MULTIVARIATE MAX-STABLE PROCESSES AND HOMOGENEOUS FUNCTIONALS

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Abstract: Multivariate max-stable processes are important for both theoretical investigations and various statistical applications motivated by the fact that these are limiting processes, for instance of stationary multivariate regularly varying time series, [1]. In this contribution we explore the relation between homogeneous functionals and multivariate max-stable processes and discuss the connections between multivariate max-stable process and zonoid / max-zonoid equivalence. We illustrate our results considering Brown-Resnick and Smith processes.

Key Words: Multivariate max-stable process; homogenoeus functions; stationary process; zonoid equivalence; max-zonoid equivalence; Brown-Resncik process;
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1. Introduction

Let $\boldsymbol{X}(t) = (X_1(t), \dots, X_d(t)), t \in \mathcal{T} = \mathbb{R}^p$ be a d-dimensional max-stable process with continuous sample paths and Fréchet marginal distribution functions (df's); here d, p are positive integers. In the light of de Haan characterisation, see e.g., [2, 3] we shall consider for simplicity $\boldsymbol{X}$ such that for some $\alpha > 0$ it has the following representation (in distribution)

$$\boldsymbol{X}(t) = \max_{i \ge 1} \Gamma_i^{-1/\alpha} \boldsymbol{Z}^{(i)}(t), \quad t \in \mathcal{T}, \tag{1.1}$$

where $\Gamma_i = \sum_{k=1}^{i} E_k$ with $E_k, k \ge 1$ unit exponential random variables (rv's) independent of $\boldsymbol{Z}^{(i)}$'s, which are independent copies of a d-dimensional process $\boldsymbol{Z}(t) = (Z_1(t), \dots, Z_d(t)), t \in \mathcal{T}$ with continuous sample paths and non-negative components. As in [4] (therein $\boldsymbol{Z}_i$'s have strictly positive components) see also [5], for any $t_i \in \mathcal{T}, \boldsymbol{x}_i \in (0, \infty)^d, i \le n$

$$\begin{aligned} (\mathbb{P}\{\boldsymbol{X}(t_i) \le \boldsymbol{x}_i, 1 \le i \le n\})^c &= \mathbb{P}\{\boldsymbol{X}(t_i) \le \boldsymbol{x}_i / c^{1/\alpha}, 1 \le i \le n\} \\ &= \exp\Big(-c\mathbb{E}\{\max_{1 \le i \le d, 1 \le j \le n} Z_i^\alpha(t_j)/x_{ij}^\alpha\}\Big) \end{aligned} \tag{1.2}$$

is valid for any $c > 0$. Eq. (1.2) is the so-called max-stability property of $\boldsymbol{X}$; here we write often $\boldsymbol{X}$ instead of $\boldsymbol{X}(t), t \in \mathcal{T}$ and similarly for other processes and refer to $\boldsymbol{Z}$ as the spectral process of $\boldsymbol{X}$.

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Write next C for the space of all continuous functions $f : \mathcal{T} \to [0,\infty)^d$. It is well-known that C can be equipped with a metric that turns it into a Polish space for which its Borel σ-field coincides with $\mathcal{C}$, the σ-field generated by projection maps $\pi_t, t \in T_0$, with T_0 a dense countable subset of $\mathcal{T}$. Given $h \in \mathcal{T}$ and some norm $\|\cdot\|$ on $\mathbb{R}^d$, define the tilted process $\boldsymbol{\Theta}^{[h]}$ (its law depends on the chosen norm $\|\cdot\|$) by

$$\mathbb{P}\{\boldsymbol{\Theta}^{[h]} \in A\} = \frac{1}{\mathbb{E}\{\|\boldsymbol{Z}(h)\|^\alpha\}}\mathbb{E}\{\|\boldsymbol{Z}(h)\|^\alpha \mathbb{I}(\boldsymbol{Z}/\|\boldsymbol{Z}(h)\| \in A)\} \tag{1.3}$$

for all $A \in \mathcal{C}$ with $\mathbb{I}(\cdot)$ the indicator function. For notational simplicity we have assumed that $\boldsymbol{\Theta}^{[h]}$ is defined in the same probability space as $\boldsymbol{Z}$.

In view of (1.2), if R is a non-negative rv with $\mathbb{E}\{R^\alpha\} = 1$ and independent of $\boldsymbol{Z}$, then clearly from (1.2)

$$\widetilde{\boldsymbol{Z}}(t) = R\boldsymbol{Z}(t) \tag{1.4}$$

is another spectral process for $\boldsymbol{X}$; in our notation $a\boldsymbol{x} = (ax_1, \ldots, ax_d), a \in \mathbb{R}, \boldsymbol{x} \in \mathbb{R}^d$.

An interesting alternative to random scaling is tilting, if for some $h \in \mathcal{T}$ we have that $\mathbb{P}\{\max_{1\le i\le n} Z_i(h) > 0\} = 1$. Indeed, (1.2) can be re-written as (set $a_h := \mathbb{E}\{\|\boldsymbol{Z}(h)\|^\alpha\}$ which is positive and finite by the assumption on Fréchet marginals of $\boldsymbol{X}$)

$$\ln \mathbb{P}\{\boldsymbol{X}(t_i) \le \boldsymbol{x}_i, 1 \le i \le n\} = -\mathbb{E}\{\|\boldsymbol{Z}(h)\|^\alpha / a_h \max_{1\le i\le d, 1\le j\le n} a_h (Z_i(t_j)/\|\boldsymbol{Z}(h)\|)^\alpha / x_{ij}^\alpha\}$$

and thus $\widetilde{\boldsymbol{Z}}(t) = a_h^{1/\alpha}\boldsymbol{\Theta}^{[h]}(t), t \in \mathcal{T}$ is also another spectral process for $\boldsymbol{X}$. By definition of $\boldsymbol{\Theta}^{[h]}$ we have that

$$\|\widetilde{\boldsymbol{Z}}(h)\|^\alpha = a_h \|\boldsymbol{\Theta}^{[h]}(h)\|^\alpha = a_h \in (0,\infty) \tag{1.5}$$

almost surely. When $d = 1$ in view of Balkema's Lemma given in [6][Lem 4.1] (see Section 4 below for details) we have that any spectral process $\widetilde{\boldsymbol{Z}}$ of $\boldsymbol{X}$ that satisfies (1.5) has the same law as $a_h^{1/\alpha}\boldsymbol{\Theta}^{[h]}$. If $\mathbb{P}\{\max_{1\le i\le d} Z_i(h) = 0\} \in (0,1)$, then $a_h^{1/\alpha}\boldsymbol{\Theta}^{[h]}$ can not be a spectral process for $\boldsymbol{X}$. It is nonetheless possible to construct a spectral process for $\boldsymbol{X}$ by utilising a family of $\boldsymbol{\Theta}^{[h]}$'s, see Section 2.

It follows from (1.2) that for given two spectral processes $\boldsymbol{Z}$ and $\widetilde{\boldsymbol{Z}}$ of $\boldsymbol{X}$ and all maps $H(f) = \max_{1\le i\le d, 1\le j\le n}(f_i(t_j))^\alpha / x_{ij}, f \in C$ with t_j's in $\mathcal{T}$ and x_{ij}'s in $(0,\infty)$

$$\mathbb{E}\{H(\boldsymbol{Z})\} = \mathbb{E}\{H(\widetilde{\boldsymbol{Z}})\}. \tag{1.6}$$

All maps H defined above belong to the class E_α of all non-negative measurable α-homogeneous maps $H : C \mapsto [0,\infty]$; here H is β-homogeneous means that $H(cf) = c^\beta H(f)$ for any $c > 0, f \in C$. In Proposition 2.1 below we show how to construct $\tilde{\boldsymbol{Z}}$ and prove further that (1.6) holds for all spectral processes $\boldsymbol{Z}, \tilde{\boldsymbol{Z}}$ and all $H \in E_\alpha$. It is known from [7] that homogeneous functions play a crucial role for the study of max-stable random vectors. Therefore the claimed validity of (1.6) does not come as a surprise.

