Insurance: Mathematics and Economics 101 (2021) 55–69

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Optimal retirement products under subjective mortality beliefs[☆]

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ARTICLE INFO

Article history:
Available online 8 July 2020

JEL classification:
G22
D81

Keywords:
Behavioral insurance
Subjective mortality beliefs
Optimal retirement product design
Tontine
Annuity

ABSTRACT

Many empirical studies confirm that policyholder's subjective mortality beliefs deviate from the information given by publicly available mortality tables. In this study, we look at the effect of subjective mortality beliefs on the perceived attractiveness of retirement products, focusing on two extreme products, conventional annuities (where the insurance company takes the longevity risk) and tontines (where a pool of policyholders shares the longevity risk). If risk loadings and charges are neglected, a standard expected utility framework, without subjective mortality beliefs, leads to the conclusion that annuities are always preferred to tontines (Yaari (1965), Milevsky and Salisbury (2015)). In the same setting, we show that this result is easily reversed if an individual perceives her peer's life expectancies to be lower than the ones used by the insurance company. We prove that, assuming such subjective beliefs, there exists a critical tontine pool size from which the tontine is always preferred over the annuity. This suggests that tontines might be perceived as much more attractive than suggested by standard expected utility theory without subjective mortality beliefs.

1. Introduction

It is well-documented in the literature that individuals tend to have subjective beliefs about their own and others' life expectancy. In several empirical studies, people were asked to provide an estimate of their life expectancy or survival probability towards a certain age. These numbers were compared to some reference data as, for example, the estimates of the government's actuary department. Typically, in such studies, people at younger ages tend to be pessimistic about their future lifetime, that is the life expectancy they report is lower than the government forecast. In contrast, at older ages, various studies document both under- and over-estimations of the life expectancy and survival probabilities. Secondly, people seem to have different subjective beliefs about the life expectancy or survival probability relative to their peers. This has already been noted in the famous book by Adam Smith back in 1776 (cf. Smith (1776)) where he stresses the "confidence which every man naturally has in his own good fortune", see also the extensive literature review provided in Section 2 of this article.

How subjective age perception influences one's economic and financial decision is an interesting and ongoing research topic. For instance, Ye and Post (2020) study how the age people feel influences their work engagement and saving profiles. In this article, we analyze the research question of how and to what extent subjective beliefs affect the optimal design and perceived attractiveness of retirement products. This is particularly interesting in view of the many novel and innovative retirement products that recently emerged from the life and pension insurance industry. We exemplarily look at two such products that can be seen as extreme in the sense that one of them (the *annuity*) leaves mortality risk with the insurance company, while the other one (the *tontine*) shares mortality risk within a pool of policyholders. The tontine used to be a popular source of retirement income from the 17th to the 19th century (see for example Milevsky (2014, 2015), Milevsky and Salisbury (2015), Milevsky and Salisbury (2016), Weinert and Gründl (2017), Chen et al. (2019) and Li and Rothschild (2019)). For a more practically oriented view on tontines, see e.g. Sabin (2010), Forman and Sabin (2015, 2016) and Fullmer and Sabin (2018).[1] Priced actuarially fair, life

☆ We want to thank the participants of the European Actuarial Journal Conference in Leuven 2018, the German Insurance Science Conference in Berlin 2019, the 23rd International Congress on Insurance: Mathematics and Economics in Munich 2019 and OICA 2020 in Lyon for fruitful discussions and comments that helped us to improve earlier versions of this manuscript. Manuel Rach acknowledges additionally the financial support given by the DFG for the research project "Zielrente: die Lösung zur alternden Gesellschaft in Deutschland".

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[1] In recent years, many products with a tontine-like structure have appeared. They are often called pooled annuity funds or group self-annuitization, and much effort has been made in recent years to explore the potential and optimal design of these products in today's world. We refer interested readers, for example, to Piggott et al. (2005), Valdez et al. (2006), Stamos (2008), Qiao and Sherris (2013), Donnelly et al. (2013, 2014) and Donnelly (2015) and Bernhardt and Donnelly (2019).

https://doi.org/10.1016/j.insmatheco.2020.07.002
0167-6687/

annuities give retirees greater lifetime utility than tontines (see also Milevsky and Salisbury (2015)). If realistic safety loadings or risk margins are taken into consideration, tontines can be preferred to annuities (cf. Milevsky and Salisbury (2015) and Chen et al. (2019)). In the present article, we analyze how subjective mortality beliefs affect the optimal design of these products. Further, we look at the perceived relative attractiveness between annuities and tontines. In particular, we aim to find out whether, under these subjective beliefs, tontines may generate a higher lifetime utility level than annuities.

We use a general, hazard-rate-type mortality model distinguishing between uncertainty in the mortality rate (systematic mortality risk) modeled, for example, by a stochastic intensity or Lee–Carter-type model, and diversifiable risks related to the pool size (unsystematic mortality risk). To incorporate subjective mortality beliefs, we fix the insurer's mortality model as reference and allow individuals to under- and overestimate the insurer's survival curve. We further allow a single individual to believe that she lives relatively longer or shorter than her peers. To be precise, we assume that there are three different mortality rates for any x-year old policyholder: For any time $t \geq 0$, we consider the (possibly stochastic) mortality rate μ_{x+t} used by the insurer and two subjective mortality rates ($\widetilde{\mu}_{x+t}$, $\widehat{\mu}_{x+t}$), representing the policyholder's subjective mortality beliefs. Here, we distinguish between the mortality rates the individual assumes for herself (denoted by $\widetilde{\mu}_{x+t}$) and the ones assumed by the individual for other policyholders (denoted by $\widehat{\mu}_{x+t}$). In the present article, we analyze the impact of subjective mortality beliefs in an expected utility framework. The policyholder we consider is not fully rational, but is a "naive" one who maximizes expected utility under biased beliefs (see Hey and Lotito (2009) for different types for utility maximizers).

Following Milevsky and Salisbury (2015) and Chen et al. (2019), we derive the optimal payout functions of an annuity and a tontine under subjective mortality beliefs and look at the effect of subjective mortality beliefs. First, a tontine's perceived attractiveness decreases in the conditional survival curve ${}_t\widehat{P}_x := e^{-\int_0^t \widehat{\mu}_{x+s} \mathrm{d}s}$ assumed for the other policyholders in the pool. As, in an annuity, mortality risks are not shared within a pool but taken by the insurance company, this subjective belief does not affect the payoff or perceived attractiveness of an annuity. If the subjective beliefs lead to an underestimation of ${}_t\widehat{P}_x$ relative to the insurer's conditional survival curve (that is if ${}_t\widehat{P}_x < {}_tP_x = e^{-\int_0^t \mu_{x+s} \mathrm{d}s}$), we find that (under a certain condition) there is a critical tontine pool size N_0 such that a tontine with initial pool size $n > N_0$ is preferred over an annuity, even if safety loadings (that are usually higher for annuities than for tontines) are ignored. Our numerical analysis confirms that this critical pool size can be as low as $N_0 = 2$. This is remarkable, since the attractiveness of tontines is strongly increasing in its pool size. This result also reverses results in similar settings without subjective mortality beliefs (Yaari (1965), Milevsky and Salisbury (2015)). Surprisingly, in this analysis, we also find that whether the policyholder believes that she lives longer or shorter than her peers does not seem to have a substantial impact on the choice between a tontine and an annuity, particularly if the pool size of the tontine is large. However, if both products are considered separately, the difference between one's own and the insurer's mortality beliefs determines the attractiveness of annuities while the difference between one's own and the peer's mortality beliefs determines the attractiveness of tontines.

It is already well-acknowledged in the literature that *annuities* seem overpriced for an individual who is pessimistic about her life expectancy (see, for example, Wu et al. (2015)). Our model is consistent with this observation as a pessimistic individual who underestimates her own survival curve relative to the one used by the insurer perceives the product as too expensive. An analysis in an expected utility framework supports this effect and shows that a lower expected utility level results if the policyholder perceives the premiums charged for annuities as "too high". Conversely, the policyholder's utility increases if she perceives the premium charged as "too low". We confirm a similar, quite obvious, effect also for *tontines*: The product is perceived as too expensive/cheap if one believes that one lives shorter/longer than the other members in the pool.

The results of this paper have an interesting implication with respect to the annuity puzzle which is a term used to describe the discrepancy between the theoretical demand for annuities (see, for instance, Yaari (1965) and Peijnenburg et al. (2016)) and the fact that only few households voluntarily purchase an annuity (see, for example, Hu and Scott (2007), Inkmann et al. (2010) and Lockwood (2012)).[2] Our article differs from the majority of the literature on the annuity puzzle by the inclusion of tontines and the effects of subjective mortality beliefs on the relative attractiveness of annuities and tontines. This allows for interesting conclusions: For example, if a policyholder perceives her own life expectancy to be higher than the one of her peers but, at the same time, is pessimistic about general life expectancy, she might perceive an annuity as *overpriced* but, at the same time, a tontine as *underpriced*. As people seem to have different mortality beliefs, offering additional innovative retirement products like tontines might encourage more people to invest into retirement products.

The remainder of this article is organized as follows: In Section 2, we provide a literature review regarding subjective mortality beliefs. In Section 3, we describe the general model setup used throughout this article. After that, we derive the optimal payout function of the annuity and tontine for a risk-averse policyholder under subjective mortality beliefs in Section 4. We also derive closed-form expressions for the individual's expected discounted lifetime utility from each product, which will enable us to compare the attractiveness of the different products. In Section 5, we analyze the effects of subjective mortality beliefs on the optimal retirement decision. Section 6 examines the robustness of the results achieved in Section 5 by explicitly considering safety loadings. Section 7 concludes the article. Most proofs are collected in the Appendix.

2. Subjective mortality beliefs

The phenomenon that people tend to have their own, subjective beliefs regarding their own and others' life expectancy is not new in the literature. Such a phenomenon is of major relevance in life insurance. For example, Bauer et al. (2014) analyze the effects of differing perceptions of mortality on the life settlement market. Individuals might systematically over- or underestimate their own and others' life expectancy, affecting their willingness to buy retirement products like annuities and tontines. Important empirical findings regarding subjective mortality beliefs include, but are not limited to, the following:

- Bucher-Koenen et al. (2013) find that, in Germany, "men as well as women are pessimistic about their life expectancy. Women (men) underestimate their life span by about 7 (6.5) years compared to the official records by the German statistical office". The sample consists of an equal share of males and females aged 26–60.

2 There is already vast literature exploring the main drivers for this puzzle. For reviews of this stream of literature, we refer the interested reader, for example, to Milevsky (2013) and Benartzi et al. (2011). Further studies related to our article in the context of behavioral insurance are, for example, Salisbury and Nenkov (2016), Chen et al. (2016, 2018), Poppe-Yanez (2017), Caliendo et al. (2017) and O'Dea et al. (2019). Note that there is more than one puzzle in life insurance, see for example Gottlieb (2012). However, the puzzle dealing with underannuitization is probably the most famous one.

- According to O'Brien et al. (2005), individuals in Great Britain underestimate their life expectancies "by 4.62 years (males), 5.95 years (females) compared to the estimates of the Government Actuary's Department". Additionally, "people are optimistic: they think they will live longer, on average, than people of their own age and sex: by 1.19 years (males), 0.76 years (females)". The sample covers ages from 16 to 99. While the underestimation is larger for young than old people, there is still an underestimation of 2.83 years for males and 4.62 years for females at ages 60–69 which is the range of typical retirement ages.
- In Greenwald and Associates (2012) the following results about American citizens are established: "When asked to estimate how long the average person their age and sex can expect to live, more than six in ten retirees (62 percent) and half of pre-retirees (57 percent) provide a response that is below the average. Only about one-quarter overestimate average life expectancy (19 percent of retirees and 28 percent of pre-retirees)".[3] Additionally, a similar observation as in O'Brien et al. (2005) is made: "Despite the tendency to underestimate population life expectancy, half of retirees (50 percent) and pre-retirees (53 percent) appear to believe that the response they provide for their personal life expectancy is within one year of average life expectancy. Three in ten think their estimate of personal life expectancy exceeds average life expectancy (31 percent of retirees and 32 percent of pre-retirees)".
- Wu et al. (2015) find that "respondents are pessimistic about overall life expectancy but optimistic about survival at advanced ages, and older respondents are more optimistic than younger". To be precise, "younger cohorts underestimate survival (the 50–54 age group underestimates life expectancy by more than eight years) while older cohorts tend to overestimate, especially males (Ludwig and Zimper (2013)). (Males in the 70–74 age group overestimate life expectancy by only one year, and females underestimate it by one year.)" These observations are based on the Retirement Plans and Retirement Incomes: Pilot Survey, conducted in May 2011 for Australian citizens.
- Elder (2013) analyzes the Health and Retirement study (HRS) which is a longitudinal survey of American citizens. The most important finding for our article is that both men and women with ages between 50 and 65 underestimate the probability of survival to age 75, but overestimate the probability of survival to age 85. A similar observation is made in Hurd and McGarry (2002) who analyze the HRS as well.

There seems to be a clear tendency for younger people to underestimate their life expectancy, while both under- and overestimations can be observed at older ages. Additional literature on this subject can, for example, be found in Wu et al. (2015). Furthermore, Payne et al. (2013) emphasize that individuals' responses to questions assessing their subjective mortality and longevity beliefs drastically depend on the framing of the question. Therefore, we consider a general model for subjective mortality beliefs which allows for both under- and overestimations of the life expectancy.

