Extremes of Chi-square Processes with Trend

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Abstract: This paper studies the supremum of chi-square processes with trend over a threshold-dependent-time horizon. Under the assumptions that the chi-square process is generated from a centered self-similar Gaussian process and the trend function is modeled by a polynomial function, we obtain the exact tail asymptotics of the supremum of the chi-square process with trend. These results are of interest in applications in engineering, insurance, queuing and statistics, etc. Some possible extensions of our results are also discussed.
Key Words: Chi-square process; Gaussian random field; safety region; tail asymptotics; first passage time; Pickands constant; Piterbarg constant; Fernique-type inequality.
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1 Introduction

Let $\{Y(t), t \geq 0\}$ be a centered self-similar Gaussian process with almost surely (a.s.) continuous sample paths and index $H \in (0,1)$, i.e., $\mathrm{Var}(Y(t)) = t^{2H}$ and for any $a > 0$ and any $s, t \geq 0$

$$\mathrm{Cov}(Y(at), Y(as)) = a^{2H}\mathrm{Cov}(Y(t), Y(s)).$$

It has been shown that self-similar Gaussian processes such as fractional Brownian motion (fBm), sub-fractional Brownian motion and bi-fractional Brownian motion are quite useful in applications in engineering, telecommunication, insurance, queueing, finance, etc., see [7, 14, 17, 20, 26, 35] and the references therein.
Let β, c be two positive constants. In this paper we are interested in the tail asymptptics of the supremum of a chi-square process with trend given by

$$\psi_T(u) = \mathbb{P}\left(\sup_{t\in[0,T]}\left(\sum_{i=1}^n b_i^2 Y_i^2(t) - ct^\beta\right) > u\right), \quad u \to \infty, \tag{1}$$

where $Y_i, i = 1, \cdots, n$ are independent copies of the centered self-similar Gaussian process Y, and $1 = b_1 = \cdots = b_k > b_{k+1} \geq b_{k+2} \geq \cdots \geq b_n > 0$. Here $T > 0$ can be a finite constant, infinity, and eventually we allow $T = T_u, u > 0$ to be a threshold-dependent positive deterministic function.
One motivation for considering (1) stems from its applications in engineering sciences, see [24] and the references therein. More precisely, let $\boldsymbol{X}(t) = (X_1(t), \cdots, X_n(t)), t \geq 0$ be a vector Gaussian load process. Of interest is the probability of exit

$$\mathbb{P}\left(\boldsymbol{X}(t) \notin \boldsymbol{S}_u(t), \quad \text{for some } t \in [0,T]\right),$$

where the *time-dependent safety region* $\boldsymbol{S}_u(t), t \geq 0$ is defined by

$$\boldsymbol{S}_u(t) = \left\{(x_1, \cdots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n x_i^2 \leq h(t,u)\right\}$$

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with $h(t,u), t,u \ge 0$ some positive function. Various models for $\boldsymbol{X}$ and $h(t,u)$ (especially, $h(t,u) \equiv u$) have been discussed in the literature (e.g., [4, 2, 28, 29, 16]) for the case that $T \in (0,\infty)$. In this framework, $\psi_T(u)$ corresponds to the model with $\boldsymbol{X} = (b_1Y_1, \cdots, b_nY_n)$ and $h(t,u) = u + ct^\beta$. As one of the new features of this contribution, we shall deal with different types of $T = T_u, u \ge 0$; see Section 4.
Another motivation stems from its applications in insurance. Specifically, the surplus process of an insurance company can be modeled by

$$R_u(t) = u + ct^\beta - \sum_{i=1}^n b_i^2 Y_i^2(t), \quad t \ge 0, \tag{2}$$

where u is the initial reserve, ct^β models the total premium received up to time t, and $\sum_{i=1}^n b_i^2 Y_i^2(t)$ represents the total amount of aggregated claims up to time t from n different types of risks. In this framework, $\psi_T(u)$ is called *ruin probability* which is the most important measure of risk of the insurance company; see, e.g., [3, 33]. Note that the model in (2) is also related with the framework of fluid queue; see, e.g., [10].
Finally, we remark that the study of $\psi_T(u)$ also gives some insight into the study of some limiting test statistics. In [13], it is shown that a test statistic converges weakly to

$$\sup_{t\in(0,1)} \left(\frac{U(t)^2}{2t(1-t)} - C(t) - \upsilon D(t) \right), \tag{3}$$

where $\{U(t), t \in [0,1]\}$ is a standard Brownian bridge, $C(t) = \ln\left(1 - \ln(1-(2t-1)^2)\right)$, $D(t) = \ln(1 + C(t)^2)$ and $\upsilon > 1$. Apparently, the above process involved is a chi-square process with trend. Asymptotical results for the tail probability of (3) is very interesting from statistical point of view; see, e.g., [19]. See also [22] and the references therein for more applications of chi-type processes in statistics.
Outline of the rest of the paper: Section 2 is concerned about some preliminary results. In Theorem 2.1 we show the tail asymptotics of the supremum of a chi-square process generated from a non-stationary Gaussian process which extends some results in [28, 16]; Lemma 2.2 derives a Fernique-type inequality for certain Gaussian random fields. In Section 3 we concentrate on the asymptotics of (1) over an infinite-time horizon (i.e., $T = \infty$). Under a local stationary condition on the correlation of the self-similar process Y (see (16)), in Theorem 3.1 we derive the asymptotics of $\psi_\infty(u)$. Section 4 is devoted to the symptotics of (1) over a threshold-dependent-time horizon (i.e., $T = T_u$ a positive deterministic function). As a corollary, we also obtain approximations of the conditional first passage time of the process defined in (2). Finally, in Section 5 possible extensions of our results are discussed. We show that general results can also be obtained for the model where Y_i's are independent but not necessarily identical and for the model with a more general correlation structure (for Y) than that in (16).

2 Preliminaries

Let $\{X(t), t \ge 0\}$ be a centered non-stationary Gaussian process with a.s. continuous sample paths. In the following, unless otherwise stated, T is considered to be a positive finite constant. We impose the following typical assumptions on the Gaussian process X (see [29]):
Assumption I: The standard deviation function $\sigma_X(\cdot) := \sqrt{\mathrm{Var}(X(\cdot))}$ of X attains its maximum (assumed to be 1) over $[0,T]$ at the unique point $t = t_0 \in [0,T]$. Further, there exist some positive constants μ, a such that

$$\sigma_X(t) = 1 - a|t - t_0|^\mu (1 + o(1)), \quad t \to t_0.$$

Assumption II: There exist some $\nu \in (0,2], d > 0$ such that

$$r_X(s,t) = \mathrm{Corr}(X(s), X(t)) = 1 - d|t-s|^\nu (1 + o(1)), \quad s,t \to t_0.$$

Assumption III: There exist some positive constants G, γ and ρ such that

$$\mathbb{E}\left((X(t)-X(s))^2\right) \le G|t-s|^{\gamma}$$

holds for all $s,t \in [t_0-\rho, t_0+\rho]\cap[0,T]$.

For such a centered non-stationary Gaussian process X, it is known that (see, e.g., [31], Theorem D.3 in [29] or Theorem 2.1 in [6])

$$\mathbb{P}\left(\sup_{t\in[0,T]} X(t) > u\right) = \mathcal{M}_{\nu,\mu,d,a}\frac{1}{\sqrt{2\pi}}u^{\left(\frac{2}{\nu}-\frac{2}{\mu}\right)_+-1}\exp\left(-\frac{u^2}{2}\right)(1+o(1)), \quad u\to\infty, \tag{4}$$

where $(x)_+ = \max(0,x)$, and, with $I_{(\cdot)}$ denoting the indicator function,

$$\mathcal{M}_{\nu,\mu,d,a} = \begin{cases} d^{1/\nu}a^{-1/\mu}\Gamma(1/\mu+1)(1+I_{(t_0\notin\{0,T\})})\mathcal{H}_\nu, & \text{if } \nu<\mu, \\ \mathcal{P}_\nu^{\frac{a}{d}}, & \text{if } \nu=\mu, \\ 1, & \text{if } \nu>\mu. \end{cases} \tag{5}$$

Here $\mathcal{H}_\nu \in (0,\infty)$ is the *Pickands constant* defined by

$$\mathcal{H}_\nu = \lim_{S\to\infty}\frac{1}{S}\mathbb{E}\left(\exp\left(\sup_{t\in[0,S]}\left(\sqrt{2}B_\nu(t)-t^\nu\right)\right)\right)$$

with $\{B_\nu(t), t\in\mathbb{R}\}$ a standard fBm defined on $\mathbb{R}$ with Hurst index $\nu/2\in(0,1]$; and $\mathcal{P}_\nu^{\frac{a}{d}} \in (0,\infty)$ is the *Piterbarg constant* defined by

$$\mathcal{P}_\nu^{\frac{a}{d}} = \widehat{\mathcal{P}}_\nu^{\frac{a}{d}} I_{(t_0\in(0,T))} + \widetilde{\mathcal{P}}_\nu^{\frac{a}{d}} I_{(t_0\in\{0,T\})} \in (0,\infty) \tag{6}$$

with

$$\widehat{\mathcal{P}}_\nu^\lambda = \lim_{S_1,S_2\to\infty}\mathcal{P}_\nu^\lambda[-S_1,S_2], \quad \widetilde{\mathcal{P}}_\nu^\lambda = \lim_{S\to\infty}\mathcal{P}_\nu^\lambda[0,S] = \lim_{S\to\infty}\mathcal{P}_\nu^\lambda[-S,0],$$

$$\mathcal{P}_\nu^\lambda[-S_1,S_2] = \mathbb{E}\left(\exp\left(\sup_{t\in[-S_1,S_2]}\left(\sqrt{2}B_\nu(t)-(1+\lambda)|t|^\nu\right)\right)\right), \quad \lambda>0, \max(S_1,S_2)>0.$$

We refer to [29, 7, 9, 12] for the properties and generalizations of the Pickands-Piterbarg and related constants. Let $\{\chi^2_{n,\boldsymbol{b}}(t), t\ge 0\}$ be a chi-square process with n degrees of freedom defined by

$$\chi^2_{n,\boldsymbol{b}}(t) = \sum_{i=1}^n b_i^2 X_i^2(t), \quad t\ge 0, \tag{7}$$

where $b_i>0$, $1\le i\le n$ and $\{X_i(t), t\ge0\}$, $1\le i\le n$, are independent copies of the centered Gaussian process X satisfying assumptions **I–III**. As an analogue of (4), [16] derived the following tail asymptotics for $\chi^2_{n,\mathbf{1}}$:

$$\mathbb{P}\left(\sup_{t\in[0,T]}\chi^2_{n,\mathbf{1}}(t) > u\right) = \mathcal{M}_{\nu,\mu,d,a}u^{\left(\frac{1}{\nu}-\frac{1}{\mu}\right)_+}\Upsilon_n(u)(1+o(1)), \quad u\to\infty, \tag{8}$$

where

$$\Upsilon_n(u) := \mathbb{P}\left(\chi^2_{n,\mathbf{1}}(0)>u\right) = \frac{2^{(2-n)/2}}{\Gamma(n/2)}u^{n/2-1}\exp\left(-\frac{u}{2}\right), \quad u\ge 0.$$

The result in (8) was derived by using a similar double-sum method as in [28]. As shown in [28, 16] the usage of the double-sum method for the chi-square process is usually technical, since we have to deal with the supremum of a Gaussian random field with variance function attaining its maximum on an infinite set; see also [5] for a recent result in this direction. Below, we present a general result on the tail asymptotics of $\chi^2_{n,\boldsymbol{b}}$ allowing for different b_i's. The next result may not be surprising (see [28, 16]), but it turns out that the proof is far from trivial. As we will see the following result is crucial when dealing with the tail asymptotics of the supremum of the chi-square process with trend; two other extensions of Theorem 2.1 will be discussed in Section 5.

Theorem 2.1 *Let $\{\chi^2_{n,\boldsymbol{b}}(t), t \ge 0\}$ be a chi-square process defined as above with generic X satisfying assumptions* **I–III**. *If $1 = b_1 = \cdots = b_k > b_{k+1} \ge b_{k+2} \ge \cdots \ge b_n > 0$, then, as $u \to \infty$,*

$$\mathbb{P}\left(\sup_{t\in[0,T]} \chi^2_{n,\boldsymbol{b}}(t) > u\right) = \prod_{i=k+1}^{n} (1-b_i^2)^{-1/2} \mathcal{M}_{\nu,\mu,d,a} u^{\left(\frac{1}{\nu}-\frac{1}{\mu}\right)_+} \Upsilon_k(u)(1+o(1)). \tag{9}$$

We conclude this section with a Fernique-type inequality, which will be used in the proof of our main result. The proof of it is quite similar to the classical Fernique's inequality (see, e.g., [23]). We refer to [25] for new developments on the Fernique-type inequality.