In Section 2 we discuss briefly the implications of (1.6) for stationary max-stable processes, whereas in Section 3 we focus on the relations between zonoid equivalence, max-zonoid equivalence and homogeneous functions. Section 4 is dedicated to Smith and Brown-Resnick max-stable processes where we derive also tractable formulas for their fidi's complementing previous results in [5]. All the proofs are relegated to Section 5.

To this end, we mention that numerous results for max-stable processes and their representations exist in the literature, see for instance [8–11]. Our findings in this paper, which have certain consequences for stationary max-stable processes, are motivated by recent contributions [1, 12–15] concerned with multivariate regularly varying time series and their relation to max-stable processes.

2. Results

Let $\boldsymbol{X}$ be a max-stable process as in the Introduction with paths in C, de Haan representation (1.1) and spectral process $\boldsymbol{Z}$ with sample paths in C such that (2.2) holds. In view of [3] for any compact set $K \subset \mathcal{T}$ we have

$$\mathbb{E}\left\{\sup_{t\in K}\|\boldsymbol{Z}(t)\|^\alpha\right\} < \infty, \tag{2.1}$$

which together with [14][Lem 7.1] implies that we can assume without loss of generality that

$$\mathbb{P}\left\{\sup_{t\in \mathcal{T}}\|\boldsymbol{Z}(t)\| > 0\right\} = 1. \tag{2.2}$$

For notational simplicity we shall suppose hereafter that

$$\mathbb{E}\{\|\boldsymbol{Z}(t)\|^\alpha\} = 1, \quad \forall t \in \mathcal{T}. \tag{2.3}$$

In the following we consider $\boldsymbol{\Theta}^{[h]}$'s to be independent and defined in the same non-atomic probability space $(\Omega, \mathcal{F}, \mathbb{P})$, this is possible in view of [16][Corr. 5.8]. Since $(C, \mathcal{C})$ is a Polish space and $\boldsymbol{X}$ has almost surely sample paths in C, in view of [17][Lem p. 1276] $\boldsymbol{X}$ can be realised also as a random process defined on $(\Omega, \mathcal{F}, \mathbb{P})$, which we shall assume in the sequel.

Hereafter $T_0 = \mathbb{Q}^p$ with $\mathbb{Q}$ the set of rational numbers, W is a T_0-valued rv with probability mass function (pmf) $p(t) > 0, t \in T_0$ being independent of everything else.

We state next our first result:

Proposition 2.1. *Let $\boldsymbol{X}$ be max-stable with sample paths in C and spectral process $\boldsymbol{Z}$ such that (2.2) and (2.3) hold. Define $\boldsymbol{\Theta}^{[h]}$ as in (1.3) and let $\widetilde{\boldsymbol{Z}}$ be a random process with sample paths in C.*

i) If $\widetilde{\boldsymbol{Z}}$ is a spectral process of $\boldsymbol{X}$ such that $\mathbb{P}\{\sup_{t\in\mathcal{T}}\|\widetilde{\boldsymbol{Z}}(t)\| > 0\} = 1$, then (1.6) holds for all $H \in E_\alpha$.

Conversely, if $\mathbb{E}\{\|\boldsymbol{Z}(h)\|^\alpha\} = \mathbb{E}\{\|\widetilde{\boldsymbol{Z}}(h)\|^\alpha\} = 1, h \in \mathcal{T}$ for some $\alpha > 0$ satisfying (1.6) with $H = H_h(f) = \|f(h)\|^\alpha \Gamma(f), f \in \mathcal{C}$ for all $\Gamma \in E_0, h \in \mathcal{T}$, then the corresponding max-stable

processes $\boldsymbol{X}$ and $\widetilde{\boldsymbol{X}}$ of $\boldsymbol{Z}$ and $\widetilde{\boldsymbol{Z}}$, respectively are equal in law.
ii) All random processes $\boldsymbol{Z}_W$ given by

$$\boldsymbol{Z}_W(t) = \frac{1}{(\sum_{s\in T_0}\|\boldsymbol{\Theta}^{[W]}(s)\|^\alpha p(s))^{1/\alpha}}\boldsymbol{\Theta}^{[W]}(t), \quad t \in \mathcal{T} \tag{2.4}$$

are spectral processes for $\boldsymbol{X}$.

Remark 2.2. *i) In view of Proposition 2.1 the law of $\boldsymbol{\Theta}^{[h]}, h \in \mathcal{T}$ does not depend on the particular choice of the spectral process $\boldsymbol{Z}$ since by (1.6)*

$$\mathbb{E}\{\mathbb{I}(\boldsymbol{\Theta}^{[h]} \in A)\} = \mathbb{E}\{\|\boldsymbol{Z}(h)\|^\alpha \mathbb{I}(\boldsymbol{Z}/\|\boldsymbol{Z}(h) \in A\|)\} = \mathbb{E}\{\|\widetilde{\boldsymbol{Z}}(h)\|^\alpha \mathbb{I}(\widetilde{\boldsymbol{Z}}/\|\widetilde{\boldsymbol{Z}}(h)\| \in A)\}, \quad \forall A \in \mathcal{C}$$

if $\widetilde{\boldsymbol{Z}}$ is another spectral process of $\boldsymbol{X}$ (recall we assume (2.3)).
ii) If $\boldsymbol{X}$ is max-stable stationary with sample paths in C, unit Fréchet marginals and spectral process $\boldsymbol{Z}$, then by definition $B^h\boldsymbol{X}$ and $\boldsymbol{X}$ have the same law for any $h \in \mathcal{T}$; here $B^h\boldsymbol{X}(t) = \boldsymbol{X}(t-h), h, t \in \mathcal{T}$. In the light of (1.2) this is equivalent with $B^h\boldsymbol{Z}$ is a spectral process for $\boldsymbol{X}$ for any $h \in \mathcal{T}$, which in view of Proposition 2.1 implies for any $\widetilde{\boldsymbol{Z}}$ a spectral process of $\boldsymbol{X}$

$$\mathbb{E}\{\Gamma_\alpha(\boldsymbol{Z})\} = \mathbb{E}\{\Gamma_\alpha(B^h\boldsymbol{Z})\} = \mathbb{E}\{\Gamma_\alpha(B^h\widetilde{\boldsymbol{Z}})\}, \quad \forall \Gamma_\alpha \in E_\alpha. \tag{2.5}$$

By Proposition 2.1 the above also implies the stationarity of $\boldsymbol{X}$. Note in passing that our claim here extends [18][Thm 4.3,6.9] to the vector-valued setup.

Example 1: Let $\boldsymbol{Z}(t), t \in \mathcal{T}$ be a BRs process as in Proposition 2.3. For a given functional $F \in E_0$ which is shift-invariant in the sense that $F(B^h f) = F(f)$ for any $h \in \mathcal{T}, f \in C$ define a new spectral process $\widetilde{\boldsymbol{Z}}(t) = \boldsymbol{Z}(t)F(\boldsymbol{Z}), t \in \mathcal{T}$. Suppose that $\mathbb{E}\{\|\widetilde{\boldsymbol{Z}}(t_0)\|^\alpha\} \in (0, \infty)$ for some $t_0 \in \mathcal{T}$. Since for given $h \in \mathcal{T}$ by (2.5)

$$\mathbb{E}\{\Gamma_\alpha(\widetilde{\boldsymbol{Z}})\} = \mathbb{E}\{\Gamma_\alpha(\boldsymbol{Z}F(\boldsymbol{Z}))\} = \mathbb{E}\{\Gamma_\alpha(B^h\boldsymbol{Z}F(B^h\boldsymbol{Z}))\} = \mathbb{E}\{\Gamma_\alpha(B^h\widetilde{\boldsymbol{Z}})\}$$

for all $\Gamma \in E_\alpha$ we have that $\widetilde{\boldsymbol{Z}}$ is also a BRs process.
We set below $\boldsymbol{\Theta} := \boldsymbol{\Theta}^{[0]}$ and present three equivalent conditions for the stationarity of a max-stable process $\boldsymbol{X}$ with sample paths in C extending thus [18][Thm 4.3].

