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BOUNDS AND APPROXIMATIONS FOR DISCRETE ASIAN OPTIONS IN A VARIANCE-GAMMA MODEL*

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Abstract

In this paper we give an overview on methods for calculating bounds and approximations for the Esscher price of European-style arithmetic and geometric average options. We consider an incomplete market model with an asset price process of exponential Lévy type with variance-gamma distributed log-returns. Numerical illustrations of the accuracy of these bounds and approximations as well as comparisons of the variance-gamma average option prices with the corresponding Black-Scholes prices and with Esscher prices in a normal inverse Gaussian model are given.

1 Introduction

In the last few years various alternatives to the Black-Scholes market models have been studied. It turned out that distributions of logarithmic asset returns can often be fitted extremely well by normal inverse Gaussian (NIG) and variance-gamma (VG) distributions (see e.g. BARNDORFF-NIELSEN [4, 5, 6], LAM, CHANG AND LEE [25], MADAN, CARR AND CHANG [28], MADAN AND MILNE [29], MADAN AND SENETA [30] or RYDBERG [34, 35]).

The normal inverse Gaussian distribution and the variance-gamma distribution are subclasses of the generalized hyperbolic (GH) distribution. Hence they are infinitely divisible and generate Lévy processes $(Z_t)_{t\geq 0}$, which give rise to the following exponential Lévy model (see e.g. EBERLEIN AND PRAUSE [16]). By setting

$$S_t = S_0 \exp(Z_t),$$

where $(S_t)_{t\geq 0}$ denotes the asset price process over time, the log-returns of this model produce exactly a normal inverse Gaussian distribution resp. a variance-gamma distribution. $(S_t)_{t\geq 0}$ is again a Lévy process.

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Since such market models are incomplete (cf. CHERNY [12]), there are many candidates of equivalent martingale measures for risk-neutral valuation of derivative securities. One mathematically tractable choice is the so-called Esscher equivalent measure, a concept which was introduced to mathematical finance by MADAN AND MILNE [29]; see also GERBER AND SHIU [18]. This particular choice of the pricing measure can be justified both within utility and equilibrium theory (cf. GERBER AND SHIU [19], BÜHLMANN ET AL. [8], CHAN [11] or GRANDITS [20]).

In [2] it was shown that the Esscher equivalent measure in the NIG model has a simple structure and this property was used to obtain easy computable approximations and bounds for the Esscher price of arithmetic and geometric average options. This paper is an extension of [2] for an exponential Lévy model with variance-gamma distributed log-returns. These two members of the generalized hyperbolic distributions are the only ones that are closed under convolution, a property which is particularly useful in connection with Lévy processes from a mathematical point of view.

Asian options are defined as options whose payoff depends on the average price of the underlying asset during a prespecified time interval. They are the most frequently traded exotic options (so far all Asian options are traded over-the-counter). Asian options have become popular financial instruments such as in retirement plans and life insurance contracts, since the arithmetic average tends to be more robust against price manipulations than the asset itself. Asian options can also be used to hedge a cooperation with a frequent cash flow in a foreign currency against uncertain exchange rates.

In the sequel we will focus on the case of average rate calls, i.e. the payoff is given as $\max(A-K,0)$, where A denotes the average and K denotes the strike price. We will assume that the average is based on finitely many asset values and distinguish between the arithmetic and the geometric average case. In most cases an analytical formula for the price of an Asian option is not available. Thus it is desirable to derive fast and accurate bounds and approximations for the corresponding prices (see e.g. BOYLE [10], KEMNA AND VORST [24], TURNBULL AND WAKEMAN [37] or LARCHER AND LEOBACHER [26]).

In this paper we will derive bounds and approximations for discrete Asian options in an exponential Lévy model with variance-gamma distributed log-returns. For that purpose we adapt several techniques developed for the Black-Scholes setting to this market model.

In Section 2 we introduce various properties of the variance-gamma distribution needed for the development of the VG asset price model and the derivation of the pricing measure in Section 3. Section 4 uses stop-loss transforms to obtain upper bounds for arithmetic average option Esscher prices. Two approximation techniques for arithmetic average option prices are developed in Section 5. Section 6 contains approximation methods for geometric average rate options and gives bounds for the arithmetic average option prices in terms of the geometric price. Finally, in Section 7 we compare the Asian Esscher option prices in the variance-gamma model with the corresponding Black-Scholes prices and with Esscher prices in the exponential NIG Lévy model showing significant price differences if the option is out of the money. The qualitative behavior of the Asian option price difference (as a function of strike price K and maturity T) turns out to be similar to the corresponding difference for plain European options.

2 The variance-gamma distribution

The variance-gamma (VG) distribution is defined by the density

$$f_{\mathrm{VG}(\alpha,\beta,\lambda,\mu)}(x) = f_{\mathrm{VG}}(x) = \frac{(\alpha^2-\beta^2)^\lambda}{\sqrt{\pi}\Gamma(\lambda)(2\alpha)^{\lambda-1/2}}|x-\mu|^{\lambda-1/2}K_{\lambda-1/2}(\alpha|x-\mu|)\,\mathrm{e}^{\beta(x-\mu)} \quad (1)$$

where $\alpha > |\beta| > 0, \lambda > 0$, $\mu \in \mathbb{R}$ and $\Gamma(x)$ denotes the gamma function. The name of this distribution stems from the fact that it is obtained as a normal variance-mean mixture where the mixing distribution is a gamma distribution.