Lemma 2.2 *Let $\{\xi(\boldsymbol{t}), \boldsymbol{t} \in [0,1]^n\}$ be a centered Gaussian process with a.s. continuous sample paths and* $\mathrm{Var}(\xi(0)) = \sigma^2 \ge 0$. *Suppose that*

$$\mathbb{E}\left((\xi(\boldsymbol{t}) - \xi(\boldsymbol{s}))^2\right) \le \mathbb{Q} \sum_{i=1}^{n} |t_i - s_i|^{\alpha_i} \tag{10}$$

holds for all $\boldsymbol{t}, \boldsymbol{s} \in [0,1]^n$, with some constants $\mathbb{Q} > 0, \alpha_i > 0, 1 \le i \le n$. Then, for all $x > 0$

$$\mathbb{P}\left(\sup_{\boldsymbol{t}\in[0,1]^n} \xi(\boldsymbol{t}) > x\right) \le 2^{n+1} \exp\left(-\frac{c^* x^2}{\mathbb{Q}}\right) + 2^{-1} \exp\left(-\frac{x^2}{8\sigma^2}\right),$$

where $c^* = \left(2n \sum_{p=0}^{\infty} \left((p+1)2^{-(p+1)\min_{1\le i\le n}\alpha_i + 1}\right)^{1/2}\right)^{-2}$, and if $\sigma^2 = 0$ then the second term on the right-hand side disappears.

3 Infinite-time Horizon

In this section we shall focus on the asymptotics of

$$\psi_\infty(u) = \mathbb{P}\left(\sup_{t\in[0,\infty)} \sum_{i=1}^{n} b_i^2 Y_i^2(t) - ct^\beta > u\right), \quad u \to \infty, \tag{11}$$

with Y_i's are the centered self-similar Gaussian processes as discussed in Section 1. Throughout the paper, for technical reasons we assume that $\beta > 2H$. As demonstrated in [17, 18] it is useful to define, for $\beta > 2H$ and $c > 0$

$$Z_i(t) = \frac{Y_i(t)}{\sqrt{1+ct^\beta}}, \quad t \ge 0, \quad 1 \le i \le n. \tag{12}$$

Indeed, by self-similarity of Y_i's, for any $u > 0$

$$\psi_\infty(u) = \mathbb{P}\left(\sup_{t\ge0} \sum_{i=1}^{n} b_i^2 Z_i^2(t) > u^{1-\frac{2H}{\beta}}\right). \tag{13}$$

Let $\sigma_Z(t) = \sqrt{\mathrm{Var}(Z_1(t))}$. It is noted that $\sigma_Z(t)$ attains its maximum on $[0,\infty)$ at the unique point

$$t_0 = \left(\frac{2H}{c(\beta-2H)}\right)^{\frac{1}{\beta}}$$

and

$$\sigma_Z(t) = A^{-1/2}\left(1 - \frac{B}{4A}(t-t_0)^2(1+o(1))\right), \quad t \to t_0 \tag{14}$$

with

$$A = \left(\frac{2H}{c(\beta-2H)}\right)^{-2H/\beta}\frac{\beta}{\beta-2H}, \quad B = 2\left(\frac{2H}{c(\beta-2H)}\right)^{-2(H+1)/\beta} H\beta. \tag{15}$$

In the rest of the paper we assume *local stationarity* for the standardized Gaussian process $\overline{Y}(t) := Y(t)/t^H, t > 0$ in a neighborhood of the point t_0, i.e.,

$$\lim_{s\to t_0, t\to t_0} \frac{\mathbb{E}\left((\overline{Y}(s)-\overline{Y}(t))^2\right)}{|s-t|^\alpha} = Q > 0 \tag{16}$$

holds for some $\alpha \in (0,2)$. Condition (16) is common in the literature; most of the known self-similar Gaussian processes (such as fBm, sub-fBm, and bi-fBm) satisfy (16), see e.g., [16]. Note that the local stationarity at t_0 and the self-similarity of the process Y imply the local stationarity at any point $r \in (0,\infty)$.

Next we present our main result concerning the tail asymptotics of the supremum of the self-similar chi-square process with trend over an infinite-time horizon.

Theorem 3.1 *Suppose that the generic process $\{Y(t), t \geq 0\}$ is a centered self-similar Gaussian process with index $H \in (0,1)$ and correlation function satisfying (16). If $\beta > 2H$, then*

$$\begin{aligned}\psi_\infty(u) &= 2^{1-1/\alpha}Q^{1/\alpha}A^{1/\alpha}B^{-1/2}\pi^{1/2}\mathcal{H}_\alpha \prod_{i=k+1}^{n}(1-b_i^2)^{-1/2} \\ &\quad \times u^{(1-2H/\beta)(1/\alpha-1/2)}\Upsilon_k(Au^{1-2H/\beta})(1+o(1)), \quad u\to\infty.\end{aligned}$$

4 Threshold-dependent-time Horizon

In this section we are concerned about the asymptotics of

$$\psi_{T_u}(u) = \mathbb{P}\left(\sup_{t\in[0,T_u]} \sum_{i=1}^{n} b_i^2 Y_i^2(t) - ct^\beta > u\right), \quad u \to \infty.$$

Throughout this section we shall adopt the same notation as in Section 3. In addition, define

$$B(u) = 2^{1/2}B^{-1/2}u^{\frac{H+1}{\beta}-\frac{1}{2}}, \quad u > 0.$$

In what follows, the following two scenarios of $T_u > 0$ will be discussed:

i) The short time horizon: $\lim_{u\to\infty}\frac{T_u}{u^{1/\beta}} = s_0 \in [0,t_0)$;

ii) The long time horizon: $\lim_{u\to\infty}\frac{T_u - t_0 u^{1/\beta}}{B(u)} = x \in (-\infty,\infty]$.

Clearly, $T=\infty$ is included in scenario ii) and $T \in (0,\infty)$ is covered by scenario i).

We present below our main result of this section.

Theorem 4.1 *Suppose that the generic process $\{Y(t), t \geq 0\}$ is a centered self-similar Gaussian process with index $H \in (0,1)$ and correlation function satisfying (16). Assume further that $\beta > 2H$. We have, as $u \to \infty$,*
i) If $\lim_{u\to\infty}\frac{T_u}{u^{1/\beta}} = s_0 \in [0,t_0)$, then

$$\psi_{T_u}(u) = \prod_{i=k+1}^{n}(1-b_i^2)^{-1/2}\mathcal{M}_{\alpha,1,\frac{Q}{2}t_0^\alpha,D}\left(\frac{u+cT_u^\beta}{T_u^{2H}}\right)^{(\frac{1}{\alpha}-1)_+}\Upsilon_k\left(\frac{u+cT_u^\beta}{T_u^{2H}}\right)(1+o(1)),$$

where the constant $\mathcal{M}_{\alpha,1,\frac{Q}{2}t_0^\alpha,D}$ *is given as in* (5) *with* $D=\frac{2H-c(\beta-2H)s_0^\beta}{2(1+cs_0^\beta)}$.
ii) If $\lim_{u\to\infty}\frac{T_u-t_0u^{1/\beta}}{B(u)}=x\in(-\infty,\infty]$, *then*

$$\psi_{T_u}(u)=\psi_\infty(u)\Phi(x)(1+o(1)),$$

where the asymptotics of $\psi_\infty(u)$ *is given in Theorem 3.1 and* $\Phi(\cdot)$ *denotes the standard normal distribution function.*

As a corollary of Theorem 4.1 we derive an approximation of the first passage time of the chi-square process with trend, which goes in line with e.g., [18, 8, 15]. Precisely, define

$$\tau_u=\inf\{t\ge 0: R_u(t)\le 0\}\quad (\text{with} \inf\{\varnothing\}=\infty)$$

to be the first passage time to 0 of the process $\{R_u(t),t\ge 0\}$ defined in (2). Denote by $\stackrel{d}{\to}$ convergence in distribution when the argument tends to infinity, and let E be a unit mean exponential random variable and $\mathcal{N}$ be a standard normal random variable. We have:

Corollary 4.2 *Under the conditions and notation of Theorem 4.1*
i) If $\lim_{u\to\infty}\frac{T_u}{u^{1/\beta}}=s_0\in[0,t_0)$, *then*

$$\frac{(2H-\frac{c\beta s_0^\beta}{1+cs_0^\beta})(u+cT_u^\beta)}{2T_u^{2H+1}}(T_u-\tau_u)\Big|(\tau_u\le T_u)\quad\stackrel{d}{\to}\quad E,\quad u\to\infty.$$

ii) If $\lim_{u\to\infty}\frac{T_u-t_0u^{1/\beta}}{B(u)}=x\in(-\infty,\infty]$, *then*

$$\frac{\tau_u-t_0u^{1/\beta}}{B(u)}\Big|(\tau_u\le T_u)\quad\stackrel{d}{\to}\quad\mathcal{N}\Big|(\mathcal{N}\le x),\quad u\to\infty.$$

5 Extensions & Discussions

In Section 3 and Section 4, we have derived asymptotical results for the case where the chi-square process is generated from a self-similar Gaussian process. In this section, we shall discuss two possible extensions: (a) instead of independent copies of a self-similar Gaussian process we shall consider independent but non-identical self-similar Gaussian processes; (b) instead of polynomial function $|t-s|^\alpha$ in (16) we consider a regularly varying function $K^2(|t-s|)$ with index $\alpha\in(0,2]$.
As we have seen, Theorem 2.1 and Theorem 6.1 are fundamental for the proofs of our results in the last two sections. Asymptotical results for the extended chi-square processes (as in the cases (a) and (b)) with trend will follow similarly if corresponding extended results for Theorems 2.1 and 6.1 are available. Therefore, it is sufficient at this point to present only an extension of Theorem 2.1; corresponding extension for Theorem 6.1 can also be obtained.

5.1 Non-identical Gaussian processes X_i's

Let $\{X_i(t),t\ge 0\},1\le i\le k$ be independent copies of the a.s. continuous Gaussian process X satisfying assumptions **I–III** with the parameters therein, and let $\{X_i(t),t\ge 0\},k+1\le i\le n$ be independent copies of another a.s. continuous Gaussian process $X^{(1)}$ satisfying assumption **III** with parameter γ_1 instead of γ. Moreover, we suppose that the standard deviation function $\sigma_{X^{(1)}}(\cdot)$ attains its maximum 1 over $[0,T]$ at t_0 as well. Besides, $\{X_i(t), t\ge 0\}, 1\le i\le k$, and $\{X_i(t), t\ge 0\}, k+1\le i\le n$ are assumed to be independent. Define also

$$\chi^2_{n,\boldsymbol{b}}(t)=\sum_{i=1}^n b_i^2 X_i^2(t),\quad t\ge 0,$$

with $1=b_1=\cdots=b_k\ge b_{k+1}\ge\cdots\ge b_n>0$.

Theorem 5.1 *Let $\{\chi^2_{n,\boldsymbol{b}}(t), t\ge 0\}$ be a chi-square process defined as above. If $\gamma\ge\nu$ and $\gamma_1\ge\nu$, then we have, as $u\to\infty$,*

$$\mathbb{P}\left(\sup_{t\in[0,T]}\chi^2_{n,\boldsymbol{b}}(t)>u\right)=\prod_{i=k+1}^n(1-b_i^2)^{-1/2}\mathcal{M}_{\nu,\mu,d,a}u^{\left(\frac{1}{\nu}-\frac{1}{\mu}\right)_+}\Upsilon_k(u)(1+o(1)).\tag{17}$$

Remarks 5.2 *a) Suppose that the generic processes X and $X^{(1)}$ are both fBm with indexes $H\in(0,1)$ and $H_1\in(0,1)$, respectively. If $H_1\ge H$, then the conditions of the last theorem are fulfilled.*
b) From the proof of the last theorem we can see the assumption that $\{X_i(t), t\ge 0\}, k+1\le i\le n$ are identical (in distribution) is not really necessary; here to simplify the notation we chose to work under this assumption.