[3] "Respondents were classified as retirees if they described their employment status as retiree, had retired from a previous career, or were not currently employed and were either age 65 or older or had a retired spouse. All other respondents were classified as pre-retirees".

3. Model setup

In this section, we describe the basic model setup used throughout the remainder of our article. In particular, we explain how the mortality and the subjective mortality beliefs are modeled and how the two retirement products under consideration, the annuity and the tontine, are designed. We ignore financial market risk to solely focus on the mortality risk.

3.1. Modeling mortality risk

There are two different kinds of mortality risk: Unsystematic, or idiosyncratic, mortality risk stems from the lifetimes of people being unknown but still following a certain mortality law. Systematic, or aggregate mortality risk stems from the fact that we cannot certainly determine the actual ("true") mortality law. In the context of retirement products, this risk is also called longevity risk. Further explanations regarding these two different aspects of mortality risk can also be found, for instance, in Piggott et al. (2005). Let us consider an x-year-old policyholder whose remaining future lifetime is denoted ζ. This remaining lifetime is affected by systematic mortality risk that is modeled by a (stochastic) mortality rate. To determine the contract's premium (see Section 3.2), the insurer uses the mortality rate $\{\mu_{x+t}\}_{t\geq 0}$.

As pointed out above, individuals tend to have their own, subjective estimates of others' and their own life expectancy. These subjective mortality beliefs will be incorporated by assuming that the individual considered and the insurance company use different mortality rates. While we denote by μ_{x+t} the mortality rate used by the insurer, we also take the viewpoint of a policyholder with subjective mortality beliefs with perceived mortality rates $\{\widetilde{\mu}_{x+t}\}_{t\geq 0}$ and $\{\widehat{\mu}_{x+t}\}_{t\geq 0}$ the policyholder assumes for herself and her peers, respectively. As mentioned in the introduction, we do not assume any "fixed" relation between these mortality rates, as both an under- or overestimation of the actual life expectancy can be observed among retirees. From the policyholder's perspective, these subjective mortality rates are applied to evaluate the attractiveness of the retirement products. As already pointed out in the introduction, the considered policyholder is not fully rational. It indicates that the considered policyholder can perceive the mortality rates for herself and her peers rather differently from the ones used by the insurer for pricing.

In Examples 3.1–3.3, we present a shocked Gompertz model, a Lee–Carter model, and an Ornstein–Uhlenbeck-type model as simple examples modeling a stochastic mortality rate $\{\mu_{x+t}\}_{t\geq 0}$. It is, of course, possible to use more sophisticated extensions of these models. Given the canonical filtration $\mathcal{E}_t := \sigma(\{\mu_{x+s}\}_{\{0\leq s\leq t\}})$, that is the filtration containing information about the mortality rates μ_{x+t}, we define the survival curve used by the insurer:

$$ {}_tp_x = \mathbb{E}\Big[\mathbb{1}_{\{\zeta>t\}}\Big] = \mathbb{E}\Big[\mathbb{E}\Big[\mathbb{1}_{\{\zeta>t\}}\,\Big|\,\mathcal{E}_t\Big]\Big] = \mathbb{E}\Big[e^{-\int_0^t \mu_{x+s}\mathrm{d}s}\Big]. \tag{1} $$

We define the conditional subjective probabilities ${}_t\widetilde{P}_x := e^{-\int_0^t \widetilde{\mu}_{x+s}\mathrm{d}s}$ and ${}_t\widehat{P}_x := e^{-\int_0^t \widehat{\mu}_{x+s}\mathrm{d}s}$ and the corresponding canonical filtration $\widetilde{\mathcal{E}}_t := \sigma(\{\widetilde{\mu}_{x+s}\}_{0\leq s\leq t}, \{\widehat{\mu}_{x+s}\}_{0\leq s\leq t})$. The policyholder with subjective beliefs obtains the following (perceived) survival probabilities:

$$ {}_t\widetilde{p}_x = \mathbb{E}\Big[e^{-\int_0^t \widetilde{\mu}_{x+s}\mathrm{d}s}\Big]\ , \text{ and } {}_t\widehat{p}_x = \mathbb{E}\Big[e^{-\int_0^t \widehat{\mu}_{x+s}\mathrm{d}s}\Big]. \tag{2} $$

To account for unsystematic mortality risk, we consider a policyholder pool of initially n members from the same cohort of age x. Given the conditional survival probabilities ${}_tP_x := e^{-\int_0^t \mu_{x+s}\mathrm{d}s}$ (respectively ${}_t\widetilde{P}_x$, ${}_t\widehat{P}_x$), the pool member's remaining lifetimes are assumed to be independent. Thus, from the insurer's perspective and from today's view, the number of pool members surviving time t follows a binomial distribution, that is,

$$N(t) \mid \{{}_tP_x\} \sim \mathrm{Bin}\big(n, {}_tP_x\big)\,. \tag{3}$$

A policyholder with subjective mortality beliefs assumes that the other pool members are ${}_t\widehat{P}_x$-conditionally binomially distributed, that is $N(t)-1 \mid \{{}_t\widehat{P}_x\} \sim \mathrm{Bin}\big(n-1, {}_t\widehat{P}_x\big)$.

Example 3.1 (*Shocked Gompertz Model*)**.** For details on the (deterministic) Gompertz model, see Gompertz (1825), Gumbel (1958) and Milevsky and Salisbury (2015). We apply a random shock ϵ to the Gompertz mortality rates assuming that ϵ is a continuous random variable with density $f_\epsilon(\cdot)$, support on $(-\infty, 1)$ and moment generating function $m_\epsilon(s) := \mathbb{E}[e^{s\epsilon}]$. Mortality rates are then given by a shocked Gompertz model

$$\mu_{x+t} = (1-\epsilon)\frac{1}{b}e^{\frac{x+t-m}{b}}\,, \tag{4}$$

where b is the dispersion coefficient and m is the modal age at death. Such a shocked mortality model is inspired by Solvency II regulation in Europe where stress scenarios are obtained by a deterministic shock of annual death rates (see, for example, Lin and Cox (2005) and Chen et al. (2019) for a more detailed motivation). We get

$$\begin{aligned} {}_tp_x &= \mathbb{E}\Big[e^{-\int_0^t \mu_{x+s}\mathrm{d}s}\Big] = \mathbb{E}\Big[\mathbb{E}\Big[e^{-\int_0^t \mu_{x+s}\mathrm{d}s} \,\Big|\, \epsilon\Big]\Big] \\ &= \mathbb{E}\Big[e^{-(1-\epsilon)e^{\frac{x-m}{b}}(e^{\frac{t}{b}}-1)}\Big] = e^{-e^{\frac{x-m}{b}}(e^{\frac{t}{b}}-1)} \cdot m_\epsilon\Big(e^{\frac{x-m}{b}}(e^{\frac{t}{b}}-1)\Big)\,. \end{aligned} \tag{5}$$

For $\epsilon \equiv 0$, we obtain the (deterministic) Gompertz model as special case.

Example 3.2 (*Lee–Carter Model*)**.** On an annual grid, for $t = 0, 1, \ldots$, the Lee–Carter model (see Lee and Carter (1992)) is given by the mortality rate

$$\mu_{x+t} = \exp(\alpha_{x+t} + \beta_{x+t}\cdot\kappa_t)\,, \tag{6}$$

where α_{x+t} is the average mortality level for age x, κ_t is the mortality improvement factor, and β_{x+t} is the relevance of mortality improvement for age x. Typically, the time-dependent $\kappa_t's$ are projected as $\kappa_{t+1} = \kappa_t + \theta + \xi_t$, where $\theta \in \mathbb{R}$ and ξ_t are identically distributed, independent normally distributed random variables with zero mean, that is $\xi_t \sim \mathcal{N}(0, \sigma_\kappa^2)$. For general $t \geq 0$, we simply set $\mu_{x+t} = \mu_{x+\lfloor t\rfloor}$, where $\lfloor x\rfloor := \max\{k \in \mathbb{Z} \mid k \leq x\}$. For $t = 0, 1, \ldots$, we obtain:

$$\begin{aligned} {}_tp_x &:= \mathbb{E}\Bigg[\exp\Big(-\sum_{s=0}^{t}\mu_{x+s}(s)\Big)\Bigg] \\ &= \mathbb{E}\Bigg[\exp\Big(-\sum_{s=0}^{t}\exp\big(\alpha_{x+s} + \beta_{x+s}\cdot\kappa_s\big)\Big)\Bigg]\,. \end{aligned} \tag{7}$$

Example 3.3 (*Ornstein–Uhlenbeck Model*)**.** The stochastic mortality rate is described by an Ornstein–Uhlenbeck (OU) process with a positive drift and no mean reversion:

$$\mathrm{d}\mu_{x+t} = \theta\mu_{x+t}\mathrm{d}t + \sigma\mathrm{d}W_t,$$

where $\theta > 0$, $\sigma > 0$, and W is a standard Brownian motion under the real world measure $\mathbb{P}$. The OU process for the mortality rate is a natural stochastic generalization of the Gompertz law for the force of mortality and was introduced by Luciano and Vigna (2008), where its properties and the conditions for its biological reasonableness have been discussed. Standard properties of affine processes allow us to write the survival probability of a policyholder in closed-form (see Luciano and Vigna (2008)):

$${}_tp_x = \mathbb{E}\Big[e^{-\int_0^t \mu_{x+s}\,\mathrm{d}s}\Big] = \exp\big(\alpha(t) + \beta(t)\mu_x\big)\,,$$

where

$$\alpha(t) = \frac{\sigma^2}{2\theta^2}t - \frac{\sigma^2}{\theta^3}e^{\theta t} + \frac{\sigma^2}{4\theta^3}e^{2\theta t} + \frac{3\sigma^2}{4\theta^3}\,, \quad \beta(t) = \frac{1}{\theta}(1-e^{\theta t})\,.$$

In the shocked Gompertz model (Example 3.1) perceived probabilities can, for example, be obtained changing the modal age at death m (see Section 3.3). In the Lee–Carter model (Example 3.2), it makes sense to adapt the average mortality level α_x and for the OU model (Example 3.3), the drift θ of the mortality rate.

3.2. Retirement products

As motivated in the introduction, we consider the annuity and the tontine as two representative retirement products. Following Yaari (1965), we assume a continuous-time stream of income for the retirement products. In an **annuity** contract, any policyholder continuously receives an annuity payment $c(t)$ until death. The payment stream of the annuity can be written as

$$b_A(t) := \mathbb{1}_{\{\zeta > t\}}\, c(t)\,. \tag{8}$$

The premium charged by the insurer (using the survival curve ${}_tp_x$) can be obtained as

$$\begin{aligned} P_0^A &= \mathbb{E}\left[\int_0^\infty e^{-rt}b_A(t)\mathrm{d}t\right] = \int_0^\infty e^{-rt}\mathbb{E}\left[\mathbb{1}_{\{\zeta>t\}}\right]c(t)\,\mathrm{d}t \\ &= \int_0^\infty e^{-rt}\mathbb{E}\left[{}_tP_x\right]c(t)\,\mathrm{d}t = \int_0^\infty e^{-rt}{}_tp_x\,c(t)\,\mathrm{d}t\,, \end{aligned} \tag{9}$$

where r is the risk-free interest rate, often also called the force of interest.[4]

While in an annuity, the longevity risk is borne by the insurance company, in a **tontine** contract it is shared among a pool of $n \geq 1$ homogeneous policyholders.[5] Denoting by $N(t)$ the number of pool members at time t, each policyholder receives $n/N(t)$ multiplied by a payment stream $d(t)$ specified at the beginning of the contract. Following Milevsky and Salisbury (2015), this yields the following continuous payment stream for each $t > 0$:

$$b_T(t) := \begin{cases} \mathbb{1}_{\{\zeta>t\}}\frac{nd(t)}{N(t)}, & \text{if } N(t) > 0\,, \\ 0, & \text{else} \end{cases}\,. \tag{10}$$

Note that, in contrast to the annuity payment (8), the tontine payment depends substantially on the number of surviving policyholders $N(t)$. In the special case where the pool consists of only one member, that is, $n = 1$, and $c(t) = d(t)$, the tontine payoff (10) and the annuity payoff (8) coincide. The premium of

4 In finance, when we value financial products with risky payoffs under the real world measure, we shall apply a risk-adjusted discounting factor, where the risk premium for the financial product shall be taken into account. This implies that the discounting factor shall be a risk-adjusted interest rate which varies among the products in which different "amounts" of risks are involved. If we apply this conventional approach used in finance to our context to determine the price for the tontines and the annuities, correctly speaking, we shall apply two different risk-adjusted discounting rates for these two products, as tontines and annuities contain different amounts of longevity risks. In the present paper, we do not apply this conventional approach in finance, but follow an actuarial approach to determine the premium of these retirement products.

