \leq \boldsymbol{\delta} \leq \boldsymbol{u}.
\end{aligned}
\tag{4.7}
$$

The constraints in (4.7) can be grouped into one equation

$$A\boldsymbol{\delta} \geq \boldsymbol{d},$$

where A is a $(2N+1) \times N$ matrix and $\boldsymbol{d}$ a vector of dimension $2N+1$ respectively given by

$$
A = \begin{pmatrix}
\pi_1 a_1 & \pi_2 a_2 & \pi_3 a_3 & \ldots & \pi_N a_N \\
1 & 0 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ldots & \vdots \\
0 & 0 & \ldots & 1 & 0 \\
0 & 0 & \ldots & 0 & 1 \\
-1 & 0 & 0 & \ldots & 0 \\
0 & -1 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ldots & \vdots \\
0 & 0 & \ldots & -1 & 0 \\
0 & 0 & \ldots & 0 & -1
\end{pmatrix}, \quad
\boldsymbol{d} = \begin{pmatrix}
-b \\
l_1 \\
l_2 \\
\vdots \\
l_{N-1} \\
l_N \\
-u_1 \\
-u_2 \\
\vdots \\
-u_{N-1} \\
-u_N
\end{pmatrix}.
$$

Since we have a convex objective function and a convex region given by constraints, the solution $\boldsymbol{\delta}^*$ is unique and we can transform the above optimisation problem to a bound constrained one using duality.

Hence, (4.7) can be rewritten as follows

(4.8)
$$\min_{\boldsymbol{\delta}} \ f(\boldsymbol{\delta}) = \frac{1}{2}\boldsymbol{\delta}^\top C \boldsymbol{\delta} - \widetilde{\boldsymbol{c}}^\top \boldsymbol{\delta},$$

$$\text{subject to } g(\boldsymbol{\delta}) = \boldsymbol{\delta} \geq \boldsymbol{0},$$

with $C = AQ^{-1}A^\top$ a square matrix of dimension $(2N+1)$ and $\widetilde{\boldsymbol{c}} = AQ^{-1}\boldsymbol{c} + \boldsymbol{d}$ a vector of dimension $(2N+1)$. Since A has rank N implying that Q is an $N \times N$ positive definite matrix, then C is also positive definite.

If $\boldsymbol{\delta^*}$ is the solution of (4.8), then $\boldsymbol{\delta^*} = Q^{-1}\left(A^\top \boldsymbol{\delta^*} - \boldsymbol{c}\right)$ is the solution of (4.7). We refer to [21] for the description of the algorithm and Appendix A.

- Hereafter we shall assume that $b_i \neq 0$ implying that Ψ_i is of the form

(4.9)
$$\Psi_i := \Psi_i(P_i, \delta_i) = \pi_i(1 + a_i\delta_i + b_i\delta_i^2).$$

Given $b_i \in (-1, 0)$ and $\delta_i \in (-1, 1)$, the condition $\Psi_i \in (0, 1)$ holds if and only if

$$a_i \in \left(\max\left(1 - \frac{1}{\pi_i}, -1 - b_i\right), \min\left(1 + b_i, \frac{1}{\pi_i} - 1\right)\right)$$

for $i \leq N$. Clearly, under (4.9) we have that (4.1) is a non-linear optimisation problem with also non-linear constraints. The most popular method discussed in the literature for solving this type of optimisation problem is the *Sequential Quadratic Programming* (SQP) method see e.g., [22–24]. It is an iterative method that generates a sequence of quadratic programs to be solved at each iteration. Typically, at a given iterate x_k (4.1) is modelled by a QP subproblem subject to linear constraints and then solution to the latter is used as a search direction to construct a new iterate x_{k+1}.

Plugging (4.9) in (4.1), the optimisation problem at hand can be reformulated as

$$\min_{\boldsymbol{\delta}} \ f(\boldsymbol{\delta}) = -\sum_{i=1}^{N} P_i\pi_i(1 + (1 + a_i)\delta_i + (a_i + b_i)\delta_i^2 + b_i\delta_i^3),$$

subject to

(4.10)
$$g(\boldsymbol{\delta}) = -\sum_{i=1}^{N}\pi_i(1 + a_i\delta_i + b_i\delta_i^2) + N\ell \leq 0,$$

$$h_1(\delta_i) = \delta_i - u_i \leq 0 \text{ for } i \leq N,$$

$$h_2(\delta_i) = -\delta_i + l_i \leq 0 \text{ for } i \leq N,$$

where f, g, h_1 and h_2 are continuous and twice differentiable.

The main steps required to solve (4.10) are described in Appendix A.

4.1.2. *Probability of renewal Ψ_i as in **Mb)***. We consider the following model for the renewal probability:

(4.11)
$$\Psi_i := \Psi_i(P_i, \delta_i) = \frac{1}{1 + c_i^{-1}e^{-T_i\delta_i}}, \quad 1 \leq i \leq N,$$

where c_i is a constant that depends on the probability of renewal for $\delta_i = 0$ denoted by π_i given by

$$c_i = \frac{\pi_i}{1 - \pi_i}$$

and $T_i < 0$ is a constant (to be estimated in applications) that measures the elasticity of the policyholder relative to the premium change. The greater $|T_i|$ the more elastic the policyholder is to premium change. Under

(4.11) we have that (4.1) is a non-linear optimisation problem subject to non-linear constraints, which can be solved by SQP algorithm described in Appendix A.

Remarks 4.1. *If δ_i are close to 0, then using Taylor expansion **Mb)** can be approximated by **Ma)** as follows*

$$\Psi_i(P_i, \delta_i) \approx \frac{c_i}{1 + c_i}\left(1 + \frac{c_i T_i}{1 + c_i}\delta_i - \frac{T_i^2(c_i - 1)}{2(1 + c_i)^2}\delta_i^2\right),$$

where

(4.12)
$$\pi_i = \frac{c_i}{1 + c_i}, \quad a_i = \frac{c_i T_i}{1 + c_i}, \quad b_i = -\frac{T_i^2(c_i - 1)}{2(1 + c_i)^2}.$$

4.1.3. *Probability of renewal Ψ_i as in **Mc)**.* In this model δ_i belongs to a discrete set, which we shall assume hereafter to be

$$\mathbf{D} = \{-20\%, -15\%, -10\%, -5\%, \ 0\%, \ 5\%, \ 10\%, \ 15\%, \ 20\%\}.$$

Also, the renewal probabilities Ψ_i's are fixed for each insured i based on δ_i for $i \leq N$ as defined in Table 3.1. In this section we deal with a *Mixed Discrete Non-Linear Programming* (MDNLP) optimisation problem. Thus, (4.1) can be reformulated as follows

(4.13)
$$\min_{\boldsymbol{\delta}} f(\boldsymbol{\delta}) = -\sum_{i=1}^{N} P_i(1 + \delta_i)\Psi_i(P_i, \delta_i),$$

$$\text{subject to } g(\boldsymbol{\delta}) = -\sum_{i=1}^{N} \Psi_i(P_i, \delta_i) + N\ell \leqslant 0,$$

$$\text{and } \delta_i \in \mathbf{D}, \quad 1 \leq i \leq N.$$

In general, this type of optimisation problem is very difficult to solve due to the fact that the discrete space is non-convex. Several methods were discussed in the literature for (4.13), see e.g., [25]. The contribution [26] proposed a new method for solving the MDNLP optimisation problem subject to non-linear constraints. It consists in approximating the original non-linear model by a sequence of mixed discrete linear problems evaluated at each point iterate $\boldsymbol{\delta_k}$. Also, a new method for solving a MDNLP was introduced by using a penalty function, see the recent contributions [18, 27] for more details. The algorithmic solution of (4.13) is described in Appendix B.