5.2 General correlation structure

First, we formulate the general assumption about the correlation structure of the generic Gaussian process X.
Assumption II': There exists some $K(\cdot)$, a regularly varying function at 0 with index $\nu/2\in(0,1]$, such that

$$r_X(s,t)=\mathrm{Corr}(X(s),X(t))=1-K^2(|t-s|)(1+o(1)),\quad s,t\to t_0.$$

Next, we introduce some further notation. Let $q(u)=\overleftarrow{K}(u^{-1/2})$ be the inverse function of $K(\cdot)$ at point $u^{-1/2}$ (assumed to exist asymptotically). It follows that $q(u)$ is a regularly function at infinity with index $-1/\nu$ which can be further expressed as $q(u)=u^{-1/\nu}L(u^{-1/2})$, with $L(\cdot)$ a slowly varying function at 0. According to the values of $L(u^{-1/2})$ as $u\to\infty$, we consider the following three scenarios:
C1: $\mu>\nu$, or $\mu=\nu$ and $\lim_{u\to\infty}L(u^{-1/2})=0$;
C2: $\mu=\nu$ and $\lim_{u\to\infty}L(u^{-1/2})=\mathcal{L}\in(0,\infty)$;
C3: $\mu<\nu$, or $\mu=\nu$ and $\lim_{u\to\infty}L(u^{-1/2})=\infty$.
We present below our second extension of Theorem 2.1.

Theorem 5.3 *Under the assumptions and conditions of Theorem 2.1 with assumption* **II** *replaced by assumption* **II'***, we have, as $u\to\infty$,*

$$\mathbb{P}\left(\sup_{t\in[0,T]}\chi^2_{n,\boldsymbol{b}}(t)>u\right)=\prod_{i=k+1}^n(1-b_i^2)^{-1/2}\widetilde{\mathcal{M}}_{\nu,\mu,1,a}(u)u^{\left(\frac{1}{\nu}-\frac{1}{\mu}\right)_+}\Upsilon_k(u)(1+o(1)),\tag{18}$$

where

$$\widetilde{\mathcal{M}}_{\nu,\mu,1,a}(u)=\begin{cases}a^{-1/\mu}\Gamma(1/\mu+1)(1+I_{(t_0\not\in\{0,T\})})\mathcal{H}_\nu\overleftarrow{L}(u^{-1/2}), & \text{for } \textbf{\textit{C1}},\\ \mathcal{P}_\nu^{a\mathcal{L}^\nu}, & \text{for } \textbf{\textit{C2}},\\ 1, & \text{for } \textbf{\textit{C3}}.\end{cases}$$

The proof of the last theorem follows by similar arguments as in the proof of Theorem 2.1, and thus we only give some remarks. Actually, it relies on a corresponding extension of Lemma 7.1, which can be done as in the proof of Theorem 8.2 in [29]. Note that the difference from the classical results in [29] is that for the case $\mu=\nu$ three sub-cases should be considered differently (depending on the property of $L(\cdot)$). This is not observed in the study of some other Gaussian random fields, e.g., [32] and [11], where it is shown that the substitution of a polynomial function $d|t-s|^\nu$ by a regularly varying function $K^2(|t-s|)$ in the correlation structure of the Gaussian random fields does not influence much on the asymptotics. However, it seems not surprising to have these sub-cases if one examines the proof of Theorem 8.2 in [29].

6 Further Results & Proofs

This section is devoted to the proofs of Theorems 2.1, 3.1, 4.1 and 5.1 and Corollary 4.2. Let in the following $\mathbb{Q}, \mathbb{Q}_i, i = 1, 2, ...$ denote positive constants whose values may change from line to line.
First, we present a result concerning the tail asymptotics of the supremum of a chi-square process over a threshold-dependent time interval, which turns out to be crucial for the proofs of Theorem 2.1, Theorem 4.1 and Corollary 4.2. The technical proof of it is deferred to Appendix.

Theorem 6.1 *Let* $\{\chi^2_{n,\boldsymbol{b}}(t), t \ge 0\}$ *be a chi-square process given as in* (7) *with generic* X *satisfying assumptions* **I–II**, *and* $1 = b_1 = \cdots = b_k > b_{k+1} \ge b_{k+2} \ge \cdots \ge b_n > 0$. *Let further* $\Delta_x(u) = [t_0 - x_1(u)u^{-2/\mu}, t_0 + x_2(u)u^{-2/\mu}]$ *with functions* $x_i(u), i = 1, 2$ *such that*

$$\lim_{u\to\infty} x_i(u) = x_i \in [-\infty, \infty], \quad \lim_{u\to\infty} x_i(u)u^{-1/\mu} = 0, \quad i = 1, 2.$$

If $-x_1 < x_2$, *then*

$$\mathbb{P}\left(\sup_{t\in\Delta_x(u)} \chi^2_{n,\boldsymbol{b}}(t) > u^2\right) = \prod_{i=k+1}^{n} (1 - b_i^2)^{-1/2} \widehat{\mathcal{M}}_{\nu,\mu,d,a}(x_1, x_2) u^{\left(\frac{2}{\nu} - \frac{2}{\mu}\right)_+} \Upsilon_k(u^2)(1 + o(1)) \tag{19}$$

as $u \to \infty$, *where*

$$\widehat{\mathcal{M}}_{\nu,\mu,d,a}(x_1, x_2) = \begin{cases} d^{1/\nu} a^{-1/\mu} \mathcal{H}_\nu\left(G_\mu(a^{1/\mu}x_2) - G_\mu(-a^{1/\mu}x_1)\right), & \text{if } \nu < \mu, \\ \mathcal{P}_\nu^{\frac{a}{d}}[-d^{1/\nu}x_1, d^{1/\nu}x_2], & \text{if } \nu = \mu, \\ 1, & \text{if } \nu > \mu, \end{cases} \tag{20}$$

with $G_\mu(x) = \int_{-\infty}^{x} e^{-|t|^\mu} dt, x > 0$ *for any* $\mu > 0$.

Proof of Theorem 2.1: Without lose of generality we shall only consider the case that $t_0 \in (0, T)$. As in the proof of Theorem 6.1, we consider the Gaussian random field

$$Y_{\boldsymbol{b}}(t, \boldsymbol{v}) = \sum_{i=1}^{n} b_i X_i(t) v_i$$

defined on $\mathcal{G}_T = [0, T] \times \mathcal{S}_{n-1}$, where $\mathcal{S}_{n-1}$ stands for the $(n-1)$-dimensional unit sphere. We refer to (34)–(36) below for some important properties of the Gaussian random field $Y_{\boldsymbol{b}}$. It follows that

$$\mathbb{P}\left(\sup_{t\in[0,T]} \chi^2_{n,\boldsymbol{b}}(t) > u^2\right) = \mathbb{P}\left(\sup_{(t,\boldsymbol{v})\in\mathcal{G}_T} Y_{\boldsymbol{b}}(t, \boldsymbol{v}) > u\right).$$

Therefore, we shall focus on the tail asymptotics of

$$\mathbb{P}\left(\sup_{(t,\boldsymbol{v})\in\mathcal{G}_T} Y_{\boldsymbol{b}}(t, \boldsymbol{v}) > u\right), \quad u \to \infty.$$

Next define $\Delta_u = [t_0 - (\ln u/u)^{2/\mu}, t_0 + (\ln u/u)^{2/\mu}]$, $C_u = \{\boldsymbol{v} \in \mathcal{S}_{n-1} : v_i \in [-\ln u/u, \ln u/u], k + 1 \le i \le n\}$ and let

$$\pi_1(u) := \mathbb{P}\left(\sup_{(t,\boldsymbol{v})\in\Delta_u\times C_u} Y_{\boldsymbol{b}}(t, \boldsymbol{v}) > u\right).$$

We have, for any $u > 0$ and any small $\rho > 0$

$$\pi_1(u) \le \mathbb{P}\left(\sup_{(t,\boldsymbol{v})\in\mathcal{G}_T} Y_{\boldsymbol{b}}(t, \boldsymbol{v}) > u\right) \le \pi_1(u) + \mathbb{P}\left(\sup_{(t,\boldsymbol{v})\in\mathcal{G}_T/([t_0-\rho,t_0+\rho]\times C_u)} Y_{\boldsymbol{b}}(t, \boldsymbol{v}) > u\right)$$

$$+\mathbb{P}\left(\sup_{(t,\boldsymbol{v})\in([t_0-\rho,t_0+\rho]/\Delta_u)\times C_u} Y_{\boldsymbol{b}}(t,\boldsymbol{v})>u\right). \tag{21}$$

Further, in view of Theorem 6.1

$$\pi_1(u)=\prod_{i=k+1}^{n}(1-b_i^2)^{-1/2}\widehat{\mathcal{M}}_{\nu,\mu,d,a}(-\infty,\infty)u^{\left(\frac{2}{\nu}-\frac{2}{\mu}\right)_+}\Upsilon_k(u^2)(1+o(1)) \tag{22}$$

as $u\to\infty$. By (34) and the Borell-TIS inequality (see, e.g., [1])

$$\mathbb{P}\left(\sup_{(t,\boldsymbol{v})\in\mathcal{G}_T/([t_0-\rho,t_0+\rho]\times C_u)} Y_{\boldsymbol{b}}(t,\boldsymbol{v})>u\right)\le \mathbb{Q}\exp\left(-\frac{(u-\mathbb{Q}_1)^2}{2(1-\delta_0)}\right) \tag{23}$$

holds for all u large, with some constants $\mathbb{Q}>0,\mathbb{Q}_1>0$ and $\delta_0\in(0,1)$. Further, in the light of (34), (36) and the Piterbarg inequality given in Theorem 8.1 in [30]

$$\mathbb{P}\left(\sup_{(t,\boldsymbol{v})\in([t_0-\rho,t_0+\rho]/\Delta_u)\times C_u} Y_{\boldsymbol{b}}(t,\boldsymbol{v})>u\right)\le \mathbb{Q}_2 u^{\frac{2(n+1)}{\gamma\wedge 2}-1}\exp\left(-\frac{u^2}{2(1-(\ln u/u)^2\mathbb{Q}_3)^2}\right) \tag{24}$$

holds for all u large, with some positive constants $\mathbb{Q}_2,\mathbb{Q}_3$. Consequently, the claim for the case that $t_0\in(0,T)$ follows from (21)–(24). This completes the proof. □

Proof of Theorem 3.1: Let $T>t_0$ be some fixed large enough integer, and let

$$\pi(u)=\mathbb{P}\left(\sup_{t\in[0,T]}\sum_{i=1}^{n}b_i^2Z_i^2(t)>u^{1-2H/\beta}\right),\quad \pi_1(u)=\mathbb{P}\left(\sup_{t\in[T,\infty)}\sum_{i=1}^{n}b_i^2Z_i^2(t)>u^{1-2H/\beta}\right).$$

Clearly

$$\pi(u)\le\psi_\infty(u)\le\pi(u)+\pi_1(u).$$

By the definition of Z_i's we have that there exist some constants $\mathbb{Q}>0,\rho\ge0$ such that

$$\mathbb{E}\left((Z_1(t)-Z_1(s))^2\right)\le\mathbb{Q}|t-s|^\alpha$$

holds for any $t,s\in[t_0-\rho,t_0+\rho]$. Thus, in view of (14), (16) and Theorem 2.1 we conclude that

$$\pi(u)=\prod_{i=k+1}^{n}(1-b_i^2)^{-1/2}\mathcal{M}_{\alpha,2,\frac{Q}{2},\frac{B}{4A}}\,(A(u))^{\frac{2}{\alpha}-1}\,\Upsilon_k((A(u))^2)(1+o(1)),\quad u\to\infty,$$

where $A(u)=A^{1/2}u^{1/2-\frac{H}{\beta}}$. Therefore, to complete the proof it is sufficient to show that

$$\pi_1(u)=o(\pi(u)),\quad u\to\infty.$$

To this end, let $\widetilde{Y_{\boldsymbol{b}}}(t,\boldsymbol{v})=\sum_{i=1}^n b_iA^{1/2}Z_i(t)v_i,(t,\boldsymbol{v})\in[T,\infty)\times[-1,1]^n$. We have

$$\pi_1(u)=\mathbb{P}\left(\sup_{(t,\boldsymbol{v})\in[T,\infty)\times S_{n-1}}\widetilde{Y_{\boldsymbol{b}}}(t,\boldsymbol{v})>A(u)\right).$$

We split the interval $[T,\infty)$ into subintervals $[k,k+1),k\ge T$. For every $k\ge T$, we have (set $Y_b^*(t,\boldsymbol{v})=\frac{\sqrt{1+ct^\beta}}{t^H}\widetilde{Y}_b(t,\boldsymbol{v})$)

$$\mathbb{P}\left(\sup_{(t,v\boldsymbol{v})\in[k,k+1)\times\mathcal{S}_{n-1}}\widetilde{Y}_b(t,\boldsymbol{v})>A(u)\right)$$

$$\leq \mathbb{P}\left(\sup_{(t,\boldsymbol{v})\in[k,k+1)\times[-1,1]^n} Y_b^*(t,\boldsymbol{v}) > \frac{\sqrt{1+c(k-1)^\beta}}{(k-1)^H}A(u)\right). \tag{25}$$