5 In a tontine, the insurer carries the longevity risk of the last living person in the pool, only. This risk is negligible expect for very small pool sizes.

this contract can then be obtained as

$$P_0^T = \mathbb{E}\left[\int_0^\infty e^{-rt} b_T(t) \mathrm{d}t\right]$$
$$= \int_0^\infty e^{-rt}\, \mathbb{E}\left[{}_tP_x\, \mathbb{E}\left[\frac{nd(t)}{N(t)} \,\middle|\, \zeta > t, \mathcal{E}_t\right]\right] \mathrm{d}t$$
$$= \int_0^\infty e^{-rt}\, \mathbb{E}\left[\sum_{k=0}^{n-1} \frac{n}{k+1}\binom{n-1}{k} ({}_tP_x)^{k+1} \cdot (1-{}_tP_x)^{n-1-k}\right] d(t)\,\mathrm{d}t$$
$$= \int_0^\infty e^{-rt}\, \mathbb{E}\left[\sum_{k=1}^{n} \binom{n}{k} ({}_tP_x)^{k} (1-{}_tP_x)^{n-k}\right] d(t)\,\mathrm{d}t$$
$$= \int_0^\infty e^{-rt}\, \mathbb{E}\left[1-(1-{}_tP_x)^n\right] d(t)\,\mathrm{d}t\,. \qquad (11)$$

As already pointed out in Chen et al. (2019), the premium for tontines (11) differs from the formula in Milevsky and Salisbury (2015), where it is assumed that the payoff to the pool $d(t)$ will always be provided by the insurer even if there are no policyholders left. The premium in (11) converges to the one in Milevsky and Salisbury (2015) if the pool size n tends to infinity.

3.3. Subjective perception of the premium

The insurer charges a premium based on its own survival probabilities (see (9) and (11)). Let us now examine how this premium is perceived from the policyholder's point of view. As the policyholder has different mortality beliefs than the insurer, she might perceive a product as over- or underpriced. We denote the expected value operator under the subjective beliefs of a policyholder by $\widetilde{\mathbb{E}}[\cdot]$. The subjective premium of the annuity, using the policyholder's subjective survival curve, is given by

$$\widetilde{P}_0^A = \widetilde{\mathbb{E}}\left[\int_0^\infty e^{-rt} b_A(t)\mathrm{d}t\right] = \int_0^\infty e^{-rt}\,{}_t\widetilde{p}_x\, c(t)\,\mathrm{d}t\,, \qquad (12)$$

where ${}_t\widetilde{p}_x$ is defined in (2). The policyholder perceives the premium charged by the insurer for the annuity as "too high" if the charged premium is higher than her perceived premium, that is, $P_0^A > \widetilde{P}_0^A$ (which is the case if ${}_tp_x > {}_t\widetilde{p}_x$). Conversely, the premium charged by the insurer is perceived as "too low" if $P_0^A < \widetilde{P}_0^A$ (which is the case if ${}_tp_x < {}_t\widetilde{p}_x$).

For the tontine, a single individual uses the mortality rate $\widetilde{\mu}_{x+t}$ for herself and $\widehat{\mu}_{x+t}$ for the other policyholders in the pool. Using the policyholder's subjective mortality rates for herself and others, the subjective premium of the tontine is given by

$$\widetilde{P}_0^T = \widetilde{\mathbb{E}}\left[\int_0^\infty e^{-rt} b_T(t)\mathrm{d}t\right]$$
$$= \int_0^\infty e^{-rt}\, \mathbb{E}\left[{}_t\widetilde{P}_x\, \widetilde{\mathbb{E}}\left[\frac{nd(t)}{N(t)} \,\middle|\, \zeta > t,\, \widetilde{\mathcal{E}}_t\right]\right] \mathrm{d}t$$
$$= \int_0^\infty e^{-rt}\, \mathbb{E}\left[{}_t\widetilde{P}_x \sum_{k=0}^{n-1} \frac{n}{k+1}\binom{n-1}{k} \left({}_t\widehat{P}_x\right)^{k} \cdot \left(1-{}_t\widehat{P}_x\right)^{n-1-k}\right] d(t)\,\mathrm{d}t$$
$$= \int_0^\infty e^{-rt}\, \mathbb{E}\left[\frac{e^{-\int_0^t \widetilde{\mu}_{x+s}\mathrm{d}s}}{e^{-\int_0^t \widehat{\mu}_{x+s}\mathrm{d}s}}\left(1-\left(1-e^{-\int_0^t \widehat{\mu}_{x+s}\mathrm{d}s}\right)^n\right)\right] d(t)\,\mathrm{d}t\,. \qquad (13)$$

Table 1
Base case parameters.

Initial wealth $v = 1$	Pool size $n = 100$	Risk-free rate $r = 0.02$
Initial age $x = 65$	Gompertz parameters $m = 88.721,\ b = 10$	Longevity shock $\epsilon \sim \mathcal{N}_{(-\infty,1]}(-0.0035, 0.0814^2)$

For a sufficiently large pool size, the term $\left(1 - e^{-\int_0^t \widehat{\mu}_{x+s}\mathrm{d}s}\right)^n$ is close to zero. The difference between the premium P_0^T and the perceived premium $\widetilde{P}_0^T$ is thus mainly driven by the term $\mathbb{E}\left[\frac{{}_t\widetilde{P}_x}{{}_tP_x}\right] = \mathbb{E}\left[\frac{e^{-\int_0^t \widetilde{\mu}_{x+s}\mathrm{d}s}}{e^{-\int_0^t \widehat{\mu}_{x+s}\mathrm{d}s}}\right]$, that is how the policyholder sees her own survival prospects relative to the other pool members.

To illustrate the patterns described above, we now consider a numerical example, working with the shocked Gompertz model (Example 3.1). Unless stated otherwise, we always use the parameters summarized in Table 1. The mortality rates μ_{x+t}, $\widetilde{\mu}_{x+t}$ and $\widehat{\mu}_{x+t}$ are assumed to follow the shocked Gompertz law as introduced in Example 3.1. To demonstrate the effects of subjective mortality beliefs, we vary the modal age at death to obtain the subjective mortality rates

$$\widetilde{\mu}_{x+t} = (1-\epsilon)\frac{1}{b}e^{\frac{x+t-\widetilde{m}}{b}}\,, \quad \widehat{\mu}_{x+t} = (1-\epsilon)\frac{1}{b}e^{\frac{x+t-\widehat{m}}{b}}$$

for subjective modal ages at death $\widetilde{m}$, $\widehat{m} > 0$. The random shock ϵ is the same shock, and we use for the insurer's mortality rates

$$\mu_{x+t} = (1-\epsilon)\frac{1}{b}e^{\frac{x+t-m}{b}}\,.$$

Recall that the resulting survival curves ${}_tp_x$, ${}_t\widetilde{p}_x$ and ${}_t\widehat{p}_x$ are given by (1) and (2), respectively. Varying the modal age at death allows us to easily control the subjective mortality beliefs: The relations $m \geq \widetilde{m}$ and $m \geq \widehat{m}$ are equivalent to the situation where the policyholder expects to have higher mortality rates than the insurer uses for pricing (that is $\widetilde{\mu}_{x+t} \geq \mu_{x+t}$ and $\widehat{\mu}_{x+t} \geq \mu_{x+t}$ and thus also ${}_t\widetilde{p}_x \leq {}_tp_x$ and ${}_t\widehat{p}_x \leq {}_tp_x$). The reason for this is that the mortality rate is decreasing in the modal age at death for all choices of x, b and t.

Note that we allow for the modal age at death to vary between the insurer's and the policyholder's perceived mortality rates, while we use, for simplicity, the same dispersion coefficient for all three curves. That is, we have tacitly assumed that the modal age at death does not depend on the dispersion coefficient.[6] For the Gompertz parameters, we follow Milevsky and Salisbury (2015), for the parameters of the shock we follow Chen et al. (2019). We assume that the insurer uses the modal age $m = 88.721$, which results in an expected remaining lifetime of $\mathbb{E}[\zeta] = \int_0^\infty {}_tp_x \mathrm{d}t \approx 20.707$ from the insurer's point of view (for an $x = 65$-year old). For our numerical illustrations, we mostly rely on subjective modal ages $\widetilde{m}$, $\widehat{m} \in \{80.5, 83, 88.721, 92, 95\}$. These parameters are in line with most of the empirical studies in Section 2. We shortly discuss these parameter choices, comparing the life expectancy of our parameter choice and some of the empirical studies cited in Section 2.

[6] Note that this is only one possible example of subjective mortality beliefs. In principle, it is possible to allow for simultaneous changes in several parameters of the Gompertz law or to change the underlying mortality law. The former can be carried out by using the so-called *Compensation Law of Mortality* (CLaM) taking into account that the lifetimes of individuals with higher mortality hazard rates are also more volatile (for further details see, e.g., Gavrilov and Gavrilova (1991, 2001) and Milevsky (2018)). In our Gompertz framework, this would imply that low modal ages should be paired with high dispersion coefficients, as explicitly stated in Milevsky (2018). Choosing, for example, $(m, b) = (88.721, 10)$ and $(\widehat{m}, \widehat{b}) = (80.5, 11)$, we still obtain ${}_t\widehat{p}_x < {}_tp_x$ on the set $x + t < 150$. That is, the parameters could be chosen in such a way that all our qualitative results in the numerical part still remain valid. Hence, we have decided to choose a rather simple way to incorporate the subjective mortality beliefs by changing the modal age at death.

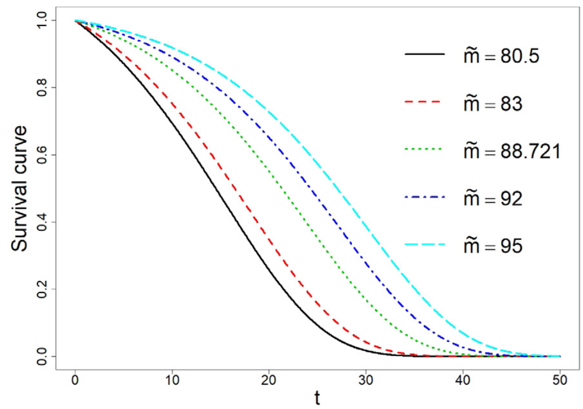


Fig. 1. Subjective survival curves ${}_t\widetilde{p}_x$ for a 65-year old individual for all the different subjective modal ages $\widetilde{m} \in \{80.5, 83, 88.721, 92, 95\}$, where the dispersion coefficient is equal to $\beta = 10$.

- $\widetilde{m} \in \{80.5, 83\}$ ($\widehat{m} \in \{80.5, 83\}$): The first value results in an underestimation of the life expectancy of 6.183 years compared to the insurer, which is consistent with the findings in Bucher-Koenen et al. (2013) and close to the findings in O'Brien et al. (2005) for females. This is the strongest underestimation we assume. The second value results in an underestimation of the life expectancy of 4.405 years compared to the insurer which is consistent with the overall findings in O'Brien et al. (2005) for males and their findings for females aged 60–69.
- $\widetilde{m} = 88.721$ ($\widehat{m} = 88.721$): In this case, there is no under- or overestimation, i.e. the policyholder's beliefs coincide with those of the insurer.
- $\widetilde{m} \in \{92, 95\}$ ($\widehat{m} \in \{92, 95\}$): These values result in an overestimation of the life expectancy of 2.705 and 5.276 years compared to the insurer. The values we use can, on some level, be connected to the findings in Ye and Post (2020) who report that people tend to feel younger than their chronological age, that is their mean and median subjective age is around 10 years below the chronological age for almost all age groups. This finding suggests that most individuals considered in Ye and Post (2020) might also overestimate their life expectancy compared to the insurer. As the insurer relies on the chronological age for pricing, it is thus possible that the insurer assigns a lower life expectancy to a policyholder than the policyholder does, due to the difference in the subjective and the chronological age.

In Fig. 1, we provide the survival curves ${}_t\widetilde{p}_x$ for the different subjective modal ages $\widetilde{m}$. We observe that the survival curves with higher subjective modal ages dominate the curves with lower modal ages for all times t. An individual assuming the modal age 80.5 believes that there is basically no chance of reaching age 100, whereas an individual assuming the modal age 95 assumes that there is a 20% chance of reaching age 100. This explains the corresponding under- and over-estimation of the life expectancy in the texts above.

In Table 2, we consider a constant annuity and provide the premium charged by the insurer and the subjective premium as expected by the policyholders. The constant annuity payoff $c(t) = c$ is chosen such that $P_0^A = 1$ with the premium P_0^A given in (9). Note that the parameter $\widehat{m}$ does not affect the premium of the annuity in any way and can therefore be omitted in Table 2. We observe that the subjective premium of the annuity increases in the modal age at death $\widetilde{m}$ assumed by the policyholder. That is, an annuity seems more and more overpriced the stronger an individual underestimates her survival probabilities. Conversely, an annuity appears underpriced to a policyholder who overestimates her survival probabilities.

Table 2
Subjective premium $\widetilde{P}_0^A$ (see (12)) of a constant annuity, given the premium $P_0^A = 1$. The parameters are as in Table 1, in particular, the insurance company's modal age at death is $m = 88.721$.

Subjective modal age	Annuity premium $\widetilde{P}_0^A$
$\widetilde{m} = 80.5$	0.7428
$\widetilde{m} = 83$	0.8197
$\widetilde{m} = 88.721$	1
$\widetilde{m} = 92$	1.1038
$\widetilde{m} = 95$	1.1979

Table 3
Subjective premium $\widetilde{P}_0^T$ (see (13)) of the natural tontine, given the premium $P_0^T = 1$. The parameters are as in Table 1, in particular, the insurance company's modal age at death is $m = 88.721$.

