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PARISIAN RUIN OF SELF-SIMILAR GAUSSIAN RISK PROCESSES

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Abstract

In this paper we derive the exact asymptotics of the probability of Parisian ruin for self-similar Gaussian risk processes. Additionally, we obtain the normal approximation of the Parisian ruin time and derive an asymptotic relation between the Parisian and the classical ruin times.

Keywords: Parisian ruin time; Parisian ruin probability; self-similar Gaussian processes; fractional Brownian motion; normal approximation; generalized Pickands constant

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1. Introduction

Let $\{X_H(t), t \geq 0\}$ be a centered self-similar Gaussian process with almost surely continuous sample paths and index $H \in (0, 1)$, i.e., $\mathrm{Var}(X_H(t)) = t^{2H}$ and for any $a > 0$ and $s, t \geq 0$

$$\mathrm{Cov}(X_H(at), X_H(as)) = a^{2H} \mathrm{Cov}(X_H(t), X_H(s)).$$

Let β, c be two positive constants. In risk theory the surplus process of an insurance company can be modeled by

$$R_u(t) = u + ct^{\beta} - X_H(t), \quad t \geq 0, \tag{1}$$

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where u is the so-called initial reserve, ct^β models the total premium received up to time t, and $X_H(t)$ represents the total amount of aggregated claims (including fluctuations) up to time t. Typically, classical risk models assume a linear premium income, meaning that $\beta = 1$. In this paper we deal with a more general case $\beta > H$ allowing for non-linear premium income. Below we shall refer to R_u as the *self-similar Gaussian risk process*. The justification for choosing self-similar processes to model the aggregated claim process comes from [35], where it is shown that the ruin probability for self-similar Gaussian risk processes is a good approximation of the ruin probability for some classical risk process. Recent contributions have shown that self-similar Gaussian processes such as fractional Brownian motion (fBm), sub-fractional Brownian motion and bi-fractional Brownian motion are useful in modeling of financial risks, see e.g., [20, 27, 28, 31, 23] and the references therein.

For any $u \geq 0$, define the *classical ruin time* of the self-similar Gaussian risk process by

$$\tau_u = \inf\{t \geq 0 : R_u(t) < 0\} \quad (\text{with } \inf\{\emptyset\} = \infty) \tag{2}$$

and thus the *probability of ruin* is defined as

$$\mathbb{P}\left\{\tau_u < \infty\right\}. \tag{3}$$

The classical ruin time and the probability of ruin for self-similar Gaussian risk processes are well studied in the literature; see, e.g., [27, 28, 17].

Recently, an extension of the classical notion of ruin, that is the *Parisian ruin*, focused substantial interest; see [10, 5, 9] and the references therein. The core of the notion of the Parisian ruin is that now one allows the surplus process to spend a pre-specified time under the level zero before the ruin is recognized. To be more precise, let T_u model the pre-specified time which is a positive deterministic function of the initial reserve u. In our setup, the *Parisian ruin time* of the self-similar Gaussian risk process R_u is defined as

$$\tau_u^* = \inf\{t \geq T_u : t - \kappa_{t,u} \geq T_u\}, \quad \text{with } \kappa_{t,u} = \sup\{s \in [0,t] : R_u(s) \geq 0\}. \tag{4}$$

Here we make the convention that $\sup\{\emptyset\} = 0$.

In this contribution we focus on the Parisian ruin probability, i.e.,

$$\mathbb{P}\left\{\tau_u^* < \infty\right\} = \mathbb{P}\left\{\inf_{t\geq 0} \sup_{s\in[t,t+T_u]} R_u(s) < 0\right\}. \tag{5}$$

We refer to [5, 33, 7, 6, 10] for recent analysis of (5) for the Lévy surplus model. In mathematical finance, Parisian stopping times have been studied initially by [4] in the context of barrier options.

Assume for the moment that X_H is a standard Brownian motion, $\beta = 1$ and $T_u = T > 0, u > 0$. Thus R_u is the Brownian motion risk process with a linear trend. As shown in [33], for any $u \geq 0$

$$\mathbb{P}\left\{\tau_u^* < \infty\right\} = \frac{\exp\left(-c^2T/2\right) - c\sqrt{2\pi T}\Phi(-c\sqrt{T})}{\exp\left(-c^2T/2\right) + c\sqrt{2\pi T}\Phi(c\sqrt{T})} \exp(-2cu), \tag{6}$$

where $\Phi(\cdot)$ is the distribution function of a standard Normal random variable. Since the case $\beta \neq 1$ seems to be completely untractable, even for the Brownian motion risk process, one has to resort to bounds and asymptotic results, allowing the initial capital u to become large, see e.g., [19].

This contribution is concerned with the asymptotic behaviour of the Parisian ruin probability as $u \to \infty$ for a large class of self-similar Gaussian risk processes. Under a local stationarity condition on the correlation of the self-similar process X_H (see (11)) and a mild condition on T_u (see (16)), in Theorem 3.1 we derive the asymptotics of the Parisian ruin probability. Interestingly, as a corollary, it appears that for the fBm risk process with a linear trend if $H > 1/2$, then

$$\mathbb{P}\left\{\tau_u^* < \infty\right\} = \mathbb{P}\left\{\tau_u < \infty\right\}(1 + o(1)), \quad u \to \infty \tag{7}$$

even if T_u grows to infinity at a specified rate, as $u \to \infty$.

The combination of (7) with the asymptotic behaviour of $\mathbb{P}\left\{\tau_u < \infty\right\}$ derived in [27] implies thus the exact asymptotic behaviour of the Parisian ruin probability.

Additionally, we derive the approximation of the conditional (scaled) Parisian ruin time and the asymptotic relation between the classical ruin time and the Parisian ruin time given that the Parisian ruin occurs. This result goes in line with, e.g., [2, 14, 19, 24, 28, 30, 22, 21, 25, 36], where the approximation of the classical ruin time is considered. The obtained normal approximation of the Parisian ruin time is a new result even for the Brownian motion risk process with a linear trend.

Brief outline of the paper: In Section 2 we introduce our notation and present a preliminary result concerning the tail of the sup-inf functional of a Gaussian random field. The asymptotics of the Parisian ruin probability is given in Section 3, while the time of the Parisian ruin is analyzed in Section 4. Proofs are relegated to Section 5.

