STRIKINGLY SIMPLE IDENTITIES RELATING EXIT PROBLEMS FOR LÉVY PROCESSES UNDER CONTINUOUS AND POISSON OBSERVATIONS

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Abstract. We consider exit problems for general Lévy processes, where the first passage over a threshold is detected either immediately or at an epoch of an independent homogeneous Poisson process. It is shown that the two corresponding one-sided problems are related through a surprisingly simple identity. Moreover, we identify a simple link between two-sided exit problems with one continuous and one Poisson exit. Finally, identities for reflected processes and a link between some Parisian type exit problems are established. For spectrally one-sided Levy processes this approach enables alternative proofs for a number of previously established identities, providing additional insight.

1. Introduction

Let $X = (X_t, t \geq 0)$ be a real-valued Lévy process, and let $T_i, i \geq 1$ be the epochs of a Poisson process with intensity $\lambda > 0$ which is independent of X; also add $T_0 = 0$. The probability law corresponding to X started at u will be denoted by $\mathbb{P}_u$ (with $\mathbb{E}_u$ denoting the expectation). When u is not mentioned explicitly we assume that $u = 0$ and write simply $\mathbb{P}$ and $\mathbb{E}$. Define

$$\tau_0^- = \inf\{t \geq 0 : X_t < 0\}, \qquad \tau_a^+ = \inf\{t \geq 0 : X_t > a\},$$
$$\widehat{\tau}_0^- = \min\{T_i, i \in \mathbb{N}_0 : X_{T_i} < 0\}, \qquad \widehat{\tau}_a^+ = \min\{T_i, i \in \mathbb{N}_0 : X_{T_i} > a\},$$

which we interpret as the first passage times under continuous and Poisson observations, respectively. Observe that $\tau_0^- < \widehat{\tau}_0^-$ and, moreover, $\widehat{\tau}_0^-$ converges in probability to τ_0^- as $\lambda \to \infty$ (the same is true for τ_a^+ and $\widehat{\tau}_a^+$). Thus exit theory under Poisson observation can be regarded as a generalization of the classical exit theory. Throughout this paper, however, we keep $\lambda > 0$ fixed.

Observation at Poisson epochs is both of theoretical and practical interest. Firstly, some exit problems with Poisson observation yield transforms of certain occupation times, e.g.

$$\mathbb{P}_u(\tau_0^- < \widehat{\tau}_a^+) = \mathbb{E}_u\left[\exp\left(-\lambda\int_0^{\tau_0^-} 1_{\{X_t > a\}}\mathrm{d}t\right); \tau_0^- < \infty\right], \qquad u \in \mathbb{R},$$

which readily follows from the void probability formula for a Poisson process. Secondly, Poisson observation is relevant in various applications such as queueing (see e.g. [5]), reliability and insurance risk theory (see e.g. [1, 2]). In particular, in

2010 *Mathematics Subject Classification.* Primary 60G51; Secondary 91B30.

Key words and phrases. Lévy processes, exit problems, Poisson observation, occupation times, Parisian ruin.

Financial support by the Swiss National Science Foundation Project 200020 143889 is gratefully acknowledged.

many applications discrete-time observation of stochastic processes would often be considered more natural, but for equidistant discrete time epochs the explicit and tractable analytical structure of continuous-time processes is typically destroyed, so that one is forced towards numerical techniques for the determination of exit probabilities and related quantities. The Poisson observation structure is a bridge between continuous-time and discrete-time observation that still leads to rather explicit, and as will be shown below, also somewhat elegant modifications of the continuous-time formulas.

1.1. **Overview and organization.** In order to stress the intuition behind the derivation of the identities, we will start with a simple case and gradually generalize the setup. Most of the results are stated in terms of relations between transforms, but can also be understood as relations between the corresponding laws in an obvious way.
Some of the wording throughout the manuscript will be in terms of the insurance application, where X is the surplus process of a portfolio of insurance contracts, τ_0^- is the time of ruin of the portfolio, $\{\tau_0^- = \infty\}$ is the event of (infinite-time) survival, and $\widehat{\tau}_0^-$ is the time of observed ruin under Poisson observation of the surplus process (in the application the Poisson epochs can for instance be interpreted as the observation times of the regulatory authority).

In Section 2 we discuss survival probabilities corresponding to the two observation types, and then proceed to the general one-sided exit problems including the time of exit and the overshoot. In Section 3 we consider more complex problems. Firstly, the two-sided exit problem with one continously observed and one (Poisson-)discretely observed boundary is related to the one where the observation types at the boundaries are interchanged. Secondly, we provide a link between Poisson exit of a reflected process and continuous exit of the process reflected at Poisson epochs. We also show that a two-sided problem with Poisson exit at both boundaries yields an identity as well, but with a non-standard first passage time. The latter quantity is then linked to a Parisian ruin problem with Erlang-distributed implementation delay. Finally, we establish a link between Parisian ruin problems with continuous and Poisson observations. We conclude with Section 4, where we specialize to the case of spectrally-one sided processes and demonstrate the use of our simple identities, providing simpler proofs and additional insight to some identities established in earlier literature.

We would like to emphasize that the identities established in this paper provide simple explicit links between ruin problems under discrete and continuous observation. These structural relationships hold for general Lévy processes, even when for neither of the two ruin problems an explicit solution is available. At the same time, in those cases for which, by other means, explicit solutions for the problems are available, these relationships reveal a deeper reason for the concrete shape of the resulting formulas, as the examples in Section 4 illustrate.