Corollary 2.3. *If $\boldsymbol{X}, \boldsymbol{Z}$ are as in Proposition 2.1, then $\boldsymbol{X}$ is stationary if and only if:*
i) For any T_0-valued rv W with pmf $p(t) > 0, t \in T_0$ being independent of everything else

$$\boldsymbol{Z}_W(t) = \frac{1}{(\sum_{s\in T_0}\|\boldsymbol{\Theta}(s-W)\|^\alpha p(s))^{1/\alpha}}\boldsymbol{\Theta}(t-W), \quad t \in \mathcal{T} \tag{2.6}$$

is a spectral process for $\boldsymbol{X}$.
ii) For all $h \in \mathcal{T}, \Gamma \in E_0$ and all positive integers $k \le d$

$$\mathbb{E}\{Z_k^\alpha(h)\Gamma(\boldsymbol{Z})\} = \mathbb{E}\{Z_k^\alpha(0)\Gamma(B^h\boldsymbol{Z})\}. \tag{2.7}$$

iii) For all $h \in \mathcal{T}$ and all positive integers $k \le d$

$$\boldsymbol{\Theta}^{[h,k]} \stackrel{d}{=} B^h\boldsymbol{\Theta}^{[0,k]}, \tag{2.8}$$

where $\boldsymbol{\Theta}^{[h,k]} = (\Theta_1^{[h,k]}, \ldots, \Theta_d^{[h,k]})$ *is defined by*

$$\mathbb{P}\{\boldsymbol{\Theta}^{[h,k]} \in A\} = \mathbb{E}\{\frac{Z_k^\alpha(h)}{\mathbb{E}\{Z_k^\alpha(h)\}}\mathbb{I}(\boldsymbol{Z}/Z_k(h) \in A)\}, \quad \forall A \in \mathcal{C},$$

with $\boldsymbol{\Theta}^{[h,k]}(t), t \in \mathcal{T}$ *equal to* $(1, \ldots, 1) \in \mathbb{R}^d$ *if* $\mathbb{E}\{Z_k^\alpha(h)\} = 0$.

Remark 2.4. *i) A general condition for the stationarity of* $\boldsymbol{X}$ *in terms of spectral processes based on the findings of [19] is derived in [20][Thm 1] for the discrete setup. When* $\boldsymbol{Z}(t), t \in \mathcal{T}$ *has strictly positive components for any* $t \in \mathbb{R}$, *a simple condition for the stationarity of* $\boldsymbol{X}$ *is given in [4].*
ii) Using (2.5) it follows that $\boldsymbol{X}$ *is stationary if and only if*

$$\boldsymbol{\Theta}^{[h]} \stackrel{d}{=} B^h \boldsymbol{\Theta}^{[0]}, \tag{2.9}$$

which is first shown for $d = 1$ *in [18][Thm 4.3, Eq. (4.6)].*
iii) Stationary of max-stable processes can be alternativly studied by relating it to the shift-invariance of the corresponding tail measure as in [1, 21].
iv) If $\mathcal{T} = \mathbb{R}$ *we can similarly consider max-stable process with cadlag sample paths. All the results derived above remain unchanged, see [21] which deals with stationary* $\boldsymbol{X}$.

3. Zonoid and max-zonoid equivalence

In this section we discuss the relations between zonoid, max-zonoid equivalence and homogeneous functions. Essentially, we state some known results presenting some short and new proofs.
We shall consider below the case $\alpha = 1$. Let $Z = (Z_1, \ldots, Z_n)$ and $Z^* = (Z_1^*, \ldots, Z_n^*)$ be two random vectors not identically equal to zero with integrable components. As in [7] we shall call Z and Z^* zonoid equivalent if

$$\mathbb{E}\{F_u(Z)\} = \mathbb{E}\{F_u(Z^*)\}, \quad \forall u = (u_1, \ldots, u_n) \in \mathbb{R}^n, \tag{3.1}$$

where $F_u(Z) = |\sum_{i=1}^n u_i Z_i|$; abbreviate the above as $Z \stackrel{z.}{=} Z^*$.
The distribution of Z, in view of Hardin [22][Thm 1.1] is uniquely determined if we know $\mathbb{E}\{|\sum_{i=0}^n u_i Z_i|\}$ for any $u_i \in \mathbb{R}, 0 \le i \le n$ assuming that $Z_0 = 1$ almost surely. Consequently, if $(1, Z) \stackrel{z.}{=} (1, Z^*)$, then $Z \stackrel{d}{=} Z^*$.

Lemma 3.1. *If* Z *and* Z^* *have non-negative components, then* $Z \stackrel{z.}{=} Z^*$ *if and only if for any* 1*-homogeneous measurable function* $H : \mathbb{R}^n \to [0, \infty)$

$$\mathbb{E}\{H(Z)\} = \mathbb{E}\{H(Z^*)\}. \tag{3.2}$$

As in [7] we shall call non-negative Z and Z^* max-zonoid equivalent, if (3.1) holds with $F_u(Z) = \max_{0 \le i \le n} u_i Z_i$, where $u_0 Z_0 = 0$. We use the abbreviation $Z \stackrel{max-z.}{=} Z^*$; note in passing that $Z \stackrel{max-z.}{=} Z^*$ is the same as $X \stackrel{d}{=} X^*$, where X and X^* are max-stable random vectors whose spectral process are Z and Z^*, respectively. The counterpart of Hardin's result is Balkema's Lemma [6][Lem 4.1], which implies that $\mathbb{E}\{\max_{0\le i\le n} u_i Z_i\}$ for any $u_i \in [0,\infty), 0 \le i \le n$ uniquely identifies the distribution of $(1, Z)$. Consequently, by Balkema's Lemma

$$(1, Z) \stackrel{max-z.}{=} (1, Z^*)$$

is equivalent with the equality in distribution $Z \stackrel{d}{=} Z^*$.

Lemma 3.2. *If Z and Z^* have non-negative components, then $Z \stackrel{max-z.}{=} Z^*$ if and only if for any 1-homogeneous measurable function $H : \mathbb{R}^n \to [0,\infty)$ (3.2) holds.*

Remark 3.3. *i) The claim of Lemma 3.1 holds for Z and Z^* that can have negative components requiring further that H is an even function $\mathbb{R}^n \mapsto [0,\infty)$, which follows from [23][Thm 1.1] and [7][Thm 2].*
ii) A direct implication of Lemma 3.1 and Lemma 3.2 is that $Z \stackrel{max-z.}{=} Z^*$ is equivalent with $Z \stackrel{z.}{=} Z^*$; see [24][Thm 2.1], [23][Thm 1.1].

4. Smith and Brown-Resnick processes

In this section we consider briefly Smith and Brown-Resncik processes, see e.g., [4, 5] for details. As shown in the aforementioned article these models are natural limiting models and therefore can be utilised for various statistical applications, which rely often on various tractable formulas for the fidi's of those processes.