4.2. **Maximise the retention level @$\mathcal{R}$.** We consider the case where the insurer would like to keep the maximum number of policyholders in the portfolio @$\mathcal{R}$. Therefore the optimisation problem of interest consists in finding the optimal retention level @$\mathcal{R}$ whilst increasing the expected premium volume by a fixed amount say C in the portfolio. Hence, the optimisation problem can be formulated as follows

(4.14)
$$\max_{\boldsymbol{\delta}} \frac{1}{N}\sum_{i=1}^{N} \Psi_i(P_i, \delta_i),$$

$$\text{subject to}$$

$$\mathbb{E}(P^*) \geqslant \mathbb{E}(P) + C,$$

$$\boldsymbol{l} \leq \boldsymbol{\delta} \leq \boldsymbol{u},$$

where $\mathbb{E}(P^*) = \sum_{i=1}^{N} P_i(1+\delta_i)\Psi_i(P_i, \delta_i)$ is the expected premium volume @$\mathcal{R}$, $\mathbb{E}(P) = \sum_{i=1}^{N} P_i\pi_i$ is the expected premium volume before premium change and C is a fixed constant which can be expressed as a percentage of the expected premium volume before premium change. We remark that C can be interpreted as a certain premium loading.

Clearly, (4.14) is a non-linear optimisation problem, which can be solved by using the SQP algorithm already described in Appendix A).

5. Insurance Applications

In this section, we consider a simulated dataset that describes the production of the motor line of business of an insurance portfolio. We simulate premiums from an exponential random variable with mean $1'204$. Also, the probability of renewal before premium change, π_i for $i = 1, \ldots, N$, are known and estimated by the insurance company for each category of policyholders based on historical data. Given that the behaviour of the policyholders is unknown at the time of renewal, the probability of renewal Ψ_i, depends on π_i and δ_i for $i = 1, \ldots, N$. If δ_i is positive, then Ψ_i decreases whereas if δ_i is negative, it is more likely that the policyholder will renew the insurance policy, thus generating a greater Ψ_i. In the following paragraphs, we are going to present some results related to the optimisation problems formulated in the last section.

5.1. **Optimisation problem Ma).**

5.1.1. *Maximise the expected premium volume @$\mathcal{R}$.* We consider, first, the optimisation problem defined in (4.1). In this case, the probability of renewal Ψ_i is defined in **Ma)** and set $b_i = 0$ for $i = 1, \ldots, N$. Given that $a_i < 0$ for $i \leq N$, the probability of renewal Ψ_i increases when δ_i is negative and decreases when δ_i is positive, thus describing perfectly the behaviour of the policyholders that are subject to a decrease, respectively increase, in their premiums @$\mathcal{R}$. The table below describes some statistics on the data for $10'000$ policyholders.

	Premium at time 0
Min	200
Q1	491
Q2	909
Q3	1'605
Max	9'061
No. Obs.	10'000
Mean	1'204
Std. Dev.	990

Table 5.1. Production statistics for the motor business.

We consider the constraint that the expected percentage of the policyholders to remain in portfolio @$\mathcal{R}$ is at least 85%. By solving (4.1) in Matlab with the function *quadprog*, we obtain the optimal δ for each policyholder. We denote by t_0 the time before premium change and by t_1 the time after premium change. Figure 5.1 below is a comparative histogram describing the number of policyholders at time t_0 and at time t_1 with respect to the different premium ranges and the average optimal δ for each premium range.
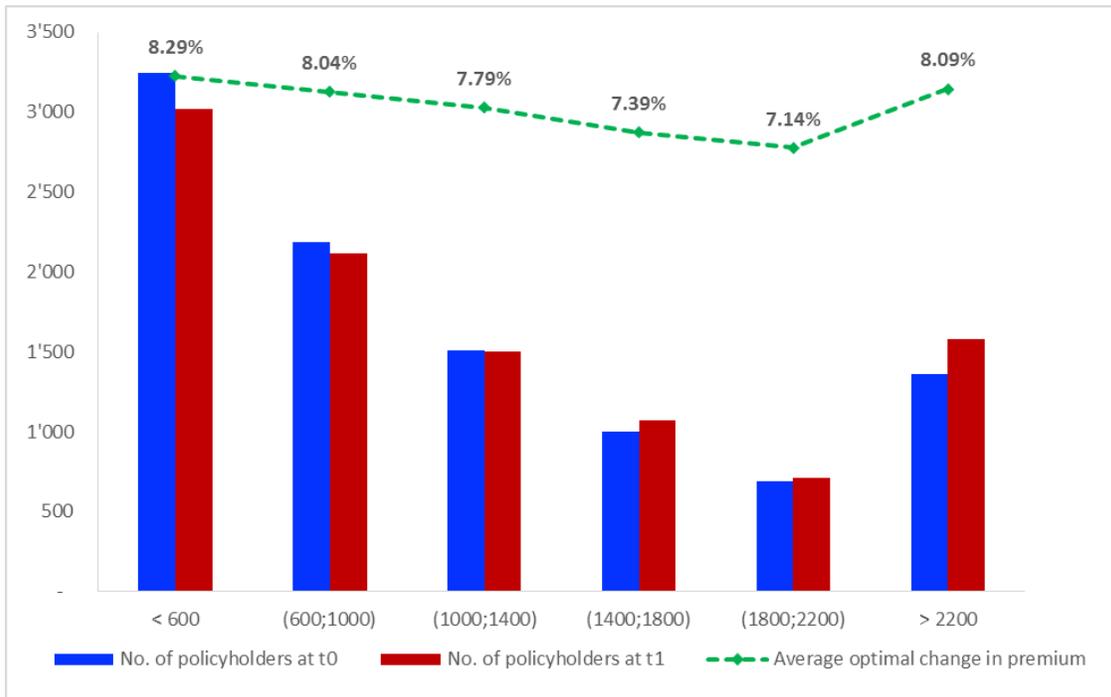
FIGURE 5.1. Number of policyholders based on premium range.

As seen in Figure 5.1, 32% of the policyholders have a premium below 600 CHF vs. 30% @$\mathcal{R}$ due to an average optimal increase in premium of 8%. The average optimal δ decreases gradually for premiums between 600 CHF and 2′200 CHF. Premiums above 2′200 CHF account for only 16% of the portfolio with an average optimal δ of 8%. Typically, in practice, insurance companies are likely to increase the tariffs of policyholders with low premiums as a small increase in the price will not have a great impact on the renewal of the policy. However, for policies with large premium amount, a small increase in the price can lead to the surrender of the policy. Therefore, the results in Figure 5.1 are accurate from the insurance company's perspective when increasing/ decreasing the premiums paid by the policyholders. It should be noted here that we neglect the cases of bad risks and large claims. We look at a homogeneous portfolio where the occurrence of a claim is low and the claim amounts are reasonable.