1.2. **Preliminaries.** The Wiener-Hopf factorization (splitting at extrema) plays a crucial role in the derivations below. Define

$$\underline{X}_t = \inf\{X_s, s \in [0,t]\}, \qquad \underline{G}_t = \inf\{s \in [0,t] : X_s \wedge X_{s-} = \underline{X}_t\},$$

the infimum and its (first) time of occurrence up to horizon t. Similarly, the supremum and its (last) time of occurrence are defined by

$$\overline{X}_t = \sup\{X_s, s \in [0,t]\}, \qquad \overline{G}_t = \sup\{s \in [0,t] : X_s \vee X_{s-} = \overline{X}_t\}.$$

Assuming $X_0 = 0$, let the pairs (D, T^D) and (U, T^U) be distributed as $(\underline{X}_{T_1}, \underline{G}_{T_1})$ and $(\overline{X}_{T_1}, \overline{G}_{T_1})$ respectively, and sampled independently of each other and of everything else (D and U stand for 'down' and 'up'). Recall that according to the Wiener-Hopf factorization we have

$$\begin{aligned}(\underline{X}_{T_1}, \underline{G}_{T_1}, X_{T_1} - \underline{X}_{T_1}, T_1 - \underline{G}_{T_1}) &\stackrel{d}{=} (D, T^D, U, T^U) \\ &\stackrel{d}{=} (X_{T_1} - \overline{X}_{T_1}, T_1 - \overline{G}_{T_1}, \overline{X}_{T_1}, \overline{G}_{T_1}),\end{aligned}$$

see, e.g., [6, Thm. VI.5], and [7] for applications of factorization embeddings. Finally, the bivariate transforms $\mathbb{E}e^{-\alpha T^D + \beta D}$ and $\mathbb{E}e^{-\alpha T^U - \beta U}$ are called the Wiener-Hopf factors corresponding to rate λ.

2. One-sided exit

2.1. **Survival probability.** Let us first consider

$$\phi(u) := \mathbb{P}_u(\tau_0^- = \infty), \qquad \widehat{\phi}(u) := \mathbb{P}_u(\widehat{\tau}_0^- = \infty), \tag{1}$$

which in the insurance application are the probabilities of survival with initial capital u under continuous and Poisson observation, respectively. In fact, the two quantities are connected by two very simple relations:

Proposition 1. *For $u \geq 0$ it holds that*

$$\widehat{\phi}(u) = \mathbb{E}\phi(u + U), \tag{2}$$

$$\phi(u) = \mathbb{E}\widehat{\phi}(u + D). \tag{3}$$

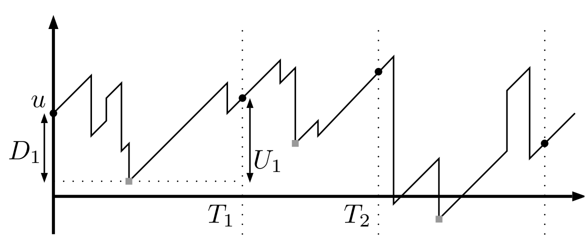

Figure 1. Schematic sample path and embeddings.

Proof. Survival under Poisson observation is determined by the sequence $u + X_{T_i}$, whereas survival under continuous observation is determined by the sequence of infima in between the observation epochs (black and grey dots in Figure 1, respectively). Let $D_1 = \underline{X}_{T_1} < 0, U_1 = X_{T_1} - \underline{X}_{T_1} > 0$ and define D_{i+1}, U_{i+1} $(i \geq 1)$ in the same way but for the shifted process $X_{T_i + t} - X_{T_i}$ and exponential time $T_{i+1} - T_i$. Let $(\widehat{S}_i, i \in \mathbb{N}_0)$ and $(S_i, i \in \mathbb{N}_0)$ be the partial sum processes corresponding to

$$(0, D_1 + U_1, D_2 + U_2, D_3 + U_3, \ldots) \quad \text{and} \quad (D_1, U_1 + D_2, U_2 + D_3, \ldots),$$

respectively; $u+\widehat{S}_i$ and $u+S_i$ are the heights of the black and grey dots in Figure 1. Observe that all U_i and D_i are independent, because of independence of increments and the Wiener-Hopf factorization. Since the D_i's have the law of D we obtain

$$(\widehat{S}_i + D, i \geq 0) \stackrel{d}{=} (S_i, i \geq 0). \tag{4}$$

Similarly,

$$(\widehat{S}_i, i \geq 1) \stackrel{d}{=} (S_i + U, i \geq 0). \tag{5}$$

Hence

$$\phi(u) = \mathbb{P}(u + \min_{i\geq 0} S_i \geq 0) = \mathbb{P}(u + D + \min_{i\geq 0} \widehat{S}_i \geq 0) = \mathbb{E}\widehat{\phi}(u + D),$$

and, since $u \geq 0$,

$$\begin{aligned}\widehat{\phi}(u) &= \mathbb{P}(u + \min_{i\geq 0} \widehat{S}_i \geq 0) = \mathbb{P}(u + \min_{i\geq 1} \widehat{S}_i \geq 0) \\ &= \mathbb{P}(u + U + \min_{i\geq 0} S_i \geq 0) = \mathbb{E}\phi(u + U).\end{aligned}$$

□

Remark 1. Relation (2) allows to interpret the transition from continuous to discrete Poisson observation simply as a (random) increase of the starting value (initial capital) u by U, as far as the survival probability is concerned; that is the structure of ϕ as a function of u is otherwise completely preserved. Likewise, Relation (3) shows that moving from discrete Poisson to continuous observation preserves the structure, reducing the initial capital by D (which has all its probability mass on the negative half-line).

Remark 2. Suppose we modify the Poisson observation model, so that there is no observation at time 0. Then (2) is still valid (even for negative u then), whereas (3) does not hold any more.

Remark 3. By the same token one can connect the finite-time survival probabilities $\phi(u, T_i) := \mathbb{P}_u(\tau_0^- > T_i) = \mathbb{P}(u + \min_{j\leq i-1} S_j \geq 0)$ and $\widehat{\phi}(u, T_i) := \mathbb{P}_u(\widehat{\tau}_0^- > T_i) = \mathbb{P}(u + \min_{j\leq i} \widehat{S}_j \geq 0)$ for $i \in \mathbb{N}$:

$$\widehat{\phi}(u, T_i) = \mathbb{E}\phi(u + U, T_i), \quad \phi(u, T_i) = \mathbb{E}\widehat{\phi}(u + D, T_{i-1}).$$

That is, survival under continuous observation up to an independent Erlang distributed time horizon is intimately related to survival under Poisson observation up to a certain arrival epoch.