In addition, there exists some global constant $\mathbb{Q}$ such that for any $k \geq T$

$$\begin{aligned}
&\mathbb{E}\left(Y_b^*(t,\boldsymbol{v}) - Y_b^*(t',\boldsymbol{v}')\right)^2 \\
&\leq 2An\mathbb{E}\left(\overline{Y}(t) - \overline{Y}(t')\right)^2 + 2A\sum_{i=1}^n (v_i - v_i^{'})^2 \\
&\leq \mathbb{Q}\left(|t-t'|^\alpha + \sum_{i=1}^n (v_i - v_i^{'})^2\right)
\end{aligned}$$

holds for all $t,t' \in [k,k+1), \boldsymbol{v},\boldsymbol{v}' \in [-1,1]^n$. Next we split $[-1,1]^n$ into 2^n subsets of the form $\prod_{i=1}^n \Delta_i^{j_i}, j_i = 1,2$, where $\Delta_i^1 = [-1,0]$ and $\Delta_i^2 = [0,1]$. By using Lemma 2.2 we derive, for $k \geq T$

$$\mathbb{P}\left(\sup_{(t,\boldsymbol{v})\in[k,k+1)\times\prod_{i=1}^n \Delta_i^{j_i}} Y_b^*(t,\boldsymbol{v}) > \frac{\sqrt{1+c(k-1)^\beta}}{(k-1)^H}A(u)\right) \leq 2^{n+2}e^{-\mathbb{Q}_1\frac{1+c(k-1)^\beta}{(k-1)^{2H}}(A(u))^2},$$

with $\mathbb{Q}_1 = \min(\frac{c^*}{\mathbb{Q}}, \frac{1}{8A})$. This together with (25) yields that

$$\mathbb{P}\left(\sup_{(t,\boldsymbol{v})\in[k,k+1)\times S_{n-1}} \widetilde{Y}_b(t,\boldsymbol{v}) > A(u)\right) \leq 2^{2n+2}e^{-\mathbb{Q}_1\frac{1+c(k-1)^\beta}{(k-1)^{2H}}(A(u))^2}.$$

Consequently, since T was chosen large enough

$$\begin{aligned}
\pi_1(u) &\leq \sum_{k=T}^\infty 2^{2n+2}e^{-\mathbb{Q}_1\frac{1+c(k-1)^\beta}{(k-1)^{2H}}(A(u))^2} \\
&\leq 2^{2n+2}\int_{T-2}^\infty e^{-\mathbb{Q}_2(A(u))^2y^{\beta-2H}}dy \\
&\leq \mathbb{Q}_3(A(u))^{-2}e^{-\mathbb{Q}_2(T-2)^{\beta-2H}(A(u))^2} = o(\pi(u))
\end{aligned}$$

as $u \to \infty$, where $\mathbb{Q}_3$ is a constant depending on T and $\mathbb{Q}_2 = c\mathbb{Q}_1$. This completes the proof. □

Proof of Theorem 4.1: Case i). We introduce a deterministic function $m(u) = \frac{u+cT_u^\beta}{T_u^{2H}}, u > 0$, and centered Gaussian processes

$$W_{u,i}(t) = \frac{Y_i(t)}{\sqrt{1-\frac{c_u}{1+c_u}(1-t^\beta)}}, \quad t \geq 0, \quad 1 \leq i \leq n,$$

with $c_u = cT_u^\beta/u, u > 0$ such that $\lim_{u\to\infty} c_u = cs_0^\beta =: c_0$. By the self-similarity of Y we have

$$\psi_{T_u}(u) = \mathbb{P}\left(\sup_{t\in[0,1]}\sum_{i=1}^n b_i^2 W_{u,i}^2(t) > m(u)\right).$$

Let further $c_0^{\pm\epsilon} = \max(c_0 \pm \epsilon, 0)$ and define

$$W_i^{\pm\epsilon}(t) = \frac{Y_i(t)}{\sqrt{1-\frac{c_0^{\pm\epsilon}}{1+c_0^{\pm\epsilon}}(1-t^\beta)}}, \quad t \geq 0, \quad 1 \leq i \leq n,$$

for any sufficiently small $\epsilon > 0$. Thus we have, for u large enough

$$\mathbb{P}\left(\sup_{t\in[0,1]}\sum_{i=1}^n b_i^2\left(W_i^{-\epsilon}(t)\right)^2 > m(u)\right) \leq \psi_{T_u}(u) \leq \mathbb{P}\left(\sup_{t\in[0,1]}\sum_{i=1}^n b_i^2\left(W_i^{+\epsilon}(t)\right)^2 > m(u)\right).$$

Next we consider the upper bound of $\psi_{T_u}(u)$. It follows that $\sigma_{W_1^{+\epsilon}}(t)$ attains its maximum over $[0,1]$ at the unique point $t_0=1$ and further

$$\sigma_{W_1^{+\epsilon}}(t) = 1-\frac{2H-(\beta-2H)c_0^{+\epsilon}}{2(1+c_0^{+\epsilon})}|t-1|(1+o(1)), \quad t\to 1,$$
$$\mathrm{Corr}(W_1^{+\epsilon}(t), W_1^{+\epsilon}(s)) = 1-\frac{t_0^{\alpha}Q}{2}|t-s|^{\alpha}(1+o(1)), \quad s,t\to 1.$$

In addition, there exists some $\mathbb{Q}>0$ such that

$$\mathbb{E}\left((W_1^{+\epsilon}(t)-W_1^{+\epsilon}(s))^2\right)\le \mathbb{Q}|t-s|^{\alpha}$$

holds for all $s,t\in[1/2,1]$. Therefore, in view of Theorem 2.1

$$\mathbb{P}\left(\sup_{t\in[0,1]}\sum_{i=1}^n b_i^2\left(W_i^{+\epsilon}(t)\right)^2>m(u)\right)=\prod_{i=k+1}^n(1-b_i^2)^{-1/2}\mathcal{M}_{\alpha,1,\frac{Q}{2}t_0^{\alpha},D_{+\epsilon}}(m(u))^{\left(\frac{1}{\alpha}-1\right)_+}\Upsilon_k(m(u))(1+o(1))$$

as $u\to\infty$, with $D_{+\epsilon}=\frac{2H-(\beta-2H)c_0^{+\epsilon}}{2(1+c_0^{+\epsilon})}$. Similar arguments give the same lower bound as above (with $+\epsilon$ replaced by $-\epsilon$) for $\psi_{T_u}(u)$, and thus letting $\epsilon\to 0$ the claim in i) follows.
Case ii). Again, using the self-similarity we derive

$$\psi_{T_u}(u)=\mathbb{P}\left(\sup_{t\in[0,T_u u^{-1/\beta}]}\sum_{i=1}^n b_i^2 AZ_i^2(t)>(A(u))^2\right),$$

where $A(u)=A^{1/2}u^{1/2-H/\beta}$. Let $t_u=t_0-u^{-1/2+H/\beta}\ln u$, and define

$$\pi_{t_u}(u)=\mathbb{P}\left(\sup_{t\in[0,t_u]}\sum_{i=1}^n b_i^2 AZ_i^2(t)>(A(u))^2\right).$$

Clearly,

$$\mathbb{P}\left(\sup_{t\in[t_u,T_u u^{-1/\beta}]}\sum_{i=1}^n b_i^2 AZ_i^2(t)>(A(u))^2\right)\le\psi_{T_u}(u)\le\mathbb{P}\left(\sup_{t\in[t_u,T_u u^{-1/\beta}]}\sum_{i=1}^n b_i^2 AZ_i^2(t)>(A(u))^2\right)+\pi_{t_u}(u).$$

In the following, we shall first derive the asymptotics of the common term on both sides of the above formula, which will give the exact asymptotics of $\psi_{T_u}(u)$. Then we show that $\pi_{t_u}(u)$ is asymptotically negligible. In view of (14), (16) and Theorem 6.1, we have, for any $x\in(-\infty,\infty)$

$$\mathbb{P}\left(\sup_{t\in[t_u,T_u u^{-1/\beta}]}\sum_{i=1}^n b_i^2 AZ_i^2(t)>(A(u))^2\right)=\psi_\infty(u)\Phi(x)(1+o(1)), \quad u\to\infty.$$

Next we show that the last formula is also valid for $x=\infty$. Since, for any fixed $y\ge 0$,

$$\mathbb{P}\left(\sup_{t\in[t_u,t_0+yB(u)]}\sum_{i=1}^n b_i^2 AZ_i^2(t)>(A(u))^2\right)\le\mathbb{P}\left(\sup_{t\in[t_u,T_u u^{-1/\beta}]}\sum_{i=1}^n b_i^2 AZ_i^2(t)>(A(u))^2\right)\le\psi_\infty(u),$$

we obtain from Theorem 6.1 that

$$\begin{aligned}\Phi(y) &\le \underline{\lim}_{u\to\infty}\frac{\mathbb{P}\left(\sup_{t\in[t_u,T_u u^{-1/\beta}]}\sum_{i=1}^n b_i^2 AZ_i^2(t)>(A(u))^2\right)}{\psi_\infty(u)}\\ &\le \overline{\lim}_{u\to\infty}\frac{\mathbb{P}\left(\sup_{t\in[t_u,T_u u^{-1/\beta}]}\sum_{i=1}^n b_i^2 AZ_i^2(t)>(A(u))^2\right)}{\psi_\infty(u)}\le 1.\end{aligned}$$

Therefore, letting $y\to\infty$ we conclude that

$$\mathbb{P}\left(\sup_{t\in[t_u,T_u u^{-1/\beta}]}\sum_{i=1}^n b_i^2 AZ_i^2(t)>(A(u))^2\right)=\psi_\infty(u)(1+o(1)),\quad u\to\infty.$$

To complete the proof we prove that $\pi_{t_u}(u)=o(\psi_\infty(u))$ as $u\to\infty$. We have

$$\mathbb{P}\left(\sup_{t\in[0,t_u]}\sum_{i=1}^n b_i^2 AZ_i^2(t)>(A(u))^2\right)=\mathbb{P}\left(\sup_{(t,\boldsymbol{v})\in[0,t_u]\times\mathcal{S}_{n-1}}\widetilde{Y_{\boldsymbol{b}}}(t,\boldsymbol{v})>A(u)\right),$$

where $\widetilde{Y_{\boldsymbol{b}}}(t,\boldsymbol{v})=\sum_{i=1}^n b_i A^{1/2}Z_i(t)v_i, (t,\boldsymbol{v})\in[0,t_0+1]\times\mathcal{S}_{n-1}$. Further, there exist some constants $\delta\in(0,1),\mathbb{Q}>0$ such that

$$\begin{aligned}\mathbb{E}\left((\widetilde{Y_{\boldsymbol{b}}}(t,\boldsymbol{v}))^2\right)&\le 1-\delta<1,\quad t\in[0,t_0-\rho],\boldsymbol{v}\in\mathcal{S}_{n-1},\\ \mathbb{E}\left((\widetilde{Y_{\boldsymbol{b}}}(t,\boldsymbol{v})-\widetilde{Y_{\boldsymbol{b}}}(t,\boldsymbol{v}))^2\right)&\le \mathbb{Q}(|t-s|^\alpha+\sum_{i=1}^n(v_i-v_i^{'})^2),\quad t\in[t_0-\rho,t_0+\rho],\boldsymbol{v}\in\mathcal{S}_{n-1}\end{aligned}$$

hold. Therefore, as in the proof of Theorem 2.1, by the Borell-TIS inequality we have

$$\mathbb{P}\left(\sup_{t\in[0,t_0-\rho]}\sum_{i=1}^n b_i^2 AZ_i^2(t)>(A(u))^2\right)\le e^{-\frac{(A(u)-\mathbb{Q}_0)^2}{2(1-\delta)^2}}=o(\psi_\infty(u)),\quad u\to\infty,\tag{26}$$

with $\mathbb{Q}_0=\mathbb{E}\left(\sup_{(t,\boldsymbol{v})\in[0,t_0-\rho]\times\mathcal{S}_{n-1}}\widetilde{Y_{\boldsymbol{b}}}(t,\boldsymbol{v})\right)<\infty$, and by the Piterbarg inequality and (14) (or by a direct application of [34], Proposition 3.2) we have

$$\mathbb{P}\left(\sup_{t\in[t_0-\rho,t_u]}\sum_{i=1}^n b_i^2 AZ_i^2(t)>(A(u))^2\right)\le\mathbb{Q}_1(A(u))^{2(n+1)/\alpha}\Psi\left(\frac{A(u)}{1-\mathbb{Q}_2(A(u)^{-1}\ln A(u))^2}\right)=o(\psi_\infty(u))\tag{27}$$

as $u\to\infty$, where $\mathbb{Q}_1$ and $\mathbb{Q}_2$ are two positive constants. Consequently, we conclude from (26) and (27) that

$$\pi_{t_u}(u)=o(\psi_\infty(u)),\quad u\to\infty.$$

This completes the proof. □

Proof of Corollary 4.2: Case i). For notational simplicity, we let

$$f(u)=\frac{2T_u^{2H+1}}{(2H-\frac{c\beta s_0^\beta}{1+cs_0^\beta})(u+cT_u^\beta)},\quad u>0.$$

By definition, for any $x>0$

$$\mathbb{P}\left(\frac{T_u-\tau_u}{f(u)}>x\Big|\tau_u\le T_u\right)=\frac{\psi_{T_u-xf(u)}(u)}{\psi_{T_u}(u)}.$$

Further, it follows from Theorem 4.1 that

$$\lim_{u\to\infty}\frac{\psi_{T_u-xf(u)}(u)}{\psi_{T_u}(u)}=\lim_{u\to\infty}e^{\frac{u+cT_u^\beta}{2T_u^{2H}}-\frac{u+c(T_u-xf(u))^\beta}{2(T_u-xf(u))^{2H}}}=e^{-x},$$

establishing the claim in i).