The moment generating function of (1) has a particular simple structure:

$$M_{\mathrm{VG}}(u) = \exp(\mu u)\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+u)^2}\right)^\lambda, \qquad |\beta+u| < \alpha. \quad (2)$$

Thus it easily follows that

$$f_{\mathrm{VG}(\alpha,\beta,\lambda_1,\mu_1)} * f_{\mathrm{VG}(\alpha,\beta,\lambda_2,\mu_2)} = f_{\mathrm{VG}(\alpha,\beta,\lambda_1+\lambda_2,\mu_1+\mu_2)}. \quad (3)$$

The first four cumulants of the VG distribution (which will be needed later) are given by

$$\begin{aligned}
\chi_1 &= \mu + \frac{2\beta\lambda}{\alpha^2-\beta^2},\\
\chi_2 &= \frac{2\lambda}{(\alpha^2-\beta^2)^2}(\alpha^2+\beta^2),\\
\chi_3 &= \frac{4\lambda}{(\alpha^2-\beta^2)^3}\beta(3\alpha^2+\beta^2),\\
\chi_4 &= \frac{12\lambda}{(\alpha^2-\beta^2)^4}(\alpha^4+6\alpha^2\beta^2+\beta^4).
\end{aligned}$$

The VG distribution was first introduced to mathematical finance by MADAN AND SENETA [30] with $\beta = 0$. In this special case the cumulants simplify to

$$\begin{aligned} \chi_1 &= \mu \\ \chi_{2n} &= \frac{2\lambda}{\alpha^{2n}}(2n-1)! \qquad & n \geq 1 \\ \chi_{2n+1} &= 0 & n \geq 1. \end{aligned}$$

The VG distribution is the limit case $\delta = 0$ of the generalized hyperbolic distribution with density

$$f_{\mathrm{GH}}(x) = \frac{\zeta^\lambda}{\sqrt{2\pi}\alpha^{2\lambda-1}\delta^{2\lambda}K_\lambda(\zeta)}\, \mathrm{e}^{\beta(x-\mu)}\, \eta(x)^{\lambda-1/2} K_{\lambda-1/2}(\eta(x)),$$

with $\zeta = \delta\sqrt{\alpha^2-\beta^2}$ and $\eta(x) = \alpha\sqrt{\delta^2+(x-\mu)^2}$, which was introduced by BARNDORFF-NIELSEN [3]. As a member of the class of generalized hyperbolic distributions the VG distribution is infinitely divisible.

Note, that the normal inverse Gaussian [2] and the variance-gamma distributions are the only subclasses of the generalized hyperbolic distribution which are closed under convolution (see e.g. BIBBY AND SØRENSEN [9]).

3 The VG Lévy asset price model

It is well-known that the classical Black-Scholes model for the asset price process, which is based on geometric Brownian motion, has various deficiencies when compared to real financial data. Among those, the normal distribution as a model for the log-returns has too little probability mass in the center and in the tails. The probability density of the VG distribution is rather peaked at the center and its tail decreases as

$$|x|^{\lambda-1}e^{-\alpha|x|+\beta x} \quad \text{for } x \to \pm\infty.$$

Thus it is a natural candidate for modelling log-returns (see e.g. LAM, CHANG AND LEE [25]). Figure 3 illustrates the situation for a data set of daily returns of OMV stocks.

In the sequel we will discuss an exponential Lévy asset price model (cf. EBERLEIN [14]). Since the VG distribution is infinitely divisible, it generates a Lévy process $(Z_t)_{t\geq 0}$ (i.e. a stochastic process with stationary and independent increments, $Z_0 = 0$ a.s. and Z_1 is VG-distributed). From the convolution property (3) it follows that the increments are VG-distributed for arbitrary time intervals. This makes the VG Lévy processes (together with the normal inverse Gaussian Lévy processes in [2]) more natural generalized hyperbolic Lévy processes than the other generalized hyperbolic Lévy processes.

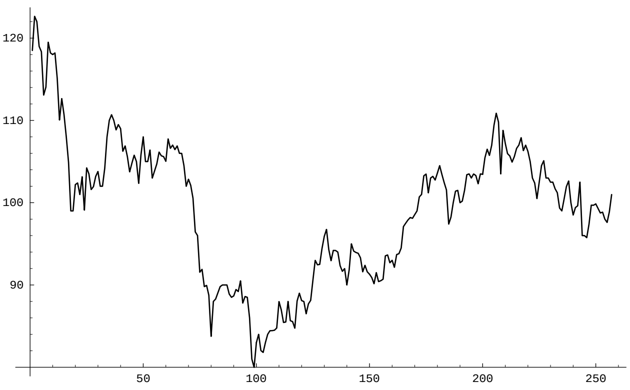

Figure 1: Price path of OMV (June 2001 - June 2002)

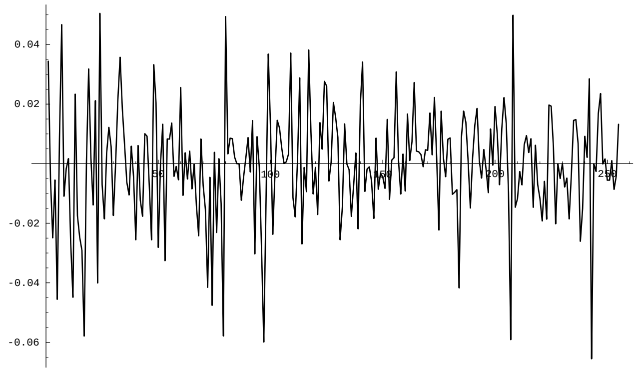

Figure 2: Log-returns of OMV (June 2001 - June 2002)