Proposition 1 provides a simple structural relation between $\phi(u)$ and $\widehat{\phi}(u)$. If the purpose is to use it for a numerical evaluation of $\widehat{\phi}(u)$, this needs the availability of the distribution of U as well as the possibility to evaluate $\phi(u)$ itself. One example is a spectrally-negative Lévy process for which U is an exponential random variable and $\phi(u)$ is available in terms of scale functions, see Section 4. In some other cases only the Laplace transforms of U and $-D$, i.e. the Wiener-Hopf factors, are available in explicit form, see [9] and references therein. Then it may be more convenient to consider the transform of $\widehat{\phi}(u)$ given in the following result in terms of the 'negative' Wiener-Hopf factors corresponding to rates λ and 0.

Corollary 1. *For $\theta > 0$ it holds that*

$$\int_0^\infty e^{-\theta u}\widehat{\phi}(u)\mathrm{d}u = \frac{\int_0^\infty e^{-\theta u}\phi(u)\mathrm{d}u}{\mathbb{E}e^{\theta D}} = \frac{\mathbb{E}e^{\theta \underline{X}_\infty}}{\theta \mathbb{E}e^{\theta D}}.$$

Proof. Let e_θ be an exponential random variable of rate θ independent of everything else. Note that

$$\mathbb{E}\phi(e_\theta) = \theta\int_0^\infty e^{-\theta u}\phi(u)\mathrm{d}u = \mathbb{P}(-\underline{X}_\infty < e_\theta) = \mathbb{E}e^{\theta \underline{X}_\infty}.$$

Letting $\underline{\widehat{X}}_\infty = \inf\{X_{T_i}, i = 0,1,\dots\}$ we similarly obtain $\mathbb{E}\widehat{\phi}(e_\theta) = \mathbb{E}e^{\theta\underline{\widehat{X}}_\infty}$. But, according to Proposition 1 we have

$$\mathbb{E}\phi(e_\theta) = \mathbb{E}\widehat{\phi}(e_\theta + D) = \mathbb{P}(-\underline{\widehat{X}}_\infty < e_\theta + D) = \mathbb{P}(-\underline{\widehat{X}}_\infty - D < e_\theta) = \mathbb{E}e^{\theta\underline{\widehat{X}}_\infty}\mathbb{E}e^{\theta D},$$

which readily yields the result. □

Remark 4. Note that $\mathbb{E}\phi(e_\theta)$ coincides with the transform of the stationary workload in a queue driven by $-X_t$, whereas $\mathbb{E}\widehat{\phi}(e_\theta)$ coincides with the transform of the minimal workload in such a queue up to time T_1 when started from stationarity, see [8, Prop. 1]; here we assume that $X_t \to \infty$ as $t \to \infty$. Therefore $\widehat{\phi}(u), u \geq 0$ is a CDF of the minimal workload, and in particular $\widehat{\phi}(0)$ coincides with the probability of the queue becoming empty before an independent exponential epoch T_1.

2.2. **The general identities.**

Proposition 2. *For $u \geq 0$ and $\alpha, \beta \geq 0$ it holds that*

$$\begin{aligned}
&\mathbb{E}_u\left(e^{-\alpha\widehat{\tau}_0^- + \beta X_{\widehat{\tau}_0^-}}; \widehat{\tau}_0^- < \infty\right) \\
&\quad = \mathbb{E}\left[e^{-\alpha T^U}\mathbb{E}_{u+U}\left(e^{-\alpha\tau_0^- + \beta X_{\tau_0^-}}; \tau_0^- < \infty\right)\right]\mathbb{E}e^{-\alpha T^D + \beta D},
\end{aligned} \tag{6}$$

$$\begin{aligned}
&\mathbb{E}_u\left(e^{-\alpha\tau_0^- + \beta X_{\tau_0^-}}; \tau_0^- < \infty\right)\mathbb{E}e^{-\alpha T^D + \beta D} \\
&\quad = \mathbb{E}\left[e^{-\alpha T^D}\mathbb{E}_{u+D}\left(e^{-\alpha\widehat{\tau}_0^- + \beta X_{\widehat{\tau}_0^-}}; \widehat{\tau}_0^- < \infty\right)\right].
\end{aligned} \tag{7}$$

Observe that the left-hand side of (7) gives the transform of the undershoot of the first grey point below 0 in Figure 1, according to the strong Markov property applied at τ_0^-. Now one can establish the relation between black and grey points as in the proof of Proposition 1, additionally taking time into account. Essentially, we just shift the picture so that we start at the first grey point.

Proof of Proposition 2. Let $T_1^D = \underline{G}_{T_1}, T_1^U = T_1 - \underline{G}_{T_1}$ and define T_{i+1}^D, T_{i+1}^U in the same way but for the shifted process $X_{T_i+t} - X_{T_i}$ and exponential time $T_{i+1} - T_i$. As in the proof of Proposition 1 we consider the sequences of black and grey dots in Figure 1, but now we also add the time component: $(\widehat{S}_i, T_i), i \in \mathbb{N}_0$ and $(S_i, G_i), i \in \mathbb{N}_0$ which are the partial sum processes corresponding to

$$\begin{aligned}
&((0,0),(D_1+U_1, T_1^D+T_1^U),(D_2+U_2,T_2^D+T_2^U),\dots) \quad \text{and} \\
&((D_1,T_1^D),(U_1+D_2,T_1^U+T_2^D),\dots),
\end{aligned}$$

respectively. Similarly to (4) and (5) we observe that

$$((\widehat{S}_i + D, T_i + T^D), i \geq 0) \stackrel{d}{=} ((S_i, G_i), i \geq 0)$$
$$((\widehat{S}_i, T_i), i \geq 1) \stackrel{d}{=} ((S_i + U, G_i + T^U), i \geq 0).$$