2. Notation and Preliminaries

Let $\{X_H(t), t \ge 0\}$ be a centered self-similar Gaussian process with almost surely continuous sample paths and index $H \in (0,1)$, as defined in the Introduction. By $\{B_\alpha(t), t \ge 0\}$ we denote a standard fBm with Hurst index $\alpha/2 \in (0,1]$.

It is useful to define, for $\beta > H$ and $c > 0$

$$Z(t) = \frac{X_H(t)}{1 + ct^\beta}, \quad t \ge 0. \tag{8}$$

Indeed, by the self-similarity of X_H, for any u positive

$$\begin{aligned} \mathbb{P}\{\tau_u^* < \infty\} &= \mathbb{P}\left\{\sup_{t\ge 0} \inf_{s\in[t,t+T_u]} \left(X_H(s) - cs^\beta\right) > u\right\} \\ &= \mathbb{P}\left\{\sup_{t\ge 0} \inf_{s\in\left[0,T_u u^{-\frac{1}{\beta}}\right]} Z(t+s) > u^{1-\frac{H}{\beta}}\right\}. \end{aligned} \tag{9}$$

If follows that (cf. [27, 28]) $\sigma_Z(t) = \sqrt{\mathrm{Var}(Z(t))}$ attains its maximum on $[0,\infty)$ at the unique point

$$t_0 = \left(\frac{H}{c(\beta - H)}\right)^{\frac{1}{\beta}}$$

and

$$\sigma_Z(t) = A - \frac{BA^2}{2}(t - t_0)^2 + o((t-t_0)^2)$$

as $t \to t_0$, where

$$A = \frac{\beta - H}{\beta}\left(\frac{H}{c(\beta - H)}\right)^{\frac{H}{\beta}}, \quad B = \left(\frac{H}{c(\beta - H)}\right)^{-\frac{H+2}{\beta}} H\beta. \tag{10}$$

In the rest of the paper we assume *the local stationarity* of the standardized Gaussian process $\overline{X}_H(t) := X_H(t)/t^H, t > 0$ in a neighborhood of the point t_0 i.e.,

$$\lim_{s\to t_0, t\to t_0} \frac{\mathbb{E}\left((\overline{X}_H(s) - \overline{X}_H(t))^2\right)}{K^2(|s-t|)} = Q > 0 \tag{11}$$

holds for some positive function $K(\cdot)$ which is assumed to be regularly varying at 0 with index $\alpha/2 \in (0,1)$. Condition (11) is common in the literature; most of the known self-similar Gaussian processes (such as fBm, sub-fBm, and bi-fBm) satisfy (11); see, e.g., [26]. Note that the local stationarity at t_0 and the self-similarity of the process X_H imply the local stationarity of X_H at any point $r > 0$ i.e.,

$$\lim_{s\to r, t\to r} \frac{\mathbb{E}\left((\overline{X}_H(s) - \overline{X}_H(t))^2\right)}{K^2(|s-t|)} = \left(\frac{t_0}{r}\right)^{\alpha} Q.$$

Throughout this paper we denote by $K^{\leftarrow}(\cdot)$ the asymptotic inverse of $K(\cdot)$; by definition

$$K^{\leftarrow}(K(t)) = K(K^{\leftarrow}(t))(1+o(1)) = t(1+o(1)), \quad t \to 0.$$

It follows that $K^{\leftarrow}(\cdot)$ is regularly varying at 0 with index $2/\alpha$; see, e.g., [19].

Let $\mathcal{H}_\alpha$ be the classical Pickands constant, defined by

$$\mathcal{H}_\alpha = \lim_{T\to\infty} \frac{1}{T} \mathbb{E}\left(\exp\left(\sup_{t\in[0,T]} (\sqrt{2}B_\alpha(t) - t^\alpha)\right)\right).$$

We refer to [1, 3, 13, 12, 16, 11, 18, 34, 39] for the basic properties of the Pickands and related constants. A new constant that shall appear in our findings below is defined as

$$\mathcal{F}_\alpha(T) = \lim_{S\to\infty} \frac{1}{S} \mathbb{E}\left(\exp\left(\sup_{t\in[0,S]} \inf_{s\in[0,T]} \left(\sqrt{2}B_\alpha(t+s) - (t+s)^\alpha\right)\right)\right) \in (0,\infty) \quad (12)$$

for any $T \in [0,\infty)$.

We conclude this section with a general result for the tail of the sup-inf functional applied to the Gaussian process Z. Recall that by $\Phi(\cdot)$ we denote the distribution function of a standard Normal random variable. In order to simplify the notation, we shall set

$$q = q(v) := K^{\leftarrow}\left(\frac{1}{v}\right), \quad v > 0. \quad (13)$$

Theorem 2.1. *Let $\{Z(t), t \geq 0\}$ be the centered Gaussian process given as in (8), and let $x_i(\cdot), i = 1,2$ be two functions such that $\lim_{v\to\infty} x_i(v) = x_i, i = 1,2$ and $\lim_{v\to\infty} x_i(v)v^{-1/2} = 0, i = 1,2$ for some $x_1, x_2 \in \mathbb{R} \cup \{\infty\}$ satisfying $x_2 > -x_1$. Further, for all v large denote $\Theta_{x_1,x_2}(v) = \left[t_0 - x_1(v)v^{-1}, t_0 + x_2(v)v^{-1}\right]$. Then, for any positive function $\lambda(\cdot)$ such that $\lim_{v\to\infty} \lambda(v) = \lambda \in [0,\infty)$ we have, as $v \to \infty$*

$$\mathbb{P}\left\{\sup_{t\in\Theta_{x_1,x_2}(v)} \inf_{s\in[0,\lambda(v)q]} Z(t+s) > v\right\} = \frac{\mathcal{F}_\alpha(D_0\lambda)}{\mathcal{H}_\alpha}\left(\Phi\left(A^{-\frac{1}{2}}B^{\frac{1}{2}}x_2\right)\right.$$

$$-\Phi\left(-A^{-\frac{1}{2}}B^{\frac{1}{2}}x_1\right)\Bigg) \times \mathbb{P}\left\{\sup_{t\geq 0} Z(t) > v\right\}(1+o(1)), \quad (14)$$

where $D_0 = 2^{-\frac{1}{\alpha}}A^{-\frac{2}{\alpha}}Q^{\frac{1}{\alpha}}$, *and* $\mathcal{F}_\alpha(\cdot)$ *defined in* (12) *is positive and finite.*

The complete proof of Theorem 2.1 is given in Section 5.