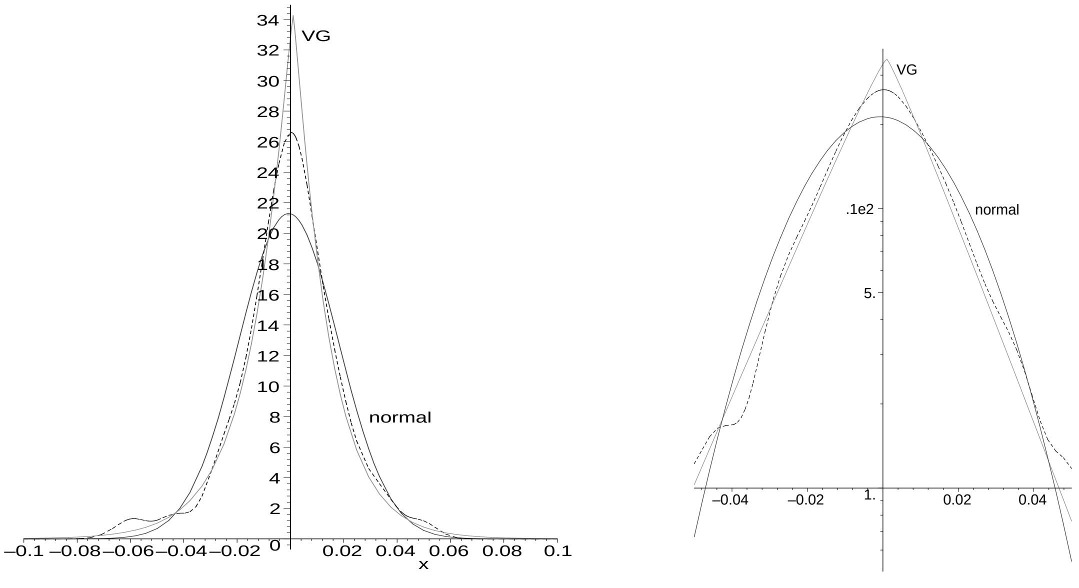

Figure 3: Kernel estimator (dotted line) of log-returns of Figure 2 with fitted densities in linear (left) and logarithmic (right) scale

We denote the price of a non-dividend-paying stock at time $t \geq 0$ by S_t and consider the following dynamics for the stock price process (see EBERLEIN [14])

$$\mathrm{d}S_t = S_{t^-}(\mathrm{d}Z_t + \mathrm{e}^{\Delta Z_t} - 1 - \Delta Z_t), \tag{4}$$

where $(Z_t)_{t\geq 0}$ denotes the VG Lévy motion, Z_{t^-} the left hand limit of the path at time t and $\Delta Z_t = Z_t - Z_{t^-}$ the jump at time t. Then, the solution of the stochastic differential equation (4) is given by

$$S_t = S_0 \exp(Z_t)$$

and it follows that the log-returns $\ln(S_t/S_{t-1})$ are indeed VG-distributed.

Due to the incompleteness of this market model (see CHERNY [12]), we have to choose an equivalent martingale measure for the risk-neutral valuation of derivative securities. A mathematical tractable choice of such a measure is the equivalent Esscher measure. It can be applied in exponential Lévy models, whenever the assumed distribution of the log-returns is infinitely divisible (see GERBER AND SHIU [18] or EBERLEIN AND KELLER [15]). Apart from its mathematical simplicity, this particular choice can also be economically justified (see e.g. PRAUSE [32]). Another motivation for this choice of measure for pricing purposes is the result of CHAN [11] that in a model very similar to the exponential Lévy model the Esscher transform is the minimal martingale measure in the sense of FÖLLMER AND SCHWEIZER [17].

From (3) we have that the density of Z_t is given by

$$f^{*t}_{\mathrm{VG}}(x) = f_{\mathrm{VG}(\alpha,\beta,t\lambda,t\mu)}(x).$$

For a real number θ let us consider the Esscher transform

$$f^{*t}(x;\theta) = \frac{\mathrm{e}^{\theta x} f^{*t}(x)}{\int_{-\infty}^{\infty} \mathrm{e}^{\theta y} f^{*t}(y)\,\mathrm{d}y} = \frac{\mathrm{e}^{\theta x}}{M(\theta)^t} f^{*t}(x), \tag{5}$$

of the one-dimensional marginal distributions $f^{*t}(x)$ of $(Z_t)_{t\geq 0}$ (this transform is well-known in the actuarial literature, cf. BÜHLMANN [7]).

For any Lévy process $(Z_t)_{t\geq 0}$ (on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}^+}, \mathbb{P})$) it is now possible to define a locally equivalent probability measure $\mathbb{P}^\theta$ through

$$\mathrm{d}\mathbb{P}^\theta = \exp(\theta Z_t - t \log M(\theta))\,\mathrm{d}\mathbb{P},$$

such that $(Z^\theta_t)_{t\geq 0}$ defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}^+}, \mathbb{P}^\theta)$ is again a Lévy process and the one-dimensional marginal distributions of $(Z^\theta_t)_{t\geq 0}$ are the Esscher transforms of the corresponding marginals of $(Z_t)_{t\geq 0}$ (see e.g. RAIBLE [33]). $\mathbb{P}^\theta$ is called the Esscher equivalent measure.

The parameter θ can now be chosen in such a way, that the discounted stock price process $(\mathrm{e}^{-rt} S_t)_{t\geq 0}$ is a $\mathbb{P}^\theta$-martingale, namely if θ is the solution of

$$r = \mu + \lambda \ln \frac{\alpha^2 - (\beta+\theta)^2}{\alpha^2 - (\beta+\theta+1)^2}. \tag{6}$$

Here r is the constant daily interest rate. The uniqueness of the solution follows from the general condition $|\beta + u| < \alpha$ in (2). Thus θ can easily be calculated numerically.