Letting $\widehat{N}_u = \min\{i \geq 0 : u + \widehat{S}_i < 0\} = \min\{i \geq 1 : u + \widehat{S}_i < 0\}$ and $N_u = \min\{i \geq 0 : u + S_i < 0\}$ be the first passage epochs we can write

$$\begin{aligned}
&\mathbb{E}_u\left(e^{-\alpha\widehat{\tau}_0^- + \beta X_{\widehat{\tau}_0^-}}; \widehat{\tau}_0^- < \infty\right) = \mathbb{E}\left(e^{-\alpha T_{\widehat{N}_u} + \beta(u + \widehat{S}_{\widehat{N}_u})}; \widehat{N}_u < \infty\right) \\
&= \mathbb{E}\left(e^{-\alpha(T^U + G_{N_{u+U}}) + \beta(u + U + S_{N_{u+U}})}; N_{u+U} < \infty\right) \\
&= \mathbb{E}\mathbb{E}_{u+U}\left(e^{-\alpha(T^U + \tau_0^-) + \beta X_{\tau_0^-}}; \tau_0^- < \infty\right)\mathbb{E}e^{-\alpha T^D + \beta D},
\end{aligned}$$

where in the last line we applied the strong Markov property at τ_0^-. Identity (7) can be derived analogously. □

3. Further exit problems

3.1. Two-sided exit with different observation types. In this section we consider two-sided exit problems with one continuous and one Poisson exit at the boundaries. It turns out that there is a simple relation between the problems when the roles of the continuous and the Poisson exit are interchanged, i.e. problems corresponding to $\{\tau_0^- < \widehat{\tau}_a^+\}$ and $\{\widehat{\tau}_0^- < \tau_a^+\}$. Here we extend the ideas of Section 2 to their full potential.

Proposition 3. *For $a \geq u \geq 0$ and $\alpha, \beta \geq 0$ it holds that*

$$\begin{aligned}
(8)\qquad &\mathbb{E}_u\left(e^{-\alpha\widehat{\tau}_0^- + \beta X_{\widehat{\tau}_0^-}}; \widehat{\tau}_0^- < \tau_a^+\right) \\
&= \mathbb{E}\left[e^{-\alpha T^U}\mathbb{E}_{u+U}\left(e^{-\alpha\tau_0^- + \beta X_{\tau_0^-}}; \tau_0^- < \widehat{\tau}_a^+\right)\right]\mathbb{E}e^{-\alpha T^D + \beta D},
\end{aligned}$$

$$\begin{aligned}
(9)\qquad &\mathbb{E}_u\left(e^{-\alpha\tau_0^- + \beta X_{\tau_0^-}}; \tau_0^- < \widehat{\tau}_a^+\right)\mathbb{E}e^{-\alpha T^D + \beta D} \\
&= \mathbb{E}\left[e^{-\alpha T^D}\mathbb{E}_{u+D}\left(e^{-\alpha\widehat{\tau}_0^- + \beta X_{\widehat{\tau}_0^-}}; \widehat{\tau}_0^- < \tau_a^+\right)\right].
\end{aligned}$$

Proof. The proof is by inspection: For (8), consider the embeddings illustrated in Figure 2. In the left picture the grey dots correspond to the observations and the black to the suprema in between two observations. In the right picture the black dots correspond to observations and the grey to the infima in between observations. Note that the position of a black point with respect to the previous grey point has the same distribution in both cases, namely (U, T^U). The same is true for the position of the grey points with respect to their previous black points with (D, T^D). So the patterns of points in each case have the same law up to a certain shifting; we illustrate this by using the same patterns of points in both pictures in Figure 2 and by drawing different sample paths. Now it follows that $\mathbb{P}_u(\widehat{\tau}_0^- < \tau_a^+)$, see the left picture, must coincide with $\mathbb{E}\mathbb{P}_{u+U}(\tau_0^- < \widehat{\tau}_a^+)$, see the right picture, because the interpretation of points was 'reversed'. Finally, we include the value of X at first passage and its time using the strong Markov property at τ_0^- as in the proof of Proposition 2. The same type of reasoning yields (9). □

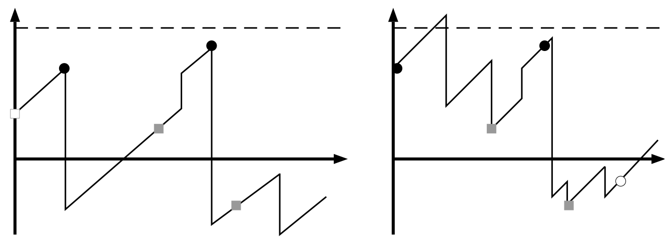

FIGURE 2. Poisson observations are in grey (left picture) and in black (right picture)

By considering the negative of X we immediately obtain the following result from Proposition 3.

Corollary 2. *For $a \geq u \geq 0$ and $\alpha, \beta \geq 0$ it holds that*

$$\begin{aligned}(10)\qquad &\mathbb{E}_u\left(e^{-\alpha\widehat{\tau}_a^+ - \beta(X_{\widehat{\tau}_a^+}-a)}; \widehat{\tau}_a^+ < \tau_0^-\right)\\ &= \mathbb{E}\left[e^{-\alpha T^D}\mathbb{E}_{u+D}\left(e^{-\alpha\tau_a^+ - \beta(X_{\tau_a^+}-a)}; \tau_a^+ < \widehat{\tau}_0^-\right)\right]\mathbb{E}e^{-\alpha T^U - \beta U},\\ (11)\qquad &\mathbb{E}_u\left(e^{-\alpha\tau_a^+ - \beta(X_{\tau_a^+}-a)}; \tau_a^+ < \widehat{\tau}_0^-\right)\mathbb{E}e^{-\alpha T^U - \beta U}\\ &= \mathbb{E}\left[e^{-\alpha T^U}\mathbb{E}_{u+U}\left(e^{-\alpha\widehat{\tau}_a^+ - \beta(X_{\widehat{\tau}_a^+}-a)}; \widehat{\tau}_a^+ < \tau_0^-\right)\right].\end{aligned}$$

3.2. **Reflected processes.** In this section we consider the process X reflected at a barrier $a > 0$ in a continuous and Poisson manner, and study its first passage below 0 in Poisson and continuous manner respectively (with always opposite manners). Again, these two problems are closely related. Note that in an insurance context reflection at a results when paying out dividends according to a *barrier strategy*, either continuously or at Poisson epochs (see e.g. [1]).