Next, we consider two scenarios:

Scenario 1 The expected percentage of the policyholders (abbreviate as EPP) to remain in portfolio @$\mathcal{R}$ is at least 75%,

Scenario 2 The EPP to remain in portfolio @$\mathcal{R}$ is at least 85%.

Table 5.2 below summarises the optimal results when solving (4.1) and examines the effect of both scenarios on the expected premium volume and the expected number of policyholders in the portfolio @$\mathcal{R}$.

Constraints on the retention level	75%		85%	
Range of δ (%)	(-10,20)	(-20,30)	(-10,20)	(-20,30)
Growth in expected premium volume @$\mathcal{R}$ (%)	15.78	23.03	8.70	12.96
Growth in expected number of policies @$\mathcal{R}$ (%)	-3.52	-5.25	-0.16	-0.16
Average optimal delta (%)	19.99	29.90	7.97	11.89
Number of increases	10'000	10'000	6'196	6'528
Number of decreases	-	-	3'804	3'472

TABLE 5.2. Scenarios testing.

Scenario 1 The optimal δ for both bounds corresponds approximately to the maximum value (upper bound) of the interval. This is mainly due to the fact that EPP @$\mathcal{R}$ to remain in portfolio is at least 75%. Therefore, the main goal is to maximise the expected premium volume at time t_1.

Scenario 2 For EPP @$\mathcal{R}$ to remain in portfolio of at least 85%, Table 5.2 shows an increase in the expected premium volume which is less important than the one observed in Scenario 1. However, the expected number of policyholders in the portfolio @$\mathcal{R}$ is higher and is approximately the same as at t_0.

Hereafter, we shall consider EPP @$\mathcal{R}$ to remain in portfolio to be at least 85%. Commonly in practice the size of a motor insurance portfolio exceeds $10'000$ policyholders. However, solving the optimisation problems for $\boldsymbol{\delta}$ using the described algorithms when N is large requires a lot of time and heavy computation and may be costly for the insurance company. Thus, an idea to overcome this problem is to split the original portfolio into subportfolios and compute the optimal $\boldsymbol{\delta}$ for the subportfolios. One criteria that can be taken into account for the split is the amount of premium in our case. However, in practice, insurance companies have a more detailed dataset, thus more information on each policyholders, so the criterion that are of interest for the split include the age and gender of the policyholders, the car type, age and value. Table 5.3 and Table 5.4 below describe the results when splitting the original portfolio into three and four subportfolios, respectively.

		Growth in % @$\mathcal{R}$	
Premium Range	Average optimal δ	Expected number of policies	Expected premium volume
< 600	8.60%	-0.27%	9.17%
(600,1'200)	7.29%	-0.03%	8.25%
> $1'200$	8.05%	-0.17%	8.99%
After the split	8.00%	-0.16%	8.84%
Before the split	7.97%	-0.16%	8.70%
Difference	-	0%	-0.13%

TABLE 5.3. Split into 3 subportfolios.

		Growth in % @$\mathcal{R}$	
Premium Range	Average optimal δ	Expected number of policies	Expected premium volume
< 500	8.99%	-0.34%	9.50%
$(500, 800)$	6.27%	0.15%	7.41%
$(800, 1'400)$	7.66%	-0.09%	8.49%
$> 1'400$	8.47%	-0.26%	9.31%
After the split	7.99%	-0.16%	8.95%
Before the split	7.97%	-0.16%	8.70%
Difference		0%	-0.23%

TABLE 5.4. Split into 4 subportfolios.

In Table 5.3 and 5.4, we consider that the insurer would like to keep 85% of the policyholders in each subportfolios, thus a total of 85% of the original portfolio. However, in practice, the constraints on the retention level @$\mathcal{R}$ are specific to each subportfolio. In this regard, the insurance company sets the constraints on the expected number of policies for each subportfolios so that the constraint of the overall portfolio is approximately equal to 85%. The error from the split into three, respectively four subportfolios is relatively small and is of -0.13%, respectively -0.23% for the expected premium volume @$\mathcal{R}$.

Remarks 5.1. *i) This application is mostly relevant when dealing with a non linear optimisation problem of a large insurance portfolio.*

ii) In the following sections, we limit the size of the insurance portfolio to 1'000 policyholders as the algorithms used thereafter to solve the optimisation problems are based on an iterative process which is computationally intensive.

5.1.2. *Maximise the expected premium volume and minimise the variance of the premium volume.* Similarly to the asset allocation optimisation problem in finance introduced by Markowitz [28], the insurer performs a trade-off between the maximum aggregate expected premiums and the minimum variance of the total earned premiums, see e.g., [29] for a different optimality criteria.

We present in Figure 5.2 the comparison of the optimal results computed with the function *gamultiobj* of *Matlab 2016a* for the following scenarios:

- **Scenario 1**: the expected premium volume and the variance of the premium volume are optimised simultaneously as in **Problem T3a**,
- **Scenario 2**: only the expected premium volume is maximised.

The same constraint on the retention level is used for both scenarios and $\delta \in (-30\%, 30\%)$. The histograms in Figure 5.2 represent the optimal variance whilst the dashed curves depict the optimal expected volume. We notice that all the optimal results are normalised with the results obtained from the assumption that the insurer will not change the premiums for next year.
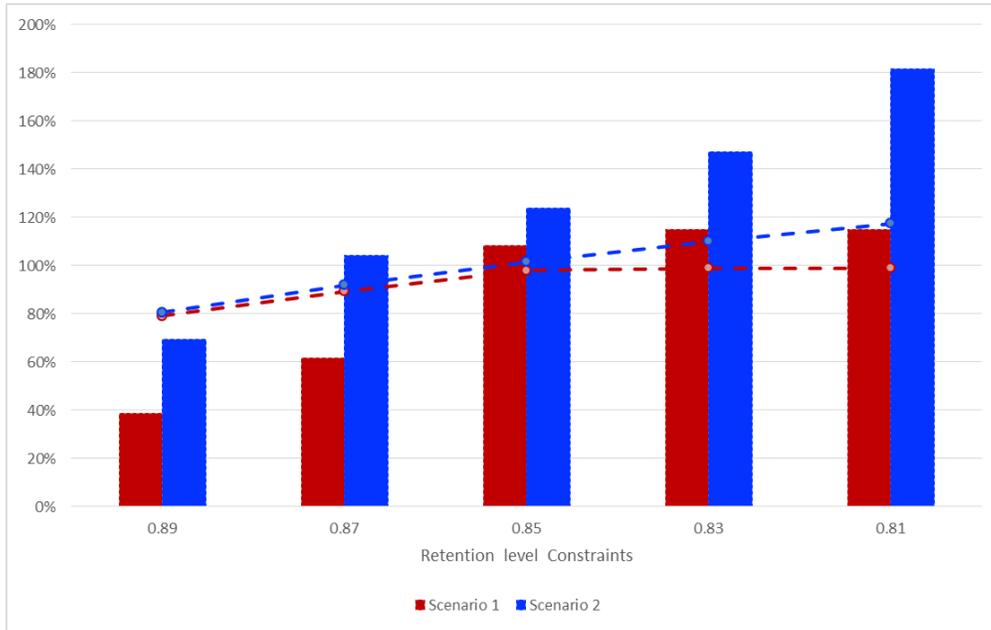
FIGURE 5.2. Optimal results with different objective function scenarios.