The following straight-forward observation will substantially simplify the calculation of Esscher prices in the VG model:

Lemma 1 *The Esscher transform of a* VG-*distributed random variable is again* VG-*distributed. In particular,*

$$f_{\mathrm{VG}(\alpha,\beta,\lambda,\mu)}(x;\theta) = f_{\mathrm{VG}(\alpha,\beta+\theta,\lambda,\mu)}(x).$$

As a first example, the value at time t of a European call option with exercise price K and maturity T can be represented by a simple analytical expression:

From $\mathrm{EC}_t = \mathbb{E}^\theta[\mathrm{e}^{-r(T-t)}(S_T - K)^+ \mid \mathcal{F}_t]$ and Lemma 1 it follows that

$$\mathrm{EC}_t = S_t \int_k^\infty f_{\mathrm{VG}(\alpha,\beta+\theta+1,(T-t)\lambda,(T-t)\mu)}(x)\,\mathrm{d}x - \mathrm{e}^{-r(T-t)}\,K \int_k^\infty f_{\mathrm{VG}(\alpha,\beta+\theta,(T-t)\lambda,(T-t)\mu)}(x)\,\mathrm{d}x \quad (7)$$

with $k = \ln(K/S_t)$. This value can be computed numerically.

4 Upper bounds for arithmetic average options using actuarial methods

In this section we consider the calculation of the Esscher price of a European-style arithmetic average call option at time t given by

$$\mathrm{AA}_t = \frac{\mathrm{e}^{-r(T-t)}}{n}\mathbb{E}^\theta\left[\left(\sum_{k=0}^{n-1} S_{T-k} - nK\right)^+ \,\middle|\, \mathcal{F}_t\right], \quad (8)$$

where n is the number of averaging days, K the strike price, T the time to expiration and r the risk-free interest rate.

The main difficulty is the determination of the distribution of the dependent sum $\sum S_i$. We will thus give bounds and approximations for (8).

A weak upper bound is the price of a plain European option with the same strike and maturity. This was proven in Kemna and Vorst [24] in the Black-Scholes case and recently generalized to arbitrary distributions of the underlying asset in Nielsen and Sandmann [31] by an elementary portfolio argument.

Another approach due to SIMON, GOOVAERTS AND DHAENE [36] for the derivation of upper bounds for arithmetic Asian options in an arbitrage-free and complete market is to interpret (8) as a stop-loss transform and use the stop-loss order to construct bounds. In the sequel we will adapt their technique to the VG model.

The stop-loss transform $\Psi_F(r)$ of a distribution function $F(x)$ with support $D \subseteq \mathbb{R}^+$ is defined by

$$\Psi_F(r) = \int_{[r,\infty)\cap D} (x-r)\,\mathrm{d}F(x).$$

and a stop-loss ordering among distribution functions $F(x)$ and $G(x)$ with support in $\mathbb{R}^+$ is given by

$$F \leq_{\mathrm{sl}} G \quad \Leftrightarrow \quad \Psi_F(r) \leq \Psi_G(r) \text{ for all } r \in \mathbb{R}^+.$$

Thus we can rewrite (8) to

$$\mathrm{AA}_t = \frac{\mathrm{e}^{-r(T-t)}}{n} \Psi_{F^s_{A_n(T)}}(nK) \tag{9}$$

for a given value $S_t = s$ with $F^s_{A_n(T)} = \mathbb{P}^\theta(A_n(T) \leq x | S_t = s)$, where

$$A_n(T) = \sum_{k=0}^{n-1} S_{T-k}.$$

In this way we have transformed the problem of pricing an arithmetic average option to calculating the stop-loss transform of a sum of dependent risks. Hence we can apply results on bounds for stop-loss transforms to our option pricing problem:

A positive random vector $(X_1, \ldots, X_n)$ with marginal distributions $F_1(x_1), \ldots, F_n(x_n)$ is called comonotone, if $F_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = \min\{F_1(x_1), \ldots, F_n(x_n)\}$ holds for every $x_1, \ldots, x_n \geq 0$. It immediately follows that a comonotone random vector $(X_1, \ldots, X_n)$ with given marginal distributions $F_1(x_1), \ldots, F_n(x_n)$ is uniquely determined.

It can easily be shown (see e.g. DHAENE ET AL. [13]) that an upper bound for the stop-loss transform of a sum of dependent random variables $\sum_{k=1}^n X_k$ with marginal distributions $F_1(x_1), \ldots, F_n(x_n)$ is now given by the stop-loss transform of the sum $\sum_{k=1}^n Y_k$, where $(Y_1, \ldots, Y_n)$ is the comonotone random vector with marginal distributions $F_1(x_1), \ldots, F_n(x_n)$, i.e.

$$\sum_{k=1}^n X_k \leq_{\mathrm{sl}} \sum_{k=1}^n Y_k.$$

Let us define

$$F_R(x) := \mathbb{P}^\theta\left(\sum_{k=1}^n Y_k \leq x\right),$$

then we have (cf. DHAENE AT AL. [13])

$$F_R^{-1}(x) = \sum_{k=1}^{n} F_k^{-1}(x) \qquad \text{for each } x \in \mathbb{R}^+$$

and

$$\Psi_{F_R}(x) = \sum_{k=1}^{n} \Psi_{F_k}\Big(F_k^{-1}(F_R(x))\Big) \qquad \text{for each } x \in \mathbb{R}^+.$$

Hence this upper bound of the arithmetic average option price can be viewed as a sum of prices of European call options with strike prices $F_k^{-1}(F_R(x))$.