Let $\mathbb{P}_u^a$ be the law of X started in u and continuously reflected at a and let R be the corresponding regulator, i.e. $(X_t, R_t), t \geq 0$ under $\mathbb{P}_u^a$ is

$$(X_t - (\overline{X}_t - a)^+, (\overline{X}_t - a)^+), t \geq 0 \text{ under } \mathbb{P}_u.$$

Similarly, let $\widehat{\mathbb{P}}_u^a$ be the law of X started at u and reflected in Poisson manner at a, i.e. $(X_t, R_t), t \geq 0$ under $\widehat{\mathbb{P}}_u^a$ is

$$(X_t - (\max\{X_{T_i} : T_i \leq t\} - a)^+, (\max\{X_{T_i} : T_i \leq t\} - a)^+), t \geq 0 \text{ under } \mathbb{P}_u.$$

Proposition 4. *For $a \geq u \geq 0$ and $\alpha, \beta, \gamma \geq 0$ it holds that*

$$\mathbb{E}_u^a\left(e^{-\alpha\widehat{\tau}_0^- + \beta X_{\widehat{\tau}_0^-} - \gamma R_{\widehat{\tau}_0^-}}; \widehat{\tau}_0^- < \infty\right) = \mathbb{E}\left[e^{-\alpha T^U}\widehat{\mathbb{E}}_{u+U}^a\left(e^{-\alpha\tau_0^- + \beta X_{\tau_0^-} - \gamma R_{\tau_0^-}}; \tau_0^- < \infty\right)\right]\mathbb{E}e^{-\alpha T^D + \beta D},$$

$$\widehat{\mathbb{E}}_u^a\left(e^{-\alpha\tau_0^- + \beta X_{\tau_0^-} - \gamma R_{\tau_0^-}}; \tau_0^- < \infty\right)\mathbb{E}e^{-\alpha T^D + \beta D} = \mathbb{E}\left[e^{-\alpha T^D}\mathbb{E}_{u+D}^a\left(e^{-\alpha\widehat{\tau}_0^- + \beta X_{\widehat{\tau}_0^-} - \gamma R_{\widehat{\tau}_0^-}}; \widehat{\tau}_0^- < \infty\right)\right].$$

Proof. Again, the proof follows merely by inspection in a similar way as for the previous results. The first relation can be seen from Figure 3, where the left picture depicts continuous reflection at a and Poisson observation at 0, and the right picture depicts the corresponding (shifted) Poisson reflection at a and continuous observation at 0. In the left picture Poisson observations yield the sequence: $u, (u+U_1)\wedge a + D_1, ((u+U_1)\wedge a + D_1 + U_2)\wedge a + D_2, \ldots$. In the right picture the infima in between Poisson reflection epochs are given by $\widehat{u}\wedge a + D_1, (\widehat{u}\wedge a + D_1 + U_1)\wedge a + D_2, \ldots$, where we choose $\widehat{u}$ to be distributed as $u + U$. These sequences can be complemented with the respective times as in the proof of Proposition 2. Finally, R enters the transforms without requiring any changes, which is easy to see by writing down the corresponding sequences. Moreover, in the right picture (scenario), R can increase only at the times of black points, and hence no correction is needed when applying the strong Markov property at τ_0^-. The second relation follows accordingly. □

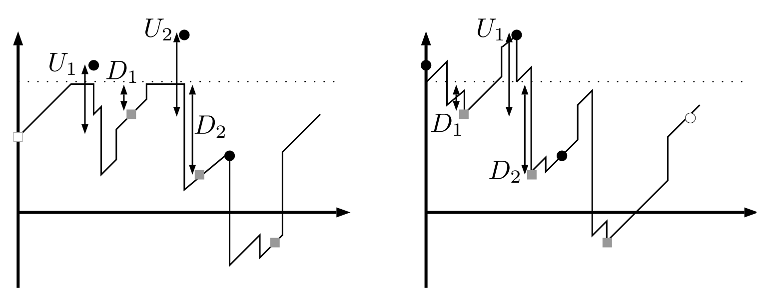

FIGURE 3. Continuous reflection and Poisson exit (left), and Poisson reflection and continuous exit (right)

Remark 5. It is easy to see that Proposition 4 can be generalized from reflection to so-called refraction. Concretely, consider the processes $X_t - \delta(\overline{X}_t - a)^+$ and $X_t - \delta(\max\{X_{T_i} : T_i \leq t\} - a)^+$ for $\delta \in [0,1]$. In the insurance context such a refraction has the interpretation of taxation according to a loss-carry-forward scheme and tax rate δ, see e.g. [4].

3.3. **Two-sided Poisson exit.** The two-sided exit with Poisson observation at both barriers can be related to a model with another type of exit time. Define the random time $\widetilde{\tau}_a^+$ of the first observation such that the process has stayed above a

during the entire preceding inter-observation period, i.e. $\widetilde{\tau}_a^+ = T_{\widetilde{N}_a^+}$ with $\widetilde{N}_a^+ := \min\{i \in \mathbb{N} : \inf\{X_t, t \in [T_{i-1}, T_i]\} > a\}$. Similarly, define $\widetilde{\tau}_0^- = T_{\widetilde{N}_0^-}$ with $\widetilde{N}_0^- := \min\{i \in \mathbb{N} : \sup\{X_t, t \in [T_{i-1}, T_i]\} < 0\}$ as the first observation time such that the process has stayed below 0 during the entire preceding inter-observation period. Then for $u \in [0, a]$ it holds that

$$\mathbb{P}_u(\widehat{\tau}_0^- < \widehat{\tau}_a^+) = \mathbb{E}\mathbb{P}_{u+U}(\tau_0^- < \widetilde{\tau}_a^+) = \mathbb{E}\mathbb{P}_{u+D}(\widetilde{\tau}_0^- < \tau_a^+). \tag{12}$$

To see this, one follows the same ideas as above: for the first equality consider infima in between two observations, see Figure 2, and for the second equality consider suprema in between two observations. Similarly, we also have the reverse identities:

$$\begin{aligned} \mathbb{P}_u(\tau_0^- < \widetilde{\tau}_a^+) &= \mathbb{E}\mathbb{P}_{u+D}(\widehat{\tau}_0^- < \widehat{\tau}_a^+), \\ \mathbb{P}_u(\widetilde{\tau}_0^- < \tau_a^+) &= \mathbb{E}\mathbb{P}_{u+U}(\widehat{\tau}_0^- < \widehat{\tau}_a^+). \end{aligned} \tag{13}$$

3.4. **Parisian ruin.** Parisian ruin is defined as the first time when an excursion of X below 0 is longer than some time $V \geq 0$ (sometimes referred to as *implementation delay*). Whereas the classical definition is in terms of a deterministic V, for analytic tractability it is often assumed that V is a random variable, and that an independent copy of V is assigned to each excursion, see e.g. [13] and [12].