4.1. **Smith processes.** For a given parameter $\alpha > 0$, we consider multivariate Smith processes that are constructed by a given deterministic $[0,\infty)^d$-valued function $\boldsymbol{L}(t), t \in \mathcal{T}$ with continuous components $L_i, i \le d$ satisfying

$$0 < \int_{\mathcal{T}} \|\boldsymbol{L}(t)\|_*^\alpha \lambda(dt) \le \int_{\mathcal{T}} \sup_{s\in K} \|\boldsymbol{L}(t-s)\|_*^\alpha \lambda(dt) < \infty \tag{4.1}$$

for any compact set $K \subset \mathbb{R}^p$ and some norm $\|\cdot\|_*$ on $\mathbb{R}^d$. We define a multivariate max-stable Smith process $\boldsymbol{X}$ with paths in C by specifying a spectral process $\boldsymbol{Z}$ of $\boldsymbol{X}$ as follows

$$\boldsymbol{Z}(t) = (1/p(W))^{1/\alpha} \boldsymbol{L}(t - W), \quad t \in \mathcal{T}, \tag{4.2}$$

with W a $\mathcal{T}$-valued rv with positive pdf $p(t) > 0, t \in \mathcal{T}$. For any norm $h \in \mathcal{T}$ and any $\|\cdot\|$ on $\mathbb{R}^d$ we have that $\boldsymbol{\Theta}^{[h]}$ calulcated with respect to this norm is given by

$$\boldsymbol{\Theta}^{[h]}(t) = \boldsymbol{L}(t - h + S)/\|\boldsymbol{L}(S)\| = B^h \boldsymbol{\Theta}^{[0]}(t), \quad t \in \mathcal{T}, \tag{4.3}$$

with S a $\mathcal{T}$-valued rv having pdf $\|\boldsymbol{L}(t)\|^\alpha / c > 0, t \in \mathcal{T}$ with $c = \int_{\mathcal{T}} \|\boldsymbol{L}(t)\|^\alpha \lambda(dt)$.
Note in passing that the inequality in (4.1) is necessary in view of (2.1). Since (4.3) implies (2.9), then the Smith max-stable process $\boldsymbol{X}$ with paths in C is stationary.
Set below $c_k = \int_{\mathcal{T}} L_k^\alpha(t) \lambda(dt) \in [0,\infty)$ for any positive integer $k \le d$. If $c_k > 0$, then

$$\boldsymbol{\Theta}^{[h,k]}(t) = \boldsymbol{L}(t - h + S_k)/L_k(S_k), \quad t \in \mathcal{T}, \tag{4.4}$$

where the $\mathcal{T}$-vaued rv S_k has pdf $L_k^\alpha(t)/c_k, t \in \mathcal{T}$. Since both $\boldsymbol{\Theta}^{[h]}$ and $\boldsymbol{\Theta}^{[h,k]}$ are known explicitly, we can calculate the fidi's of $\boldsymbol{X}$ as shown next. Hereafter $\|\boldsymbol{x}\|_\infty = \max_{1 \le i \le d} |x_i|, \boldsymbol{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$.

Lemma 4.1. *Both* (4.3) *and* (4.4) *are valid and for any* $t_i \in \mathcal{T}, \boldsymbol{x}_i \in [0,\infty)^d, 1 \le i \le n$, *if further* $c_k \in (0,\infty), k \le d$ *and* $c_\infty = \int_{\mathcal{T}} \|\boldsymbol{L}(t)\|_\infty^\alpha \lambda(dt)$ *we have*

$$
\begin{aligned}
&-\ln \mathbb{P}\{\boldsymbol{X}(t_i) \le \boldsymbol{x}_i, 1 \le i \le n\} \\
(4.5) \quad &= c_\infty \sum_{1 \le l \le n} \mathbb{E}\{\|\boldsymbol{L}(S)/\boldsymbol{x}_l\|_\infty^\alpha / \|\boldsymbol{L}(S)\|_\infty^\alpha \mathbb{I}(infargmax_{1 \le j \le n} \|\boldsymbol{L}(t_j - t_l + S)/\boldsymbol{x}_j\|_\infty = l)\} \\
(4.6) \quad &= \sum_{(k,l) \in J} \frac{c_k}{x_{kl}^\alpha} \mathbb{P}\Bigg\{ infargmax_{(i,j) \in J} \frac{L_i(t_j - t_l + S_k)}{x_{ij} L_i(S_k)} = (k,l) \Bigg\},
\end{aligned}
$$

where $J = \{(i,j), 1 \le i \le d, 1 \le j \le p\}$ *and the infargmax funcitonal in* (4.6) *is defined (also in the sequel) taking the maximum with respect to the lexicographical order in* $\mathbb{Z}^2$.

Example 2: Assume that $\alpha = 1$ and $\boldsymbol{L}(t), t \in \mathcal{T}$ has deterministic components $L_i, i \le d$ being equal to the pdf of a Gaussian random vector $\boldsymbol{Y} = (Y_1, \ldots, Y_p)$ with independent $N(0,1)$ components. In view of Lemma 4.1 for $t_i = (t_{i1}, \ldots, t_{ip}) \in \mathcal{T}, \boldsymbol{x}_i \in [0,\infty)^d, 1 \le i \le n$ we have

$$
\begin{aligned}
&-\ln \mathbb{P}\{\boldsymbol{X}(t_i) \le \boldsymbol{x}_i, 1 \le i \le n\} \\
&= \sum_{(k,l) \in J} \frac{1}{x_{kl}} \mathbb{P}\Bigg\{ (i,j) \in J \setminus \{(k,l)\} : \ (\sum_{r=1}^p (t_{lr} - t_{jr}) Y_r - (t_{jr} - t_{lr})^2/2) \le \ln(x_{ij}/x_{kl}) \Bigg\},
\end{aligned}
$$

where $J = \{(i,j), 1 \le j \le d, 1 \le j \le p\}$.

4.2. **Brown-Resnick processes.** In the Brown-Resnick model, $\alpha = 1$ and the spectral process is the log-Gaussian one

$$
(4.7) \qquad \boldsymbol{Z}(t) = e^{\boldsymbol{Y}(t) - \boldsymbol{V}(t)/2}, \quad \boldsymbol{V}(t) = (Var(Y_1(t), \ldots, Var(Y_d(t)), \quad t \in \mathcal{T},
$$

where $\boldsymbol{Y}(t), t \in \mathcal{T}$ is a $\mathbb{R}^d$-valued Gaussian process with centered components.

Lemma 4.2. *If* $\boldsymbol{X}$ *is a Brown-Resnick max-stable process as defined in* (4.7), *then the law of* $\boldsymbol{X}$ *depends only on the matrix-valued function with* ij*th entry equal* $\gamma_{ij}(t,s) = Var(Y_i(t) - Y_j(s)), s, t \in \mathcal{T}$ *where* $1 \le i, j \le d$. *In particular, if* $\gamma_{ij}(t,s), i,j \le d$ *depend only on the difference* $t - s$, *then* $\boldsymbol{X}$ *is stationary. Further, for any* $t_i \in \mathcal{T}, \boldsymbol{x}_i \in (0,\infty)^d, 1 \le i \le n$

$$
\begin{aligned}
&-\ln \mathbb{P}\{\boldsymbol{X}(t_i) \le \boldsymbol{x}_i, 1 \le i \le n\} \\
(4.8) \quad &= \sum_{(k,l) \in J} \frac{1}{x_{kl}} \mathbb{P}\{\forall (i,j) \in J \setminus \{(k,l)\} : Z_i(t_j) - Z_k(t_l) - Var(Z_i(t_j) - Z_k(t_l))/2 \le \ln(x_{ij}/x_{kl})\},
\end{aligned}
$$

provided that $(\boldsymbol{Y}(t_1), \ldots, \boldsymbol{Y}(t_n))$ *possesses a density.*

Remark 4.3. *A similar formula to* (4.8) *is shown for the special case that* $\boldsymbol{X}$ *has stationary increments in [5]; that assumption is not needed in our case. It is worth noting that the formula*

[?] derived for Hüsler-Reiss distributions appears in [25]. Note further that the other claims of Lemma 4.2 are stated without proof in [20].