3. Asymptotics of the Parisian ruin probability

In this section we display the main result of the paper, which is the asymptotics of the Parisian ruin probability $\mathbb{P}\{\tau_u^* < \infty\}$, as $u \to \infty$, for the self-similar Gaussian risk model in (1). First, we note that in the light of the seminal contribution [27]

$$\mathbb{P}\{\tau_u < \infty\} = \frac{A^{\frac{3}{2}-\frac{2}{\alpha}}Q^{\frac{1}{\alpha}}\mathcal{H}_\alpha}{2^{\frac{1}{\alpha}}B^{\frac{1}{2}}} \frac{u^{\frac{2H}{\beta}-2}}{K^{\leftarrow}(u^{\frac{H}{\beta}-1})} \exp\left(-\frac{u^{2\left(1-\frac{H}{\beta}\right)}}{2A^2}\right)(1+o(1)) \quad (15)$$

holds as $u \to \infty$. In order to control the growth of T_u, we shall assume that

$$\lim_{u\to\infty} \frac{T_u u^{-\frac{1}{\beta}}}{K^{\leftarrow}(u^{\frac{H}{\beta}-1})} = T \in [0, \infty). \quad (16)$$

Theorem 3.1. *Let* $\{R_u(t), t \geq 0\}$ *be the self-similar Gaussian risk process given as in* (1) *with* X_H *satisfying* (11) *and* $T_u, u > 0$ *satisfying* (16)*. If* τ_u^* *denotes the Parisian ruin time of* R_u*, then as* $u \to \infty$

$$\mathbb{P}\{\tau_u^* < \infty\} = \frac{\mathcal{F}_\alpha(D_0 T)}{\mathcal{H}_\alpha}\mathbb{P}\{\tau_u < \infty\}(1+o(1)), \quad (17)$$

where $D_0 = 2^{-\frac{1}{\alpha}}A^{-\frac{2}{\alpha}}Q^{\frac{1}{\alpha}}$ *with* $\mathcal{F}_\alpha(T)$ *defined in* (12)*.*

The proof of Theorem 3.1 is deferred to Section 5; it relies on the general result for the asymptotics of sup-inf functional of the Gaussian process Z, given in Theorem 2.1.

Remark 1. Observe that the Pickands constant $\mathcal{H}_\alpha = \mathcal{F}_\alpha(0)$ and $\mathcal{H}_1 = 1$ (cf. [39]). It is not clear how to calculate $\mathcal{F}_\alpha(T)$ using the definition in (12). However for the special case $\alpha = 1$, (6) and (19) below imply

$$\mathcal{F}_1(T) = \frac{\exp(-T/4) - \sqrt{\pi T}\Phi(-\sqrt{T/2})}{\exp(-T/4) + \sqrt{\pi T}\Phi(-\sqrt{T/2})}, \quad T > 0. \quad (18)$$

In this paper we shall refer to $\mathcal{F}_\alpha(T)$ as the generalized Pickands constant.

As a corollary of the last theorem we present next a result for the fBm risk processes with a linear trend where X_H is assumed to be a standard fBm B_{2H}. Specifically, for any $H \in (0,1]$ we have

$$\mathrm{Cov}(X_H(t), X_H(s)) = \frac{1}{2}(t^{2H} + s^{2H} - \mid t-s \mid^{2H}), \quad t,s \geq 0$$

and thus (11) holds with $K(t) = t^H, t \geq 0$ and $Q = t_0^{-2H} = [H/(c(\beta - H))]^{-2H/\beta}$ if further $\beta > H$.

Corollary 3.2. *Let $R_u(t) = u + ct - B_{2H}(t), t \geq 0$ and let $T_u, u > 0$ be such that $\lim_{u\to\infty} T_u u^{1/H-2} = T \in [0,\infty)$. If $c > 0$ and $H \in (0,1)$, then as $u \to \infty$*

$$\begin{aligned}\mathbb{P}\{\tau_u^* < \infty\} &= \mathcal{F}_{2H}(D_0 T)\frac{2^{-\frac{1}{2H}}}{\sqrt{H(1-H)}}\left(\frac{c^H u^{1-H}}{H^H(1-H)^{1-H}}\right)^{\frac{1}{H}-2}\\ &\quad \times \exp\left(-\frac{c^{2H}u^{2(1-H)}}{2H^{2H}(1-H)^{2(1-H)}}\right)(1+o(1)),\end{aligned} \tag{19}$$

where $D_0 = 2^{-\frac{1}{2H}} c^2 H^{-2}(1-H)^{2-\frac{1}{H}}$.

Remark 2. Using the fact that $\mathcal{F}_{2H}(0) = \mathcal{H}_{2H}$, Corollary 3.2 implies that

$$\mathbb{P}\{\tau_u^* < \infty\} = \mathbb{P}\{\tau_u < \infty\}(1+o(1))$$

as $u \to \infty$, if $T = 0$ (i.e. $T_u = o(u^{(2H-1)/H})$). Thus, if $H > 1/2$, the asymptotics of the Parisian ruin probability coincides with the asymptotics of the classical ruin probability even if T_u grows to infinity, provided that $T = 0$. This property is another manifestation of the long-range dependence structure of fBm with Hurst index $H > 1/2$.
For the boundary case $T_u = Tu^{1/H-2}$ with $T > 0$, the Parisian ruin probability and the classical ruin probability are not asymptotically equivalent, as the initial capital u tends to infinity.

In [32] a different type of Parisian ruin is considered, where the deterministic pre-specified time T_u is replaced by an independent random variable (in particular, an exponential random variable is dealt with therein, see also [8]). In the following corollary we calculate the Parisian ruin probability of this model.

Corollary 3.3. *Let $\{R_u(t), t \geq 0\}$ be the self-similar Gaussian risk process given as in (1) with X_H satisfying (11). If $\mathcal{T}$ is a positive random variable independent of*

$\{R_u(t), t \geq 0\}$, *then*

$$\mathbb{P}\left\{\inf_{t\geq 0}\sup_{s\in[t,t+\mathcal{T}]} R_u(s) < 0\right\} = \mathbb{P}\left\{\tau_u < \infty\right\}(1+o(1)), \quad u\to\infty \tag{20}$$

holds, provided that $2H+\alpha > 2\beta$.

4. Normal approximation of the Parisian ruin time

In this section we present a normal approximation for the conditional (scaled) Parisian ruin time. Additionally, we derive an asymptotic relation between the classical ruin time and the Parisian ruin time, given that the Parisian ruin occurs.
Hereafter $\stackrel{d}{\to}$ and $\stackrel{p}{\to}$ stand for convergence in distribution and convergence in probability, respectively.