Case ii). Similarly as above, in the light of Theorem 4.1 we have, for any $y\le x$

$$\lim_{u\to\infty}\mathbb{P}\left(\frac{\tau_u-t_0u^{1/\beta}}{B(u)}<y\Big|\tau_u\le T_u\right)=\lim_{u\to\infty}\frac{\psi_{t_0u^{1/\beta}+yB(u)}(u)}{\psi_{T_u}(u)}=\frac{\Phi(y)}{\Phi(x)}.$$

Thus, the proof is complete. $\square$

Proof of Theorem 5.1: One approach is to follow a similar proof as Theorem 2.1 by using the double-sum method. Here, we give another proof based on the ideas and results in [16], [28] and [27]. We first show that

$$\mathbb{P}\left(\sup_{t\in[0,T]} \chi^2_{n,\boldsymbol{b}}(t) > u\right) = \mathbb{P}\left(\sup_{t\in[0,T]} \chi^2_{k,\mathbf{1}}(t) + \sum_{i=k+1}^{n} b_i^2 X_i^2(t_0) > u\right)(1+o(1)) \tag{28}$$

holds as $u \to \infty$, which in view of Lemma 2.1 in [27] is sufficient. Indeed, letting $G(u) = \mathbb{P}\left(\sup_{t\in[0,T]} \chi^2_{k,\mathbf{1}}(t) \le u\right)$ we have from (8) that

$$\lim_{u\to\infty} \frac{1-G(u+y)}{1-G(u)} = \exp\left(-\frac{1}{2}y\right), \quad \forall y \in \mathbb{R}.$$

Further let $H(u) = \mathbb{P}\left(\sum_{i=k+1}^n b_i^2 X_i^2(t_0) \le u\right)$. It is known (cf. Example 2 in [21]) that

$$1 - H(u) = O\left(u^r \exp\left(-\frac{u}{2b_{k+1}}\right)\right) = o(1-G(u))$$

for some $r \in \mathbb{N}$. Moreover, by choosing some $\theta \in (1/2, 1/(2b_{k+1}))$ we have that

$$\int_0^\infty e^{\theta x} dH(x) < \infty.$$

Therefore, by Lemma 2.1 in [27] the claim in (9) follows from (28).

It remains to show (28). To this end, we introduce the following two Gaussian random fields:

$$Y_{\boldsymbol{b}}(t,\boldsymbol{v}) = \sum_{i=1}^n b_i v_i X_i(t), \quad Z_{\boldsymbol{b}}(t,\boldsymbol{v}) = \sum_{i=1}^k v_i X_i(t) + \sum_{i=k+1}^n b_i v_i X_i(t_0), \quad t \ge 0,\ \boldsymbol{v} \in \mathbb{R}^n.$$

As in the proof of Theorem 2.1 it is sufficient that

$$\mathbb{P}\left(\sup_{(t,\boldsymbol{v})\in\mathcal{G}_T} Y_{\boldsymbol{b}}(t,\boldsymbol{v}) > u\right) = \mathbb{P}\left(\sup_{(t,\boldsymbol{v})\in\mathcal{G}_T} Z_{\boldsymbol{b}}(t,\boldsymbol{v}) > u\right)(1+o(1)) \tag{29}$$

holds as $u \to \infty$. Next we have that the standard deviations $\sigma_{Y_{\boldsymbol{b}}}(t,\boldsymbol{v})$ and $\sigma_{Z_{\boldsymbol{b}}}(t,\boldsymbol{v})$ attain their absolute maximum (equal to 1) over $\mathcal{G}_T$ at all points of C_0 given as

$$C_0 = \{t_0\} \times \{\boldsymbol{v} \in \mathcal{S}_{n-1} : v_1^2 + \cdots + v_k^2 = 1\} \subset \mathcal{G}_T.$$

Further we consider the expansions of the standard deviations and the correlations of the Gaussian random fields $Y_{\boldsymbol{b}}$ and $Z_{\boldsymbol{b}}$ around the sphere C_0. By direct calculations we have

$$\sigma_{Y_{\boldsymbol{b}}}(t,\boldsymbol{v}) = 1 - a|t-t_0|^\mu(1+o(1)) - \frac{1}{2}\sum_{i=k+1}^n (1-b_i^2)v_i^2(1+o(1)),$$

$$\sigma_{Z_{\boldsymbol{b}}}(t,\boldsymbol{v}) = 1 - a|t-t_0|^\mu(1+o(1)) - \frac{1}{2}\sum_{i=k+1}^n (1-b_i^2)v_i^2(1+o(1)) \tag{30}$$

hold as $t \to t_0$ and $\sum_{i=k+1}^n v_i^2 \to 0$. Further, since $\gamma > \nu, \gamma_1 \ge \nu$,

$$r_{Y_{\boldsymbol{b}}}(t,\boldsymbol{v},s,\boldsymbol{u}) = \mathrm{Corr}(Y_{\boldsymbol{b}}(t,\boldsymbol{v}), Y_{\boldsymbol{b}}(s,\boldsymbol{u})) = 1 - d|t-s|^\nu(1+o(1)) - \frac{1}{2}\sum_{i=1}^n b_i^2(v_i-u_i)^2(1+o(1)),$$

$$r_{Z_{\boldsymbol{b}}}(t,\boldsymbol{v},s,\boldsymbol{u}) = \mathrm{Corr}(Z_{\boldsymbol{b}}(t,\boldsymbol{v}), Y_{\boldsymbol{b}}(s,\boldsymbol{u})) = 1 - d|t-s|^\nu(1+o(1)) - \frac{1}{2}\sum_{i=1}^n b_i^2(v_i-u_i)^2(1+o(1)) \tag{31}$$

hold as $s,t\to t_0$, $\sum_{i=k+1}^n v_i^2\to 0$ and $\sum_{i=k+1}^n u_i^2\to 0$. The technical proof of (31) is relegated to the Appendix. Define a neighborhood C_u of C_0 as

$$C_u=\{(t,\boldsymbol{v}): a|t-t_0|^\mu+\frac{1}{2}\sum_{i=k+1}^n(1-b_i^2)v_i^2<\ln u/u\}\cap\mathcal{G}_T.$$

By an application of Borell inequality and Piterbarg inequality as in the proof of Lemma 8.1 in [29] we can show that

$$\mathbb{P}\left(\sup_{(t,\boldsymbol{v})\in\mathcal{G}_T}Y_{\boldsymbol{b}}(t,\boldsymbol{v})>u\right)=\mathbb{P}\left(\sup_{(t,\boldsymbol{v})\in C_u}Y_{\boldsymbol{b}}(t,\boldsymbol{v})>u\right)(1+o(1)), \tag{32}$$

$$\mathbb{P}\left(\sup_{(t,\boldsymbol{v})\in\mathcal{G}_T}Z_{\boldsymbol{b}}(t,\boldsymbol{v})>u\right)=\mathbb{P}\left(\sup_{(t,\boldsymbol{v})\in C_u}Z_{\boldsymbol{b}}(t,\boldsymbol{v})>u\right)(1+o(1)) \tag{33}$$

hold as $u\to\infty$. Moreover, since we are concerned about the asymptotic results it follows that the expansions of the standard deviations and the correlations of the Gaussian random field $Y_{\boldsymbol{b}}$ (or $Z_{\boldsymbol{b}}$) around the sphere C_0 are the only necessary properties influencing the asymptotics of (32) (or (33)); this is due to the fact that $Y_{\boldsymbol{b}}$ and $Z_{\boldsymbol{b}}$ are Gaussian and $C_u\to C_0$ as $u\to\infty$. Therefore, it follows from (30) and (31) that (29) is established. This completes the proof. □

7 Appendix

This section is devoted to the proofs of Theorem 6.1 and Eq. (31).
We first present a lemma concerning the tail asymptotics of the supremum of a Gaussian random field over a threshold-dependent-time interval, which is crucial for the proof of Theorem 6.1.

Lemma 7.1 *Let $\{X(\boldsymbol{v}),\boldsymbol{v}\in\mathbb{R}^n\}$ be a centered stationary Gaussian random field with a.s. continuous sample paths and covariance function $r(\cdot)$ satisfying*

$$r(\boldsymbol{v})=1-d_1|v_1|^\alpha(1+o(1))-\sum_{i=2}^n d_iv_i^2(1+o(1)),\quad v_1^2+v_2^2+\cdots+v_n^2\to 0,$$

with $\alpha\in(0,2]$ and $d_i>0, i\le n$. Define

$$X^*(\boldsymbol{v})=\frac{X(\boldsymbol{v})}{(1+c_1|v_1|^\beta)\left(1+\sum_{i=k+1}^n c_iv_i^2\right)},\quad \boldsymbol{v}\in\mathbb{R}^n$$

with some $1<k<n$ and $c_1>0, c_i>0, k+1\le i\le n, \beta>0$. Let $A\subset\mathbb{R}^{k-1}$ be a Jordan measurable set with positive Lebesgue measure mes(A)*. Let further $\widetilde{\Delta}_x(u)=[-x_1(u)u^{-2/\beta},x_2(u)u^{-2/\beta}]$ with functions $x_i(u), i=1,2$ such that*

$$\lim_{u\to\infty}x_i(u)=x_i\in[-\infty,\infty],\quad \lim_{u\to\infty}x_i(u)u^{-1/\beta}=0,\quad i=1,2.$$

Denote $D(u)=\widetilde{\Delta}_x(u)\times A\times[-\ln u/u),\ln u/u]^{n-k}$. If $-x_1<x_2$, then

$$\mathbb{P}\left(\sup_{\boldsymbol{v}\in D(u)}X^*(\boldsymbol{v})>u\right) = 2^{-1/2}\pi^{-k/2}\prod_{i=2}^k d_i^{1/2}\prod_{i=k+1}^n\sqrt{1+d_i/c_i}\,\mathrm{mes}(A) \times\widehat{\mathcal{M}}_{\alpha,\beta,d_1,c_1}(x_1,x_2)\ u^{k-2+(2/\alpha-2/\beta)_+}e^{-\frac{u^2}{2}}(1+o(1))$$

as $u\to\infty$, with $\widehat{\mathcal{M}}_{\alpha,\beta,d_1,c_1}(x_1,x_2)$ given as in (20).