5. Proofs

Proof of Proposition 2.1 *i*) Let $\boldsymbol{Z}$ and $\widetilde{\boldsymbol{Z}}$ be two spectral processes of $\boldsymbol{X}$ with sample paths in C and let $H_h(f) = \|f(h)\|^\alpha \Gamma(f/\|f(h)\|), f \in C, h \in \mathcal{T}$ for some norm $\|\cdot\|$ on $\mathbb{R}^d$ and Γ a measurable functional $C \mapsto [0,\infty]$. Considering the case that $\|\boldsymbol{x}\| = \|\boldsymbol{x}\|_\infty = \max_{1\le i\le d} |x_i|$ we have

$$\mathbb{E}\{H_h(\boldsymbol{Z})\} = \mathbb{E}\{\|\boldsymbol{Z}(h)\|^\alpha \Gamma(\boldsymbol{Z}/\|\boldsymbol{Z}(h)\|)\} = \mathbb{E}\{\Gamma(\boldsymbol{\Theta}^{[h]})\}, \quad \forall h \in \mathbb{R}, \tag{5.1}$$

with $\boldsymbol{\Theta}^{[h]}$ defined in (1.3). Since Γ can be approximated by simple functions, the claim in (1.6) for $H = H_h$ follows by showing that the law of $\boldsymbol{\Theta}^{[h]}$ is uniquely defined by that of $\boldsymbol{X}$ and this does not depend on the spectral process $\boldsymbol{Z}$. We shall assume for simplicity that

$$\mathbb{E}\{\|\boldsymbol{Z}(h)\|^\alpha\} = 1, \quad \alpha = 1.$$

By the definition $\|\boldsymbol{\Theta}^{[h]}(h)\| = \max_{1\le j\le d} \Theta_j^{[h]}(h) = 1$ almost surely. Using (1.2) for any $s > 1, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in (0,\infty)^d, t_i \in \mathcal{T}, i \le n$ and for any $u > 0$ we have

$$\begin{aligned}
&\mathbb{P}\{\|\boldsymbol{X}(h)\| \le su, \boldsymbol{X}(t_i) \le u\boldsymbol{x}_i, 1 \le i \le n | \|\boldsymbol{X}(h)\| > u\}\\
&= \frac{u\mathbb{P}\{\|\boldsymbol{X}(h)\| \le su, \boldsymbol{X}(t_i) \le u\boldsymbol{x}_i, 1 \le i \le n\} - \mathbb{P}\{\|\boldsymbol{X}(h)\| \le u, \boldsymbol{X}(t_i) \le u\boldsymbol{x}_i, 1 \le i \le n\}}{u\mathbb{P}\{\|\boldsymbol{X}(h)\| > u\}}\\
&\to \mathbb{E}\{\max(\|\boldsymbol{Z}(h)\|, \max_{1\le j\le d, 1\le i\le n} Z_j(t_i)/x_{ij}) - \max(\|\boldsymbol{Z}(h)\|/s, \max_{1\le j\le d, 1\le i\le n} Z_j(t_i)/x_{ij})\}\\
&= \mathbb{E}\{(\|\boldsymbol{Z}(h)\| - \max(\|\boldsymbol{Z}(h)\|/s, \max_{1\le j\le d, 1\le i\le n} Z_j(t_i)/x_{ij})_+\}\\
&= \mathbb{E}\{(1 - \max(1/s, \max_{1\le j\le d, 1\le i\le n} \Theta_j^{[h]}(t_i)/x_{ij})_+\}\\
&= \mathbb{E}\{(1 - \max_{1\le j\le d, 0\le i\le n} \Theta_j^{[h]}(t_i)/x_{ij})_+\}\\
&= \mathbb{P}\{R \le s, R\boldsymbol{\Theta}^{[h]}(t_i) \le \boldsymbol{x}_i, 0 \le i \le n\}
\end{aligned}$$

as $u \to \infty$, with R a Pareto rv with survival function $1/r, r > 1$ independent of $\boldsymbol{\Theta}^{[h]}$ and we set $x_{0j} = 1/s, t_0 = h$. Consequently, we have the convergence in distribution

$$\Big(\frac{\boldsymbol{X}(t_1)}{\|\boldsymbol{X}(h)\|}, \ldots, \frac{\boldsymbol{X}(t_n)}{\|\boldsymbol{X}(h)\|}\Big)\Big|(\|\boldsymbol{X}(h)\| > n) \overset{d}{\to} (\boldsymbol{\Theta}^{[h]}(t_1), \ldots, \boldsymbol{\Theta}^{[h]}(t_n)), \quad n \to \infty.$$

Hence the fidi's of $\boldsymbol{\Theta}^{[h]}$ are uniquely determined by those of $\boldsymbol{X}$. Hence (5.1) holds and further for all $\Gamma \in E_0, h \in \mathcal{T}$

$$\mathbb{E}\{\|\boldsymbol{Z}(h)\|_\infty^\alpha \Gamma(\boldsymbol{Z})\} = \mathbb{E}\{\|\widetilde{\boldsymbol{Z}}(h)\|_\infty^\alpha \Gamma(\widetilde{\boldsymbol{Z}})\}. \tag{5.2}$$

Note in passing that the above convergence in distribution follows also from [12] for any norm on $\mathbb{R}^d$.

Next consider a general norm $\|\cdot\|$ on $\mathbb{R}^d$. Since $\mathbb{E}\{\|\boldsymbol{Z}(h)\|\} \in (0,\infty)$ for any $h \in \mathcal{T}$ implies by the equivalence of the norms on $\mathbb{R}^d$, then also $\mathbb{E}\{\|\boldsymbol{Z}(h)\|_\infty\} \in (0,\infty)$ and by (2.2), (2.3) we can find a non-negative measurable function $q(t), t \in \mathcal{T}$ such that

$$S(\boldsymbol{Z}) = \int_{\mathcal{T}} \|\boldsymbol{Z}(t)\|_\infty q(t)\lambda(dt) \in (0,\infty)$$

almost surely, with $\lambda(dt)$ the Lebesgue measure on $\mathcal{T}$. Moreover, by the assumption on $\widetilde{\boldsymbol{Z}}$ we also have $\mathbb{P}\{S(\widetilde{\boldsymbol{Z}}) \in (0,\infty)\} = 1$. Consequently, for any $H \in E_\alpha$, i.e., $H : C \mapsto [0,\infty]$ is an α-homogeneous measurable functional Fubini-Tonelli theorem yields

$$\begin{aligned}
\mathbb{E}\{H(\boldsymbol{Z})\} &= \mathbb{E}\left\{H(\boldsymbol{Z})\frac{S(\boldsymbol{Z})}{S(\boldsymbol{Z})}\right\} = \int_{\mathcal{T}} \mathbb{E}\left\{\|\boldsymbol{Z}(t)\|_\infty \frac{H(\boldsymbol{Z})}{S(\boldsymbol{Z})}\right\} q(t)\lambda(dt)\\
&=: \int_{\mathcal{T}} \mathbb{E}\{\|\boldsymbol{Z}(t)\|_\infty F(\boldsymbol{Z})\} q(t)\lambda(dt)\\
&= \int_{\mathcal{T}} \mathbb{E}\{\|\widetilde{\boldsymbol{Z}}(t)\|_\infty F(\widetilde{\boldsymbol{Z}})\} q(t)\lambda(dt) = \mathbb{E}\{H(\widetilde{\boldsymbol{Z}})\},
\end{aligned}$$

where we used the fact that $F \in E_0$ and (5.2) for the derivation of the second last equality above. We show below the converse, i.e., for a given norm $\|\cdot\|$ on $\mathbb{R}^d$ we assume that (1.6) holds for any $H_h = \|f(h)\|^\alpha \Gamma(f), h \in \mathcal{T}, \Gamma \in E_0$ and prove that the max-stable processes $\boldsymbol{X}$ and $\widetilde{\boldsymbol{X}}$ with spectral processes $\boldsymbol{Z}$ and $\widetilde{\boldsymbol{Z}}$, respectively have the same fidi's. For given $t_1,\dots,t_n \in \mathcal{T}$ let $H(\boldsymbol{Z}) = \max_{1\le i\le n}\|\boldsymbol{Z}(t_i)/\boldsymbol{x}_i\|_\infty$, with $\boldsymbol{x}_1,\dots,\boldsymbol{x}_n \in (0,\infty)^d$. Putting $S(\boldsymbol{Z}) = \sum_{i=1}^n \|\boldsymbol{Z}(t_i)\|_\infty$ which is positive whenever $H(\boldsymbol{Z})$ is positive, then as above

$$\mathbb{E}\{H(\boldsymbol{Z})\} = \mathbb{E}\left\{H(\boldsymbol{Z})\frac{S(\boldsymbol{Z})}{S(\boldsymbol{Z})}\right\} = \sum_{i=1}^n \mathbb{E}\left\{\|\boldsymbol{Z}(t_i)\|_\infty \frac{H(\boldsymbol{Z})}{S(\boldsymbol{Z})}\right\} = \sum_{i=1}^n \mathbb{E}\{\|\widetilde{\boldsymbol{Z}}(t_i)\|_\infty F(\widetilde{\boldsymbol{Z}})\} = \mathbb{E}\{H(\widetilde{\boldsymbol{Z}})\}$$

since $F \in E_0$ by the assumption on H. Consequently, in view of (1.2) $\boldsymbol{X}$ and $\widetilde{\boldsymbol{X}}$ have the same fidi's.