Theorem 4.1. *Let* τ_u, τ_u^* *be the classical ruin time and the Parisian ruin time for the self-similar Gaussian risk process* $\{R_u(t), t \geq 0\}$ *given as in* (1). *If* X_H *satisfies* (11) *and* $T_u, u > 0$ *satisfies* (16), *then as* $u \to \infty$

$$\frac{\tau_u^* - t_0 u^{\frac{1}{\beta}}}{A^{\frac{1}{2}}B^{-\frac{1}{2}}u^{\frac{H}{\beta}+\frac{1}{\beta}-1}}\Big|(\tau_u^*<\infty)\stackrel{d}{\to}\mathcal{N}, \tag{21}$$

where A, B *are as in* (10) *and* $\mathcal{N}$ *is a standard Normal random variable. Moreover, as* $u \to \infty$,

$$\frac{\tau_u^* - \tau_u}{u^{\frac{H}{\beta}+\frac{1}{\beta}-1}}\Big|(\tau_u^*<\infty)\stackrel{p}{\to} 0. \tag{22}$$

The complete proof of Theorem 4.1 is given in Section 5.

As a straightforward implication of Theorem 4.1 it follows that if $H+1=\beta$, then

$$(\tau_u^*-\tau_u)\Big|(\tau_u^*<\infty)\stackrel{p}{\to}0, \quad u\to\infty. \tag{23}$$

Remark 3. In [28] a slightly more general class of Gaussian processes was considered. Under additional technical conditions as A1 and A3 therein similar results as in Theorem 3.1 and Theorem 4.1 also hold for that class of Gaussian processes; the only difference is that in (21) and (22) we shall have $\sqrt{\mathrm{Var}(X_H(u^{1/\beta}))}$ instead of $u^{H/\beta}$ and $s_0(u)$ (in their notation) instead of t_0.
We note that extensions of our result to Gaussian processes with random variance under similar conditions as in [29] are also possible.

5. Proofs

This section is dedicated to proofs of Theorems 2.1, 3.1 and 4.1 and Corollary 3.3. We first present a crucial lemma which can be seen as an extension of the celebrated Pickands lemma; see, e.g., [37, 38, 39]. We refer to [15] for recent developments in this direction.

Let λ_1, λ_2 be two given positive constants. Consider a family of a.s. continuous centered Gaussian random fields

$$\{X_v(t,s), (t,s) \in [0,\lambda_1] \times [0,\lambda_2]\}$$

indexed by $v > 0$. We shall assume that its variance equals 1 and the correlation functions $r_v(t,s,t',s') = \mathrm{Cov}(X_v(t,s), X_v(t',s')), (t,s), (t',s') \in [0,\lambda_1] \times [0,\lambda_2], v > 0$ satisfy the following two conditions:

C1. There exist constants $D > 0, \alpha \in (0,2]$ and a positive function $f(\cdot)$ defined in $(0,\infty)$ such that

$$\lim_{v\to\infty} (f(v))^2 (1 - r_v(t,s,t',s')) = D\,|s + t - s' - t'|^{\alpha}$$

holds for any $(t,s), (t',s') \in [0,\lambda_1] \times [0,\lambda_2]$.

C2. There exist constants $C > 0, v_0 > 0, \gamma \in (0,2]$ such that, for any $v > v_0$, with $f(\cdot)$ given in C1

$$(f(v))^2 (1 - r_v(t,s,t',s')) \le C(|s - s'|^{\gamma} + |t - t'|^{\gamma})$$

holds uniformly with respect to $(t,s), (t',s') \in [0,\lambda_1] \times [0,\lambda_2]$.

Lemma 5.1. *Let $\{X_v(t,s), (t,s) \in [0,\lambda_1] \times [0,\lambda_2]\}, v > 0$ be the family of centered Gaussian random fields with variance equal to 1 defined above. If both* **C1** *and* **C2** *hold, then for any positive function $\theta(\cdot)$ satisfying $\lim_{v\to\infty} f(v)/\theta(v) = 1$ we have*

$$\begin{aligned} \mathbb{P}\left\{ \sup_{t\in[0,\lambda_1]} \inf_{s\in[0,\lambda_2]} X_v(t,s) > \theta(v) \right\} &= \mathcal{H}_\alpha(D^{\frac{1}{\alpha}}\lambda_1, D^{\frac{1}{\alpha}}\lambda_2)(1 + o(1)) \\ &\quad \times \frac{1}{\sqrt{2\pi}\theta(v)} \exp\left(-\frac{(\theta(v))^2}{2}\right) \qquad (24) \end{aligned}$$

as $u \to \infty$, where

$$\mathcal{H}_\alpha(\lambda_1,\lambda_2) = \mathbb{E}\left(\exp\left(\sup_{t\in[0,\lambda_1]} \inf_{s\in[0,\lambda_2]} \left(\sqrt{2}B_\alpha(t+s) - (t+s)^{\alpha}\right)\right)\right) \in (0,\infty).$$

Proof. Note that the sup-inf functional satisfies **F1-F2** in [15]. The proof follows by similar arguments as the proof of Lemma 1 therein, and therefore we omit the technical details.

The next result plays an important role in the proof of Theorem 3.1. We refer to [27] for its proof.

Lemma 5.2. *Let $\{Z(t), t \geq 0\}$ be defined as in (8) and set $v(u) = u^{1-H/\beta}$. If $c > 0$ and $\beta > H$, then for any $G > t_0$ we have as $u \to \infty$*

$$\begin{aligned} \mathbb{P}\{\tau_u < \infty\} &= \mathbb{P}\left\{\sup_{t\in[0,G]}\left(X_H(t) - ct^{\beta}\right) > u\right\}(1+o(1)) \\ &= \mathbb{P}\left\{\sup_{t\in\left[t_0-\frac{\ln v(u)}{v(u)}, t_0+\frac{\ln v(u)}{v(u)}\right]} Z(t) > v(u)\right\}(1+o(1)). \end{aligned} \tag{25}$$

Further, as $u \to \infty$

$$\mathbb{P}\left\{\sup_{|t-t_0|>\frac{\ln v(u)}{v(u)}} Z(t) > v(u)\right\} = o\left(\mathbb{P}\left\{\sup_{t\geq 0} Z(t) > v(u)\right\}\right). \tag{26}$$