For both Scenarios, the maximum expected volume is associated with higher variance. In this respect the lower the retention level the higher the expected volume and the higher the variance. Furthermore, compared to Scenario 2, Scenario 1 results in smaller expected volume but yields a smaller variance. We show next in Table 5.5 the optimal results for the different constraints on the retention level and the possible range of premium changes.

Retention level constraints	75%		85%	
Range of δ (%)	(-10,20)	(-20,30)	(-10,20)	(-20,30)
Aggregate expected future premiums @$\mathcal{R}$ (%)	103.57	103.66	99.90	103.90
Variance of the aggregate future premiums @$\mathcal{R}$ (%)	109.76	113.08	98.41	101.05
Expected number of policies @$\mathcal{R}$ (%)	98.95	98.84	99.98	99.99
Average optimal δ (%)	6.13	6.82	1.68	1.98
Average optimal increase (%)	18.50	26.92	11.82	20.32
Average optimal decrease (%)	-8.33	-16.32	-8.92	-17.10
Number of increases	539	535	511	510
Number of decreases	461	465	489	490

TABLE 5.5. **Scenario 1 optimal results based on different retention levels** .

It can be seen that the optimal variance @$\mathcal{R}$ increases with the range of the possible premium changes δ. For instance when the insurer would like to keep at least 75% of the policyholders, the variance @$\mathcal{R}$ increases from 109.76 for $\delta \in (-10\%, 20\%)$ to 113.08 for $\delta \in (-20\%, 30\%)$, respectively. Furthermore, the increase in variance @$\mathcal{R}$ is associated with an increase of the expected volume @$\mathcal{R}$. This means that the riskier the portfolio the more the insurance company earns premiums.

5.1.3. *Maximise the retention level* @$\mathcal{R}$. We consider here that the insurer would like to maximise the EPP @$\mathcal{R}$ to remain in portfolio whilst increasing the expected premium volume @$\mathcal{R}$ by a certain amount C needed to cover, for instance, the operating costs and other expenses of the insurance company. Figure 5.3 below

describes the results obtained from solving the optimisation problem (4.14) defined in Section 4.2 using the *fmincon* function in *Matlab* for $C = 95'000$ and $\delta \in (-10\%, 20\%)$.
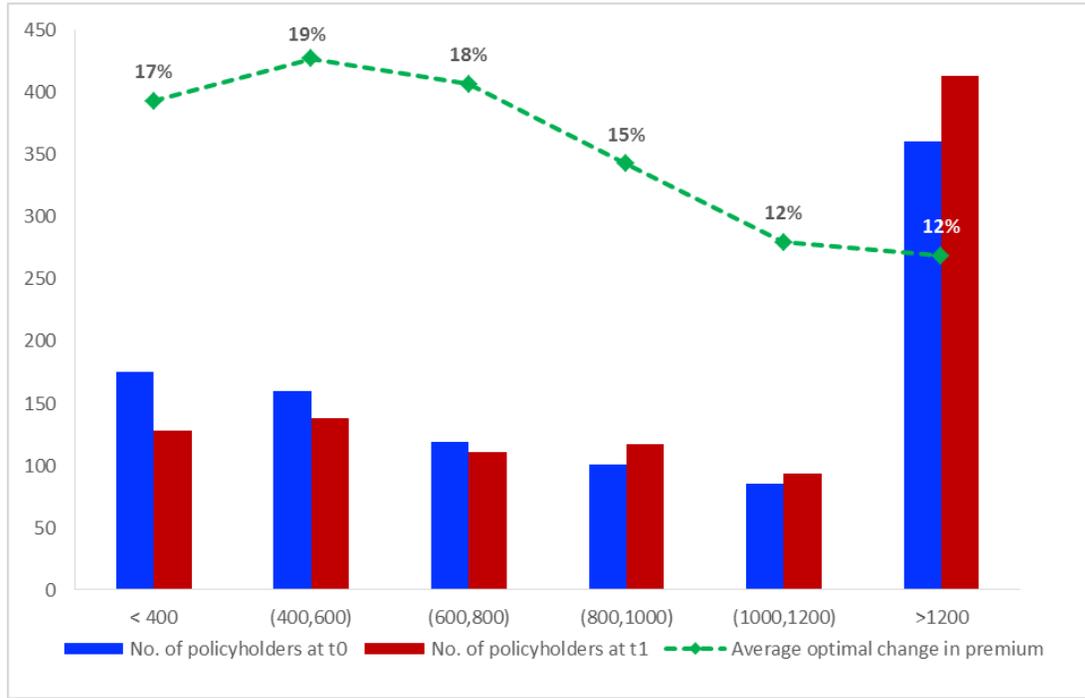
FIGURE 5.3. Number of policyholders based on premium range.

In practice, the amount C needed to cover the expenses of the company is set by the insurers. In fact, C can be expressed as a percentage of the expected premium volume at time t_0. Therefore, we consider three different loadings: 9%, 10% and 11% thus adding an amount of $85'000$, respectively $95'000$ and $105'000$ to the expected premium volume at time t_0. We consider two ranges for δ, namely $\delta \in (-10\%, -20\%)$ and $\delta \in (-20\%, -30\%)$.

Values for C	$C = 85'000$		$C = 95'000$		$C = 105'000$	
Range of δ (%)	(-10,20)	(-20,30)	(-10,20)	(-20,30)	(-10,20)	(-20,30)
Growth in expected number of policies(%)	-2.19	-2.06	-2.50	-2.36	-2.82	-2.67
Growth in expected premium volume(%)	8.90	8.90	9.95	9.95	11.00	11.00
Average optimal δ (%)	13.92	15.82	15.12	17.23	16.60	18.64

TABLE 5.6. Scenario testing - Retention

Table 5.6 shows that when C increases, the expected number of policyholders $@\mathcal{R}$ decreases whereas the average optimal δ increases.