ii) It follows easily that (recall $\|\boldsymbol{\Theta}^{[h]}(h)\| = 1$ almost surely for all $h \in \mathcal{T}$)

$$I(W) = \sum_{s\in T_0} \|\boldsymbol{\Theta}^{[W]}(s)\|^\alpha p(s) \in (0,\infty)$$

almost surely and thus $\boldsymbol{Z}_W$ is well-defined and with sample paths in C. In order to establish the proof we utilise the claim of statement *i*). Let therefore $H \in E_\alpha$ be given. Using again Fubini-Tonelli theorem we have

$$\begin{aligned}
\mathbb{E}\{H(\boldsymbol{Z}_W)\} &= \mathbb{E}\left\{\sum_{x\in T_0} \frac{1}{I(x)} H(\boldsymbol{\Theta}^{[x]}) p(x)\right\}\\
&= \sum_{x\in T_0} \mathbb{E}\left\{\|\boldsymbol{Z}(x)\|^\alpha \frac{1}{\sum_{s\in T_0}(\|\boldsymbol{Z}(s)\|/\|\boldsymbol{Z}(x)\|)^\alpha p(s)} H(\boldsymbol{Z}/\|\boldsymbol{Z}(x)\|) p(x)\right\}\\
&= \mathbb{E}\left\{H(\boldsymbol{Z}) \sum_{x\in T_0} \frac{\|\boldsymbol{Z}(x)\|^\alpha}{\sum_{s\in T_0}\|\boldsymbol{Z}(s)\|^\alpha p(s)} p(x)\right\} = \mathbb{E}\{H(\boldsymbol{Z})\}
\end{aligned}$$

establishing the proof. □

PROOF OF COROLLARY 2.3 *i)* Assume that $\boldsymbol{X}$ is stationary, i.e., (2.5) holds and recall that $\|\boldsymbol{\Theta}^{[0]}(0)\| = \|\boldsymbol{\Theta}(0)\| = 1$ almost surely for all $h \in \mathcal{T}$. It follows that

$$J(\boldsymbol{\Theta}) = \sum_{s \in T_0} \|B^W \boldsymbol{\Theta}(s)\|^\alpha p(s) \in (0, \infty)$$

almost surely, where W is a T_0-valued rv with pdf $p(t) > 0, t \in T_0$ being further independent of $\boldsymbol{Z}$. Next, for $\boldsymbol{Z}_W$ defined in (2.6), using (2.5) for the derivation of the third last equality, for all $F \in E_0, h \in T_0$ we have

$$\begin{aligned}
\mathbb{E}\{\|\boldsymbol{Z}_W(h)\|^\alpha F(\boldsymbol{Z}_W)\} &= \mathbb{E}\left\{\sum_{x \in T_0} \|\boldsymbol{Z}(0)\|^\alpha \|B^x \boldsymbol{Z}(h)\|^\alpha \frac{1}{J(\boldsymbol{Z})} F(B^x \boldsymbol{Z}) p(x)\right\} \\
&=: \sum_{x \in T_0} \mathbb{E}\{\|\boldsymbol{Z}(0)\|^\alpha \Gamma(B^x \boldsymbol{Z})\} p(x) \\
&= \sum_{x \in T_0} \mathbb{E}\{\|\boldsymbol{Z}(x)\|^\alpha \Gamma(\boldsymbol{Z})\} p(x) \\
&= \mathbb{E}\left\{\|\boldsymbol{Z}(h)\|^\alpha F(\boldsymbol{Z}) \sum_{x \in T_0} \frac{\|\boldsymbol{Z}(x)\|^\alpha p(x)}{\sum_{s \in T_0} \|\boldsymbol{Z}(s)\|^\alpha p(s)}\right\} \\
&= \mathbb{E}\{\|\boldsymbol{Z}(h)\|^\alpha F(\boldsymbol{Z})\}.
\end{aligned}$$

Since T_0 is dense in $\mathcal{T}$ and $\boldsymbol{Z}$ has sample paths in C and thus it is stochastic continuous, then by the above equality and (2.1), applying the dominated convergence theorem yields for all $h \in \mathcal{T}, k > 0$

$$\mathbb{E}\{\|\boldsymbol{Z}_W(h)\|^\alpha F(\boldsymbol{Z}_W) \mathbb{I}(F(\boldsymbol{Z}_W) \le k)\} = \mathbb{E}\{\|\boldsymbol{Z}(h)\|^\alpha F(\boldsymbol{Z}) \mathbb{I}(F(\boldsymbol{Z}_W) \le k)\}.$$

Letting k to infinity, by Proposition 2.1 we conclude that $\boldsymbol{Z}_W$ is a spectral process for $\boldsymbol{X}$. Note in passing that if $F \in E_0$, then also $F(f)\mathbb{I}(F(f) \le k), f \in C$ belongs to E_0.
Conversely, if $\boldsymbol{Z}_W$ defined in (2.6) is a spectral process for $\boldsymbol{X}$ with W independent of $\boldsymbol{\Theta}$ with pdf $p(t) > 0, t \in T_0$, then for any $h \in T_0$

$$\begin{aligned}
B^h \boldsymbol{Z}_W &= B^{W+h} \boldsymbol{\Theta}(t) \frac{1}{(\sum_{s \in T_0} \|B^W \boldsymbol{\Theta}(s)\|^\alpha p(s))^{1/\alpha}} \\
&\stackrel{d}{=} B^{W_h} \boldsymbol{\Theta}(t) \frac{1}{(\sum_{s \in T_0} \|B^{W_h} \boldsymbol{\Theta}(s)\|^\alpha p_h(s))^{1/\alpha}} =: \boldsymbol{Z}_{W_h},
\end{aligned}$$

where $W_h = W + h$ has pdf $p_h(t) = p(t-h)$. Hence $B^h \boldsymbol{Z}_W$ is a spectral process for $\boldsymbol{X}$ since it is equal in law with $\boldsymbol{Z}_{W_h}$, which by the assumption is a spectral process for $\boldsymbol{X}$. Using the stochastic continuity of $\boldsymbol{Z}_W$ and (2.1), then $\boldsymbol{Z}_{W_h}$ is a spectral process of $\boldsymbol{X}$ for all $h \in \mathcal{T}$, thus $\boldsymbol{X}$ is stationary establishing the proof.
If (2.8) holds, then by the definition (2.7) follows. If the latter is satisfied, then for $\|\boldsymbol{x}\|_\alpha = (\sum_{1 \le i \le d} |x_i|^\alpha)^{1/\alpha}$ and all $\Gamma \in E_0$

$$\mathbb{E}\{\|\boldsymbol{Z}(t+h)\|_\alpha^\alpha \Gamma(\boldsymbol{Z})\} = \mathbb{E}\{\|\boldsymbol{Z}(t)\|_\alpha^\alpha \Gamma(B^h \boldsymbol{Z})\}, \quad \forall h, t \in \mathcal{T}. \tag{5.3}$$