The following proposition is an adaption of the result of SIMON, GOOVAERTS AND DHAENE [36] to our situation. Let in the sequel $F(x_2, t_2, x_1, t_1)$ denote the conditional distribution function of S_{t_2} under the equivalent Esscher martingale measure $\mathbb{P}^\theta$ given $S_{t_1} = x_1$, i.e.

$$F(x_2, t_2, x_1, t_1) \;=\; \int_{-\infty}^{\ln(x_2/S_{t_1})} f_{\mathrm{VG}(\alpha,\beta+\theta,(t_2-t_1)\lambda,(t_2-t_1)\mu)}(z)\,\mathrm{d}z \quad (t_1 \le t_2).$$

Proposition 1 ([36]) *Let k^* be such that $T - k^* \le t < T - k^* + 1$ and $K_j = nK - \sum_{k=j}^{n-1} S_{T-k}$ for $j < n$, $K_n = nK$. Let AA_t be the price of an arithmetic average option at time t as given in* (8) *and let furthermore $\mathrm{EC}_t(\kappa_k, T-k)$ be the price of a European option with strike price κ_k and time to expiration $T-k$. Then we have for $K_{k^*} > 0$*

$$\mathrm{AA}_t \le \frac{\mathrm{e}^{-r(T-t)}}{n} \sum_{k=0}^{k^*-1} \Psi_{F(\cdot,T-k,s,t)}(\kappa_k) = \frac{1}{n} \sum_{k=0}^{k^*-1} \mathrm{e}^{-kr}\,\mathrm{EC}_t(\kappa_k, T-k), \tag{10}$$

where

$$\kappa_k = F^{-1}\Big(F_R(K_{k^*}), T-k, s, t\Big), \quad k = 0, \dots, k^*. \tag{11}$$

Moreover, this choice of the strike prices κ_k is best possible.

In case $K_{k^*} \le 0$, we have

$$\mathrm{AA}_t = \frac{S_t}{n} \sum_{k=0}^{k^*-1} \mathrm{e}^{-kr} + \frac{\mathrm{e}^{-r(T-t)}}{n} \sum_{k=k^*}^{n-1} S_{T-k} - \mathrm{e}^{-r(T-t)}\,K.$$

In order to obtain a bound for the arithmetic average option price we thus have to calculate k^* strike prices κ_k using (11) and then evaluate (10) using (7).

Table 1 compares the stop-loss upper bound for the Esscher price of the European-style arithmetic average option with a Monte Carlo simulated price $\mathrm{AA}_0^{\mathrm{MC}}$ obtained by generating 1 million sample paths (the European call option price EC_0 is also given). Since

Stock	model	T	K	EC_0	AA_0^{MC}	SL bound	r.e. (%)
OMV	VG	10	94	6.7814	6.2674	6.3316	1.01
$S_0 = 100$			97	4.4098	3.5803	3.7319	4.06
			98.5	3.4119	2.4538	2.6457	7.25
			100	2.5619	1.5538	1.7633	11.88
			101.5	1.8652	0.9063	1.1050	17.98
			103	1.3168	0.4901	0.6545	25.12
			106	0.6006	0.1192	0.1998	40.34
		20	94	7.6898	6.6063	6.7389	1.97
			97	5.5257	4.1180	4.3537	5.41
			98.5	4.5895	3.0781	3.3509	8.14
			100	3.7582	2.2092	2.4997	11.62
			101.5	3.0334	1.5209	1.8063	15.80
			103	2.4130	1.0058	1.2650	20.49
			106	1.4620	0.3892	0.5679	31.47
	NIG	10	94	6.7877	6.2642	6.3376	1.16
			97	4.4105	3.5864	3.7335	3.94
			98.5	3.4085	2.4554	2.6436	7.12
			100	2.5546	1.5508	1.7574	11.76
			101.5	1.8551	0.9031	1.0956	17.57
			103	1.3053	0.4854	0.6437	24.59
			106	0.5906	0.1160	0.1936	40.11
		20	94	7.6941	6.5989	6.7444	2.16
			97	5.5250	4.1090	4.3543	5.63
			98.5	4.5860	3.0832	3.3480	7.91
			100	3.7519	2.1998	2.4934	11.77
			101.5	3.0247	1.5178	1.7972	15.55
			103	2.4026	1.0004	1.2542	20.23
			106	1.4503	0.3805	0.5580	31.80

Table 1: Comparison of simulated Asian option prices and the SL upper bound ($r = 0.1$)

this paper is an extension of [2], where the exponential Lévy model with normal inverse Gaussian distributed log-returns was investigated, we will also give the corresponding prices in the NIG model for convenience of the reader. In all subsequent tables the number of averaging days n equals the number of days T until maturity. The inverse distribution function needed for the calculation of κ_k is interpolated, since there is no analytic expression available. The numerical values indicate that the accuracy of the upper bound is satisfying if the option is in the money.

5 Two approximations for the distribution of the arithmetic mean

In this section we study approximations of the arithmetic option price (8) using Edgeworth series expansions. For notational simplicity we will assume $t = 0$ and $n = T$ in (8), i.e. the averaging starts at time $t = 1$ and we determine the price at time $t = 0$

$$\mathrm{AA}_0 = \frac{\mathrm{e}^{-rT}}{n}\mathbb{E}^{\theta}\left[\left(\sum_{k=1}^{n} S_k - nK\right)^{+}\middle|\mathcal{F}_0\right]. \tag{12}$$

The extension to the general case is straightforward. In the sequel we will make use of the following classical result:

Lemma 2 ([23]) *Let F and G be two continuous distribution functions with $G \in \mathcal{C}^5$ and $\chi_1(F) = \chi_1(G)$, and assume that the first five moments of both distributions exist. Then we can expand the density $f(x)$ in terms of the density $g(x)$ as follows*

$$\begin{aligned} f(x) =& g(x) + \frac{\chi_2(F)-\chi_2(G)}{2}\frac{\partial^2 g}{\partial x^2}(x) - \frac{\chi_3(F)-\chi_3(G)}{3!}\frac{\partial^3 g}{\partial x^3}(x) \\ &+ \frac{\chi_4(F)-\chi_4(G)+3\left(\chi_2(F)-\chi_2(G)\right)^2}{4!}\frac{\partial^4 g}{\partial x^4}(x) + \varepsilon(x), \end{aligned}$$

where $\varepsilon(x)$ is a residual error term.