Subjective modal age	Tontine premium $\widetilde{P}_0^T$		
	$n = 10$	$n = 100$	$n = 1000$
$\widetilde{m} = \widehat{m} = 80.5$	0.9472	0.9873	0.9966
$\widetilde{m} = \widehat{m} = 83$	0.9704	0.9944	0.9988
$\widetilde{m} = \widehat{m} = 88.721$	1	1	1
$\widetilde{m} = \widehat{m} = 92$	1.0068	1.0005	1.0000
$\widetilde{m} = \widehat{m} = 95$	1.0097	1.0006	1.0000

Next, we turn to a tontine. Table 3 provides the premium charged by the insurer and the subjective premium as expected by the policyholders for a so-called natural tontine as introduced in Milevsky and Salisbury (2015). The payoff of the natural tontine is, in our model setup, given by $d(t) = {}_tp_x \cdot d_0$, where d_0 is a constant. Note that the tontine payoff to a single individual in the pool remains constant over time if deaths in the pool occur exactly as expected.[7] The constant d_0 is chosen such that $P_0^T = 1$ with the premium P_0^T given in (11). We first proceed similar to Table 2 and change the subjective mortality rates, keeping the insurer's rate constant. If we assume that the two subjective mortality rates are equal, that is $\widetilde{m} = \widehat{m}$, we observe from Table 3, that the perceived premium is for pool sizes of $n = 100$ and $n = 1\,000$ hardly different from 1. This result is obvious if we again compare the tontine premium (11) to the perceived tontine premium (13): In the shocked Gompertz model, $\widetilde{m} = \widehat{m}$ implies that the two mortality rates $\widetilde{\mu}_{x+t}$ and $\widehat{\mu}_{x+t}$ are equal for all $t > 0$. Further, for a sufficiently large pool size, the terms $\left(1-e^{-\int_0^t \mu_{x+s} ds}\right)^n$ and $\left(1-e^{-\int_0^t \widehat{\mu}_{x+s} ds}\right)^n$ are close to zero and disappear. This leaves us to conclude that for sufficiently large pool sizes, the perceived premium $\widetilde{P}_0^T$ is close to the insurer's premium $P_0^T = 1$ if $\widetilde{m} = \widehat{m}$. That is why, as a next step, in Table 4, we only vary $\widehat{m}$ while keeping the relation between $\widetilde{m}$ and m constant at $\widetilde{m} = m - 4 = 84.721$. As expected, we now observe significant changes of the subjective premium $\widetilde{P}_0^T$ if $\widetilde{m}$ differs from $\widehat{m}$. If the policyholder thinks to live longer than the other pool members ($\widehat{m} < \widetilde{m}$, see the first two lines of Table 4), the perceived premium is higher than the premium $P_0^T = 1$. For the policyholder's perceived tontine premium, it seems to be relevant how the policyholder sees her own survival prospects relative to the other pool members. In contrast, the perceived premium of an annuity is driven by the relation between one's own perceived survival curve (in the shocked Gompertz model given by the parameter $\widetilde{m}$) and the insurer's survival curve (that is the parameter m). The first two lines in Table 4 correspond to a policyholder optimistic about her own life expectancy compared to her peers but pessimistic compared to the insurer (i.e. $\widehat{m} <$

[7] Due to its nice structure, Milevsky and Salisbury (2015) recommend this tontine design for an implementation of tontines in today's world which is also why we have decided to choose this design for our numerical demonstration.

Table 4
Subjective premium $\widetilde{P}_0^T$ (see (13)) of the natural tontine, given the premium $P_0^T = 1$. The parameters are as in Table 1, in particular, the insurance company's modal age at death is $m = 88.721$ which is equal to $\widetilde{m} = m - 4 = 84.721$.

Subjective modal age	Tontine premium $\widetilde{P}_0^T$		
	$n = 10$	$n = 100$	$n = 1000$
$\widehat{m} = 81$	1.1412	1.2515	1.2993
$\widehat{m} = 83$	1.0466	1.0896	1.1006
$\widehat{m} = 84.721$	0.9824	0.9972	0.9995
$\widehat{m} = 86$	0.9432	0.9471	0.9475
$\widehat{m} = 88$	0.8940	0.8897	0.8893

$\widetilde{m} < m$), a perception that is in line with most empirical studies, see Section 2. From Table 2, we observe that this policyholder believes an annuity to be overpriced ($\widetilde{P}_0^A < P_0^A$). At the same time, from Table 4, this policyholder believes that a tontine is underpriced ($\widetilde{P}_0^T > P_0^T$).

To conclude, the perceived attractiveness of an annuity is fully determined by how the policyholder sees her survival prospects relative to the survival prospects used by the insurance company (m vs. $\widetilde{m}$ and thus ${}_tp_x$ vs. ${}_t\widetilde{p}_x$). In contrast, the perceived attractiveness of a tontine is mainly determined by how the policyholder sees her survival prospects relative to the other pool members ($\widetilde{m}$ vs. $\widehat{m}$ and thus ${}_t\widetilde{p}_x$ vs. ${}_t\widehat{p}_x$).

As it is usual in this stream of literature (for example Yaari (1965), Yagi and Nishigaki (1993), Mitchell (2002), Davidoff et al. (2005), Milevsky and Salisbury (2015), Peijnenburg et al. (2016) and Chen et al. (2019)), the attractiveness of a retirement product is frequently examined in an expected utility framework. To confirm our results in such a framework, we, in the following sections, consider an expected utility framework to figure out how subjective mortality beliefs affect the relative attractiveness of annuities and tontines and whether a tontine is preferable over an annuity under certain subjective mortality beliefs. We analyze whether the arguments about the premium perception in this section still hold true if the policyholder's expected lifetime utility is used for the product comparison. We start by deriving optimal payoff functions of annuities and tontines in Section 4 and then compare the resulting attractiveness of both products in Section 5.

4. Optimal payoff and expected utility

In this section, we derive the optimal payoff and the corresponding expected lifetime utility of the annuity and the tontine under subjective mortality beliefs. Our results can be seen as a generalization of the theorems given in Milevsky and Salisbury (2015) and Chen et al. (2019). We assume that the considered policyholder is endowed with an initial wealth v, which can be used to invest in one of the retirement products, and introduce the policyholder's expected discounted lifetime utility as

$$U\big(\{\alpha(t)\}_{t\geq 0}\big) := \widetilde{\mathbb{E}}\left[\int_0^\infty e^{-\rho t}\cdot u\left(\alpha(t)\right)\cdot \mathbb{1}_{\{\zeta>t\}}\mathrm{d}t\right], \tag{14}$$

where $\{\alpha(t)\}_{t\geq 0}$ denotes the insurance product's payoff, $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$ is a CRRA utility function with a risk aversion parameter $\gamma > 0$, $\gamma \neq 1$ and ρ is the subjective discount factor of the policyholder. Note that the expected discounted lifetime utility is taken under the policyholder's subjective expectation $\widetilde{\mathbb{E}}[\,\cdot\,]$. In the following two subsections, we first consider an annuity with payoff $\alpha(t) = b_A(t)$, then a tontine with payoff $\alpha(t) = b_T(t)$.

4.1. Annuity

We assume that the individual aims to maximize her expected discounted lifetime utility under the constraint that her initial wealth equals the premium charged by the insurer. The expected discounted lifetime utility of an annuity is given by

$$U\big(\{b_A(t)\}_{t\geq 0}\big) = \int_0^\infty e^{-\rho t}\,{}_t\widetilde{p}_x\, u\big(c(t)\big)\,\mathrm{d}t.$$

To be more precise, we solve the following optimization problem to determine the optimal annuity payment $c(t)$:

$$\begin{aligned}\max_{c(t)} U\big(\{b_A(t)\}_{t\geq 0}\big) &= \max_{c(t)} \int_0^\infty e^{-\rho t}\,{}_t\widetilde{p}_x\, u\big(c(t)\big)\,\mathrm{d}t\\ \text{subject to}\quad v = P_0^A &:= \int_0^\infty e^{-rt}\,{}_tp_x\, c(t)\,\mathrm{d}t\,,\end{aligned} \tag{15}$$

where P_0^A is the premium charged by the insurer, and v is the initial wealth of the individual, available to buy the annuity product. Strictly speaking, we shall put $P_0^A \leq v$ in the constraint. As typically in this kind of optimization problems, the budget constraint is binding in the optimal solution, we start immediately with an "equality" in the constraint.

The solution of optimization problem (15) is given in Theorem 4.1.

Theorem 4.1 (*Optimal Annuity Payoff*). *For an annuity contract, the solution to problem (15) is given by the optimal payout function*

$$c^*(t) = \frac{e^{\frac{(r-\rho)t}{\gamma}}}{\lambda_A^{1/\gamma}}\left(\frac{{}_t\widetilde{p}_x}{{}_tp_x}\right)^{1/\gamma}, \tag{16}$$

where λ_A is the optimal Lagrangian multiplier given by

$$\lambda_A = \left(\frac{1}{v}\int_0^\infty e^{\left(\frac{1}{\gamma}-1\right)rt-\frac{1}{\gamma}\rho t}\,{}_tp_x\left(\frac{{}_t\widetilde{p}_x}{{}_tp_x}\right)^{1/\gamma}\mathrm{d}t\right)^{\gamma}.$$

The optimal level of expected utility is then given by

$$U_A := \frac{\lambda_A}{1-\gamma}\,v\,. \tag{17}$$

Proof. See Appendix A.1. □

Note that if there are no subjective mortality beliefs and if $r = \rho$, the optimal annuity payment reduces to the constant $\lambda_A^{-\frac{1}{\gamma}}$ which is in line with Yaari (1965). In all the other cases, the optimal annuity payoff is not constant and may increase or decrease over time. This implies that constant annuities are sub-optimal for individuals whose subjective discount rate differs from the risk-free interest rate, consistent with, for example, Yagi and Nishigaki (1993). Before we analyze the effects of subjective mortality beliefs on the optimal payoff $c^*(t)$ and the optimal level of expected utility U_A in Section 5, we derive the optimal tontine payoff in the following subsection.

4.2. Tontine

In this section, we determine the optimal withdrawal payment $d(t)$ for the tontine. The expected discounted lifetime utility of a tontine is given by

$$U\big(\{b_T(t)\}_{t\geq 0}\big) = \int_0^\infty e^{-\rho t}\,u\big(d(t)\big)\,\widetilde{\mathbb{E}}\left[\mathbb{1}_{\{\zeta>t\}}\left(\frac{n}{N(t)}\right)^{1-\gamma}\right]\mathrm{d}t,$$

where, from a perceived policyholder's perspective $N(t) - 1 \mid \{{}_t\widehat{P}_x\} \sim \mathrm{Bin}\big(n-1, {}_t\widehat{P}_x\big)$, see also Section 3.1. Based on this, we obtain

$$
\begin{aligned}
\kappa_{n,\gamma}\left({}_t\widehat{P}_x, {}_t\widetilde{P}_x\right) &:= \widetilde{\mathbb{E}}\left[\mathbb{1}_{\{\zeta>t\}}\left(\frac{n}{N(t)}\right)^{1-\gamma}\right] \\
&= \mathbb{E}\left[{}_t\widetilde{P}_x\,\widetilde{\mathbb{E}}\left[\left(\frac{n}{N(t)}\right)^{1-\gamma}\,\middle|\,\zeta>t,\,\widetilde{\mathcal{E}}_t\right]\right] \\
&= \mathbb{E}\left[{}_t\widetilde{P}_x\sum_{k=0}^{n-1}\left(\frac{n}{k+1}\right)^{1-\gamma}\binom{n-1}{k}\left({}_t\widehat{P}_x\right)^k\right. \\
&\quad\left.\cdot\left(1-{}_t\widehat{P}_x\right)^{n-1-k}\right] \\
&= \mathbb{E}\left[e^{-\int_0^t\widetilde{\mu}_{x+s}\mathrm{d}s}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{\gamma}\binom{n}{k}\left(e^{-\int_0^t\widehat{\mu}_{x+s}\mathrm{d}s}\right)^{k-1}\right. \\
&\quad\left.\cdot\left(1-e^{-\int_0^t\widehat{\mu}_{x+s}\mathrm{d}s}\right)^{n-k}\right] \\
&= \sum_{k=1}^{n}\binom{n}{k}\left(\frac{k}{n}\right)^{\gamma}\mathbb{E}\left[e^{-\int_0^t\left(\widetilde{\mu}_{x+s}+(k-1)\widehat{\mu}_{x+s}\right)\mathrm{d}s}\right. \\
&\quad\left.\cdot\left(1-e^{-\int_0^t\widehat{\mu}_{x+s}\mathrm{d}s}\right)^{n-k}\right].
\end{aligned} \tag{18}
$$

That is, we solve the following optimization problem:

$$
\begin{aligned}
&\max_{d(t)} U\left(\{b_T(t)\}_{t\geq 0}\right) = \max_{d(t)}\int_0^\infty e^{-\rho t}\,u\big(d(t)\big)\,\kappa_{n,\gamma}\left({}_t\widehat{P}_x, {}_t\widetilde{P}_x\right)\,\mathrm{d}t \\
&\quad\text{subject to}\quad v = P_0^T := \int_0^\infty e^{-rt} \\
&\quad\cdot\,\mathbb{E}\left[\left(1-\left(1-e^{-\int_0^t\mu_{x+s}\mathrm{d}s}\right)^n\right)\right]d(t)\,\mathrm{d}t\,.
\end{aligned} \tag{19}
$$

The solution to problem (19) is given in Theorem 4.2.