5.2. **Optimisation problem Mb).** We consider the probability of renewal Ψ_i as defined in **Mb)**. As discussed in Section 4.1.2, T_i describes the behaviour of the policyholders subject to premium change. For instance, let us consider a policyholder whose probability of renewal without premium change π_i is 0.95. Figure 5.4 shows that the greater T_i the more the curve of the renewal probability goes to the right thus the less elastic the policyholder is to premium change. Conversely as T_i decreases, the more elastic the policyholder is to premium change.
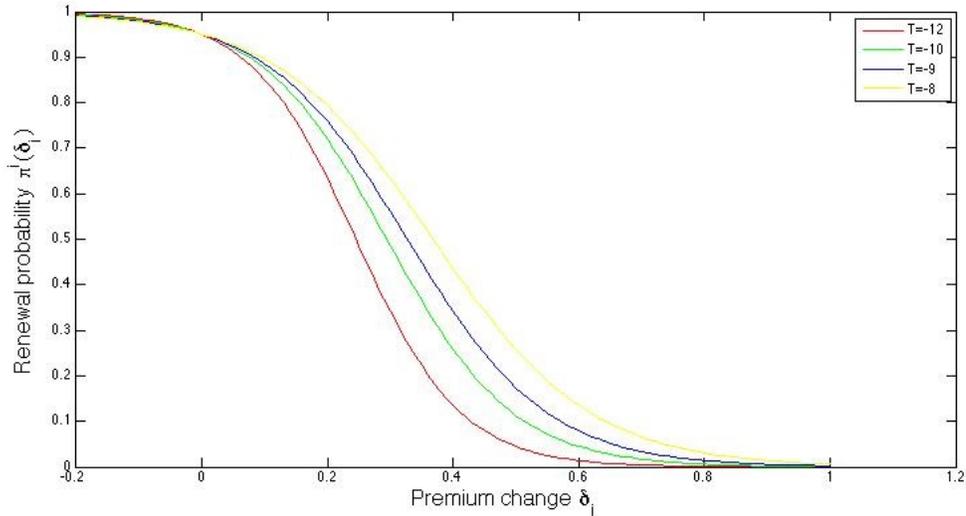
FIGURE 5.4. Renewal probability with respect to premium change for different values of T_i

In this section, we will only consider the case where the insurer would like to maximise the expected premium volume $@\mathcal{R}$. The constraint on the retention level is assumed to be of 85%.
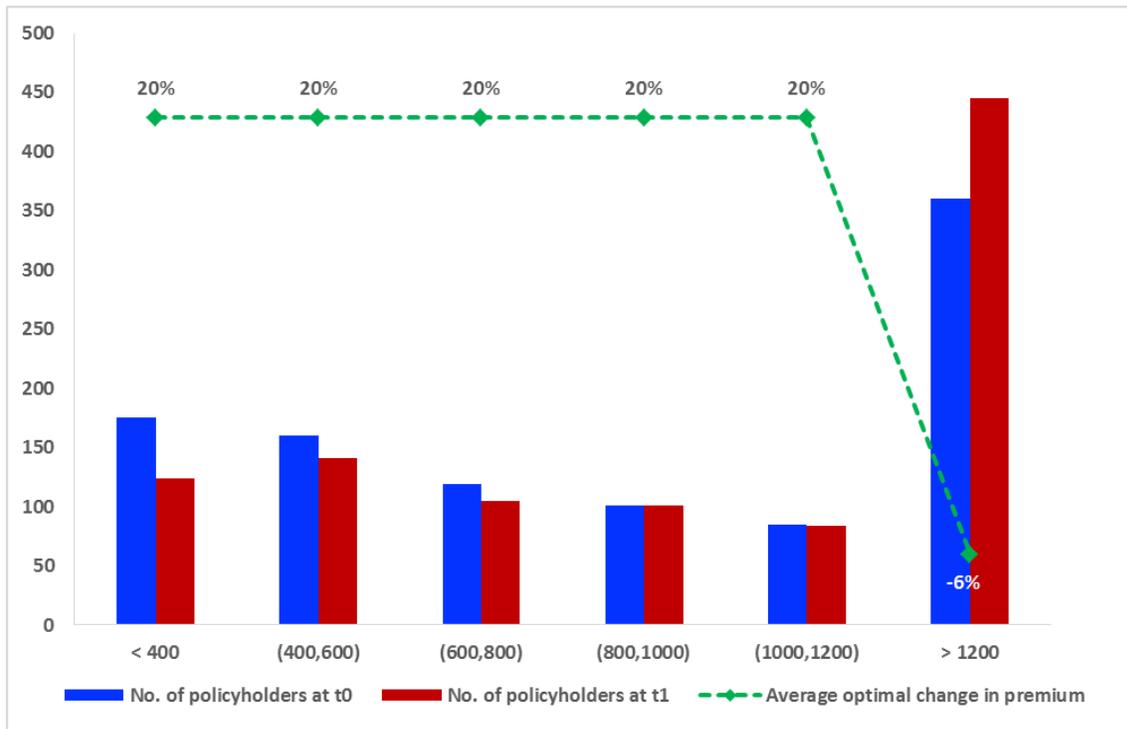
FIGURE 5.5. Number of policyholders based on premium range.

Figure 5.5 shows that the average optimal δ for premiums less than 1'200 CHF is constant for the different premium ranges at 20% which corresponds to the maximum value that δ can take. However, for premiums greater than 1'200 CHF, the average optimal δ decreases to -6%. As stated in Section 5.1.1, insurers are more likely to increase the premiums of policyholders with small premium amounts and decrease the premiums of policyholders with large premium amounts. Thus, the results obtained in Figure 5.5 are accurate as they

describe the behaviour of the insurer when increasing, respectively decreasing the premiums of the policyholders.

At the time of renewal, the insurer sets the constraints on EPP to remain in portfolio. Typically, when the retention level is low, the expected premium volume @$\mathcal{R}$ is greater compared to the case when the retention level is high. Therefore, we consider two different scenarios:

Scenario 1 The EPP @$\mathcal{R}$ to remain in portfolio is at least 75%,

Scenario 2 The EPP @$\mathcal{R}$ to remain in portfolio is at least 85%.

The table below summarises the optimal results when solving (4.1) for the different constraints.

Constraints on the retention level	75%		85%	
Range of δ (%)	(-10,20)	(-20,30)	(-10,20)	(-20,30)
Growth in expected premium volume @$\mathcal{R}$ (%)	17.84	26.45	4.50	6.48
Growth in expected number of policies @$\mathcal{R}$ (%)	-0.93	-1.41	-0.02	-0.02
Average optimal delta (%)	20.00	30.00	10.70	16.09
Number of increases	1'000	1'000	703	736
Number of decreases	-	-	297	264

TABLE 5.7. Scenarios testing.

Scenario 1 Table 5.7 shows that all policyholders are subject to an increase in their premiums and the average optimal δ for the whole portfolio corresponds to the maximum change in premium for both bounds of δ.

Scenario 2 As seen in Table 5.7, the expected number of policyholders @$\mathcal{R}$ is approximately the same as the one before premium change. However, the growth in expected premium volume is lower than in Scenario 1 due to the fact that the average optimal δ for both bounds is lower.

Remarks 5.2. *It should be noted that the probability of renewal defined in **Mb)** can be approximated by the probability of renewal defined in **Ma)** for δ relatively small (refer to Remark 4.1). Therefore, consider $\delta \in (-5\%, 5\%)$ and a retention level $\ell = 85\%$ @$\mathcal{R}$. The table below describes the optimal results when using the logit model **Mb)** and the polynomial model defined in **Ma)**.*

Model	*Logit*	*Polynomial*	*Difference**
Growth in expected premium volume @$\mathcal{R}$	*1.53%*	*0.47%*	*1.04%*
Growth in expected number of policies @$\mathcal{R}$	*-0.02%*	*-0.02%*	*0%*
Average optimal delta	*2.97%*	*1.30%*	*NR*
Number of increases	*796*	*619*	*NR*
Number of decreases	*204*	*381*	*NR*

TABLE 5.8. Comparison between **Ma)** and **Mb)**.