5.1. Proof of Theorem 2.1

We shall give only the proof for the case $\infty > x_2 > 0 > -x_1 > -\infty$. The other cases can be established by similar arguments. Since our approach is of asymptotic nature, we assume in the following that v is sufficiently large so that $x_i(v) > 0, i = 1, 2$. Let $S > 2\lambda$ be any positive constant. With $q = q(v)$ defined in (13) we denote

$$\triangle_k = [kSq, (k+1)Sq], \ k \in \mathbb{Z}, \quad \text{and} \quad N_i(v) = \left\lfloor S^{-1}x_i(v)q^{-1}v^{-1}\right\rfloor, \ i = 1, 2,$$

where $\lfloor\cdot\rfloor$ is the ceiling function. For any small $\varepsilon_0 > 0$, denote $\lambda_{\varepsilon_0}^+ = \lambda + \varepsilon_0$ and $\lambda_{\varepsilon_0}^- = \max(0, \lambda - \varepsilon_0)$. It follows by Bonferroni's inequality that

$$\begin{aligned} \sum_{k=-N_1(v)-1}^{N_2(v)+1} Q_k^+(v) &\geq \mathbb{P}\left\{\sup_{t\in\Theta_{x_1,x_2}(v)} \inf_{s\in[0,\lambda(v)q]} Z(t+s) > v\right\} \\ &\geq \sum_{k=-N_1(v)}^{N_2(v)} Q_k^-(v) - \Sigma_1(v) \end{aligned} \tag{27}$$

for large enough u, where

$$Q_k^+(v) = \mathbb{P}\left\{\sup_{t\in\triangle_k} \inf_{s\in\left[0,\lambda_{\varepsilon_0}^- q\right]} Z(t_0+t+s) > v\right\}, \quad k \in \mathbb{Z},$$

$$Q_k^-(v) \quad = \quad \mathbb{P}\left\{\sup_{t\in\triangle_k}\inf_{s\in\left[0,\lambda_{\varepsilon_0}^+q\right]} Z(t_0+t+s)>v\right\}, \quad k\in\mathbb{Z},$$

and

$$\Sigma_1(v) \quad = \quad \sum_{-N_1(v)\le k<l\le N_2(v)} \mathbb{P}\Bigg\{\sup_{t\in\triangle_k}\inf_{s\in\left[0,\lambda_{\varepsilon_0}^+q\right]} Z(t_0+t+s)>v, \\ \sup_{t\in\triangle_l}\inf_{s\in\left[0,\lambda_{\varepsilon_0}^+q\right]} Z(t_0+t+s)>v\Bigg\}.$$

Next, we shall derive upper bounds for $Q_k^+(v)$ and lower bounds for $Q_k^-(v)$. First, note that

$$Q_k^+(v) \quad \le \quad \mathbb{P}\left\{\sup_{t\in\triangle_k}\inf_{s\in\left[0,\lambda_{\varepsilon_0}^-q\right]} \overline{Z}(t_0+t+s)>\frac{v}{\sigma_Z^+(k,v)}\right\}$$

$$Q_k^-(v) \quad \ge \quad \mathbb{P}\left\{\sup_{t\in\triangle_k}\inf_{s\in\left[0,\lambda_{\varepsilon_0}^+q\right]} \overline{Z}(t_0+t+s)>\frac{v}{\sigma_Z^-(k,v)}\right\},$$

where $\overline{Z}(t):=Z(t)/\sigma_Z(t), t\ge 0$ and

$$\sigma_Z^-(k,v)=\inf_{t\in\triangle_k}\inf_{s\in\left[0,\lambda_{\varepsilon_0}^+q\right]}\sigma_Z(t_0+t+s),\quad \sigma_Z^+(k,v)=\sup_{t\in\triangle_k}\sup_{s\in\left[0,\lambda_{\varepsilon_0}^-q\right]}\sigma_Z(t_0+t+s).$$

Furthermore, since

$$\sigma_Z(t)=A-\frac{A^2B}{2}(t-t_0)^2(1+o(1)),\quad t\to t_0 \tag{28}$$

for any small $\varepsilon_1>0$ there exists v_0 such that for any $v>v_0$ (set below $B^\pm=B(1\pm\varepsilon_1)$)

$$\frac{1}{\sigma_Z^-(k,v)}\le\frac{1}{A}+\frac{B^+}{2}\left(((k+1)S+\lambda_{\varepsilon_0}^+)q\right)^2,\quad \frac{1}{\sigma_Z^+(k,v)}\ge\frac{1}{A}+\frac{B^-}{2}\left(kSq\right)^2$$

hold for $k=0,\cdots,N_2(v)+1$, and also

$$\frac{1}{\sigma_Z^-(k,v)}\le\frac{1}{A}+\frac{B^+}{2}\left(kSq\right)^2,\quad \frac{1}{\sigma_Z^+(k,v)}\ge\frac{1}{A}+\frac{B^-}{2}\left(((k+1)S+\lambda_{\varepsilon_0}^-)q\right)^2$$

hold for $k=-N_1(v)-1,\cdots,-1$. Moreover, for any $k=-N_1(v)-1,\cdots,N_2(v)+1$, set $\overline{Z}_{k,v}(t,s)=\overline{Z}(t_0+kSq+tq+sq),(t,s)\in[0,S]\times[0,\lambda_{\varepsilon_0}^+]$. It follows from (11) that, for the correlation function $r_{\overline{Z}_{k,v}}(\cdot,\cdot,\cdot,\cdot)$ of $\overline{Z}_{k,v}$

$$\lim_{v\to\infty}2v^2(1-r_{\overline{Z}_{k,v}}(t,s,t',s'))=Q\left|s+t-s'-t'\right|^\alpha \tag{29}$$

holds for any $(t,s),(t',s')\in[0,S]\times[0,\lambda_{\varepsilon_0}^+]$. Furthermore, for sufficiently large v

$$2v^2(1-r_{\overline{Z}_{k,v}}(t,s,t',s')) \quad \le \quad G_0\frac{K^2(q\left|s+t-s'-t'\right|)}{K^2(q)},$$

for all $(t,s),(t',s')\in[0,S]\times[0,\lambda_{\varepsilon_0}^+]$, with some positive constant G_0. Set $S_{\max}=\max\{|s+t-s'-t'|:(t,s),(t',s')\in[0,S]\times[0,\lambda_{\varepsilon_0}^+]\}$. Using Potter bounds (cf. [19]), for any small $\delta>0$ we have, when v is sufficiently large