Proof: Note that

$$\mathcal{H}_2 = \frac{1}{\sqrt{\pi}}, \quad \widehat{\mathcal{P}}_2^{c/d} = \sqrt{1+d/c}.$$

The proof follows by a little modification of the proof of Theorem 8.2 in [29]; see also Lemma 6 in [28] or Theorem 3.2 in [16]. □

Proof of Theorem 6.1: The key idea is to work with Gaussian random fields instead of analyzing chi-square processes. The first step is standard (see, e.g., [29]). We consider the Gaussian random field

$$Y_{\boldsymbol{b}}(t,\boldsymbol{v}) = \sum_{i=1}^{n} b_i X_i(t) v_i,$$

defined on $\mathcal{G}_x(u) = \Delta_x(u) \times \mathcal{S}_{n-1}$, where $\mathcal{S}_{n-1}$ stands for the $(n-1)$-dimensional unit sphere. Since from [29]

$$\sup_{t\in\Delta_x(u)} \chi_{n,\boldsymbol{b}}(t) = \sup_{(t,\boldsymbol{v})\in\mathcal{G}_x(u)} Y_{\boldsymbol{b}}(t,\boldsymbol{v})$$

we have that

$$\mathbb{P}\left(\sup_{t\in\Delta_x(u)} \chi^2_{n,\boldsymbol{b}}(t) > u^2\right) = \mathbb{P}\left(\sup_{(t,\boldsymbol{v})\in\mathcal{G}_x(u)} Y_{\boldsymbol{b}}(t,\boldsymbol{v}) > u\right).$$

It follows that the standard deviation $\sigma_{Y_{\boldsymbol{b}}}$ of $Y_{\boldsymbol{b}}$ attains its maximum (equal to 1) over $\mathcal{G}_x(u)$ only at points on $\{(t_0,\boldsymbol{v}), \boldsymbol{v}\in\mathcal{S}_{n-1}, v_i = 0, k+1 \le i \le n\}$. Furthermore, following the arguments as in [28] we conclude that $\sigma_{Y_{\boldsymbol{b}}}$ and the correlation function $r_{Y_{\boldsymbol{b}}}$ of $Y_{\boldsymbol{b}}$ have the following asymptotic expansions:

$$\sigma_{Y_{\boldsymbol{b}}}(t,\boldsymbol{v}) = 1 - a|t-t_0|^{\mu}(1+o(1)) - \sum_{i=k+1}^{n} \frac{1-b_i^2}{2} v_i^2 (1+o(1)) \tag{34}$$

as $t \to t_0$ and $v_{k+1}^2 + \cdots + v_n^2 \to 0$, and

$$r_{Y_{\boldsymbol{b}}}(t,\boldsymbol{v},t',\boldsymbol{v}') = 1 - d|t-t'|^{\nu}(1+o(1)) - \sum_{i=1}^{n} \frac{b_i^2}{2}(v_i - v_i')^2(1+o(1)) \tag{35}$$

as $t,t' \to t_0$, $v_{k+1}^2 + \cdots + v_n^2 \to 0$, and ${v'}_{k+1}^2 + \cdots + {v'}_n^2 \to 0$.
In addition, there exist $\delta > 0, \mathbb{Q} > 0$ such that

$$\mathbb{E}\left((Y_{\boldsymbol{b}}(t,\boldsymbol{v}) - Y_{\boldsymbol{b}}(t',\boldsymbol{v}'))^2\right) \le \mathbb{Q}(|t-t'|^{\gamma} + \sum_{i=1}^{n}(v_i - v_i')^2) \tag{36}$$

holds for all $(t,\boldsymbol{v}) \in ([t_0-\rho, t_0+\rho]\cap[0,T])\times\mathcal{S}_{n-1}$. Next define $C_u := \{\boldsymbol{v}\in\mathcal{S}_{n-1} : v_i \in [-\ln u/u, \ln u/u], k+1 \le i \le n\}$. Let

$$\pi(u) := \mathbb{P}\left(\sup_{(t,\boldsymbol{v})\in\Delta_x(u)\times C_u} Y_{\boldsymbol{b}}(t,\boldsymbol{v}) > u\right).$$

We have, for any $u > 0$

$$\pi(u) \le \mathbb{P}\left(\sup_{(t,\boldsymbol{v})\in\mathcal{G}_x(u)} Y_{\boldsymbol{b}}(t,\boldsymbol{v}) > u\right) \le \pi(u) + \mathbb{P}\left(\sup_{(t,\boldsymbol{v})\in\Delta_x(u)\times(\mathcal{S}_{n-1}/C_u)} Y_{\boldsymbol{b}}(t,\boldsymbol{v}) > u\right).$$

Further, in the light of (34), (36) and the Piterbarg inequality given in Theorem 8.1 in [30]

$$\mathbb{P}\left(\sup_{(t,\boldsymbol{v})\in(t,\boldsymbol{v})\in\Delta_x(u)\times(\mathcal{S}_{n-1}/C_u)} Y_{\boldsymbol{b}}(t,\boldsymbol{v}) > u\right) \le \mathbb{Q}u^{\frac{2(n+1)}{\gamma\wedge 2}-1}\exp\left(-\frac{u^2}{2(1-(\ln u/u)^2\mathbb{Q}_1)^2}\right) \tag{37}$$

holds for all u large, with some positive constants $\mathbb{Q}, \mathbb{Q}_1$. In the following we shall give the asymptotics of $\pi(u)$, from which we shall see that the right-hand side of (37) is asymptotically negligible. Thus we conclude that

$$\mathbb{P}\left(\sup_{(t,\boldsymbol{v})\in\mathcal{G}_x(u)} Y_{\boldsymbol{b}}(t,\boldsymbol{v})>u\right)=\pi(u)(1+o(1)),\quad u\to\infty.$$

To this end, we partition C_u into sets of small diameters. To make it more precise, we resort to the following polar coordinates, i.e., for any $\boldsymbol{v}\in\mathcal{S}_{n-1}$

$$v_n=\sin(\theta_1), v_{n-1}=\cos(\theta_1)\sin(\theta_2),\ldots,v_{k+1}=\sin(\theta_{n-k})\prod_{i=1}^{n-k-1}\cos(\theta_i),\ldots,v_1=\prod_{i=1}^{n-1}\cos(\theta_i),$$

with $\theta_i\in[-\pi/2,\pi/2], 1\le i\le n-2, \theta_{n-1}\in[0,2\pi)$. We divide $[-\pi/2,\pi/2]^{k-2}\times[0,2\pi]$ into several small cubes with length of edge $R>0$, denoted by $\{B_j\}_{j\in\mathcal{N}}$, with

$$\mathcal{N}=\{j\in\mathbb{Z}: B_j\subset[-\pi/2,\pi/2]^{k-2}\times[0,2\pi]\}.$$

Therefore, the corresponding partition of C_u can be represented as $\{D_j\}_{j\in\mathcal{N}}$ with

$$D_j=\left\{\boldsymbol{v}=\boldsymbol{v}(\theta):(\theta_{n-k+1},\cdots,\theta_{n-1})\in B_j, v_i\in[-\ln u/u,\ln u/u], k+1\le i\le n\right\},$$

where

$$\boldsymbol{v}(\theta)=(\cos(\theta_1)\cdots\cos(\theta_{n-2})\cos(\theta_{n-1}),\cdots,\sin(\theta_1)).$$

Further, set

$$\mathcal{N}_1=\{j\in\mathbb{Z}: B_j\subset[-\pi/2+R,\pi/2-R]^{k-2}\times[R,2\pi-R]\}.$$

It follows from Bonferroni's inequality that

$$\sum_{j\in\mathcal{N}_1}\mathbb{P}\left(\sup_{\Delta_x(u)\times D_j} Y_{\boldsymbol{b}}(t,\boldsymbol{v})>u\right)-\Pi(u)\le\pi(u)\le\sum_{j\in\mathcal{N}}\mathbb{P}\left(\sup_{\Delta_x(u)\times D_j} Y_{\boldsymbol{b}}(t,\boldsymbol{v})>u\right),\tag{38}$$

where

$$\Pi(u)=\sum_{i<j\in\mathcal{N}_1}\mathbb{P}\left(\sup_{\Delta_x(u)\times D_i} Y_{\boldsymbol{b}}(t,\boldsymbol{v})>u,\sup_{\Delta_x(u)\times D_j} Y_{\boldsymbol{b}}(t,\boldsymbol{v})>u\right).$$

Since

$$\mathrm{Cov}\left(Y_{\boldsymbol{b}}(t,\boldsymbol{v}),Y_{\boldsymbol{b}}(t',\boldsymbol{v}')\right)=\sum_{i=1}^n b_iv_iv_i'\mathrm{Cov}(X(t),X(t'))$$

the Gaussian random field $Y_{\boldsymbol{b}}(t,\boldsymbol{v})$ is rotational invariant in law with respect to $\boldsymbol{v}$ under the orthogonal matrix

$$A=\begin{pmatrix}\tilde{A}_k & O\\ O & E_{n-k}\end{pmatrix},\tag{39}$$

where $\widetilde{A}_k$ is any $k\times k$ orthogonal matrix and E_{n-k} is the $(n-k)\times(n-k)$ unit matrix. Hence, for any $j\in\mathcal{N}$ there exists a orthogonal matrix A_j of the form (39) such that $(1,0,\cdots,0)\in A_jD_j$, and thus

$$\begin{aligned}\mathbb{P}\left(\sup_{\Delta_x(u)\times D_j} Y_{\boldsymbol{b}}(t,\boldsymbol{v})>u\right) &= \mathbb{P}\left(\sup_{\Delta_x(u)\times D_j} Y_{\boldsymbol{b}}(t,A_j\boldsymbol{v})>u\right)\\ &= \mathbb{P}\left(\sup_{\Delta_x(u)\times A_jD_j} Y_{\boldsymbol{b}}(t,\boldsymbol{v})>u\right).\end{aligned}$$

Suppose $(1,0,\cdots,0)\in D_0$. Therefore, for the summand in (38) it is sufficient to consider the case $j=0$. Define projections $g_l:\mathbb{R}^l\to\mathbb{R}^{l-1}, 1<l\le n$ with $g_l(v_1,\dots,v_l)=(v_2,\cdots,v_l)$. For u large enough and R sufficient small, we have

$$\left(g_k(\sqrt{1-(n-k)(\ln u/u)^2}D_0^k)\right)\times[-\ln u/u,\ln u/u]^{n-k}\subset g_n(D_0)\subset\left(g_kD_0^k\right)\times[-\ln u/u,\ln u/u]^{n-k}:=\mathcal{E}_u(D_0^k),$$

where $D_0^k=\{(\cos(\theta_1)\cdots\cos(\theta_{k-1}),\cdots,\cos(\theta_1)\sin(\theta_2),\sin(\theta_1)),(\theta_1,\cdots,\theta_{k-1})\in B_0\}$ is a subset of the $(k-1)$-dimensional unit sphere. Clearly, for any $R>0$ small enough there exists $\epsilon_R\in(0,1)$ such that

$$\begin{aligned}&(1-\epsilon_R)\mathrm{mes}\left(D_0^k\right)\le\mathrm{mes}\left(g_kD_0^k\right)\le(1+\epsilon_R)\mathrm{mes}\left(D_0^k\right),\\&\mathrm{mes}\left(g_k(\sqrt{1-(n-k)(\ln u/u)^2}D_0^k)\right)\to\mathrm{mes}\left(g_kD_0^k\right)\end{aligned}\tag{40}$$

hold as $u\to\infty$. Consequently, since the projection g_n on D_0 is one-to-one for u large enough and R sufficient small, we conclude that

$$\mathbb{P}\left(\sup_{(t,\tilde{\boldsymbol{v}})\in\Delta_x^{(1)}(u)}Y_{\boldsymbol{b}}(t,\tilde{\boldsymbol{v}})>u\right)\le\mathbb{P}\left(\sup_{(t,\boldsymbol{v})\in\Delta_x(u)\times D_0}Y_{\boldsymbol{b}}(t,\boldsymbol{v})>u\right)\le\mathbb{P}\left(\sup_{(t,\tilde{\boldsymbol{v}})\in\Delta_x^{(2)}(u)}Y_{\boldsymbol{b}}(t,\tilde{\boldsymbol{v}})>u\right),$$

where $\tilde{\boldsymbol{v}}=(v_2,\dots,v_n)$ and

$$\Delta_x^{(1)}(u)=\Delta_x(u)\times\mathcal{E}_u(\sqrt{1-(n-k)(\ln u/u)^2}D_0^k),\quad\Delta_x^{(2)}(u)=\Delta_x(u)\times\mathcal{E}_u(D_0^k).$$

Define next independent centered stationary Gaussian process $\{X_1^{\pm\epsilon}(t),t\ge0\}$ and centered homogeneous (stationary) Gaussian random field $\{X_2^{\pm\epsilon}(\tilde{\boldsymbol{v}}),\tilde{\boldsymbol{v}}\in\mathbb{R}^{n-1}\}$ with unit variances and correlation functions satisfying

$$\rho_1^{\pm\epsilon}(t)=1-(1\pm\epsilon)2d|t|^\nu(1+o(1)),t\to0,\quad\rho_2^{\pm\epsilon}(\tilde{\boldsymbol{v}})=1-(1\pm\epsilon)\sum_{i=2}^nb_i^2v_i^2(1+o(1)),\tilde{\boldsymbol{v}}\to\mathbf{0}.$$