Theorem 4.2 (*Optimal Tontine Payoff*)**.** *For a tontine, the solution to problem (19) is given by the optimal payout function*

$$
d^*(t) = \frac{e^{\frac{(r-\rho)t}{\gamma}}\left(\kappa_{n,\gamma}\left({}_t\widehat{P}_x, {}_t\widetilde{P}_x\right)\right)^{1/\gamma}}{\lambda_T^{1/\gamma}\,\mathbb{E}\left[\left(1-\left(1-e^{-\int_0^t\mu_{x+s}\mathrm{d}s}\right)^n\right)\right]^{1/\gamma}}\,, \tag{20}
$$

where λ_T is the optimal Lagrangian multiplier given by

$$
\lambda_T = \left(\frac{1}{v}\int_0^\infty e^{\left(\frac{1}{\gamma}-1\right)rt-\frac{1}{\gamma}\rho t}\frac{\left(\kappa_{n,\gamma}\left({}_t\widehat{P}_x, {}_t\widetilde{P}_x\right)\right)^{1/\gamma}}{\mathbb{E}\left[\left(1-\left(1-e^{-\int_0^t\mu_{x+s}\mathrm{d}s}\right)^n\right)\right]^{1/\gamma-1}}\mathrm{d}t\right)^{\gamma}.
$$

The expected discounted lifetime utility is then given by

$$
U_T := \frac{\lambda_T}{1-\gamma}v\,. \tag{21}
$$

Proof. See Appendix A.2. □

Although the optimal payout structure $d^*(t)$ is much more complex than the optimal annuity payment $c^*(t)$ from (16), the optimal expected utility in (21) differs from (17) only through the Lagrangian multiplier. In the following section, we will have a closer look at the effect of the subjective mortality beliefs on the optimal payoff and expected utility of both annuities and tontines.

5. Effects of subjective mortality beliefs

In this section, we analyze the effect of the subjective mortality beliefs on the optimal retirement decision. As explained in Section 3.1, we denote by μ_{x+t} the mortality rate used by the insurer and by $(\widetilde{\mu}_{x+t}, \widehat{\mu}_{x+t})$ the policyholder's subjective mortality rates used for herself and the remaining policyholders in the pool, respectively.

5.1. Subjective mortality beliefs concerning oneself

We start by analyzing the effects of the individual's subjective mortality beliefs about herself on the optimal payoff and the optimal level of expected utility of the two products. In Fig. 2, we illustrate the effects of the subjective mortality rate $\widetilde{\mu}_{x+t}$ on $c^*(t)$ and $d^*(t)$. In the following analysis, we always consider a policyholder with a risk aversion of $\gamma = 3$ and a subjective discount factor of $\rho = r = 0.02$. We basically make the same observation in the two panels in Fig. 2: If the individual believes that she lives shorter than the insurer has estimated, that is, $\widetilde{m} < m$ (here $\widetilde{m} = 80.5 < m$ and $\widetilde{m} = 83 < m$, respectively), the individual will buy a product which provides a higher payment at the early retirement ages and a lower payment at the more advanced retirement ages (compared to the case with $m = \widetilde{m}$). For the annuity, this leads to decreasing payoffs. For the tontine, a more steeply declining payoff results (compared to the case with $m = \widetilde{m}$). In the reverse case, that is, for a policyholder believing to live longer than the insurer has estimated (here $\widetilde{m} = 92 > m$ and $\widetilde{m} = 95 > m$, respectively), the individual buys a product which provides a lower payment at the early retirement ages and a higher payment at the more advanced retirement ages. For the annuity, increasing payoffs result. For the tontine, less steeply declining payoffs (compared to the case with $m = \widetilde{m}$), which slightly increase at very old ages, are obtained. Thus, living shorter in expectation has the same effect as being less patient about the future: Individuals tend to consume more at earlier retirement ages. Older ages are given less importance than earlier ages and therefore, lower payments result at older ages. If, on the other hand, the individual expects to live longer than the insurer assumes, the opposite is true.

We want to verify whether the perceived overpricing (underpricing) of annuities leads to a lower (higher) utility level for the policyholders. For this purpose, we introduce problem (22). A policyholder who assumes her own subjective survival curve to be ${}_t\widetilde{p}_x$ wants to choose the optimal retirement product following the optimization problem:

$$
\begin{aligned}
&\max_{c(t)} U\left(\{b_A(t)\}_{t\geq 0}\right) = \max_{c(t)}\int_0^\infty e^{-\rho t}\,{}_t\widetilde{p}_x\,u\big(c(t)\big)\,\mathrm{d}t \\
&\quad\text{subject to}\quad v = \widetilde{P}_0^A := \int_0^\infty e^{-rt}\,{}_t\widetilde{p}_x\,c(t)\,\mathrm{d}t\,.
\end{aligned} \tag{22}
$$

Note that, in contrast to the optimization problem (15), the constraint in the optimization problem (22) is given in terms of the policyholder's subjective premium $\widetilde{P}_0^A$ instead of the insurer's premium P_0^A.

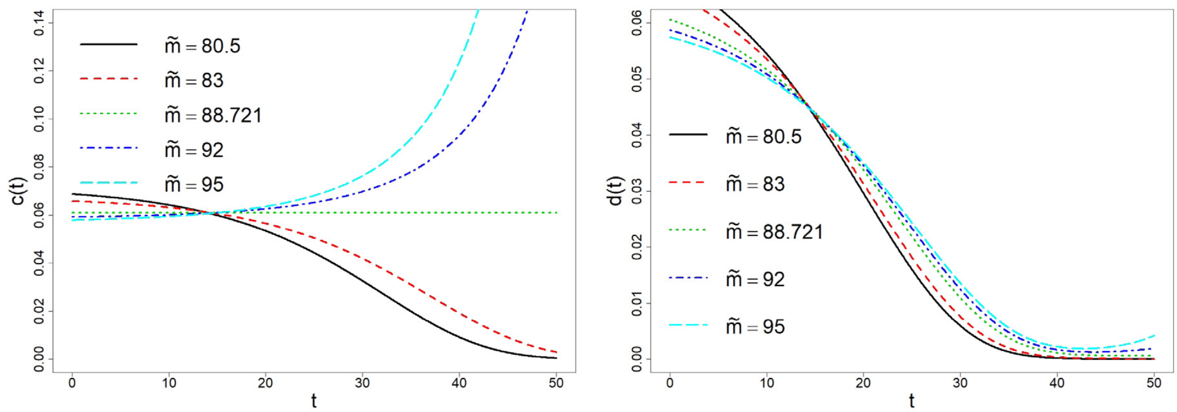
For a more thorough analysis of the retirement products, we introduce certainty equivalents CE defined as the level of constant retirement benefits that yield the same expected utility as the annuity and tontine, respectively. In other words, we determine $\mathrm{CE} > 0$ such that

$$
U\left(\{\mathrm{CE}\}_{t\geq 0}\right) = U\left(\{\alpha(t)\}_{t\geq 0}\right), \tag{23}
$$

or equivalently,

$$
\mathrm{CE} = \left((1-\gamma)\left(\int_0^\infty e^{-\rho t}\,{}_t\widetilde{p}_x\,\mathrm{d}t\right)^{-1}\cdot U\left(\{\alpha(t)\}_{t\geq 0}\right)\right)^{\frac{1}{1-\gamma}},
$$

where $U\left(\{\alpha(t)\}_{t\geq 0}\right)$ is the expected discounted lifetime utility of the individual as defined in (14).



(a) Annuity (b) Tontine, $m = \widehat{m} = 88.721$

Fig. 2. Optimal payoff of the annuity and the tontine for different choices of the modal age at death $\widetilde{m}$, where the parameters are chosen as in Table 1 with $\gamma = 3$ and $\rho = r$.

Table 5
Certainty equivalents (CE) of the annuity for problems (15) and (22), where the payoff is the (subjective) utility maximizing payoff (16). The parameters are as in Table 1 with risk aversion $\gamma = 3$, subjective discount factor $\rho = r$ and $m = 88.721$.

Subjective modal age	CE with premium P_0^A Problem (15)	CE with premium $\widetilde{P}_0^A$ Problem (22)
$\widetilde{m} = 80.5$	0.0629	0.0822
$\widetilde{m} = 83$	0.0619	0.0745
$\widetilde{m} = 88.721$	0.0611	0.0611
$\widetilde{m} = 92$	0.0613	0.0553
$\widetilde{m} = 95$	0.0618	0.0510

Comparing the results of optimization problems (15) and (22), we can see whether "perceived" overpricing (underpricing) leads to a lower (higher) certainty equivalent CE for the policyholder. Table 5 presents the results for an annuity. The column "CE with premium P_0^A" gives the certainty equivalent obtained via optimization problem (15). The third column "CE with premium $\widetilde{P}_0^A$" gives the CE obtained via optimization problem (22). If the third column gives a higher value, problem (22) leads to a higher CE to the policyholder than problem (15) does. The reason for this is that the policyholder perceives the premium charged by the insurer in problem (15) as too expensive. We observe that this is the case if the policyholder underestimates her survival curve ($\widetilde{m} < m$), in line with the results in Table 2. The reverse results hold for a policyholder that is optimistic with respect to her survival curve ($\widetilde{m} > m$). These results confirm the intuition about the premium perception analyzed in Section 3.3.

5.2. Subjective mortality beliefs concerning others

In the previous subsection, we have analyzed how the policyholder's perceived mortality rates ($\widetilde{\mu}_{x+t}$, $\widehat{\mu}_{x+t}$) affect the perceived attractiveness of annuity and tontine. Note, however, that the mortality rate the policyholder assumes for everyone else $\widehat{\mu}_{x+t}$ does not affect the payoffs or the expected utility of the annuity in any way. Therefore, if we want to compare the relative attractiveness of both products under subjective mortality beliefs, we need to analyze the influence of $\widehat{\mu}_{x+t}$ on the expected utility of the tontine. In this sense, this subsection serves to analyze the relative attractiveness of the annuity and the tontine, focusing on the effect of subjective mortality beliefs.

Proposition 5.1. *The optimal level of expected utility of the tontine U_T decreases in the perceived conditional survival curve for the policyholder's peers ${}_t\widehat{P}_x := e^{-\int_0^t \widehat{\mu}_{x+s} ds}$ for all $\gamma > 0$, $\gamma \neq 1$. As a consequence, the certainty equivalent of the tontine is also decreasing in ${}_t\widehat{P}_x$.*

Proof. See Appendix A.3. □

The expected utility decreases in the perceived conditional survival curve for the other policyholders in the tontine pool, that is, the more the individual under consideration underestimates ${}_t\widehat{P}_x$, the higher the perceived expected utility of the tontine. This is logical as the tontine pool payoff $d(t)$ is shared among the survivors of the pool. Believing that other pool members have a lower survival probability, the surviving individual receives (at least on average) a higher payout. This raises the "perceived" expected utility. In the shocked Gompertz model (Example 3.1), increases in the conditional survival rates are achieved by increasing the perceived modal age at death $\widehat{m}$.

This leads us to our main result, the comparison between the certainty equivalents (CE) of annuity and tontine under subjective mortality beliefs, see Theorem 5.2.

Theorem 5.2 (*Certainty Equivalent Comparison Under Subjective Beliefs*)**.**

*(a) If beliefs do not differ between policyholder and insurance company, that is if $\mu_{x+t} = \widetilde{\mu}_{x+t} = \widehat{\mu}_{x+t}$, we find that the CE of a tontine **never** (that is for any portfolio size $n \in \mathbb{N}$) **exceeds** the CE of an annuity.*

*(b) Consider the case **with systematic mortality risk**. If*

$$
{}_tp_x > \left(\mathbb{E}\left[\frac{e^{-\int_0^t \widetilde{\mu}_{x+s} ds}}{{}_t\widehat{p}_x} \left(e^{-\int_0^t \widehat{\mu}_{x+s} ds} \right)^{\gamma-1} \right] \right)^{\frac{1}{\gamma-1}}, \tag{24}
$$

*there exists a pool size $N_0 \in \mathbb{N}$ such that the subjective CE of a tontine is (for any portfolio size $n \geq N_0$) **higher** than the subjective CE of an annuity.*

*(c) Consider the case **without systematic mortality risk** (deterministic mortality rates μ, $\widetilde{\mu}$ and $\widehat{\mu}$). In this case, assumption (24) simplifies to ${}_tp_x > {}_t\widehat{p}_x$.*

Remark 5.3 (*Theorem 5.2*)**.** Part (a) of Theorem 5.2 is not an unknown result and is presented by Milevsky and Salisbury (2015) in a scholar setting. In contrast to Milevsky and Salisbury (2015), we also include systematic mortality risks to our setting. Further, we have a slightly different definition of the tontine premium. Parts (b) and (c) of Theorem 5.2 are, to the best of our knowledge, new results that are not available in the literature.

Proof.