We will now approximate the distribution function of $\sum_{k=1}^{n} S_k$ (which we denote by F) by a lognormal distribution G (see TURNBULL AND WAKEMAN [37] and LEVY [27] for a similar procedure in the Black-Scholes case). Therefore let us define

$$R_i = \frac{S_i}{S_{i-1}}, \quad i = 1, \dots, n$$

and

$$\begin{aligned} L_n &= 1 \\ L_{i-1} &= 1 + R_i L_i, \quad i = 2, \dots, n. \end{aligned}$$

Then we have

$$\sum_{k=1}^{n} S_k = S_0(R_1 + R_1R_2 + \cdots + R_1R_2 \ldots R_n) = S_0R_1L_1.$$

Since we can rewrite equation (12) to

$$\mathrm{AA}_0 = \frac{\mathrm{e}^{-rT}\, S_0}{n}\mathbb{E}^{\theta}\left[\left(R_1L_1 - \frac{nK}{S_0}\right)^{+}\middle|\, \mathcal{F}_0\right],$$

it remains to determine $\mathbb{E}^{\theta}[(R_1L_1)^m]$ for $m = 1, 2, 3, 4$. Because of the independent increments property of a Lévy process, we have $\mathbb{E}^{\theta}[(R_1L_1)^m] = \mathbb{E}^{\theta}[R_1^m]\mathbb{E}^{\theta}[L_1^m]$ and

$$\mathbb{E}^{\theta}[L_{i-1}^m] = \mathbb{E}^{\theta}[(1 + L_iR_i)^m] = \sum_{k=0}^{m}\binom{m}{k}\mathbb{E}^{\theta}[L_i^k]\mathbb{E}^{\theta}[R_i^k]. \tag{13}$$

In order to apply recursion (13), we need to determine the moments $\mathbb{E}^{\theta}[R_i^k]$:

Lemma 3 *For all $k \in \mathbb{N}$ we have*

$$\mathbb{E}^{\theta}[R_i^k] = \exp(\mu k)\left(\frac{\alpha^2 - (\beta+\theta)^2}{\alpha^2 - (\beta+\theta+k)^2}\right)^{\lambda}. \tag{14}$$

The moments $\mathbb{E}^{\theta}[L_1^m]$, $m = 1, 2, 3, 4$, and subsequently the cumulants $\chi_i(F)$ can now be calculated recursively starting with $\mathbb{E}^{\theta}[L_n^k] = 1\ \forall\ k \in \{0, \ldots, m\}$ by using (13) and (14).

The parameters $\tilde{\mu}$ and $\tilde{\sigma}^2$ of the approximating lognormal distribution are determined so that the first two moments of the approximating and the original distribution coincide, i.e.

$$\begin{aligned}\tilde{\mu} &= 2\ln(\chi_1(F)) - \frac{1}{2}\ln\left(\chi_1^2(F) + \chi_2(F)\right)\\ \tilde{\sigma}^2 &= \ln\left(\chi_1^2(F) + \chi_2(F)\right) - 2\ln(\chi_1(F)).\end{aligned}$$

In this way we have derived a lognormal approximation pricing formula for an arithmetic average option in the VG model, which we call the Turnbull-Wakeman price $\mathrm{AA}_0^{\mathrm{TW}}$ at time $t = 0$:

Proposition 2 *The price at time 0 of a European-style arithmetic average option in the* VG *model with maturity T and strike price K can be approximated by*

$$\begin{aligned}\mathrm{AA}_0^{\mathrm{TW}} = {} & \mathrm{e}^{-rT}\,\frac{S_0}{n}\left(\mathrm{e}^{\tilde{\mu}+\frac{\tilde{\sigma}^2}{2}}\,\Phi\left(\frac{\tilde{\mu}+\tilde{\sigma}^2 - \ln\frac{nK}{S_0}}{\tilde{\sigma}}\right) - \frac{nK}{S_0}\Phi\left(\frac{\tilde{\mu} - \ln\frac{nK}{S_0}}{\tilde{\sigma}}\right)\right)\\ & - \mathrm{e}^{-rT}\,\frac{S_0}{n}\left(\frac{\chi_3(F) - \chi_3(G)}{3!}\frac{\partial g\left(\frac{nK}{S_0}\right)}{\partial x} + \frac{\chi_4(F) - \chi_4(G)}{4!}\frac{\partial^2 g\left(\frac{nK}{S_0}\right)}{\partial x^2}\right).\end{aligned}$$

If we omit the terms of third and fourth order, then we call the corresponding approximation the Levy price

$$\mathrm{AA}_0^{\mathrm{L}} = \mathrm{e}^{-rT}\,\frac{S_0}{n}\left(\mathrm{e}^{\tilde{\mu}+\frac{\tilde{\sigma}^2}{2}}\,\Phi\left(\frac{\tilde{\mu}+\tilde{\sigma}^2-\ln\frac{nK}{S_0}}{\tilde{\sigma}}\right)-\frac{nK}{S_0}\Phi\left(\frac{\tilde{\mu}-\ln\frac{nK}{S_0}}{\tilde{\sigma}}\right)\right).$$

Another natural way to obtain an appproximation of $\sum_{k=1}^n S_k$ is to match its the first four moments with a VG distribution. This leads to a system of equations

$$\chi_3(F) = \frac{2\tilde{\beta}\chi_2(F)(3\tilde{\alpha}^2+\tilde{\beta}^2)}{\tilde{\alpha}^4-\tilde{\beta}^4},$$

$$\chi_4(F) = \frac{6\chi_2(F)(\tilde{\alpha}^4+6\tilde{\alpha}^2\tilde{\beta}^2+\tilde{\beta}^4)}{(\tilde{\alpha}^4-\tilde{\beta}^4)(\tilde{\alpha}^2-\tilde{\beta}^2)},$$

which we can solve numerically to obtain $\tilde{\alpha}$ and $\tilde{\beta}$ and then the other two parameters are determined by

$$\tilde{\lambda} = \frac{\chi_2(F)(\tilde{\alpha}^2-\tilde{\beta}^2)^2}{2(\tilde{\alpha}^2+\tilde{\beta}^2)},$$

$$\tilde{\mu} = \chi_1(F) - \frac{2\tilde{\beta}\tilde{\lambda}}{\tilde{\alpha}^2-\tilde{\beta}^2}.$$