Firstly, from the memoryless property it follows that the time $\widehat{\tau}_0^-$ of Poisson ruin is also the time of Parisian ruin in the case where V is an exponential random variable with rate λ. Secondly, $\widetilde{\tau}_0^-$ as defined in Section 3.3 is the time of Parisian ruin in the case where V is Erlang$(2, \lambda)$ distributed (since the latter is the sum of two independent exponential variables). Similarly to (6), Equation (13) can easily be extended to

$$\begin{aligned} &\mathbb{E}_u\left(e^{-\alpha\widetilde{\tau}_0^- + \beta X_{\widetilde{\tau}_0^-}}; \widetilde{\tau}_0^- < \infty\right) \\ &\qquad = \mathbb{E}\left[e^{-\alpha T^U}\mathbb{E}_{u+U}\left(e^{-\alpha\widehat{\tau}_0^- + \beta X_{\widehat{\tau}_0^-}}; \widehat{\tau}_0^- < \infty\right)\right]\mathbb{E}e^{-\alpha T^D + \beta D}, \end{aligned} \tag{14}$$

which under the present interpretation relates Parisian ruin quantities with exponential and Erlang$(2, \lambda)$-distributed implementation delay (here we took $a = \infty$ for simplicity).

More generally, consider Parisian ruin with Erlang(k, λ) implementation delay and let ρ_k denote the corresponding ruin time. So in particular $\rho_0 = \tau_0^-$, $\rho_1 = \widehat{\tau}_0^-$ and $\rho_2 = \widetilde{\tau}_0^-$. On the other hand, define $\widehat{\rho}_k$ for $k = 0, 1, \ldots$ as the first epoch $T_i, i \geq k$ such that $X(T_{i-k}), \ldots, X(T_i) < 0$, i.e. the process has been observed negative at the last $k+1$ Poisson epochs. In other words, $\widehat{\rho}_k$ is the time of Parisian ruin in the model with discrete Poisson observations corresponding to a delay of k steps. Then, along the same line of arguments, we can extend (14) (cf. Figure 4):

Proposition 5. *For $k \geq 2$ and $u \geq 0$ we have*

$$\begin{aligned} &\mathbb{E}_u\left(e^{-\alpha\rho_k + \beta X_{\rho_k}}; \rho_k < \infty\right) \\ &\qquad = \mathbb{E}\left[e^{-\alpha T^U}\mathbb{E}_{u+U}\left(e^{-\alpha\widehat{\rho}_{k-2} + \beta X_{\widehat{\rho}_{k-2}}}; \widehat{\rho}_{k-2} < \infty\right)\right]\mathbb{E}e^{-\alpha T^D + \beta D}. \end{aligned} \tag{15}$$

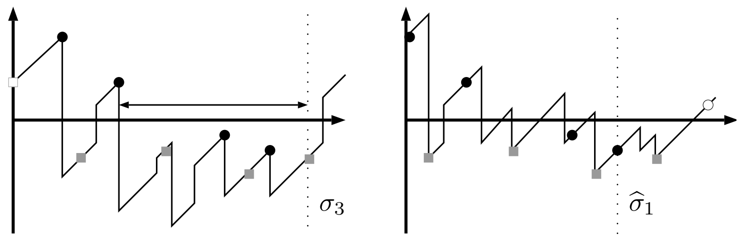

FIGURE 4. Poisson observations are in grey (left picture) and in black (right picture)

and

$$\mathbb{E}_u\left(e^{-\alpha\widehat{\rho}_{k-2}+\beta X_{\widehat{\rho}_{k-2}}};\widehat{\rho}_{k-2}<\infty\right)\mathbb{E}e^{-\alpha T^D+\beta D} = \mathbb{E}\left[e^{-\alpha T^D}\mathbb{E}_{u+D}\left(e^{-\alpha\rho_k+\beta X_{\rho_k}};\rho_k<\infty\right)\right]. \tag{16}$$

The above result connects Parisian ruin in the continuous observation model (with Erlang distributed delay), and Parisian ruin in the discrete Poisson observation model. Here ρ_2 is related to $\widehat{\rho}_0$ (which can be interpreted as ρ_1 using the memoryless property). Such an interpretation, however, can not to be extended to larger indices, so there is no recursive formula for an Erlang(k,λ) implementation delay. Note that the case of $k=1$ corresponds to Proposition 2.

4. Spectrally one-sided Lévy process

If X is a one-sided Lévy process, some of the identities lead to more explicit forms, and this will allow to retrieve a number of results previously obtained in the literature, now with alternative proofs, revealing some more structure of the formulas. Without loss of generality assume that X is a spectrally-negative Lévy process, i.e. it may only have negative jumps and it is not a non-increasing process. Consider its Laplace exponent

$$\psi(\theta)=\log\mathbb{E}e^{\theta X_1},\quad \theta\geq 0,$$

and put $\psi_\alpha(\theta)=\psi(\theta)-\alpha$ for $\alpha\geq 0$.