(*NR = Not Relevant)

*Table 5.8 shows that for a small range of δ, the difference between the exact results obtained from **Mb)** and the approximate results obtained from **Ma)** is relatively small and is of around 1% for the expected premium volume @$\mathcal{R}$ and is of 0% for the expected number of policyholders @$\mathcal{R}$. Thus, the approximate values tend to the real ones when the range of δ tends to 0.*

5.3. **Optimisation problem Mc) and Simulation studies.** In this Section, we consider the case where the renewal probabilities Ψ_i are fixed for each insured i, as defined in Table 3.1. To solve the optimisation problem (4.13), we use the MDNLP method described in Appendix B. The table below summarises the optimal results

for a portfolio of 100'000 policyholders with respect to different constraints on the retention level at renewal.

Retention level constraints (%)	85	87.5	90	92.5	95	97.5
Growth in expected premium volume @$\mathcal{R}$ (%)	5.92	5.92	5.34	4.19	2.22	-1.24
Growth in expected number of policies @$\mathcal{R}$ (%)	-7.89	-7.89	-5.26	-2.63	0.00	2.63
Average optimal delta (%)	15.00	15.00	10.00	4.82	-0.51	-6.37

TABLE 5.9. Scenario testing-Discrete optimisation

Table 5.9 shows that when the retention level increases, the expected number of policies increases whereas the expected premium volume @$\mathcal{R}$ decreases. In fact, the average optimal δ decreases gradually from 15% for a retention level of 85% to -6% for a retention level of 97.5%. Also, it can be seen that for a retention level of 95% the optimisation has a negligible effect on the expected number of policies and premium volume @$\mathcal{R}$ as the average optimal δ is approximately null. Hence, no optimisation is needed in this case.

In addition to the MDNLP approach, we have implemented a simulation technique which consists in simulating the premium change δ for each policyholder as described in the following pseudo algorithm:

- **Step 1:** Based on a chosen prior distribution for δ, sample the premium change for each policyholder,

- **Step 2:** Repeat **Step 1** until the constraint on the retention level is satisfied,

- **Step 3:** Repeat **Step 2** m times,

- **Step 4:** Among the m simulations take the simulated δ which gives out the maximum expected profit.

Next, we present the optimal results obtained through 1'000 simulations for the same portfolio. We shall consider three different assumptions on the prior distribution of δ, namely:

- **Case 1: Simulation based on the Uniform distribution**

  In this simulation approach, we assume that the prior distribution of δ is uniform. As highlighted in Table 9.1- 9.2, the parameters of the uniform distribution and the possible values of the premium change are chosen so that the constraint on the retention level is fulfilled. Actually, this choice is based on many simulations trials that we have implemented in which for a fixed range of δ, the parameter of the Uniform distribution is modified at each trial so that the retention level is reached. It should be noted that the more the elements of δ, the smaller the bounds of the Uniform distribution. We present in Table 5.10 the simulation results.

Retention level constraints (%)	85	87.5	90	92.5	95	97.5
Growth in expected premium volume @$\mathcal{R}$ (%)	5.13	5.11	4.02	1.87	-0.55	-4.10
Growth in expected number of policies @$\mathcal{R}$ (%)	-10.32	-6.64	-5.24	-2.61	0.04	2.73
Average optimal delta (%)	17.30	12.62	9.95	4.87	-0.36	-6.52

TABLE 5.10. Scenario testing- simulation approach: Uniform distribution.

- **Case 2: Simulation based on practical experience**

  Next, we assume a prior distribution for δ which is based on the historical premium change of each policyholder. Those prior distributions are presented in Figure 9.1 and the results are described in Table

5.11 below.

Retention level constraints (%)	85	87.5	90	92.5	95	97.5
Growth in expected premium volume @$\mathcal{R}$ (%)	5.50	5.08	4.10	1.98	-0.87	-3.75
Growth in expected number of policies @$\mathcal{R}$ (%)	-9.19	-7.70	-5.26	-2.63	0.47	2.77
Average optimal delta (%)	16.2	13.9	10.0	4.92	-1.17	-6.25

TABLE 5.11. Scenario testing- simulation approach: practical experience

- **Case 3: Simulation based on the results of the MDNLP**

  We use the empirical distribution of the optimal δ obtained from the MDNLP algorithm as a prior distribution. The chosen distribution are shown in Figure 9.1 with different constraints on the retention level. Table 5.3 below summarises the optimal results.

Retention level constraints (%)	85	87.5	90	92.5	95	97.5
Growth in expected premium volume @$\mathcal{R}$ (%)	5.92	5.92	3.90	1.61	-0.91	-4.05
Growth in expected number of policies @$\mathcal{R}$ (%)	-7.89	-7.89	-5.26	-2.63	0.00	2.63
Average optimal delta (%)	15.00	15.00	10.00	4.82	-0.51	-6.35

TABLE 5.12. Scenario testing- simulation approach.

It can be seen that the simulation approaches yield approximately to the same results obtained from the MDNLP algorithm presented in Table 5.9.

6. APPENDIX A: SOLUTION OF (4.8)

Let $I = \{1, \ldots, N\}$. The Lagrangian function related to (4.8) is

$$\mathcal{L}(\boldsymbol{\delta}, \boldsymbol{\lambda}) = f(\boldsymbol{\delta}) + \boldsymbol{\lambda} g(\boldsymbol{\delta}),$$

where $\boldsymbol{\lambda} := (\lambda_1, \ldots, \lambda_N)$ are the Lagrangian multipliers. In this case, the KKT conditions below

$$(6.1) \quad \begin{cases} C\boldsymbol{\delta}^* - \widetilde{\boldsymbol{c}} + \boldsymbol{\lambda}^* &= \mathbf{0}, \\ \lambda_i^* \delta_i^* &= 0, \forall i \in I \\ \delta_i^* &\geq 0, \forall i \in I \\ \lambda_i^* &\geq 0, \forall i \in I \end{cases}$$

hold for $(\boldsymbol{\delta^*}, \boldsymbol{\lambda^*})$.

Let L be the subset of I such that $\lambda_i > 0$ and $\delta_i^* = 0$ for all $i \in L$ if L is non-empty. Note that $\lambda_i = 0$ for $i \notin L$. We have that if L is empty, then by (6.1)

$$(6.2) \qquad \qquad \boldsymbol{\delta}^* = C^{-1}\widetilde{\boldsymbol{c}}.$$

Next suppose that L is non-empty and set $R = I \setminus L$. If R is empty, then the solution is found to be on the boundary as above, i.e, $\delta_i^* = 0$ for all i. If R is non-empty, we have that $\lambda_i = 0$ for any $i \in R$. We need to determine $\boldsymbol{\delta}_R^*$ which is the subvector of $\boldsymbol{\delta}^*$ determined by dropping the components with indices not in R. Since C is positive-definite, then $C_{R,R}$ the submatrix of C determined by dropping the rows and columns with indices not in R is positive definite and therefore non-singular. In view of (6.1) we obtain the solution

$$\boldsymbol{\delta}_R^* = (C_{R,R})^{-1}(-C_{R,L}\boldsymbol{\delta}_L^* + \widetilde{\boldsymbol{c}}_R) = (C_{R,R})^{-1}\widetilde{\boldsymbol{c}}_R$$

and
$$\boldsymbol{\lambda}_L^* = \widetilde{\boldsymbol{c}}_L - C_{L,L}\boldsymbol{\delta}_L^* - C_{L,R}\boldsymbol{\delta}_R^* = \widetilde{\boldsymbol{c}}_L - C_{L,R}\left((C_{R,R})^{-1}\widetilde{\boldsymbol{c}}_R\right).$$

For practical implementation, it is necessary to determine the index set L and this can be achieved by an iterative approach, see [21].