Then $\tilde{X}^{\pm\epsilon}(t,\tilde{\boldsymbol{v}})=\frac{X_1^{\pm\epsilon}(t)+X_2^{\pm\epsilon}(\tilde{\boldsymbol{v}})}{\sqrt{2}},t\ge0,\tilde{\boldsymbol{v}}\in\mathbb{R}^{n-1}$, is a centered homogeneous Gaussian random field with unit variance and correlation function satisfying

$$\rho^{\pm\epsilon}(t,\tilde{\boldsymbol{v}})=1-(1\pm\epsilon)d|t|^\nu(1+o(1))-(1\pm\epsilon)\sum_{i=2}^n\frac{b_i^2}{2}v_i^2(1+o(1)),\quad(t,\tilde{\boldsymbol{v}})\to(t_0,\mathbf{0}).$$

Further, we have that

$$r_{Y_{\boldsymbol{b}}}(t,\boldsymbol{v},t',\boldsymbol{v}')=1-d|t-t'|^\nu(1+o(1))-\sum_{i=2}^n\frac{b_i^2}{2}(v_i-v_i')^2(1+o(1))\tag{41}$$

holds as $t,t'\to t_0,\boldsymbol{v},\boldsymbol{v}'\to(1,0,\cdots,0)$. This can be established as in Lemma 9 in [28]. Therefore, by Slepian lemma (cf. [29]) we derive, for u sufficient large and $R>0$ small enough

$$\begin{aligned}\mathbb{P}\left(\sup_{(t,\tilde{\boldsymbol{v}})\in\Delta_x^{(2)}(u)}Y_{\boldsymbol{b}}(t,\tilde{\boldsymbol{v}})>u\right)&\le\mathbb{P}\left(\sup_{(t,\tilde{\boldsymbol{v}})\in\Delta_x^{(2)}(u)}\frac{\tilde{X}^{+\epsilon}(t,\tilde{\boldsymbol{v}})}{(1+(1-\epsilon)a|t-t_0|^\mu)(1+(1-\epsilon)\sum_{i=k+1}^n\frac{1-b_i^2}{2}v_i^2)}>u\right)\\\mathbb{P}\left(\sup_{(t,\tilde{\boldsymbol{v}})\in\Delta_x^{(1)}(u)}Y_{\boldsymbol{b}}(t,\tilde{\boldsymbol{v}})>u\right)&\ge\mathbb{P}\left(\sup_{(t,\tilde{\boldsymbol{v}})\in\Delta_x^{(1)}(u)}\frac{\tilde{X}^{-\epsilon}(t,\tilde{\boldsymbol{v}})}{(1+(1+\epsilon)a|t-t_0|^\mu)(1+(1+\epsilon)\sum_{i=k+1}^n\frac{1-b_i^2}{2}v_i^2)}>u\right).\end{aligned}$$

Consequently, we have from Lemma 7.1 that

$$\mathbb{P}\left(\sup_{(t,\tilde{\boldsymbol{v}})\in\Delta_x^{(2)}(u)}Y_{\boldsymbol{b}}(t,\tilde{\boldsymbol{v}})>u\right)\le a(\epsilon)(2\pi)^{-k/2}\prod_{i=k+1}^n(1-b_i^2)^{-1/2}\mathrm{mes}(g_kD_0^k)$$

$$\times \widehat{\mathcal{M}}_{\nu,\mu,d,a}(x_1,x_2) u^{k-2+(2/\nu-2/\mu)_+} e^{-\frac{u^2}{2}}(1+o(1))$$

holds as $u\to\infty$, where $a(\epsilon)\to 1$ as $\epsilon\to 0$. Since further $\mathrm{mes}(S_{k-1})=\sum_{j\in\mathcal{N}}\mathrm{mes}(D_j)=2\pi^{k/2}/\Gamma(k/2)$ we conclude that

$$\limsup_{u\to\infty}\frac{\sum_{j\in\mathcal{N}}\mathbb{P}\left(\sup_{\Delta_x(u)\times D_j} Y_{\boldsymbol{b}}(t,\boldsymbol{v})>u\right)}{\prod_{i=k+1}^n(1-b_i^2)^{-1/2}\widehat{\mathcal{M}}_{\nu,\mu,d,a}(x_1,x_2)u^{(2/\nu-2/\mu)_+}\Upsilon_k(u^2)}\le a'(\epsilon,\epsilon_R) \tag{42}$$

with $a'(\epsilon,\epsilon_R)\to 1$ as $\epsilon\to 0$ and $R\to 0$. Similarly, we have

$$\liminf_{u\to\infty}\frac{\sum_{j\in\mathcal{N}_1}\mathbb{P}\left(\sup_{\Delta_x(u)\times D_j} Y_{\boldsymbol{b}}(t,\boldsymbol{v})>u\right)}{\prod_{i=k+1}^n(1-b_i^2)^{-1/2}\widehat{\mathcal{M}}_{\nu,\mu,d,a}(x_1,x_2)u^{(2/\nu-2/\mu)_+}\Upsilon_k(u^2)}\ge b'(\epsilon,\epsilon_R)\frac{\sum_{j\in\mathcal{N}_1}\mathrm{mes}(D_j)}{\mathrm{mes}(\mathcal{S}_{n-1})} \tag{43}$$

with $b'(\epsilon,\epsilon_R)\to 1$ as $\epsilon\to 0$ and $R\to 0$.

Next we show that

$$\limsup_{R\to 0}\limsup_{u\to\infty}\frac{\Pi(u)}{u^{(2/\nu-2/\mu)_+}\Upsilon_k(u^2)}=0. \tag{44}$$

We have for any $u>0$

$$\Pi(u)\le\sum_{i<j\in\mathcal{N}_1, D_i\cap D_j=\emptyset}p_{ij}(u)+\sum_{i<j\in\mathcal{N}_1, D_i\cap D_j\neq\emptyset}p_{ij}(u)=:\Pi_1(u)+\Pi_2(u),$$

where

$$p_{ij}(u):=\mathbb{P}\left(\sup_{(t,\boldsymbol{v})\in\Delta_x(u)\times D_i}Y_{\boldsymbol{b}}(t,\boldsymbol{v})>u,\ \sup_{(t,\boldsymbol{v})\in\Delta_x(u)\times D_j}Y_{\boldsymbol{b}}(t,\boldsymbol{v})>u\right).$$

We first consider the sum taking over $D_i\cap D_j=\emptyset$. The following standard upper bound

$$p_{ij}(u)\le\mathbb{P}\left(\sup_{(t,\boldsymbol{v},t',\boldsymbol{w})\in\Delta_x(u)\times D_i\times\Delta_x(u)\times D_j}\left(\overline{Y_{\boldsymbol{b}}}(t,\boldsymbol{v})+\overline{Y_{\boldsymbol{b}}}(t',\boldsymbol{w})\right)>2u\right)$$

will be crucial for the proof, where $\overline{Y_{\boldsymbol{b}}}(t,\boldsymbol{v})=Y_{\boldsymbol{b}}(t,\boldsymbol{v})/\sigma_{Y_{\boldsymbol{b}}}(t,\boldsymbol{v})$. Since $D_i\cap D_j=\emptyset$, in view of (35) we have that, for any $(t,\boldsymbol{v})\in\Delta_x(u)\times D_i$, $(t',\boldsymbol{w})\in\Delta_x(u)\times D_j$,

$$\mathbb{E}\left((\overline{Y_{\boldsymbol{b}}}(t,\boldsymbol{v})+\overline{Y_{\boldsymbol{b}}}(s,\boldsymbol{w}))^2\right)=4-2(1-r_{Y_{\boldsymbol{b}}}((t,\boldsymbol{v},t',\boldsymbol{w}))\le 4(1-\delta_1)$$

holds with some $\delta_1\in(0,1)$ independent of j. Thus, in the light of Borell-TIS inequality, there exists a common positive constant $\mathbb{Q}$ such that, for all $u>\mathbb{Q}$

$$\mathbb{P}\left(\sup_{(t,\boldsymbol{v},s,\boldsymbol{w})\in\Delta_x(u)\times D_i\times\Delta_x(u)\times D_j}\left(\overline{Y_{\boldsymbol{b}}}(t,\boldsymbol{v})+\overline{Y_{\boldsymbol{b}}}(s,\boldsymbol{w})\right)>2u\right)\le e^{-\frac{(2u-\mathbb{Q})^2}{8(1-\delta_1)}}$$

holds for all $D_i, D_j, i,j\in\mathcal{N}$ satisfying $D_i\cap D_j=\emptyset$. Consequently,

$$\Pi_1(u)\le N_R^2 e^{-\frac{(2u-\mathbb{Q})^2}{8(1-\delta_1)}} \tag{45}$$

with N_R representing the number of i in $\mathcal{N}_1$. Next, for the other sum taking over $D_i\cap D_j\neq\emptyset$ we consider first the special summand when $i=0$, so that $(1,0,\cdots,0)\in D_0\cup D_j$. By using the projection g_n, we have

$$p_{0j}(u)\quad=\quad\mathbb{P}\left(\sup_{(t,\widetilde{\boldsymbol{v}})\in\Delta_x(u)\times(g_n(D_0))}Y_{\boldsymbol{b}}(t,\widetilde{\boldsymbol{v}})>u,\ \sup_{(s,\widetilde{w})\in\Delta_x(u)\times(g_n(D_j))}Y_{\boldsymbol{b}}(s,\widetilde{\boldsymbol{w}})>u\right)$$

$$\le \quad \mathbb{P}\left(\sup_{(t,\widetilde{\boldsymbol{v}})\in\Delta_x(u)\times\mathcal{E}_u(D_0^k)} Y_{\boldsymbol{b}}(t,\widetilde{\boldsymbol{v}})>u\right)+\mathbb{P}\left(\sup_{(t,\widetilde{\boldsymbol{v}})\in\Delta_x(u)\times\mathcal{E}_u(D_0^k)} Y_{\boldsymbol{b}}(t,\widetilde{\boldsymbol{v}})>u\right)$$
$$-\mathbb{P}\left(\sup_{(t,\widetilde{\boldsymbol{v}})\in\Delta_x(u)\times\mathcal{E}_u(\sqrt{1-(n-k)(\ln u/u)^2}D_0^k\cup D_j^k)} Y_{\boldsymbol{b}}(t,\widetilde{\boldsymbol{v}})>u\right)$$

with $\widetilde{\boldsymbol{w}}=(w_2,\cdots,w_n)$, and $D_j^k=\{(\cos(\theta_1)\cdots\cos(\theta_{k-1}),\cdots,\cos(\theta_1)\sin(\theta_2),\sin(\theta_1)),(\theta_1,\cdots,\theta_{k-1})\in B_j\}$. Further, with the aid of Lemma 7.1 we get

$$\limsup_{u\to\infty}\frac{p_{0j}(u)}{(2\pi)^{-k/2}\prod_{i=k+1}^n(1-b_i^2)^{-1/2}\widehat{\mathcal{M}}_{\nu,\mu,d,a}(x_1,x_2)u^{k-2+(2/\nu-2/\mu)_+}e^{-\frac{u^2}{2}}}$$
$$\le a(\epsilon)\mathrm{mes}(g_k(D_0^k))+a(\epsilon)\mathrm{mes}(g_k(D_j^k))-b(\epsilon)\mathrm{mes}(g_k(D_0^k\cup D_j^k)),$$

where $a(\epsilon),b(\epsilon)$ are two positive functions such that $a(\epsilon)\to 1, b(\epsilon)\to 1$ as $\epsilon\to 0$. Since for any fixed $i\in\mathcal{N}_1$, $\#\{j\in\mathcal{N}_1, D_j^k\cap D_i^k\neq\emptyset\}=\#\ \{j\in\mathcal{N}_1, B_j\cap B_i\neq\emptyset\}\le 3^{k-1}$. Thus we conclude that

$$\limsup_{u\to\infty}\frac{\Pi_2(u)}{(2\pi)^{-k/2}\prod_{i=k+1}^n(1-b_i)^{-1/2}\widehat{\mathcal{M}}_{\nu,\mu,d,a}(x_1,x_2)u^{k-2+(2/\nu-2/\mu)_+}e^{-\frac{u^2}{2}}}$$
$$\le 3^k(c(\epsilon,\epsilon_R)-c'(\epsilon,\epsilon_R))\mathrm{mes}(S_{k-1}) \tag{46}$$

with $c(\epsilon,\epsilon_R)\to 1$ and $c'(\epsilon,\epsilon_R)\to 1$ as $\epsilon\to 0$ and $R\to 0$. Therefore, we obtain from (45) and (46) that (44) holds. Consequently, the claim follows from (42), (43) and (44) by letting $\epsilon\to 0$ and $R\to 0$. This completes the proof. ☐