4.1. Preliminaries.

Let us first recall some basic functions which play a fundamental role in exit theory, see e.g. [10, Ch. 8]. Let Φ_α be the largest (non-negative) zero of ψ_α, and let $W_\alpha(u), u\geq 0$ be the so-called scale function: a continuous non-negative function determined by its Laplace transform $\int_0^\infty e^{-\theta u}W_\alpha(u)\mathrm{d}u = 1/\psi_\alpha(\theta), \theta>\Phi_\alpha$. In addition, we need a second scale function

$$Z_\alpha(u,\theta)=e^{\theta u}\left(1-\psi_\alpha(\theta)\int_0^u e^{-\theta y}W_\alpha(y)\mathrm{d}y\right),\quad u\geq 0,$$

which can be rewritten as

$$Z_\alpha(u,\theta)=\psi_\alpha(\theta)\int_0^\infty e^{-\theta y}W_\alpha(u+y)\mathrm{d}y \tag{17}$$

for $\theta > \Phi_\alpha$, see also [3]. The two basic one-sided exit identities under continuous observation are

$$\mathbb{E}_u e^{-\alpha\tau_a^+} = e^{-\Phi_\alpha(a-u)}, \quad u \le a \tag{18}$$

$$\mathbb{E}_u \left(e^{-\alpha\tau_0^- + \beta X(\tau_0^-)}, \tau_0^- < \infty\right) = Z_\alpha(u,\beta) - W_\alpha(u)\frac{\psi_\alpha(\beta)}{\beta - \Phi_\alpha}, \quad u \ge 0, \tag{19}$$

and the Wiener-Hopf factors are given by

$$\mathbb{E}e^{-\alpha T^U - \beta U} = \frac{\Phi_\lambda}{\Phi_{\lambda+\alpha} + \beta}, \qquad \mathbb{E}e^{-\alpha T^D + \beta D} = \frac{\lambda}{\Phi_\lambda}\frac{\Phi_{\lambda+\alpha} - \beta}{\lambda - \psi_\alpha(\beta)},$$

see e.g. [10, Ch. 8]. In order to apply Formula (6) of Proposition 2 to (19), we first need the following identities:

Lemma 1. *For $u \ge 0$ it holds that*

$$\mathbb{E}\left(e^{-\alpha T^U} W_\alpha(u+U)\right) = \frac{\Phi_\lambda}{\lambda} Z_\alpha(u, \Phi_{\lambda+\alpha}),$$

$$\mathbb{E}\left(e^{-\alpha T^U} Z_\alpha(u+U,\beta)\right) = \frac{\Phi_\lambda}{\Phi_{\lambda+\alpha} - \beta}\left(Z_\alpha(u,\beta) - \frac{\psi_\alpha(\beta)}{\lambda} Z_\alpha(u, \Phi_{\lambda+\alpha})\right).$$

Proof. Firstly,

$$\mathbb{E}(e^{-\alpha T^U}; U \in \mathrm{d}y) = \Phi_\lambda e^{-\Phi_{\lambda+\alpha} y}\mathrm{d}y, \tag{20}$$

which can be checked by taking transforms and comparing to the Wiener-Hopf factor. Hence

$$\mathbb{E}\left(e^{-\alpha T^U} W_\alpha(u+U)\right) = \int_0^\infty W_\alpha(u+y)\Phi_\lambda e^{-\Phi_{\lambda+\alpha} y}\mathrm{d}y = \frac{\Phi_\lambda}{\psi_\alpha(\Phi_{\lambda+\alpha})} Z_\alpha(u, \Phi_{\lambda+\alpha})$$

according to (17). For the first identity it is left to note that $\psi_\alpha(\Phi_{\lambda+\alpha}) = \lambda$.

Using (17) several times we can write for $\mu > \beta > \Phi_\alpha$:

$$\int_0^\infty Z_\alpha(u+x,\beta)e^{-\mu x}\mathrm{d}x = \psi_\alpha(\beta)\int_0^\infty e^{-(\mu-\beta)x}\int_0^\infty e^{-\beta(y+x)}W_\alpha(u+y+x)\mathrm{d}y\mathrm{d}x$$
$$= \int_0^\infty e^{-(\mu-\beta)x}\psi_\alpha(\beta)\int_x^\infty e^{-\beta y}W_\alpha(u+y)\mathrm{d}y\mathrm{d}x$$
$$= \int_0^\infty e^{-(\mu-\beta)x}\left(Z_\alpha(u,\beta) - \psi_\alpha(\beta)\int_0^x e^{-\beta y}W_\alpha(u+y)\mathrm{d}y\right)\mathrm{d}x$$
$$= Z_\alpha(u,\beta)\frac{1}{\mu-\beta} - \psi_\alpha(\beta)\frac{1}{\mu-\beta}\int_0^\infty e^{-\mu y}W_\alpha(u+y)\mathrm{d}y$$
$$= \frac{1}{\mu-\beta}\left(Z_\alpha(u,\beta) - \frac{\psi_\alpha(\beta)}{\psi_\alpha(\mu)}Z_\alpha(u,\mu)\right).$$

Plugging in $\mu = \Phi_{\lambda+\alpha}$ and multiplying by Φ_λ we obtain the second identity. By analytic extension $\beta \ge 0$ can be chosen arbitrarily. □

4.2. **One-sided exit.** Assume that $\mathbb{E}X_1 = \psi'(0) > 0$ and consider the survival probabilities (1). It is well known that $\phi(u) = \psi'(0)W_0(u)$, see also (19). According to Proposition 1 and Lemma 1 we have

$$\widehat{\phi}(u) = \mathbb{E}\phi(u+U) = \psi'(0)\mathbb{E}W_0(u+U) = \psi'(0)\frac{\Phi_\lambda}{\lambda}Z_0(u,\Phi_\lambda),$$

which is Corollary 1 of [11].

Remark 6. Note that due to (20), for the spectrally negative Lévy process the identity (2) simplifies to the pleasant form

$$\widehat{\phi}(u) = \mathbb{E}\phi(u + e_{\Phi_\lambda}),$$

where e_{Φ_λ} is an exponential random variable with parameter Φ_λ. This for instance immediately explains why for a compound Poisson process X with exponential jump sizes the discrete Poisson observation changes the classical ruin probability formula just by a multiplicative factor $\Phi_\lambda/(\Phi_\lambda + R_0)$, where R_0 is the Lundberg adjustment coefficient (cf. [2, Eq.2.18]).