(a) Consider the following optimization problem (with *no subjective mortality beliefs*):

$$\max_{\alpha\in[0,1]} \mathbb{E}\left[\int_0^\infty e^{-\rho t}\mathbb{1}_{\{\zeta>t\}}u\left(\alpha c^*(t)+(1-\alpha)\frac{n}{N(t)}d^*(t)\right)\,\mathrm{d}t\right], \tag{25}$$

where $c^*(t)$ is the optimal annuity payoff (16) and $d^*(t)$ is the optimal tontine payoff (20) with no subjective mortality beliefs, i.e. $\mu_{x+t} = \widetilde{\mu}_{x+t} = \widehat{\mu}_{x+t}$. In particular, (25) states that the policyholder optimally splits the initial budget $v = \alpha P_0^A + (1-\alpha)P_0^T$ between annuities and tontines. The premiums P_0^A and P_0^T are as defined in Problems (15) and (19), respectively. The objective function to Problem (25) can be written as

$$\begin{aligned}\mathcal{F}(\alpha) &= \int_0^\infty e^{-\rho t}\mathbb{E}\left[\mathbb{1}_{\{\zeta>t\}}u\left(\alpha c^*(t)+(1-\alpha)\frac{nd^*(t)}{N(t)}\right)\right]\mathrm{d}t\\
&= \int_0^\infty e^{-\rho t}\mathbb{E}\left[e^{-\int_0^t \mu_{x+s}ds}\mathbb{E}\left[u\left(\alpha c^*(t)+(1-\alpha)\right.\right.\right.\\
&\qquad \left.\left.\left.\cdot\frac{n}{N(t)}d^*(t)\right)\middle|\zeta>t,\mathcal{E}_t\right]\right]\mathrm{d}t\\
&= \int_0^\infty e^{-\rho t}\mathbb{E}\left[e^{-\int_0^t \mu_{x+s}ds}\sum_{k=0}^{n-1}u\left(\alpha c^*(t)\right.\right.\\
&\qquad \left.+(1-\alpha)\frac{n}{k+1}d^*(t)\right)\\
&\qquad \cdot\binom{n-1}{k}\left(e^{-\int_0^t \mu_{x+s}ds}\right)^k\\
&\qquad \left.\cdot\left(1-e^{-\int_0^t \mu_{x+s}ds}\right)^{n-1-k}\right]\mathrm{d}t\\
&= \int_0^\infty e^{-\rho t}\sum_{k=0}^{n-1}u\left(\alpha c^*(t)+(1-\alpha)\frac{n}{k+1}d^*(t)\right)\\
&\qquad \cdot\binom{n-1}{k}\mathbb{E}\left[\left(e^{-\int_0^t \mu_{x+s}ds}\right)^{k+1}\right.\\
&\qquad \left.\cdot\left(1-e^{-\int_0^t \mu_{x+s}ds}\right)^{n-1-k}\right]\mathrm{d}t\,.\end{aligned}$$

We determine the first-order derivative to find a solution to this optimization problem. The first-order condition with respect to α is given by

$$\begin{aligned}\frac{\partial\mathcal{F}(\alpha)}{\partial\alpha} &= \int_0^\infty e^{-\rho t}\sum_{k=0}^{n-1}u'\left(\alpha c^*(t)+(1-\alpha)\frac{n}{k+1}d^*(t)\right)\\
&\quad\cdot\left(c^*(t)-\frac{n}{k+1}d^*(t)\right)\\
&\quad\cdot\binom{n-1}{k}\mathbb{E}\left[\left(e^{-\int_0^t \mu_{x+s}ds}\right)^{k+1}\right.\\
&\quad\left.\cdot\left(1-e^{-\int_0^t \mu_{x+s}ds}\right)^{n-1-k}\right]\mathrm{d}t \overset{!}{=} 0\,.\end{aligned} \tag{26}$$

Using (16) and some effort, we can verify that $\alpha^* = 1$ fulfills the first-order condition (26). We still need to verify that $\alpha^* = 1$ is a maximum and that it is the only maximum of the objective function. We can do this by taking a look at the second-order derivative:

$$\begin{aligned}\frac{\partial^2\mathcal{F}(\alpha)}{\partial\alpha^2} &= \int_0^\infty e^{-\rho t}\sum_{k=0}^{n-1}u''\left(\alpha c^*(t)+(1-\alpha)\frac{n}{k+1}d^*(t)\right)\\
&\quad\cdot\left(c^*(t)-\frac{n}{k+1}d^*(t)\right)^2\\
&\quad\cdot\binom{n-1}{k}\mathbb{E}\left[\left(e^{-\int_0^t \mu_{x+s}ds}\right)^{k+1}\right.\\
&\quad\left.\cdot\left(1-e^{-\int_0^t \mu_{x+s}ds}\right)^{n-1-k}\right]\mathrm{d}t<0\,,\end{aligned}$$

since $u''\left(\alpha c^*(t)+(1-\alpha)\frac{n}{k+1}d^*(t)\right) < 0$ for all $\alpha\in[0,1]$. If the second-order derivative is strictly negative, this implies that the first-order derivative is strictly decreasing in α. Hence, $\partial\mathcal{F}(\alpha)/\partial\alpha$ can only be equal to zero for exactly one value of α, which we have already found above ($\alpha^* = 1$). From this, we also see that the first-order derivative has to be greater than zero for all $\alpha < 1$. Consequently, the expected utility in Problem (25) is increasing in α until it reaches its maximum at $\alpha = 1$. Particularly, a 100% investment of initial wealth in the optimal annuity delivers a higher expected lifetime utility than a 100% investment in the optimal tontine.

(b) For the limiting ($n\to\infty$) tontine, we follow the proof of Theorem 4.2. Note first that by the conditional law of large numbers (see, for example, Majerek et al. (2005) and Hanbali et al. (2019)), we obtain, given the filtration of systematic mortality risk factors $\widetilde{\mathcal{E}}_t$, that the share of survivors under subjective beliefs equals:

$$\begin{aligned}\lim_{n\longrightarrow\infty}\left(\frac{N(t)}{n}\middle|\widetilde{\mathcal{E}}_t\right) &= \lim_{n\longrightarrow\infty}\frac{e^{-\int_0^t\widetilde{\mu}_{x+s}\,\mathrm{d}s}+(n-1)\cdot e^{-\int_0^t\widehat{\mu}_{x+s}\,\mathrm{d}s}}{n}\Bigg|_{\widetilde{\mathcal{E}}_t}\\
&\longrightarrow e^{-\int_0^t\widehat{\mu}_{x+s}\,\mathrm{d}s}.\end{aligned}$$

We obtain, applying the dominated convergence theorem:

$$\begin{aligned}\kappa_{\infty,\gamma}\left({}_t\widehat{P}_x,{}_t\widetilde{P}_x\right) &:= \lim_{n\to\infty}\kappa_{n,\gamma}\left({}_t\widehat{P}_x,{}_t\widetilde{P}_x\right)\\
&= \lim_{n\to\infty}\widetilde{\mathbb{E}}\left[\mathbb{1}_{\{\zeta>t\}}\left(\frac{n}{N(t)}\right)^{1-\gamma}\right]\\
&= \lim_{n\to\infty}\mathbb{E}\left[e^{-\int_0^t\widetilde{\mu}_{x+s}\,\mathrm{d}s}\right.\\
&\qquad\left.\cdot\widetilde{\mathbb{E}}\left[\left(\frac{n}{N(t)}\right)^{1-\gamma}\middle|\widetilde{\mathcal{E}}_t,\zeta>t\right]\right]\\
&= \mathbb{E}\left[e^{-\int_0^t\widetilde{\mu}_{x+s}\,\mathrm{d}s}\left(\frac{1}{e^{-\int_0^t\widehat{\mu}_{x+s}\,\mathrm{d}s}}\right)^{1-\gamma}\right]\\
&= \mathbb{E}\left[e^{-\int_0^t\left(\widetilde{\mu}_{x+s}-(1-\gamma)\widehat{\mu}_{x+s}\right)\mathrm{d}s}\right].\end{aligned}$$

For the annuity, we obtain the Lagrangian multiplier (see Theorem 4.1):

$$\lambda_A = \left(\frac{1}{P_0^A}\int_0^\infty e^{\left(\frac{1}{\gamma}-1\right)rt-\frac{1}{\gamma}\rho t}\,{}_tp_x\left(\frac{{}_t\widetilde{p}_x}{{}_tp_x}\right)^{1/\gamma}\mathrm{d}t\right)^\gamma.$$

In the limit $n\to\infty$, we obtain the Lagrangian multiplier for the limiting tontine (see Theorem 4.2):

$$\lambda_{T,n\to\infty} = \left(\frac{1}{P_0^T}\int_0^\infty e^{\left(\frac{1}{\gamma}-1\right)rt-\frac{1}{\gamma}\rho t}\cdot\left(\kappa_{\infty,\gamma}\left({}_t\widehat{P}_x,{}_t\widetilde{P}_x\right)\right)^{\frac{1}{\gamma}}\mathrm{d}t\right)^\gamma.$$

Following Theorems 4.1 and 4.2, we know that the certainty equivalents of annuity and tontine can be written as functions of the Lagrangian multipliers λ_A and λ_T, that is for $i \in \{T, A\}$:

$$\mathrm{CE}_i = \left(P_0^i \cdot \lambda_i \cdot \left(\int_0^\infty e^{-\rho t} \cdot {}_t\widetilde{p}_x \, \mathrm{d}t\right)^{-1}\right)^{\frac{1}{1-\gamma}} . \quad (27)$$

Comparing the certainty equivalents of annuity and limiting tontine is thus equivalent to comparing $\lambda_A^{\frac{1}{1-\gamma}}$ and $\lambda_{T,n\to\infty}^{\frac{1}{1-\gamma}}$. Using assumption (24), we obtain:

$$\begin{aligned}({}_tp_x)^\gamma \cdot \frac{{}_t\widetilde{p}_x}{{}_tp_x} &= {}_t\widetilde{p}_x \cdot ({}_tp_x)^{\gamma-1} \\ &\begin{cases} < {}_t\widetilde{p}_x \cdot \dfrac{\mathbb{E}\left[e^{-\int_0^t \widetilde{\mu}_{x+s}\,\mathrm{d}s}\left(\dfrac{1}{e^{-\int_0^t \widehat{\mu}_{x+s}\,\mathrm{d}s}}\right)^{1-\gamma}\right]}{{}_t\widetilde{p}_x}, & \text{if } \gamma \in (0,1) \\ > {}_t\widetilde{p}_x \cdot \dfrac{\mathbb{E}\left[e^{-\int_0^t \widetilde{\mu}_{x+s}\,\mathrm{d}s}\left(\dfrac{1}{e^{-\int_0^t \widehat{\mu}_{x+s}\,\mathrm{d}s}}\right)^{1-\gamma}\right]}{{}_t\widetilde{p}_x}, & \text{if } \gamma > 1 \end{cases} \\ &= \mathbb{E}\left[e^{-\int_0^t \left(\widetilde{\mu}_{x+s} - (1-\gamma)\widehat{\mu}_{x+s}\right)\mathrm{d}s}\right] = \kappa_{\infty,\gamma}\left({}_t\widehat{P}_x, {}_t\widetilde{P}_x\right) .\end{aligned}$$

This is equivalent to

$$\lambda_A \begin{cases} < \lambda_{T,n\to\infty}, & \text{if } \gamma \in (0,1) \\ > \lambda_{T,n\to\infty}, & \text{if } \gamma > 1 \end{cases}$$

which is again equivalent to $\lambda_A^{\frac{1}{1-\gamma}} < \lambda_{T,n\to\infty}^{\frac{1}{1-\gamma}}$. From (27), we can immediately conclude that the *certainty equivalent of the limiting tontine exceeds the certainty equivalent of the annuity*. Denoting by $\mathrm{CE}_{T,n}$ an optimal tontine's certainty equivalent with pool size n, we can use basic properties of a converging series $\mathrm{CE}_{T,n} \longrightarrow \mathrm{CE}_{T,n\to\infty}$ that there exists a pool size $N_0 \in \mathbb{N}$ such that the CE of a tontine $\mathrm{CE}_{T,n}$ is (for any portfolio size $n \geq N_0$) higher than the CE of an annuity (this basic convergence result can be found in any mathematical textbook covering the convergence of a sequence of real numbers, like, for example, Schulz (2011)). □

We now analyze for which individuals a tontine might be preferable to an annuity, where the individuals are distinguished by their relative risk aversion. For our numerical analysis, we focus, again, on the findings of Greenwald and Associates (2012) and O'Brien et al. (2005) who state that people tend to underestimate their own and others' life expectancy, that is, they assign ${}_t\widetilde{P}_x < {}_tP_x$ and ${}_t\widehat{P}_x < {}_tP_x$, respectively. We consider the parameter setup summarized in Table 1 along with the following three cases of subjective mortality beliefs:

- Case 1: $\widetilde{m} = 82$, $\widehat{m} = 80.5$: In this case, the policyholder underestimates others' life expectancy by 6.183 years and her own by 5.128 years compared to the insurer. In particular, the individual believes that she lives in expectation 1.055 years longer than her peers.
- Case 2: $\widetilde{m} = 80.5$, $\widehat{m} = 82$: In this case, the policyholder underestimates others' life expectancy by 5.128 years and her own by 6.183 years compared to the insurer. In particular, the individual believes that she lives in expectation 1.055 years less than her peers.
- Case 3: $\widetilde{m} = \widehat{m} = 88.721$: In this case, there are no subjective mortality beliefs, that is, $\widetilde{\mu}_{x+t} = \mu_{x+t} = \widehat{\mu}_{x+t}$. This corresponds to the setting analyzed in Milevsky and Salisbury (2015) and we mainly include this case to emphasize the importance of our results.