To this end, we remark that in the proof above we used the fact that Q and C are positive definite, but did not use the fact that Q is diagonal matrix.

7. APPENDIX B: SOLUTION OF (4.10)

Step 1: Let

$$\mathcal{L}(\boldsymbol{\delta}, \lambda, \boldsymbol{\mu}, \boldsymbol{\gamma}) = f(\boldsymbol{\delta}) + \lambda g(\boldsymbol{\delta}) + \sum_{i=1}^{N} \mu_i h_1(\delta_i) + \sum_{i=1}^{N} \gamma_i h_2(\delta_i)$$

be the Lagrangian function of (4.10) where $\lambda \in \mathbb{R}$ and $\boldsymbol{\mu}, \boldsymbol{\gamma} \in \mathbb{R}^N$ are the Lagrangian multipliers and $(\boldsymbol{\delta_0}, \lambda_0, \boldsymbol{\mu_0}, \boldsymbol{\gamma_0})$ an initial estimate of the solution. It should be noted that the SQP is not a feasible point method. This means that neither the initial point nor the subsequent iterate ought to satisfy the constraints of the optimisation problem.

Step 2: In order to find the next point iterate $(\boldsymbol{\delta_1}, \lambda_1, \boldsymbol{\mu_1}, \boldsymbol{\gamma_1})$, the SQP determines a step vector $\boldsymbol{s} = (\boldsymbol{s_\delta}, s_\lambda, \boldsymbol{s_\mu}, \boldsymbol{s_\gamma})$ solution of the QP subproblem evaluated at $(\boldsymbol{\delta_0}, \lambda_0, \boldsymbol{\mu_0}, \boldsymbol{\gamma_0})$ and defined below

$$\min_{\boldsymbol{s}} \frac{1}{2}\boldsymbol{s}^\top H\boldsymbol{s} + \nabla f(\boldsymbol{\delta_0})^\top \boldsymbol{s},$$

subject to

(7.1)
$$\nabla g(\boldsymbol{\delta_0})^\top \boldsymbol{s} + g(\boldsymbol{\delta_0}) \leq 0,$$
$$\nabla h_1(\delta_{0,i})^\top \boldsymbol{s} + h_1(\delta_{0,i}) \leq 0 \text{ for } i \leq N,$$
$$\nabla h_2(\delta_{0,i})^\top \boldsymbol{s} + h_2(\delta_{0,i}) \leq 0 \text{ for } i \leq N,$$

where H is an approximation of the Hessian matrix of $\mathcal{L}$, ∇f the gradient of the objective function and ∇g, ∇h_1 and ∇h_2 the gradient of the constraint functions.

The Hessian matrix H is updated at each iteration by the BFGS quazi Newton formula. The SQP method maintains the sparsity of the approximation of the Hessian matrix and its positive definetness, a necessary condition for a unique solution.

Step 3: In order to ensure the convergence of the SQP method to a global solution, the latter uses a merit function ϕ whose reduction implies progress towards a solution. Thus, a step length, denoted by $\alpha \in (0,1)$, is chosen in order to guarantee the reduction of ϕ after each iteration such that

$$\phi(\delta_k + \alpha s_k) \leq \phi(\delta_k),$$

with

$$\phi(x) = f(x) + r\left(g(x) + \sum_{i=1}^{N} h_1(x_i) + \sum_{i=1}^{N} h_2(x_i)\right) \text{ and } r > \max_{1 \leq i \leq N}(|\lambda|, |\mu_i|, |\gamma_i|).$$

Step 4: The new point iterate is given by

$$(\boldsymbol{\delta_1}, \lambda_1, \boldsymbol{\mu_1}, \boldsymbol{\gamma_1}) = (\boldsymbol{\delta_0} + \alpha\boldsymbol{s_\delta}, \lambda_0 + \alpha s_\lambda, \boldsymbol{\mu_0} + \alpha\boldsymbol{s_\mu}, \boldsymbol{\gamma_0} + \alpha\boldsymbol{s_\gamma}).$$

If the latter satisfies the KKT conditions (6.1), the SQP converges at that point. If not, set $k = k + 1$ and go back to **Step 2**.

Remarks 7.1. *It should be noted that the KKT conditions defined in (6.1) are known as the first order optimality conditions, see e.g., [18]. Hence if for a given vector $(\boldsymbol{\delta^*}, \lambda^*, \boldsymbol{\mu^*}, \boldsymbol{\gamma^*})$, the KKT conditions are satisfied, then $(\boldsymbol{\delta^*}, \lambda^*, \boldsymbol{\mu^*}, \boldsymbol{\gamma^*})$ is a local minimum of (4.10).*

8. Appendix C: MDNLP optimisation problem (4.13)

Step 1: Given that Ψ_i is discrete and depends on the values of δ_i, we assume that Ψ_i can be written as a function of δ_i as follows

$$\Psi_i(\delta_i) = -0.9775\delta_i^2 - 0.4287\delta_i + 0.9534 \text{ for } \delta_i \in \mathbf{D}.$$

(4.13) is then treated as a continuous optimisation problem and the optimal solution is found by using one of the methods described previously. We denote by $\boldsymbol{\delta^*}$ the continuous optimal solution.

Step 2: Let $\boldsymbol{\delta_0}$ be the rounded up vector of $\boldsymbol{\delta^*}$ to the nearby discrete values of the set $\mathbf{D}$. $\boldsymbol{\delta_0}$ is considered to be the initial point iterate. If $\boldsymbol{\delta_0}$ is not a feasible point of (4.13), then (4.13) is approximated by a mixed discrete linear optimisation problem at $\boldsymbol{\delta_0}$ and is given by

$$
\begin{aligned}
\text{(8.1)} \qquad &\min_{\boldsymbol{\delta}} \ \nabla f(\boldsymbol{\delta_0})^\top (\boldsymbol{\delta} - \boldsymbol{\delta_0}), \\
&\text{subject to } g(\boldsymbol{\delta_0}) + \nabla g(\boldsymbol{\delta_0})^\top (\boldsymbol{\delta} - \boldsymbol{\delta_0}) \leqslant 0, \\
&\text{and } \boldsymbol{\delta} \in \mathbf{D}^N.
\end{aligned}
$$

Step 3: (8.1) is solved by using a linear programming method and the branch and bound method, see [30] for more details. We denote by $\boldsymbol{\delta_k}$ the new point iterate. If $\boldsymbol{\delta_k}$ is feasible and $||\boldsymbol{\delta_k} - \boldsymbol{\delta_{k-1}}|| < \epsilon$ with $\epsilon > 0$ small, then the iteration is stopped. Else $k = k + 1$ and go back to **Step 2**.