$$\begin{aligned}\frac{K^2(q\,|s+t-s'-t'|)}{K^2(q)} &\le G_1\max\left(S_{\max}^{\alpha-\delta},S_{\max}^{\alpha+\delta}\right)\left(\frac{|s+t-s'-t'|}{S_{\max}}\right)^{\alpha-\delta}\\ &\le G_2(|t-t'|^{\alpha-\delta}+|s-s'|^{\alpha-\delta})\end{aligned}$$

holds uniformly with respect to $(t,s),(t',s')\in[0,S]\times[0,\lambda_{\varepsilon_0}^+]$, where G_1,G_2 are two positive constants. Hence, by an application of Lemma 5.1, where we set

$$f(v)=\frac{v}{A},\quad \theta_k(v)=\left(\frac{1}{A}+\frac{B^+}{2}\left(((k+1)S+\lambda_{\varepsilon_0}^+)q\right)^2\right)v,\quad D=\frac{Q}{2A^2},$$

we obtain, for any $k=0,\cdots,N_2(v)+1$

$$Q_k(v)\ge\mathcal{H}_\alpha(D_0S,D_0\lambda_{\varepsilon_0}^+)\frac{1}{\sqrt{2\pi}\theta_k(v)}\exp\left(-\frac{(\theta_k(v))^2}{2}\right)(1+o(1)),\quad u\to\infty,$$

where $D_0=D^{\frac{1}{\alpha}}=2^{-\frac{1}{\alpha}}A^{-\frac{2}{\alpha}}Q^{\frac{1}{\alpha}}$. Therefore, as $v\to\infty$ (set $\zeta(v)=v^{-2}q^{-1}\exp(-\frac{v^2}{2A^2})$)

$$\begin{aligned}\sum_{k=0}^{N_2(v)}Q_k(v)&\ge\mathcal{H}_\alpha(D_0S,D_0\lambda_{\varepsilon_0}^+)\frac{A}{\sqrt{2\pi}v}\sum_{k=0}^{N_2(v)}\exp\left(-\frac{(\theta_k(v))^2}{2}\right)(1+o(1))\\ &=\frac{1}{S}\mathcal{H}_\alpha(D_0S,D_0\lambda_{\varepsilon_0}^+)\frac{A}{\sqrt{2\pi}}\zeta(v)\int_0^{x_2}\exp\left(-\frac{B^+}{2A}x^2\right)dx(1+o(1)),\end{aligned}\tag{30}$$

where we used that $\lim_{v\to\infty}vq=\lim_{v\to\infty}vK^{\leftarrow}\left(\frac{1}{v}\right)=0$ and $\lim_{v\to\infty}x_2(v)v^{-1/2}=0$.

Similarly, as $v\to\infty$

$$\sum_{k=-N_1(v)}^{-1}Q_k(v)\ge\frac{1}{S}\mathcal{H}_\alpha(D_0S,D_0\lambda_{\varepsilon_0}^+)\frac{A}{\sqrt{2\pi}}\zeta(v)\int_{-x_1}^{0}\exp\left(-\frac{B^+}{2A}x^2\right)dx(1+o(1)).\tag{31}$$

Furthermore, with the same arguments as above for any $S_1>2\lambda$

$$\begin{aligned}\sum_{k=-N_1(v)-1}^{N_2(v)+1}Q_k(v)&\le\frac{1}{S_1}\mathcal{H}_\alpha(D_0S_1,D_0\lambda_{\varepsilon_0}^-)\frac{A}{\sqrt{2\pi}}\zeta(v)\\ &\quad\times\int_{-x_1}^{x_2}\exp\left(-\frac{B^-}{2A}x^2\right)dx(1+o(1)).\end{aligned}\tag{32}$$

Consequently, (27) and (30-32) imply (set $\bar\zeta(v):=D_0A^{\frac{3}{2}}\zeta(v)/\sqrt{B^+}$)

$$\frac{1}{D_0S_1}\mathcal{H}_\alpha(D_0S_1,D_0\lambda_{\varepsilon_0}^-)\left(\Phi\left(\left(\frac{B^-}{A}\right)^{\frac{1}{2}}x_2\right)-\Phi\left(-\left(\frac{B^-}{A}\right)^{\frac{1}{2}}x_1\right)\right)$$

$$
\begin{aligned}
&\geq \limsup_{v\to\infty} \mathbb{P}\left\{ \sup_{t\in\Theta_{x_1,x_2}(v)} \inf_{s\in[0,\lambda^-_{\varepsilon_0} q]} Z(t+s) > v \right\} / \bar{\zeta}(v) \\
&\geq \limsup_{v\to\infty} \mathbb{P}\left\{ \sup_{t\in\Theta_{x_1,x_2}(v)} \inf_{s\in[0,\lambda(v) q]} Z(t+s) > v \right\} / \bar{\zeta}(v) \\
&\geq \liminf_{v\to\infty} \mathbb{P}\left\{ \sup_{t\in\Theta_{x_1,x_2}(v)} \inf_{s\in[0,\lambda(v) q]} Z(t+s) > v \right\} / \bar{\zeta}(v) \\
&\geq \liminf_{v\to\infty} \mathbb{P}\left\{ \sup_{t\in\Theta_{x_1,x_2}(v)} \inf_{s\in[0,\lambda^+_{\varepsilon_0} q]} Z(t+s) > v \right\} / \bar{\zeta}(v) \qquad (33)\\
&\geq \frac{1}{D_0 S}\mathcal{H}_\alpha(D_0 S, D_0\lambda^+_{\varepsilon_0}) \left(\Phi\left(\left(\frac{B^+}{A}\right)^{\frac{1}{2}} x_2\right) - \Phi\left(-\left(\frac{B^+}{A}\right)^{\frac{1}{2}} x_1\right)\right) \\
&\quad - \limsup_{v\to\infty} \Sigma_1(v)/\bar{\zeta}(v).
\end{aligned}
$$

Moreover, since

$$
\Sigma_1(v) \quad \leq \sum_{-N_1(v)\leq k<l\leq N_2(v)} \mathbb{P}\left\{ \sup_{t\in\triangle_k} Z(t_0+t) > v, \sup_{t\in\triangle_l} Z(t_0+t) > v \right\}
$$

similar arguments as in the proof of Eqs. (31) and (32) in [24] imply

$$
\lim_{S\to\infty} \limsup_{v\to\infty} \Sigma_1(v)/\bar{\zeta}(v) = 0. \qquad (34)
$$