In Fig. 3, the corresponding certainty equivalents are given for the annuity and the tontine. The risk aversion parameters are equidistantly placed in the interval [0.1, 10]. Here, we consider two different tontines, one with $n = 10$ policyholders and another one with $n = 100$ members. We use very small pool sizes to emphasize that tontines with a low number of policyholders can already be preferred to annuities. We make the following observation from Fig. 3:

- In both panels, we can see that the tontine is preferred by all individuals whose relative risk aversion falls in the interval [0.1, 10] in cases 1 and 2, whereas in case 3 all the policyholders prefer the annuity over the tontine. The reason behind the tontine's superiority for the cases with subjective mortality beliefs is the underestimation of the survival curve used for the remaining policyholders ${}_t\widehat{P}_x$. As we have already seen in Proposition 5.1, a decrease of this conditional survival curve leads to a higher certainty equivalent of the tontine, while the certainty equivalent of the annuity remains completely unchanged.
- In both Panels (a) and (b), the individual's survival curve assumed for herself ${}_t\widetilde{P}_x$ has almost no effect on the tontine's superiority over the annuity. This can be seen from the fact that both in case 1 (${}_t\widetilde{P}_x > {}_t\widehat{P}_x$) and case 2 (${}_t\widetilde{P}_x < {}_t\widehat{P}_x$) the tontine is preferred to the annuity for all risk aversion parameters. In case 1, the tontine is preferred stronger over the annuity than in case 2, which is only due to ${}_t\widehat{P}_x$ being smaller in case 1 than in case 2.
- One last effect we can observe here is that the number of policyholders in a tontine also largely impacts the attractiveness of tontines. In both figures, the investors prefer the tontine over the annuity. Comparing Panel (a) with $n = 10$ investors to Panel (b) with $n = 100$ investors, we can see that the certainty equivalent of the tontine is significantly increased if the pool is larger. We have already argued at the beginning of this article that the unsystematic risk in a tontine can be diversified by a sufficiently large pool size and it is well-known already that the attractiveness of a tontine increases with its pool size (see for example Milevsky and Salisbury (2015) and Chen et al. (2019)).

We conclude our numerical analysis by providing the critical values N_0 from Theorem 5.2 for our base case parameter setup. We consider a policyholder with a risk aversion $\gamma = 3$. We can check numerically that condition (24) is fulfilled. Table 6 provides the critical pool sizes N_0 under the three cases considered in Fig. 3.

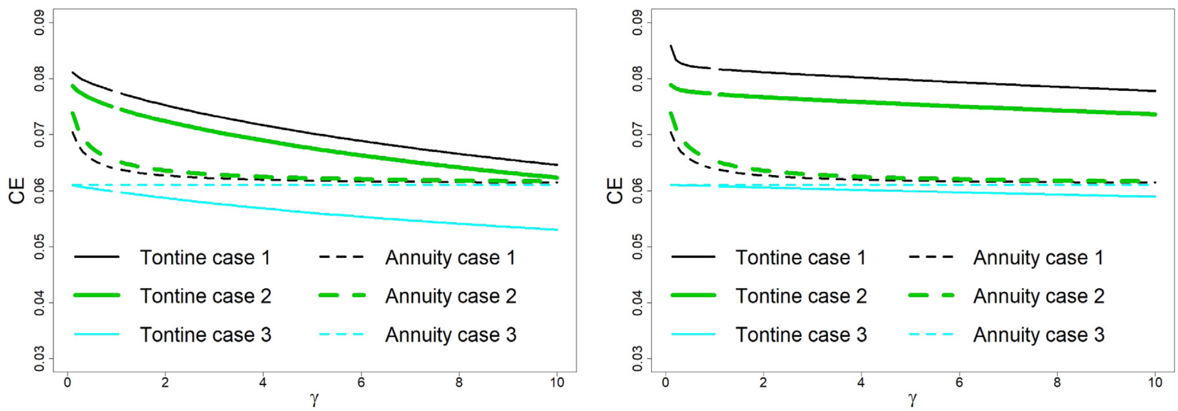
Under case 1, the critical pool size N_0 that leads to a larger certainty equivalent of the tontine compared to the annuity is already equal to 2. Under case 2, the critical pool size equals 3. Note that for case 3 no critical pool size N_0 exists, due to Theorem 5.2(a).[8]

6. Inclusion of safety loadings

In the previous sections of this article, we have tacitly excluded safety loadings. In this section, we want to set ourselves in a more realistic framework taking account of safety loadings. With the life annuity, there is a guarantee provided for mortality risk. For the tontine, the insurer is left with risks related to the time of death of the last survivor only. Therefore, it is reasonable to assume that the annuity comes with a higher safety loading than the tontine. For the following theoretical derivations, however, this assumption is not necessary.

The relative attractiveness of tontines compared to annuities increases if safety loadings are included, see, e.g., Milevsky and

[8] Figures of the certainty equivalents of the annuity and the tontine depending on n are available from the authors upon request.



(a) Portfolio size $n = 10$ (b) Portfolio size $n = 100$

Fig. 3. Certainty equivalent for different investors with the three cases explained above and the remaining parameters chosen as in Table 1 with $\rho = r = 0.02$.

Salisbury (2015) and Chen et al. (2019, 2020). In the following, we still consider an individual endowed with an initial wealth v. However, now, the expected value premium principle is used to incorporate safety loadings which fund risk capital charges and administration expenses. The total gross premium is given by $(1+\delta_X)P_0^X$, where δ_X is a proportional loading and P_0^X is again equal to the expected present value of future benefits for both products $X \in \{A, T\}$. To optimally determine the annuity and tontine payoffs with loadings, we need to adjust the budget constraint in (15) and (19) to:

$$v = (1+\delta_X)P_0^X, \quad \text{with } X = A, T. \tag{28}$$

Theorem 6.1 presents the optimal annuity and tontine payoffs under loaded premiums.

Theorem 6.1. *For an annuity contract, we obtain the optimal annuity payoff solving (15) with the budget constraint (28) incorporating safety loadings. This yields:*

$$c_s^*(t) = \frac{c^*(t)}{1+\delta_A}\,. \tag{29}$$

The optimal level of expected utility is then given by

$$U_A^s := (1+\delta_A)^{\gamma-1} U_A\,,$$

where $c^*(t)$ and U_A are as in Theorem 4.1. Similarly, for a tontine with loadings, the optimal tontine payoff is obtained from (19) with budget (28) incorporating safety loadings:

$$d_s^*(t) = \frac{d^*(t)}{1+\delta_T}, \tag{30}$$

The optimal level of expected utility is then given by

$$U_T^s := (1+\delta_T)^{\gamma-1} U_T\,,$$

where $d^*(t)$ and U_T are as in Theorem 4.2.

Proof. Follow the same steps as in Theorems 4.1 and 4.2. See also, e.g., Milevsky and Salisbury (2015) and Chen et al. (2020). □

Next, we present an alternative version to Theorem 5.2 which takes account of the safety loadings.

Theorem 6.2 (*Comparison Under Subjective Beliefs and Safety Loadings*)**.**

(a) Assume that beliefs do not differ between policyholder and insurance company, that is $\mu_{x+t} = \widetilde{\mu}_{x+t} = \widehat{\mu}_{x+t}$. If and only if

$$\delta_A \le \frac{(1+\delta_T)\,\mathrm{CE}_A}{\mathrm{CE}_T} - 1, \tag{31}$$

*where CE_X is the CE with no loading, we find that the CE of a tontine **never** (that is for any portfolio size $n \in \mathbb{N}$) **exceeds** the CE of an annuity.*

*(b) Consider the case **with systematic mortality risk**. If*

$$ {}_tp_x > \left(\mathbb{E}\left[\frac{e^{-\int_0^t \widetilde{\mu}_{x+s}\,ds}}{{}_t\widetilde{p}_x}\left(e^{-\int_0^t \widehat{\mu}_{x+s}\,ds}\right)^{\gamma-1}\right]\right)^{\frac{1}{\gamma-1}}, \tag{32}$$

*and $\delta_A \ge \delta_T$, there exists a pool size $N_0^s \in \mathbb{N}$ such that the subjective CE of a tontine is (for any portfolio size $n \ge N_0^s$) **higher** than the subjective CE of an annuity.*

Table 6
Critical pool size N_0 as described in Theorem 5.2. The parameters are as in Table 1 with risk aversion $\gamma = 3$ and subjective discount factor $\rho = r$.

Case	N_0
Case 1	2
Case 2	3
Case 3	–

Proof.

(a) For the two products $X \in \{A, T\}$, we can express the certainty equivalents with loadings by the certainty equivalents without loadings, using Theorem 6.1:

$$\mathrm{CE}_{X,s} = \left((1-\gamma)\left(\int_0^\infty e^{-\rho t}\,{}_t\widetilde{p}_x\,dt\right)^{-1}\cdot U_X^s\right)^{\frac{1}{1-\gamma}} = \frac{\mathrm{CE}_X}{1+\delta_X}.$$

Now it holds that

$$\begin{aligned}
&\mathrm{CE}_{A,s} \ge \mathrm{CE}_{T,s}\\
\Leftrightarrow\ &\frac{\mathrm{CE}_A}{1+\delta_A} \ge \frac{\mathrm{CE}_T}{1+\delta_T}\\
\Leftrightarrow\ &(1+\delta_A) \le \frac{(1+\delta_T)\mathrm{CE}_A}{\mathrm{CE}_T}\\
\Leftrightarrow\ &\delta_A \le \frac{(1+\delta_T)\mathrm{CE}_A}{\mathrm{CE}_T} - 1\,.
\end{aligned}$$

(b) In Theorem 5.2(b), we show that there exists a critical pool size $N_0 \in \mathbb{N}$ such that $\mathrm{CE}_{T,n} > \mathrm{CE}_A$ for $n \ge N_0$. If we consider safety loadings $\delta_A \geq \delta_T$, we obviously have that

$$\frac{\mathrm{CE}_{T,n}}{1+\delta_T} > \frac{\mathrm{CE}_A}{1+\delta_A}$$

for $n \geq N_0$. This suggests that there exists $N_0^s \leq N_0$ such that the proposed claim holds.

The upper bound for δ_A provided in (31) in Theorem 6.2(a) is known from Milevsky and Salisbury (2015) as *annuity indifference loading*, where the difference from our setting to theirs is that we consider a proportional loading, a general stochastic mortality setting and allow for a tontine loading δ_T to be greater than 0. Assuming the absence of subjective mortality beliefs, the annuity loading is equal to this upper bound if and only if the policyholder is indifferent between an annuity and a tontine, meaning that both products deliver the same certainty equivalent. For numerical analyses of this indifference loading, we refer once again to Milevsky and Salisbury (2015).

Note that Theorem 5.2(c) still holds if safety loadings are included. With Theorem 6.2, we confirm our previous results if loadings are included. In Theorem 6.2(b) we obtain a possibly even lower critical pool size $N_0^s \leq N_0$ such that a policyholder with subjective beliefs favors a tontine over an annuity. However, since Table 6 already shows that without loadings the critical pool size can get as small as two, we omit a similar analysis for the critical pool size with loadings at this point. Our main focus in this article is laid on the subjective mortality, and its importance should already be clear by now.

7. Conclusion

In this article, we study the effects of subjective mortality beliefs on the optimal design of annuities and tontines and their (relative) attractiveness to risk-averse policyholders. We find that subjective mortality beliefs have a substantial impact on the choice between a tontine and an annuity. In Section 3.3, we have shown that the relation between the individual's perceived life expectancy and the life expectancy used by the insurer in the premium calculation determines whether an annuity is perceived as too expensive or not, consistent with, for example, Wu et al. (2015). Whether a tontine is too expensive or not is determined by how the individual perceives her own life expectancy relative to the one of her peers. The obvious reason for this is that tontine payments are shared within the survivors: If somebody expects to live longer than the other pool members, the share distributed to this person's account is perceived to be (on average) higher than the share of the other pool members. We confirm this result in an expected utility framework. Further, we find that the *relative* attractiveness of annuities and tontines is determined by how the individual perceives the difference between the life expectancy of her peers and the life expectancy assumed by the insurance company. Surprisingly, if we assume that this difference is 4–6 years (a value that is consistent with most empirical studies in Section 2 for the age group 60–70 years), the certainty equivalent of tontines is 20% higher than the certainty equivalent of annuities. This completely reverses findings in a standard expected utility framework without administration and risk charges, where an annuity is always preferred over a tontine product (see, e.g., Yaari (1965)). Interestingly, one's own perceived life expectancy is no longer important in the relative comparison of the two products. Since annuitization rates remain low and are unlikely to increase in the current low interest environment, this result is of high relevance for the life insurance market as it shows that, under subjective mortality beliefs, a tontine might be an attractive alternative to a conventional annuity. An interesting generalization of our subjective belief model is the inclusion of "money illusion", that is, the empirically observed tendency to think in nominal rather than in real monetary terms (see, for example, Basak and Yan (2010)). Although the real terms matter, people tend to think in nominal terms. Additionally, it would be interesting to analyze the effect of an additional drawdown option. We leave these questions for future research.