The approximated option price is then given by

$$\mathrm{AA}_0^{\star} = \mathrm{e}^{-rT}\,\frac{S_0}{n}\int_d^{\infty}(x-d)f_{\mathrm{VG}(\tilde{\alpha},\tilde{\beta},\tilde{\lambda},\tilde{\mu})}(x)\,\mathrm{d}x \qquad \text{with} \quad d = nK/S_0.$$

In Table 2 we compare the performance of these approximation techniques for the arithmetic average option and give the relative error with respect to the Monte Carlo price $\mathrm{AA}_0^{\mathrm{MC}}$. The results show that the VG approximation outperforms the Turnbull-Wakeman approximation, which itself is superior to the Levy approximation in most cases.

6 Arithmetic and geometric average options

We now turn to a geometric average option with Esscher price given by

$$\mathrm{GA}_0 = e^{-rT}\,\mathbb{E}^{\theta}\left[(G_T-K)^+\middle|\,\mathcal{F}_0\right], \tag{15}$$

where $G_T = \left(\prod_{k=1}^n S_k\right)^{1/n}$ and K denotes the strike price. Again, we have chosen $t=0$ and $n=T$ (the generalization to arbitrary $t\geq 0$ and arbitrary starting times of the averaging period is straightforward). Here we have

$$\ln G_T = \ln S_0 + X_1 + \frac{n-1}{n}X_2 + \ldots + \frac{1}{n}X_n$$

model	T	K	AA_0^{MC}	AA_0^{L}	r.e.(%)	AA_0^{TW}	r.e.(%)	$AA_0^{\star}$	r.e.(%)
VG	10	94	6.2674	6.2573	0.16	6.2710	0.06	6.2694	0.03
		97	3.5803	3.5879	0.22	3.5799	0.01	3.5838	0.10
		98.5	2.4538	2.4797	1.05	2.4563	0.10	2.4594	0.23
		100	1.5538	1.5899	2.27	1.5610	0.46	1.5614	0.49
		101.5	0.9063	0.9373	3.31	0.9161	1.07	0.9147	0.92
		103	0.4901	0.5046	2.87	0.4981	1.61	0.4963	1.25
		106	0.1192	0.1094	8.96	0.1210	1.49	0.1196	0.33
	20	94	6.6063	6.5916	0.22	6.5932	0.20	6.5951	0.17
		97	4.1180	4.1242	0.15	4.1088	0.22	4.1123	0.14
		98.5	3.0781	3.0930	0.48	3.0721	0.20	3.0743	0.12
		100	2.2092	2.2280	0.84	2.2067	0.11	2.2071	0.10
		101.5	1.5209	1.5369	1.04	1.5205	0.03	1.5193	0.11
		103	1.0058	1.0129	0.70	1.0047	0.11	1.0025	0.33
		106	0.3892	0.3813	2.07	0.3880	0.31	0.3859	0.86
NIG	10	94	6.2642	6.2569	0.12	6.2784	0.23	6.2748	0.17
		97	3.5864	3.5867	0.01	3.5824	0.11	3.5879	0.04
		98.5	2.4554	2.4781	0.92	2.4522	0.13	2.4593	0.16
		100	1.5508	1.5881	2.35	1.5517	0.06	1.5558	0.32
		101.5	0.9031	0.9356	3.47	0.9054	0.25	0.9051	0.22
		103	0.4854	0.5032	3.54	0.4897	0.88	0.4867	0.27
		106	0.1160	0.1088	6.62	0.1194	2.85	0.1164	0.34
	20	94	6.5989	6.5905	0.13	6.5990	0.00	6.6011	0.03
		97	4.1090	4.1223	0.32	4.1091	0.00	4.1153	0.15
		98.5	3.0832	3.0908	0.25	3.0684	0.48	3.0735	0.32
		100	2.1998	2.2256	1.16	2.1997	0.01	2.2020	0.10
		101.5	1.5178	1.5345	1.09	1.5118	0.40	1.5108	0.46
		103	1.0004	1.0108	1.03	0.9958	0.46	0.9925	0.80
		106	0.3805	0.3799	0.16	0.3821	0.42	0.3784	0.56

Table 2: Comparison of simulated Asian option prices and approximations on OMV

with $X_k \overset{\text{iid}}{\sim}$ VG. Thus the distribution of $\ln G_T$ is not VG anymore. However, we will approximate it with a $\mathrm{VG}(\tilde{\alpha}, \tilde{\beta}, \tilde{\lambda}, \tilde{\mu})$ distribution by matching the first four cumulants (which for $\ln G_T$ are easy to obtain in terms of the cumulants of X_i and functions of n, see [2]).

Then, the price of a European-style geometric average option at time 0 is approximated by

$$\mathrm{GA}_0^\star = \mathrm{e}^{-rT} \int_K^\infty (x - K) f_{\mathrm{LogVG}(\tilde{\alpha},\tilde{\beta},\tilde{\lambda},\tilde{\mu})}(x)\,\mathrm{d}x. \tag{16}$$

This value can be calculated numerically.