Let us assume for the moment that

$$
\limsup_{S\to\infty} \frac{1}{S}\mathcal{H}_\alpha(S, D_0\lambda) > 0. \qquad (35)
$$

Letting first $\varepsilon_0, \varepsilon_1 \to 0$ and then $S, S_1 \to \infty$ we get from (33) and the definition of $\mathcal{H}_\alpha$

$$
\infty > \mathcal{H}_\alpha \geq \liminf_{S\to\infty} \frac{1}{S}\mathcal{H}_\alpha(S, D_0\lambda) \geq \limsup_{S\to\infty} \frac{1}{S}\mathcal{H}_\alpha(S, D_0\lambda) > 0.
$$

Further, in view of (15) and (25) we have

$$
\mathbb{P}\left\{ \sup_{t\geq 0} Z(t) > v \right\} = D_0 A^{\frac{3}{2}} B^{-\frac{1}{2}} \mathcal{H}_\alpha \zeta(v)(1+o(1)), \qquad v\to\infty.
$$

Therefore, the claim of Theorem 2.1 follows with $\mathcal{F}_\alpha(\lambda) \in (0,\infty)$.

Next, we prove (35). Define

$$
E_v = \bigcup_k \left(\triangle_{2k} \cap \Theta_{x_1,x_2}(v) \right), \quad N^*(v) = \sharp\{k\in\mathbb{Z} : \triangle_{2k}\cap\Theta_{x_1,x_2}(v) \neq \emptyset\}.
$$

For any v positive

$$
\mathbb{P}\left\{ \sup_{t\in\Theta_{x_1,x_2}(v)} \inf_{s\in[0,\lambda^+_{\varepsilon_0} q]} Z(t,s) > v \right\} \geq \mathbb{P}\left\{ \sup_{t\in E_v} \inf_{s\in[0,\lambda^+_{\varepsilon_0} q]} Z(t,s) > v \right\}. \qquad (36)
$$

Using Bonferroni's inequality and the same arguments as in the derivation of (30) yield

$$\begin{aligned}&\mathbb{P}\left\{\sup_{t\in E_v}\inf_{s\in[0,\lambda^+_{\varepsilon_0}q]} Z(t,s)>v\right\}\\&\geq \frac{1}{2S}\mathcal{H}_\alpha(D_0S,D_0\lambda^+_{\varepsilon_0})\frac{A}{\sqrt{2\pi}}\zeta(v)\int_{-x_1}^{x_2}\exp\left(-\frac{B^+}{2A}x^2\right)dx-\Sigma_2(v),\end{aligned} \tag{37}$$

where

$$\begin{aligned}\Sigma_2(v) &= \sum_{k,l\in N^*(v),k>l}\mathbb{P}\Bigg\{\sup_{t\in\triangle_{2k}}\inf_{s\in\left[0,\lambda^+_{\varepsilon_0}q\right]} Z(t_0+t+s)>v,\\ &\qquad\qquad \sup_{t\in\triangle_{2l}}\inf_{s\in\left[0,\lambda^+_{\varepsilon_0}q\right]} Z(t_0+t+s)>v\Bigg\}\\ &\leq \sum_{k,l\in N^*(v),k>l}\mathbb{P}\left\{\sup_{t\in\triangle_{2k}} Z(t_0+t)>v,\ \sup_{t\in\triangle_{2l}} Z(t_0+t)>v\right\}.\end{aligned}$$

Similar arguments as in the proof of Eq. (32) in [24] show that

$$\limsup_{v\to\infty}\Sigma_2(v)/\bar{\zeta}(v)\leq G_3S\sum_{k\geq 1}\exp\left(-G_4(kS)^\alpha\right) \tag{38}$$

for some positive constants G_3,G_4. Therefore, combining (33), (36-38) we conclude that

$$\liminf_{S_1\to\infty}\frac{1}{S_1}\mathcal{H}_\alpha(S_1,D_0\lambda)\geq\frac{1}{S}\left(\frac{1}{2D_0}\mathcal{H}_\alpha(D_0S,D_0\lambda)-G_5S^2\sum_{k\geq1}\exp\left(-G_4(kS)^\alpha\right)\right),$$

with some positive constant G_5. Since $\mathcal{H}_\alpha(D_0S,D_0\lambda)$ is positive and increasing as S increases, then for S sufficiently large the right hand side in the last formula is strictly positive, implying thus (35). This completes the proof.

5.2. Proof of Theorem 3.1

The proof is based on an application of Theorem 2.1. From (9) we have that

$$\mathbb{P}\left\{\tau_u^*<\infty\right\}=\mathbb{P}\left\{\sup_{t\geq0}\inf_{s\in[0,S_v]}Z(t+s)>v\right\},$$

with

$$v=v(u)=u^{1-\frac{H}{\beta}}\quad S_v=S_{v(u)}=T_uu^{-\frac{1}{\beta}},\quad u>0.$$

Further, condition (16) implies $\lim_{v\to\infty}S_v/q=T\in[0,\infty)$, and

$$\Pi(v)\leq\mathbb{P}\left\{\sup_{t\geq0}\inf_{s\in[0,S_v]}Z(t+s)>v\right\}\leq\Pi(v)+\Sigma(v), \tag{39}$$

where

$$\Pi(v) = \mathbb{P}\left\{ \sup_{t\in\left[t_0-\frac{\ln v}{v}, t_0+\frac{\ln v}{v}\right]} \inf_{s\in[0,S_v]} Z(t+s) > v \right\},$$

$$\Sigma(v) = \mathbb{P}\left\{ \sup_{|t-t_0|\geq \frac{\ln v}{v}} Z(t) > v \right\}.$$

Taking $x_1(v) = x_2(v) = \ln v$ and $\lambda(v) = S_v/q$ in Theorem 2.1 we conclude that

$$\begin{aligned} \Pi(v) &= \frac{\mathcal{F}_\alpha(D_0T)}{\mathcal{H}_\alpha}\mathbb{P}\left\{\sup_{t\geq 0} Z(t) > v\right\}(1+o(1)) \\ &= \frac{\mathcal{F}_\alpha(D_0T)}{\mathcal{H}_\alpha}\mathbb{P}\left\{\tau_u < \infty\right\}(1+o(1)), \quad u\to\infty. \end{aligned}$$

Moreover, from (26) we have $\Sigma(v) = o(\Pi(v))$ as $u \to \infty$, establishing thus the proof.