Since $\|\boldsymbol{x}\|_\alpha$ vanishes if and only if $\|\boldsymbol{x}\|$ vanishes, where $\|\cdot\|$ is some norm on $\mathbb{R}^d$, as in the proof of Proposition 2.1 for given $t_i h \in \mathcal{T}, \boldsymbol{x}_n \in (0,\infty)^d, 1 \le i \le n$ and

$$H(\boldsymbol{Z}) = \max_{1\le i\le n} \|\boldsymbol{Z}(t_i+h)/\boldsymbol{x}_i\|_\infty$$

by (5.3) we have with $S_\alpha(\boldsymbol{Z}) = \sum_{i=1}^n \|\boldsymbol{Z}(t_i)\|_\alpha$, which is positive whenever $H(\boldsymbol{Z})$ is positive

$$\begin{aligned}\mathbb{E}\{H(\boldsymbol{Z})\} &= \sum_{i=1}^n \mathbb{E}\Big\{\|\boldsymbol{Z}(t_i+h)\|_\alpha^\alpha \frac{H(\boldsymbol{Z})}{S_\alpha(B^{-h}\boldsymbol{Z})}\Big\} =: \sum_{i=1}^n \mathbb{E}\{\|\boldsymbol{Z}(t_i+h)\|_\alpha^\alpha F(\boldsymbol{Z})\} \\ &= \sum_{i=1}^n \mathbb{E}\{\|\boldsymbol{Z}(t_i)\|_\alpha^\alpha F(B^h\boldsymbol{Z})\} = \mathbb{E}\{H(B^h\boldsymbol{Z})\}\end{aligned}$$

since $F \in E_0$ by the assumption on H. Consequently, $\boldsymbol{X}$ and $B^h\boldsymbol{X}$ have the same fidi's and thus $\boldsymbol{X}$ is stationary. □

PROOF OF LEMMA 3.1 First note that $Z \stackrel{z.}{=} Z^*$ is equivalent with

$$(S, Z) \stackrel{z.}{=} (S^*, Z^*),$$

where $S = \sum_{i=1}^n Z_i$ and $S^* = \sum_{i=1}^n Z_i^*$ and $\mathbb{E}\{S\} = \mathbb{E}\{S^*\} = a$. The assumption that Z and Z^* are non-negative, not identical to zero almost surely and with integrable components yields $a \in (0,\infty)$. Assume for simplicity that $a = 1$. For any $H : \mathbb{R}^n \mapsto [0,\infty)$ a 1-homogeneous, measurable function that vanishes at the origin

$$\mathbb{E}\{H(Z)\} = \mathbb{E}\{H(Z)\mathbb{I}(S>0)\} = \mathbb{E}\{H(Z/S)S\} = \mathbb{E}\{H(\widetilde{Z})\}, \tag{5.4}$$

where $\widetilde{Z}$ is the tilted random vector with respect to S. From $Z \stackrel{z.}{=} Z^*$ we have that $\widetilde{Z} \stackrel{z.}{=} \widetilde{Z^*}$ and consequently by the definition of zonoid equivalence

$$(\sum_{i=1}^n \widetilde{Z}_i, \widetilde{Z}) \stackrel{z.}{=} (\sum_{i=1}^n \widetilde{Z^*}_i, \widetilde{Z^*}).$$

Since by the definition $\sum_{i=1}^n \widetilde{Z}_i = \sum_{i=1}^n \widetilde{Z^*}_i = 1$ almost surely, then Hardin's result yields that $\widetilde{Z} \stackrel{d}{=} \widetilde{Z^*}$ and consequently using (5.4), by the measurability of H

$$\mathbb{E}\{H(Z)\} = \mathbb{E}\{H(\widetilde{Z})\} = \mathbb{E}\{H(\widetilde{Z^*})\} = \mathbb{E}\{H(Z^*)\}, \tag{5.5}$$

hence the proof is complete. □

PROOF OF LEMMA 3.2 First note that $Z \stackrel{max-z.}{=} Z^*$ is equivalent with $(M, Z) \stackrel{z.}{=} (M^*, Z^*)$, where $M = \max_{1\le i\le n} Z_i$ and $M^* = \max_{1\le i\le n} Z_i^*$ and $\mathbb{E}\{M\} = \mathbb{E}\{M^*\} = a$. Since $\mathbb{E}\{M\} \le \sum_{i=1}^n \mathbb{E}\{Z_i\}$ and Z and Z^* are non-negative integrable and not zero almost surely we have that $a \in (0,\infty)$. Suppose for simplicity that $a = 1$. For any $H : \mathbb{R}^n \mapsto [0,\infty)$ a 1-homogeneous, measurable function that vanishes at the origin

$$\mathbb{E}\{H(Z)\} = \mathbb{E}\{H(Z)\mathbb{I}(M>0)\} = \mathbb{E}\{H(Z/M)M\} = \mathbb{E}\{H(\widetilde{M})\},$$

where $\widetilde{Z}$ is the tilted random vector (tilted with respect to M). From the above and $Z \stackrel{max-z.}{=} Z^*$ we have that $\widetilde{Z} \stackrel{max-z.}{=} \widetilde{Z^*}$ and consequently

$$(\max_{1\le i\le n} \widetilde{Z}_i, \widetilde{Z}) \stackrel{max-z.}{=} (\max_{1\le i\le n} \widetilde{Z^*}_i, \widetilde{Z^*}).$$

Since by the definition $\max_{1\le i\le n} \widetilde{Z}_i = \max_{1\le i\le n} \widetilde{Z^*}_i = 1$ almost surely, then Balkema's Lemma yields that $\widetilde{Z} \stackrel{d}{=} \widetilde{Z^*}$ and consequently, the measurability of H implies (5.5), hence the proof is complete. □

Proof of Lemma 4.1 We give first the proofs of (4.3) and (4.4). For any $h \in T$ by the definition of $\boldsymbol{Z}$ in (4.2), i.e., $\boldsymbol{Z}(t) = (1/p(W))^{1/\alpha} B^W \boldsymbol{L}(t), t \in \mathcal{T}$, with W a $\mathcal{T}$-valued rv with positive pdf $p(t) > 0, t \in \mathcal{T}$ using further the translation invariance of the Lebesgues measure on $\mathcal{T}$ for all $h \in \mathcal{T}$

$$\mathbb{E}\{\|\boldsymbol{Z}(h)\|^\alpha\} = \int_{\mathcal{T}} \|\boldsymbol{L}(t)\|^\alpha \lambda(dt) = c.$$

Since $\int_{\mathcal{T}} \|\boldsymbol{L}(t)\|_*^\alpha \lambda(dt) \in (0,\infty)$, by the equivalence of the norms on $\mathbb{R}^d$ we have that $c \in (0,\infty)$. Given $A \in \mathcal{C}$ for any $h \in \mathcal{T}$ Fubini-Tonelli theorem implies

$$\begin{aligned}
\mathbb{P}\{\boldsymbol{\Theta}^{[h]} \in A\} &= \frac{1}{c}\mathbb{E}\{\|\boldsymbol{Z}(h)\|^\alpha \mathbb{I}(\boldsymbol{Z}/\|\boldsymbol{Z}(h)\| \in A)\}\\
&= \frac{1}{c}\int_{\mathcal{T}} \|(B^{-t}\boldsymbol{L})(h)\|^\alpha \mathbb{I}(B^{-t}\boldsymbol{L}/\|(B^{-t}\boldsymbol{L})(h)\| \in A)\lambda(dt)\\
&= \int_{\mathcal{T}} \mathbb{I}(B^{h-s}\boldsymbol{L}/\|\boldsymbol{L}(s)\| \in A)\|\boldsymbol{L}(s)\|^\alpha/c\lambda(ds)\\
&= \mathbb{P}\{B^{h-S}\boldsymbol{L}/\|\boldsymbol{L}(S)\| \in A\},
\end{aligned}$$