Remarks 8.1. *If, for a certain point iterate $\boldsymbol{\delta}$, the constraint of (4.13) is satisfied and $\boldsymbol{\delta} \in \mathbf{D}^N$ then $\boldsymbol{\delta}$ is a feasible solution of the optimisation problem.*

In general, it is very hard to find the global minimum of a MDNLP optimisation problem due to the fact that there are multiple local minimums. Therefore, $\boldsymbol{\delta^*}$ is said to be a global minimum if $\boldsymbol{\delta^*}$ is feasible and $f(\boldsymbol{\delta^*}) \leq f(\boldsymbol{\delta})$ for all feasible $\boldsymbol{\delta}$.

9. Appendix D: Prior distribution for simulation

9.1. **Simulation based on the Uniform distribution (simulation Case 1).** The tables below describe the range of δ with their respective distribution based on the different retention levels.

Retention level (%)	85	87.5	90
Range of $\boldsymbol{\delta}$(%)	$\{15, 20\}$	$\{10, 15\}$	$\{0, 5, 10, 15\}$
Prior distribution	$U(0.85, 0.99)$	$U(0.90, 0.99)$	$U(0.04, 0.68)$

TABLE 9.1. Possible range of $\boldsymbol{\delta}$ and prior distribution uniformly distributed.

Retention level (%)	92.5	95	97.5
Range of $\boldsymbol{\delta}$(%)	$\{-5, 0, 5, 10, 15\}$	$\{-5, 0, 5, 10, 15\}$	$\{-20, -10, -5, 0, 5, 10, 15\}$
Prior distribution	$U(0.05, 0.40)$	$U(0.04, 0.21)$	$U(0.002, 0.47)$

TABLE 9.2. Possible range of $\boldsymbol{\delta}$ and prior distribution uniformly distributed.

9.2. **Simulation based on practical experience and on the optimal premium changes from the MDNLP algorithm.** We depict in Figure 9.1 the prior distributions used in the simulation approach described in Section 5.3. The red curves represent the prior distribution from practical experience (simulation Case 2) while the blue curves are the empirical distribution of the optimal premium changes obtained with the MDNLP algorithm (simulation Case 3).

(A) Retention level 85%

(B) Retention level 87.5%

(C) Retention level 90%

(D) Retention level 92.5%

(E) Retention level 95%

(F) Retention level 97.5%
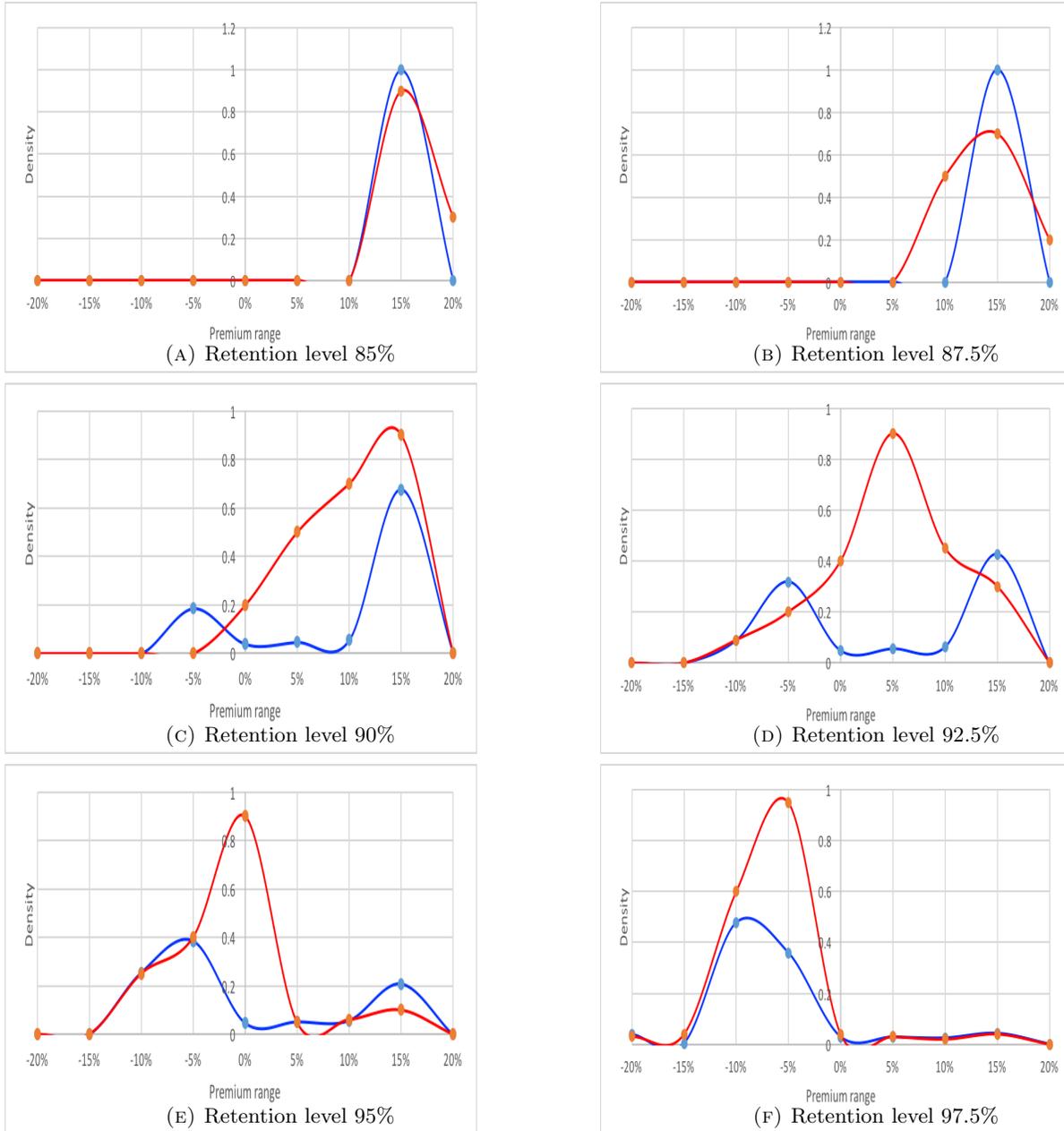
FIGURE 9.1. Prior distribution used in simulations studies: case 2 and case 3.

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ENKELEJD HASHORVA, DEPARTMENT OF ACTUARIAL SCIENCE, UNIVERSITY OF LAUSANNE, UNIL-DORIGNY 1015 LAUSANNE, SWITZERLAND

GILDAS RATOVOMIRIJA, DEPARTMENT OF ACTUARIAL SCIENCE, UNIVERSITY OF LAUSANNE, UNIL-DORIGNY 1015 LAUSANNE, SWITZERLAND, AND VAUDOISE ASSURANCES, PLACE DE MILAN CP 120, 1001 LAUSANNE, SWITZERLAND

MAISSA TAMRAZ, DEPARTMENT OF ACTUARIAL SCIENCE, UNIVERSITY OF LAUSANNE, UNIL-DORIGNY 1015 LAUSANNE, SWITZERLAND

YIZHOU BAI, SCHOOL OF MATHEMATICAL SCIENCES, NANKAI UNIVERSITY, PR CHINA AND DEPARTMENT OF ACTUARIAL SCIENCE, UNIVERSITY OF LAUSANNE, UNIL-DORIGNY 1015 LAUSANNE, SWITZERLAND