Appendix. Proofs

A.1. Proof of Theorem 4.1

We obtain the following Lagrangian function for our optimization problem:

$$\mathcal{L} = \int_0^\infty e^{-\rho t}\, {}_t\widetilde{p}_x\, u\big(c(t)\big)\,\mathrm{d}t + \lambda_A \left(v - \int_0^\infty e^{-rt}\, {}_tp_x\, c(t)\,\mathrm{d}t \right).$$

Rearranging the first order condition delivers

$$c^*(t) = \frac{e^{\frac{(r-\rho)t}{\gamma}}}{\lambda_A^{1/\gamma}} \left(\frac{{}_t\widetilde{p}_x}{{}_tp_x} \right)^{1/\gamma}.$$

Now we can use the budget constraint to determine the Lagrangian multiplier λ_A. We have

$$\begin{aligned} v &= \int_0^\infty e^{-rt}\, {}_tp_x\, c^*(t)\,\mathrm{d}t \\ &= \int_0^\infty e^{\left(\frac{1}{\gamma}-1\right)rt - \frac{1}{\gamma}\rho t}\, {}_tp_x \left(\frac{{}_t\widetilde{p}_x}{{}_tp_x} \right)^{1/\gamma} \frac{1}{\lambda_A^{1/\gamma}}\,\mathrm{d}t\,. \end{aligned}$$

As a consequence, we obtain

$$\lambda_A = \left(\frac{1}{v} \int_0^\infty e^{\left(\frac{1}{\gamma}-1\right)rt - \frac{1}{\gamma}\rho t}\, {}_tp_x \left(\frac{{}_t\widetilde{p}_x}{{}_tp_x} \right)^{1/\gamma} \mathrm{d}t \right)^{\gamma}.$$

The expected discounted lifetime utility is then given by

$$\begin{aligned} U_A &= \widetilde{\mathbb{E}}\left[\int_0^\infty e^{-\rho t}\, \mathbb{1}_{\{t<\zeta\}}\, u\big(c^*(t)\big)\,\mathrm{d}t \right] \\ &= \int_0^\infty e^{-\rho t}\, {}_t\widetilde{p}_x\, u\big(c^*(t)\big)\,\mathrm{d}t \\ &= \frac{1}{1-\gamma} \int_0^\infty e^{-\rho t}\, {}_t\widetilde{p}_x \frac{e^{\frac{1-\gamma}{\gamma}(r-\rho)t}}{\lambda_A^{\frac{1-\gamma}{\gamma}}} \left(\frac{{}_t\widetilde{p}_x}{{}_tp_x} \right)^{\frac{1-\gamma}{\gamma}} \mathrm{d}t \\ &= \frac{\left(\lambda_A^{\frac{1}{\gamma}}\right)^{\gamma-1}}{1-\gamma} \int_0^\infty e^{-\frac{1}{\gamma}\rho t + \frac{1-\gamma}{\gamma} rt}\, {}_tp_x \left(\frac{{}_t\widetilde{p}_x}{{}_tp_x} \right)^{\frac{1}{\gamma}} \mathrm{d}t \\ &= \frac{\left(\lambda_A^{\frac{1}{\gamma}}\right)^{\gamma-1}}{1-\gamma} \lambda_A^{\frac{1}{\gamma}}\, v = \frac{\lambda_A}{1-\gamma} v\,. \quad \square \end{aligned}$$

A.2. Proof of Theorem 4.2

We obtain the following Lagrangian function for our optimization problem:

$$\begin{aligned} \mathcal{L} &= \int_0^\infty e^{-\rho t}\, u\big(d(t)\big)\, \widetilde{\mathbb{E}}\left[\mathbb{1}_{\{\zeta>t\}} \left(\frac{n}{N(t)} \right)^{1-\gamma} \right] \mathrm{d}t \\ &\quad + \lambda_T \left(v - \int_0^\infty e^{-rt}\, \mathbb{E}\Big[\Big(1 - \Big(1 - e^{-\int_0^t \mu_{x+s}\mathrm{d}s} \Big)^n \Big) \Big] d(t)\,\mathrm{d}t \right) \\ &= \int_0^\infty e^{-\rho t}\, u\big(d(t)\big)\, \kappa_{n,\gamma}\left({}_t\widehat{P}_x, {}_t\widetilde{P}_x \right) \mathrm{d}t \\ &\quad + \lambda_T \left(v - \int_0^\infty e^{-rt}\, \mathbb{E}\Big[\Big(1 - \Big(1 - e^{-\int_0^t \mu_{x+s}\mathrm{d}s} \Big)^n \Big) \Big] d(t)\,\mathrm{d}t \right). \end{aligned}$$

with $\kappa_{n,\gamma}\left({}_t\widehat{P}_x, {}_t\widetilde{P}_x\right)$ defined as in (18). The first order condition is equivalent to

$$d^*(t) = \frac{e^{\frac{(r-\rho)t}{\gamma}}\left(\kappa_{n,\gamma}\left({}_t\widehat{P}_x, {}_t\widetilde{P}_x\right)\right)^{1/\gamma}}{\lambda_T^{1/\gamma}\,\mathbb{E}\left[\left(1-\left(1-e^{-\int_0^t \mu_{x+s}\mathrm{d}s}\right)^n\right)\right]^{1/\gamma}},$$

Now we can use the budget constraint to determine the Lagrangian multiplier λ_T:

$$\begin{aligned} v &= \int_0^\infty e^{-rt}\,\mathbb{E}\left[\left(1-\left(1-e^{-\int_0^t \mu_{x+s}\mathrm{d}s}\right)^n\right)\right] d^*(t)\,\mathrm{d}t \\ &= \int_0^\infty e^{\left(\frac{1}{\gamma}-1\right)rt-\frac{1}{\gamma}\rho t} \\ &\quad \cdot \frac{\left(\kappa_{n,\gamma}\left({}_t\widehat{P}_x, {}_t\widetilde{P}_x\right)\right)^{1/\gamma}}{\lambda_T^{1/\gamma}\left(\mathbb{E}\left[\left(1-\left(1-e^{-\int_0^t \mu_{x+s}\mathrm{d}s}\right)^n\right)\right]\right)^{1/\gamma-1}}\,\mathrm{d}t\,. \end{aligned}$$

As a consequence, we obtain

$$\lambda_T = \left(\frac{1}{v}\int_0^\infty e^{\left(\frac{1}{\gamma}-1\right)rt-\frac{1}{\gamma}\rho t} \cdot \frac{\left(\kappa_{n,\gamma}\left({}_t\widehat{P}_x, {}_t\widetilde{P}_x\right)\right)^{1/\gamma}}{\left(\mathbb{E}\left[\left(1-\left(1-e^{-\int_0^t \mu_{x+s}\mathrm{d}s}\right)^n\right)\right]\right)^{1/\gamma-1}}\,\mathrm{d}t\right)^\gamma.$$

The expected discounted lifetime utility is then given by

$$\begin{aligned} U_T &:= \widetilde{\mathbb{E}}\left[\int_0^\infty e^{-\rho t}\mathbb{1}_{\{t<\zeta\}}\left(\frac{n}{N(t)}\right)^{1-\gamma} u\left(d^*(t)\right)\mathrm{d}t\right] \\ &= \int_0^\infty e^{-\rho t}\,\kappa_{n,\gamma}\left({}_t\widehat{P}_x, {}_t\widetilde{P}_x\right)\,u\left(d^*(t)\right)\mathrm{d}t \\ &= \frac{1}{1-\gamma}\int_0^\infty e^{-\rho t}\kappa_{n,\gamma}\left({}_t\widehat{P}_x, {}_t\widetilde{P}_x\right) \\ &\quad \cdot \frac{e^{\frac{1-\gamma}{\gamma}(r-\rho)t}\left(\kappa_{n,\gamma}\left({}_t\widehat{P}_x, {}_t\widetilde{P}_x\right)\right)^{\frac{1-\gamma}{\gamma}}}{\mu_T^{\frac{1-\gamma}{\gamma}}\left(\mathbb{E}\left[\left(1-\left(1-e^{-\int_0^t \mu_{x+s}\mathrm{d}s}\right)^n\right)\right]\right)^{\frac{1-\gamma}{\gamma}}}\,\mathrm{d}t \\ &= \frac{\left(\lambda_T^{\frac{1}{\gamma}}\right)^{\gamma-1}}{1-\gamma}\int_0^\infty e^{-\frac{1}{\gamma}\rho t+\frac{1-\gamma}{\gamma}rt} \\ &\quad \cdot \frac{\left(\kappa_{n,\gamma}\left({}_t\widehat{P}_x, {}_t\widetilde{P}_x\right)\right)^{\frac{1}{\gamma}}}{\left(\mathbb{E}\left[\left(1-\left(1-e^{-\int_0^t \mu_{x+s}\mathrm{d}s}\right)^n\right)\right]\right)^{\frac{1-\gamma}{\gamma}}}\,\mathrm{d}t \\ &= \frac{\left(\lambda_T^{\frac{1}{\gamma}}\right)^{\gamma-1}}{1-\gamma}\lambda_T^{\frac{1}{\gamma}}v = \frac{\lambda_T}{1-\gamma}v\,. \quad \square \end{aligned}$$

A.3. Proof of Proposition 5.1

Recall that the policyholder perceives the number of other members in the pool as $N(t)-1 \mid \{{}_t\widehat{P}_x\} \sim \mathrm{Bin}(n-1, {}_t\widehat{P}_x)$, where ${}_t\widehat{P}_x = e^{-\int_0^t \widehat{\mu}_{x+s}\,\mathrm{d}s}$. Note that the optimal level of expected utility of the tontine U_T given in (21) depends on $N(t)$ only in terms of

$$\kappa_{n,\gamma}\left({}_t\widehat{P}_x, {}_t\widetilde{P}_x\right) = \widetilde{\mathbb{E}}\left[{}_t\widetilde{P}_x\,\widetilde{\mathbb{E}}\left[\left(\frac{n}{N(t)}\right)^{1-\gamma}\,\middle|\,\zeta>t, \widetilde{\mathcal{E}}_t\right]\right].$$

Define $p := {}_t\widehat{P}_x$. To figure out how U_T changes with the conditional survival curve p, it thus suffices to determine the behavior of $\kappa_{n,\gamma}\left({}_t\widehat{P}_x, {}_t\widetilde{P}_x\right)$ in terms of p:

- It is decreasing in p for $\gamma > 1$. This can be seen as follows: It is shown in Milevsky and Salisbury (2015) that for any random variable $N(p)$ with $N(p)-1 \sim \mathrm{Bin}(n-1,p)$, it holds that

$$\frac{\mathrm{d}}{\mathrm{d}p}\widetilde{\mathbb{E}}\left[f\left(N(p)\right)\right] = \frac{1}{p}\,\widetilde{\mathbb{E}}\left[\left(N(p)-1\right)\left(f\left(N(p)\right)-f\left(N(p)-1\right)\right)\right].$$

  Therefore, we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}p}\,\widetilde{\mathbb{E}}\left[\left(\frac{n}{N_\epsilon(t)}\right)^{1-\gamma}\,\middle|\,\zeta>t, \widetilde{\mathcal{E}}_t\right] \\ &= n^{1-\gamma}\frac{\mathrm{d}}{\mathrm{d}p}\,\widetilde{\mathbb{E}}\left[N(t)^{\gamma-1}\,\middle|\,\zeta>t, \widetilde{\mathcal{E}}_t\right] \\ &= \frac{n^{1-\gamma}}{p}\widetilde{\mathbb{E}}\Big[\underbrace{\left(N(t)-1\right)}_{\geq 0}\underbrace{\left(N(t)^{\gamma-1}-\left(N(t)-1\right)^{\gamma-1}\right)}_{\geq 0} \\ &\qquad \Big|\,\zeta>t, \widetilde{\mathcal{E}}_t\Big] \geq 0\,. \end{aligned}$$

  This implies that $\kappa_{n,\gamma}\left({}_t\widehat{P}_x, {}_t\widetilde{P}_x\right)$ is increasing in p. Since $\gamma > 1$, the utility decreases as p increases.
- Now let us consider the case $\gamma \in (0,1)$: We obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}p}\,\widetilde{\mathbb{E}}\left[\left(\frac{n}{N(t)}\right)^{1-\gamma}\,\middle|\,\zeta>t, \widetilde{\mathcal{E}}_t\right] \\ &= \frac{n^{1-\gamma}}{p}\,\widetilde{\mathbb{E}}\left[\left(N(t)-1\right)\left(N(t)^{\gamma-1}-\left(N(t)-1\right)^{\gamma-1}\right)\,\middle|\,\zeta>t, \widetilde{\mathcal{E}}_t\right] \\ &= \frac{n^{1-\gamma}}{p}\,\widetilde{\mathbb{E}}\left[\left(N(t)-1\right)N(t)^{\gamma-1}-\left(N(t)-1\right)^{\gamma}\,\middle|\,\zeta>t, \widetilde{\mathcal{E}}_t\right] \\ &= \frac{n^{1-\gamma}}{p}\,\widetilde{\mathbb{E}}\Big[\left(\left(N(t)-1\right)N(t)^{\gamma-1}-\left(N(t)-1\right)^{\gamma}\right)\mathbb{1}_{\{N(t)=1\}} \\ &\qquad \Big|\,\zeta>t, \widetilde{\mathcal{E}}_t\Big] \\ &\quad + \frac{n^{1-\gamma}}{p}\,\widetilde{\mathbb{E}}\Big[\left(N(t)-1\right)\left(N(t)^{\gamma-1}-\left(N(t)-1\right)^{\gamma-1}\right) \\ &\quad \cdot \mathbb{1}_{\{N(t)\geq 2\}}\,\Big|\,\zeta>t, \widetilde{\mathcal{E}}_t\Big] \\ &= \frac{n^{1-\gamma}}{p}\,\widetilde{\mathbb{E}}\Big[\underbrace{\left(N(t)-1\right)}_{>0}\underbrace{\left(N(t)^{\gamma-1}-\left(N(t)-1\right)^{\gamma-1}\right)}_{\leq 0} \\ &\quad \cdot \mathbb{1}_{\{N(t)\geq 2\}}\,\Big|\,\zeta>t, \widetilde{\mathcal{E}}_t\Big] \leq 0\,. \end{aligned}$$

  This implies that $\kappa_{n,\gamma}\left({}_t\widehat{P}_x, {}_t\widetilde{P}_x\right)$ is increasing in p. Since $1-\gamma > 0$, the utility decreases as p increases.

The certainty equivalent defined in (23) increases in the expected utility, so it decreases in $p = {}_t\widehat{P}_x = e^{-\int_0^t \widehat{\mu}_{x+s}\,\mathrm{d}s}$ as well. $\square$

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