where the d-dimensional random vector S has pdf $\|\boldsymbol{L}(t)\|^\alpha/c, t \in \mathcal{T}$, hence (4.3) follows.
Since for any positive integer $k \le d$ by the translation invariance of the Lebesgue measure on $\mathcal{T}$

$$\mathbb{E}\{Z_k^\alpha(h)\} = \int_{\mathcal{T}} L_k^\alpha(t)\lambda(dt) = c_k$$

for all $A \in \mathcal{C}, h \in \mathcal{T}$, if further $c_k > 0$

$$\begin{aligned}
\mathbb{P}\{\boldsymbol{\Theta}^{[h,k]} \in A\} &= \mathbb{E}\{Z_k^\alpha(h)\mathbb{I}(\boldsymbol{Z}/Z_k(h) \in A)/c_k\}\\
&= \int_{\mathcal{T}} \mathbb{I}(B^{-t}\boldsymbol{L}/L_k(t) \in A)L_k^\alpha(t)/c_k\lambda(dt)\\
&= \mathbb{P}\{B^{h-S_k}\boldsymbol{L}/L_k(S_k) \in A\},
\end{aligned}$$

where the $\mathcal{T}$-valued random vector S_k has pdf $L_k^\alpha(t)/c_k, t \in \mathcal{T}$, hence (4.4) follows.
Next we show that the fidi's of a max-stable process $\boldsymbol{X}$ as in the Introduction can be determined by infargmax functional in terms of $\boldsymbol{\Theta}^{[k]}$, where the maximum and minimum are taken with respect to the lexicographical order. In view of (1.2) for any $t_i \in \mathcal{T}, \boldsymbol{x}_i \in (0,\infty)^d, 1 \le i \le n$ we have (set $\|\boldsymbol{x}\| =: \|\boldsymbol{x}\|_\infty = \max_{1\le i\le d}|x_i|$)

$$-\ln \mathbb{P}\{\boldsymbol{X}(t_i) \le \boldsymbol{x}_i, 1 \le i \le n\}$$

$$
\begin{aligned}
&= \mathbb{E}\{\max_{1\le j\le n} \|\boldsymbol{Z}(t_j)/\boldsymbol{x}_j\|^\alpha\}\\
&= \sum_{1\le l\le n} \mathbb{E}\{\|\boldsymbol{Z}(t_l)/\boldsymbol{x}_l\|^\alpha\ \mathbb{I}(infargmax_{1\le j\le n}\|\boldsymbol{Z}(t_j)/\boldsymbol{x}_j\|^\alpha = l)\}\\
&= \sum_{1\le l\le n} \mathbb{E}\{\|\boldsymbol{Z}(t_l)\|^\alpha\}\mathbb{E}\{\frac{\|\boldsymbol{Z}(t_l)\|^\alpha}{\mathbb{E}\{\|\boldsymbol{Z}(t_l)\|^\alpha\}}\frac{\|\boldsymbol{Z}(t_l)/\boldsymbol{x}_l\|^\alpha}{\|\boldsymbol{Z}(t_l)\|^\alpha}\mathbb{I}(infargmax_{1\le j\le n}\|\boldsymbol{Z}(t_j)/\boldsymbol{x}_j\| = l)\}\\
&= c\sum_{1\le l\le n} \mathbb{E}\{\|\boldsymbol{\Theta}^{[t_l]}(t_l)/\boldsymbol{x}_l\|^\alpha\mathbb{I}(infargmax_{1\le j\le n}\|\boldsymbol{\Theta}^{[t_l]}(t_j)/\boldsymbol{x}_j\| = l)\},
\end{aligned}
$$

hence (4.5) follows from (4.3). Next, if $c_k = \mathbb{E}\{Z_k^\alpha(t)\}, t \in \mathcal{T}$ is positive for any $k \le d$, setting $J = \{(i,j), 1 \le j \le d, 1 \le j \le p\}$ for all $t_i \in \mathcal{T}, \boldsymbol{x}_i \in (0,\infty)^d, 1 \le i \le n$ we can write using (1.2)

$$
\begin{aligned}
&-\ln \mathbb{P}\{\boldsymbol{X}(t_i) \le \boldsymbol{x}_i, 1 \le i \le n\}\\
&= \mathbb{E}\{\max_{(i,j)\in J} Z_i^\alpha(t_j)/x_{ij}^\alpha\}\\
&= \sum_{(k,l)\in J} \frac{1}{x_{kl}^\alpha}\mathbb{E}\Big\{Z_k^\alpha(t_l)\mathbb{I}\Big(infargmax_{(i,j)\in J}\big(Z_i^\alpha(t_j)/x_{ij}^\alpha\big) = (k,l)\Big)\Big\}\\
&= \sum_{(k,l)\in J} \frac{\mathbb{E}\{Z_k^\alpha(t_l)\}}{x_{kl}^\alpha}\mathbb{E}\Big\{\frac{Z_k^\alpha(t_l)}{\mathbb{E}\{Z_k^\alpha(t_l)\}}\mathbb{I}\Big(infargmax_{(i,j)\in J}\big(Z_i^\alpha(t_j)/x_{ij}^\alpha\big) = (k,l)\Big)\Big\}
\end{aligned}
$$

$$
(5.6)\qquad = \sum_{(k,l)\in J} \frac{\mathbb{E}\{Z_k^\alpha(t_l)\}}{x_{kl}^\alpha}\mathbb{P}\{infargmax_{(i,j)\in J}(\Theta_i^{[t_l,k]}(t_j)/x_{ij}) = (k,l)\}.
$$

Note in passing that the above calculations hold also when some c_k's are equal to zero. Applying the above formula and utilising further (4.4) establishes the proof. □

Proof of Lemma 4.2 In view of [18][Lem 6.1] we have that $\boldsymbol{\Theta}^{[t_l,k]}$ is again log-Gaussian, but there is an additional deterministic trend function which can be calculated for each component separately as therein. Specifically, for any positive integer $k \le d$ and all $h \in \mathcal{T}$ we have

$$
(5.7)\qquad (\Theta_1^{[h,k]}(t), \ldots, \Theta_d^{[h,k]}(t)) \stackrel{d}{=} \Big(e^{Y_1(t)-Y_k(h)-\frac{Var(Y_1(t)-Y_k(h))}{2}}, \ldots, e^{Y_d(t)-Y_k(h)-\frac{Var(Y_d(t)-Y_k(h))}{2}}\Big),
$$

hence in view of (5.6) the fidi's of $\boldsymbol{X}$ depends only on the matrix-valued functions $P^{h,k}(t,s)$ with ijth entry equal to $p_{ij}^{h,k}(t,s) = cov(Y_i(t) - Y_k(h), Y_j(s) - Y_k(h))$. Since

$$
p_{ij}^{h,k}(t,s) = [\gamma_{ik}(t,h) + \gamma_{jk}(s,h) - \gamma_{ij}(t,s)]/2, \quad 1 \le i,j \le d, h,s,t \in \mathcal{T},
$$

with $\gamma_{ij}(t,s) = Var(Y_i(t) - Y_j(s))$, then the law of $\boldsymbol{X}$ depends only on γ_{ij}'s. If $\gamma_{ij}(t,s)$ depends only on the difference $t-s$, then

$$
p_{ij}^{h+a,k}(t+a, s+a) = [\gamma_{ik}(t,h) + \gamma_{jk}(s,h) - \gamma_{ij}(t,s)]/2, \quad 1 \le i,j \le d, a,h,s,t \in \mathcal{T}
$$

and hence by (5.6) we have that $(\boldsymbol{X}(t_1), \ldots, \boldsymbol{X}(t_n))$ has the same law as $(\boldsymbol{X}(t_1+a), \ldots, \boldsymbol{X}(t_n+a))$ for any $a \in \mathcal{T}$ implying $\boldsymbol{X}$ is stationary. Using (5.6) the formula (4.8) follows easily, hence the proof is complete. □

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