Proof of (31)**:** We only present the proof for $r_{Y_{\boldsymbol{b}}}(t,\boldsymbol{v},s,\boldsymbol{u})$, since the proof for $r_{Z_{\boldsymbol{b}}}(t,\boldsymbol{v},s,\boldsymbol{u})$ follows similarly. In the following, all the asymptotics are meant for $s,t\to t_0$, $\sum_{i=k+1}^n v_i^2\to 0$ and $\sum_{i=k+1}^n u_i^2\to 0$. Denoting $\boldsymbol{v}_1=(v_1,\cdots,v_k,0,\cdots,0)$ and $\boldsymbol{u}_1=(u_1,\cdots,u_k,0,\cdots,0)$ we have

$$\begin{aligned}1-r_{Y_{\boldsymbol{b}}}(t,\boldsymbol{v},s,\boldsymbol{u}) &= \frac{\sum_{i=1}^n\mathbb{E}\left((b_iX_i(t)v_i-b_iX_i(s)u_i)^2\right)-(\sigma_{Y_{\boldsymbol{b}}}(t,\boldsymbol{v})-\sigma_{Y_{\boldsymbol{b}}}(s,\boldsymbol{u}))^2}{2\sigma_{Y_{\boldsymbol{b}}}(t,\boldsymbol{v})\sigma_{Y_{\boldsymbol{b}}}(s,\boldsymbol{u})}\\ &= \frac{\sigma_{Y_{\boldsymbol{b}}}(t,\boldsymbol{v}_1)\sigma_{Y_{\boldsymbol{b}}}(s,\boldsymbol{u}_1)}{\sigma_{Y_{\boldsymbol{b}}}(t,\boldsymbol{v})\sigma_{Y_{\boldsymbol{b}}}(s,\boldsymbol{u})}\xi_1(t,\boldsymbol{v},s,\boldsymbol{u})+\xi_2(t,\boldsymbol{v},s,\boldsymbol{u})+\xi_3(t,\boldsymbol{v},s,\boldsymbol{u}),\end{aligned} \tag{47}$$

where

$$\xi_1(t,\boldsymbol{v},s,\boldsymbol{u})=\frac{\sum_{i=1}^k\mathbb{E}\left((X_i(t)v_i-X_i(s)u_i)^2\right)-(\sigma_{Y_{\boldsymbol{b}}}(t,\boldsymbol{v}_1)-\sigma_{Y_{\boldsymbol{b}}}(s,\boldsymbol{u}_1))^2}{2\sigma_{Y_{\boldsymbol{b}}}(t,\boldsymbol{v}_1)\sigma_{Y_{\boldsymbol{b}}}(s,\boldsymbol{u}_1)},$$
$$\xi_2(t,\boldsymbol{v},s,\boldsymbol{u})=\frac{\sum_{i=k+1}^n\mathbb{E}\left((X_i(t)v_i-X_i(s)u_i)^2\right)}{2\sigma_{Y_{\boldsymbol{b}}}(t,\boldsymbol{v})\sigma_{Y_{\boldsymbol{b}}}(s,\boldsymbol{u})},$$
$$\xi_3(t,\boldsymbol{v},s,\boldsymbol{u})=\frac{(\sigma_{Y_{\boldsymbol{b}}}(t,\boldsymbol{v}_1)-\sigma_{Y_{\boldsymbol{b}}}(s,\boldsymbol{u}_1))^2-(\sigma_{Y_{\boldsymbol{b}}}(t,\boldsymbol{v})-\sigma_{Y_{\boldsymbol{b}}}(s,\boldsymbol{u}))^2}{2\sigma_{Y_{\boldsymbol{b}}}(t,\boldsymbol{v})\sigma_{Y_{\boldsymbol{b}}}(s,\boldsymbol{u})}.$$

By assumption **II** we have

$$\begin{aligned}\xi_1(t,\boldsymbol{v},s,\boldsymbol{u}) &= 1-\frac{\mathbb{E}\left(X_1(t)X_1(s)\right)}{\sigma_{X_1}(t)\sigma_{X_1}(s)}\frac{\sum_{i=1}^k v_iu_i}{\sqrt{\sum_{i=1}^k v_i^2}\sqrt{\sum_{i=1}^k u_i^2}}\\ &= d|t-s|^\nu(1+o(1))+\frac{1}{2}\sum_{i=1}^k(v_i-u_i)^2(1+o(1))+o\left(\sum_{i=k+1}^n(u_i-v_i)^2\right).\end{aligned} \tag{48}$$

Further, we have

$$\sum_{i=k+1}^n\mathbb{E}\left((b_iX_i(t)v_i-b_iX_i(s)u_i)^2\right) \;=\; \sum_{i=k+1}^n b_i^2\Big(\mathbb{E}\left((X_i(t)-X_i(s))^2\right)v_i^2+\mathbb{E}\left((X_i(s)v_i-X_i(s)u_i)^2\right)$$

$$+2\mathbb{E}\left((X_i(t)v_i - X_i(s)v_i)(X_i(s)v_i - X_i(s)u_i)\right)\Big).$$

Next we deal with the last three terms on the right-hand side in turn. By assumption **III** and the fact that $\gamma_1 \geq \nu$ we get

$$\sum_{i=k+1}^{n} b_i^2 \mathbb{E}\left((X_i(t) - X_i(s))^2\right) v_i^2 \leq G_1 \sum_{i=k+1}^{n} b_i^2 v_i^2 |t-s|^{\gamma_1} = o(|t-s|^{\nu}),$$

and

$$\sum_{i=k+1}^{n} b_i^2 \mathbb{E}\left((X_i(s)v_i - X_i(s)u_i)^2\right) = \sum_{i=k+1}^{n} b_i^2 \sigma_{X_{k+1}}^2(s)(u_i - v_i)^2 = \sum_{i=k+1}^{n} b_i^2 (u_i - v_i)^2 (1 + o(1)).$$

In addition, we have

$$\begin{aligned}
&2\sum_{i=k+1}^{n} b_i^2 \mathbb{E}\left((X_i(t)v_i - X_i(s)v_i)(X_i(s)v_i - X_i(s)u_i)\right)\\
&\leq 2\sum_{i=k+1}^{n} b_i^2 \left(\mathbb{E}\left((X_i(t)v_i - X_i(s)v_i)^2\right)\right)^{1/2} \left(\mathbb{E}\left((X_i(s)v_i - X_i(s)u_i)^2\right)\right)^{1/2}\\
&\leq 2\sum_{i=k+1}^{n} b_i^2 G_1^{1/2} |t-s|^{\gamma_1/2} |v_i||v_i - u_i|\\
&\leq \sum_{i=k+1}^{n} b_i^2 G_1^{1/2} (|t-s|^{\gamma_1}|v_i| + |v_i|(v_i - u_i)^2)\\
&= o(|t-s|^{\nu}) + o\left(\sum_{i=k+1}^{n} (v_i - u_i)^2\right).
\end{aligned}$$

Therefore, we obtain

$$\xi_2(t, \boldsymbol{v}, s, \boldsymbol{u}) = \frac{1}{2}\sum_{i=k+1}^{n} b_i^2 (u_i - v_i)^2 (1 + o(1)) + o(|t-s|^{\nu}). \tag{49}$$

Moreover, it follows that

$$\begin{aligned}
\left(\sigma_{Y_{\boldsymbol{b}}}(t, \boldsymbol{v}_1) - \sigma_{Y_{\boldsymbol{b}}}(s, \boldsymbol{u}_1)\right)^2 \quad &= \quad \left(\sigma_{Y_{\boldsymbol{b}}}(t, \boldsymbol{v}_1) - \sigma_{Y_{\boldsymbol{b}}}(t, \boldsymbol{u}_1)\right)^2 + \left(\sigma_{Y_{\boldsymbol{b}}}(t, \boldsymbol{u}_1) - \sigma_{Y_{\boldsymbol{b}}}(s, \boldsymbol{u}_1)\right)^2\\
&\quad +2(\sigma_{Y_{\boldsymbol{b}}}(t, \boldsymbol{v}_1) - \sigma_{Y_{\boldsymbol{b}}}(t, \boldsymbol{u}_1))(\sigma_{Y_{\boldsymbol{b}}}(t, \boldsymbol{u}_1) - \sigma_{Y_{\boldsymbol{b}}}(s, \boldsymbol{u}_1)).
\end{aligned}$$

Direct calculation yields that

$$\begin{aligned}
\left(\sigma_{Y_{\boldsymbol{b}}}(t, \boldsymbol{v}_1) - \sigma_{Y_{\boldsymbol{b}}}(t, \boldsymbol{u}_1)\right)^2 \quad &\leq \quad \left(\left(\sum_{i=1}^{k} v_i^2\right)^{1/2} - \left(\sum_{i=1}^{k} u_i^2\right)^{1/2}\right)^2\\
&= \quad a(\boldsymbol{v}, \boldsymbol{u}) \sum_{i=k+1}^{n} (v_i - u_i)^2
\end{aligned}$$

with $a(\boldsymbol{v}, \boldsymbol{u}) \to 0$, and

$$\left(\sigma_{Y_{\boldsymbol{b}}}(t, \boldsymbol{u}_1) - \sigma_{Y_{\boldsymbol{b}}}(s, \boldsymbol{u}_1)\right)^2 = \left(\sigma_{X_1}(t) - \sigma_{X_1}(s)\right)^2 (1 + o(1))$$

hold. Since further by assumption **III**

$$\left(\sigma_{X_1}(t) - \sigma_{X_1}(s)\right)^2 \leq G|t-s|^{\gamma}$$

we conclude, by the fact that $\gamma \geq \nu$,

$$
\begin{aligned}
&2(\sigma_{Y_{\boldsymbol{b}}}(t,\boldsymbol{v}_1)-\sigma_{Y_{\boldsymbol{b}}}(t,\boldsymbol{u}_1))(\sigma_{Y_{\boldsymbol{b}}}(t,\boldsymbol{u}_1)-\sigma_{Y_{\boldsymbol{b}}}(s,\boldsymbol{u}_1))\\
&\leq (a(\boldsymbol{v},\boldsymbol{u}))^{-1/2}\left(\sigma_{Y_{\boldsymbol{b}}}(t,\boldsymbol{v}_1)-\sigma_{Y_{\boldsymbol{b}}}(t,\boldsymbol{u}_1)\right)^2+(a(\boldsymbol{v},\boldsymbol{u}))^{1/2}\left(\sigma_{Y_{\boldsymbol{b}}}(t,\boldsymbol{u}_1)-\sigma_{Y_{\boldsymbol{b}}}(s,\boldsymbol{u}_1)\right)^2\\
&= o\left(|t-s|^{\nu}\right)+o\left(\sum_{i=k+1}^{n}(v_i-u_i)^2\right)
\end{aligned}
$$

implying that

$$
\left(\sigma_{Y_{\boldsymbol{b}}}(t,\boldsymbol{v}_1)-\sigma_{Y_{\boldsymbol{b}}}(s,\boldsymbol{u}_1)\right)^2=\left(\sigma_{X_1}(t)-\sigma_{X_1}(s)\right)^2(1+o(1))+o\left(|t-s|^{\nu}\right)+o\left(\sum_{i=k+1}^{n}(v_i-u_i)^2\right).
$$

Similarly,

$$
\left(\sigma_{Y_{\boldsymbol{b}}}(t,\boldsymbol{v})-\sigma_{Y_{\boldsymbol{b}}}(s,\boldsymbol{u})\right)^2=\left(\sigma_{X_1}(t)-\sigma_{X_1}(s)\right)^2(1+o(1))+o\left(|t-s|^{\nu}\right)+o\left(\sum_{i=k+1}^{n}(v_i-u_i)^2\right).
$$

Therefore,

$$
\begin{aligned}
\xi_3(t,\boldsymbol{v},s,\boldsymbol{u}) &= o((\sigma_{X_1}(t)-\sigma_{X_1}(s))^2)+o\left(|t-s|^{\nu}\right)+o\left(\sum_{i=k+1}^{n}(v_i-u_i)^2\right)\\
&= o\left(|t-s|^{\nu}\right)+o\left(\sum_{i=k+1}^{n}(v_i-u_i)^2\right). \qquad (50)
\end{aligned}
$$

Consequently, the claim in (31) follows by combining (47)–(50). ☐

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