5.3. Proof of Corollary 3.3

For any $u > 0$ we have

$$\begin{aligned} \mathbb{P}\left\{\sup_{t\geq 0}\inf_{s\in[t,t+\mathcal{T}]}\Big(X_H(s) - cs^\beta\Big) > u\right\} &\leq \mathbb{P}\left\{\sup_{t\geq 0}\Big(X_H(s) - cs^\beta\Big) > u\right\} \\ &= \mathbb{P}\left\{\tau_u < \infty\right\}. \end{aligned}$$

Further, for any small positive $\varepsilon \in (0, 2H+\alpha-2\beta)$ by the independence of $\mathcal{T}$ and X_H

$$\begin{aligned} &\mathbb{P}\left\{\sup_{t\geq 0}\inf_{s\in[t,t+\mathcal{T}]}\Big(X_H(s) - cs^\beta\Big) > u\right\} \\ &\geq \mathbb{P}\left\{\sup_{t\geq 0}\inf_{s\in[t,t+\mathcal{T}]}\Big(X_H(s) - cs^\beta\Big) > u, \mathcal{T} < u^{\frac{2H+\alpha-2\beta-\varepsilon}{\alpha\beta}}\right\} \\ &\geq \mathbb{P}\left\{\sup_{t\geq 0}\inf_{s\in\left[t,t+u^{\frac{2H+\alpha-2\beta-\varepsilon}{\alpha\beta}}\right]}\Big(X_H(s) - cs^\beta\Big) > u\right\}\mathbb{P}\left\{\mathcal{T} < u^{\frac{2H+\alpha-2\beta-\varepsilon}{\alpha\beta}}\right\}. \end{aligned}$$

Hence, the claim follows from Theorem 3.1, by letting $u \to \infty$.

5.4. Proof of Theorem 4.1

We use the same notation as in the proof of Theorem 3.1. For any $x \in \mathbb{R}$ and $u > 0$

$$\mathbb{P}\left\{\tau_u^* < \infty\right\}\mathbb{P}\left\{\frac{\tau_u^* - t_0u^{\frac{1}{\beta}}}{A^{\frac{1}{2}}B^{-\frac{1}{2}}u^{\frac{H}{\beta}+\frac{1}{\beta}-1}} \leq x \middle| \tau_u^* < \infty\right\} = \mathbb{P}\left\{\tau_u^* \leq t_0u^{\frac{1}{\beta}} + A^{\frac{1}{2}}B^{-\frac{1}{2}}xu^{\frac{H}{\beta}+\frac{1}{\beta}-1}\right\}.$$

Next we focus on the asymptotics of

$$\mathbb{P}\left\{\tau_u^* \leq t_0u^{\frac{1}{\beta}} + A^{\frac{1}{2}}B^{-\frac{1}{2}}xu^{\frac{H}{\beta}+\frac{1}{\beta}-1}\right\}$$

$$= \mathbb{P}\left\{ \sup_{t\in[0,t_0u^{\frac{1}{\beta}}+A^{\frac{1}{2}}B^{-\frac{1}{2}}xu^{\frac{H}{\beta}+\frac{1}{\beta}-1}]} \inf_{s\in[t,t+T_u]} \left(X_H(s)-cs^\beta\right) > u \right\}$$

$$= \mathbb{P}\left\{ \sup_{t\in[0,t_0+A^{\frac{1}{2}}B^{-\frac{1}{2}}xv^{-1}]} \inf_{s\in[0,S_v]} Z(t+s) > v \right\},$$

where

$$v = v(u) = u^{1-\frac{H}{\beta}}, \quad S_v = S_{v(u)} = T_u u^{-\frac{1}{\beta}}, \quad u > 0.$$

Similarly to the proof of Theorem 3.1, we have

$$\Pi_0(v) \le \mathbb{P}\left\{ \sup_{t\in[0,t_0+A^{\frac{1}{2}}B^{-\frac{1}{2}}xv^{-1}]} \inf_{s\in[0,S_v]} Z(t+s) > v \right\} \le \Pi_0(v) + \Sigma_0(v),$$

where

$$\begin{aligned} \Pi_0(v) &= \mathbb{P}\left\{ \sup_{t\in\left[t_0-\frac{\ln v}{v},t_0+A^{\frac{1}{2}}B^{-\frac{1}{2}}xv^{-1}\right]} \inf_{s\in[0,S_v]} Z(t+s) > v \right\} \\ \Sigma_0(v) &= \mathbb{P}\left\{ \sup_{t\in[0,t_0-\frac{\ln v}{v}]} Z(t) > v \right\}. \end{aligned}$$

In the light of Theorem 2.1 and (26) we conclude that, as $u\to\infty$

$$\mathbb{P}\left\{\tau_u^* \le t_0 u^{\frac{1}{\beta}} + A^{\frac{1}{2}}B^{-\frac{1}{2}}xu^{\frac{H}{\beta}+\frac{1}{\beta}-1}\right\} = (1+o(1))\frac{\mathcal{F}_\alpha(D_0T)}{\mathcal{H}_\alpha}\mathbb{P}\left\{\tau_u<\infty\right\}\Phi(x).$$

Therefore, the claim of (21) follows by applying Theorem 3.1. Moreover, as shown in [28], Theorem 1

$$\frac{\tau_u - t_0u^{\frac{1}{\beta}}}{A^{\frac{1}{2}}B^{-\frac{1}{2}}u^{\frac{H}{\beta}+\frac{1}{\beta}-1}}\Big|(\tau_u<\infty) \stackrel{d}{\to} \widetilde{\mathcal{N}}, \quad u\to\infty,$$

with $\widetilde{\mathcal{N}}$ an $N(0,1)$ random variable. Consequently, by Lemma 2.3 in [24]

$$\left(\frac{\tau_u - t_0u^{\frac{1}{\beta}}}{A^{\frac{1}{2}}B^{-\frac{1}{2}}u^{\frac{H}{\beta}+\frac{1}{\beta}-1}}, \frac{\tau_u^* - t_0u^{\frac{1}{\beta}}}{A^{\frac{1}{2}}B^{-\frac{1}{2}}u^{\frac{H}{\beta}+\frac{1}{\beta}-1}}\right)\Big|(\tau_u^*<\infty) \stackrel{d}{\to} (\widetilde{\mathcal{N}},\widetilde{\mathcal{N}}), \quad u\to\infty$$

implying thus (22). This completes the proof.

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