Using the standard identity (19), Proposition 2 and Lemma 1 we obtain

(21)

$$\begin{aligned}
&\mathbb{E}_u\left(e^{-\alpha\widehat{\tau}_0^- + \beta X(\widehat{\tau}_0^-)}, \widehat{\tau}_0^- < \infty\right) \\
&= \mathbb{E}\left(e^{-\alpha T^U} Z_\alpha(u+U,\beta) - e^{-\alpha T^U} W_\alpha(u+U)\frac{\psi_\alpha(\beta)}{\beta - \Phi_\alpha}\right)\mathbb{E}e^{-\alpha T^D + \beta D} \\
&= \left(\frac{\Phi_\lambda}{\Phi_{\lambda+\alpha} - \beta}\left(Z_\alpha(u,\beta) - \frac{\psi_\alpha(\beta)}{\lambda}Z_\alpha(u,\Phi_{\lambda+\alpha})\right) - \frac{\Phi_\lambda}{\lambda}Z_\alpha(u,\Phi_{\lambda+\alpha})\frac{\psi_\alpha(\beta)}{\beta - \Phi_\alpha}\right) \\
&\times \frac{\lambda}{\Phi_\lambda}\frac{\Phi_{\lambda+\alpha} - \beta}{\lambda - \psi_\alpha(\beta)} = \frac{\lambda}{\lambda - \psi_\alpha(\beta)}\left(Z_\alpha(u,\beta) - Z_\alpha(u,\Phi_{\lambda+\alpha})\frac{\psi_\alpha(\beta)}{\lambda}\frac{\Phi_{\lambda+\alpha} - \Phi_\alpha}{\beta - \Phi_\alpha}\right).
\end{aligned}$$

Also, by considering Proposition 2 for the negative of X, see also Corollary 2, we arrive at

$$\begin{aligned}
&\mathbb{E}_u\left(e^{-\alpha\widehat{\tau}_a^+ - \beta(X_{\widehat{\tau}_a^+} - a)}; \widehat{\tau}_a^+ < \infty\right) \\
&= \mathbb{E}\left[e^{-\alpha T^D}\mathbb{E}_{u+D}\left(e^{-\alpha\tau_a^+ - \beta(X_{\tau_a^+} - a)}; \tau_a^+ < \infty\right)\right]\mathbb{E}e^{-\alpha T^U - \beta U} \\
&= \mathbb{E}e^{-\alpha T^D - \Phi_\alpha(a-u-D)}\mathbb{E}e^{-\alpha T^U - \beta U} = e^{-\Phi_\alpha(a-u)}\frac{\lambda}{\Phi_\lambda}\frac{\Phi_{\lambda+\alpha} - \Phi_\alpha}{\lambda - \psi_\alpha(\Phi_\alpha)}\frac{\Phi_\lambda}{\Phi_{\lambda+\alpha} + \beta} \\
&= e^{-\Phi_\alpha(a-u)}\frac{\Phi_{\lambda+\alpha} - \Phi_\alpha}{\Phi_{\lambda+\alpha} + \beta},
\end{aligned}$$

because of (18) and the fact that $X_{\tau_a^+} = a$ on τ_a^+. These two identities were obtained in [3, Thm. 3.1] by virtue of a rather technical argument using the expression for the potential density of X.

4.3. **Parisian ruin.** Finally, we relate our results to previous literature on Parisian ruin. Firstly, taking $u = 0$ and $\beta = 0$ in (21) and noting that $Z_\alpha(0,\beta) = 1$ one retrieves Corollary 3.2 of [12], which is based on exponential implementation delay. Furthermore, from the form of [12, Eq.49] one can, after some lengthy calculations, obtain the following expression for Erlang$(2,\lambda)$ implementation delay:

$$\mathbb{E}_0\left(e^{-\alpha\rho_2}; \rho_2 < \infty\right) = \frac{\lambda}{\lambda+\alpha} - \frac{\alpha}{(\lambda+\alpha)^2}\frac{\Phi_{\lambda+\alpha} - \Phi_\alpha}{\Phi_\alpha}\frac{\Phi_{\lambda+\alpha}}{\Phi'_{\lambda+\alpha}}, \tag{22}$$

where the derivative is with respect to the subindex. We can alternatively obtain (22) directly using the results of this paper: (15) and (21) imply

$$\begin{aligned}
&\mathbb{E}_0\left(e^{-\alpha\rho_2}; \rho_2 < \infty\right) = \mathbb{E}\left[e^{-\alpha T^U}\mathbb{E}_U\left(e^{-\alpha\widehat{\tau}_0^-}; \widehat{\tau}_0^- < \infty\right)\right]\mathbb{E}e^{-\alpha T^D} \\
&= \frac{\lambda}{\lambda+\alpha}\mathbb{E}\left[e^{-\alpha T^U}\left(Z_\alpha(U,0) - Z_\alpha(U,\Phi_{\lambda+\alpha})\frac{\alpha}{\lambda}\frac{\Phi_{\lambda+\alpha} - \Phi_\alpha}{\Phi_\alpha}\right)\right]\mathbb{E}e^{-\alpha T^D}.
\end{aligned}$$

Using Lemma 1 and noting again that $Z_\alpha(0,\beta)=1$ we get

$$\mathbb{E}\left(e^{-\alpha T^U} Z_\alpha(U,0)\right) = \frac{\Phi_\lambda}{\Phi_{\lambda+\alpha}}\frac{\alpha+\lambda}{\lambda},$$

$$\mathbb{E}\left(e^{-\alpha T^U} Z_\alpha(U,\Phi_{\lambda+\alpha})\right) = \lim_{h\downarrow 0}\frac{\Phi_\lambda}{\Phi_{\lambda+\alpha}-\Phi_{\lambda+\alpha-h}}\frac{h}{\lambda} = \frac{\Phi_\lambda}{\lambda\Phi'_{\lambda+\alpha}},$$

which together with the expression for the Wiener-Hopf factor $\mathbb{E}e^{-\alpha T^D}$ readily yields (22).

Acknowledgements

The authors would like to thank Ton Dieker for stimulating discussions on the topic.

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