Next, we derive bounds for AA_0 using approximation (16) for the geometric average option price. Since the geometric average is always less or equal the arithmetic average, we have

$$\mathrm{GA}_0 = \mathrm{e}^{-rT}\,\mathbb{E}^\theta\left[(G_T - K)^+ \middle| \mathcal{F}_0\right] \le \mathrm{e}^{-rT}\,\mathbb{E}^\theta\left[(A_T - K)^+ \middle| \mathcal{F}_0\right] = \mathrm{AA}_0,$$

where $A_T = \frac{1}{n}\sum_{k=1}^n S_k$. Following VORST [38], we obtain the upper bound

$$\mathrm{AA}_0 \le \mathrm{GA}_0 + \mathrm{e}^{-rT}\left(\mathbb{E}^\theta[A_T|\mathcal{F}_0] - \mathbb{E}^\theta[G_T|\mathcal{F}_0]\right) =: \mathrm{AA}_0^{\mathrm{U}}. \tag{17}$$

The expected value $\mathbb{E}^\theta[A_T|\mathcal{F}_0] = \frac{S_0}{n}\mathbb{E}^\theta[R_1 L_1]$ in (17) can be calculated by recursion (13), and $\mathbb{E}^\theta[G_T|\mathcal{F}_0] \approx M_{VG}(1)$, since $\ln G_T$ is approximated by a VG distribution.

VORST [38] also proposed the approximation

$$\mathrm{AA}_0^{\mathrm{V}} = \mathrm{e}^{-rT}\,\mathbb{E}^\theta\left[(G_T - K')^+ \middle| \mathcal{F}_0\right] \qquad \text{with} \quad K' = K - \left(\mathbb{E}^\theta[A_T|\mathcal{F}_0] - \mathbb{E}^\theta[G_T|\mathcal{F}_0]\right)$$

for the price of an arithmetic average value option, leading to

$$\mathrm{AA}_0^{\mathrm{V}} = \mathrm{e}^{-rT} \int_{K'}^\infty (x - K') f_{\mathrm{LogVG}}(x)\,\mathrm{d}x.$$

Remark. Since the asset price model is arbitrage-free, the above techniques can also directly be used for the derivation of prices of put option of Asian type due to put-call parity.

The numerical values for the Vorst approximation $\mathrm{AA}_0^{\mathrm{V}}$ and its relative error w.r.t. $\mathrm{AA}_0^{\mathrm{MC}}$ are depicted in Table 3. Moreover, the approximation $\mathrm{GA}_0^\star$ is compared with the simulated geometric price $\mathrm{GA}_0^{\mathrm{MC}}$. And finally the upper bound $\mathrm{AA}_0^{\mathrm{U}}$ is given.

model	T	K	$\mathrm{AA}_0^{\mathrm{MC}}$	$\mathrm{AA}_0^{\mathrm{V}}$	r.e.(%)	$\mathrm{GA}_0^{\mathrm{MC}}$	$\mathrm{GA}_0^{\star}$	r.e.(%)	$\mathrm{AA}_0^{\mathrm{U}}$	r.e.(%)
VG	10	94	6.2674	6.2713	0.06	6.2407	6.2435	0.05	6.2726	0.08
		97	3.5803	3.5892	0.25	3.5584	3.5655	0.20	3.5946	0.40
		98.5	2.4538	2.4640	0.41	2.4348	2.4441	0.38	2.4732	0.78
		100	1.5538	1.5619	0.52	1.5378	1.5468	0.58	1.5758	1.40
		101.5	0.9063	0.9105	0.46	0.8933	0.9003	0.78	0.9293	2.48
		103	0.4901	0.4901	0.00	0.4802	0.4839	0.77	0.5129	4.45
		106	0.1192	0.1163	2.49	0.1148	0.1146	0.18	0.1436	16.99
	20	94	6.6063	6.6006	0.09	6.5564	6.5479	0.13	6.6064	0.00
		97	4.1180	4.1188	0.02	4.0760	4.0750	0.03	4.1335	0.38
		98.5	3.0781	3.0782	0.00	3.0404	3.0407	0.01	3.0991	0.68
		100	2.2092	2.2065	0.12	2.1758	2.1759	0.01	2.2344	1.13
		101.5	1.5209	1.5139	0.46	1.4917	1.4903	0.09	1.5487	1.80
		103	1.0058	0.9935	1.24	0.9811	0.9763	0.49	1.0348	2.80
		106	0.3892	0.3762	3.46	0.3735	0.3685	1.36	0.4270	8.85
NIG	10	94	6.2642	6.2768	0.20	6.2380	6.2486	0.17	6.2781	0.22
		97	3.5864	3.5922	0.16	3.5649	3.5681	0.09	3.5976	0.31
		98.5	2.4554	2.4626	0.29	2.4369	2.4424	0.23	2.4719	0.67
		100	1.5508	1.5559	0.33	1.5352	1.5405	0.34	1.5700	1.22
		101.5	0.9031	0.9018	0.14	0.8904	0.8914	0.11	0.9209	1.93
		103	0.4854	0.4816	0.79	0.4759	0.4754	0.11	0.5049	3.86
		106	0.1160	0.1132	2.47	0.1117	0.1116	0.09	0.1411	17.79
	20	94	6.5989	6.6063	0.11	6.5497	6.5527	0.05	6.6122	0.20
		97	4.1090	4.1207	0.28	4.0680	4.0760	0.20	4.1355	0.64
		98.5	3.0832	3.0765	0.22	3.0462	3.0382	0.26	3.0976	0.47
		100	2.1998	2.2012	0.06	2.1672	2.1700	0.13	2.2294	1.33
		101.5	1.5178	1.5060	0.78	1.4894	1.4819	0.51	1.5414	1.53
		103	1.0004	0.9847	1.59	0.9761	0.9672	0.92	1.0267	2.56
		106	0.3805	0.3696	2.95	0.3653	0.3619	0.94	0.4214	9.71

Table 3: Comparison of simulated Asian option prices and approximations on OMV

7 Comparison with the Black-Scholes and the NIG model

In this section we compare the numerical results for the Esscher option prices obtained in the VG model to the corresponding values in the NIG model and in the Black-Scholes setting. In the Black-Scholes model there is an explicit pricing formula for the geometric average option (see KEMNA AND VORST [24]):

$$^{(BS)}\mathrm{GA}_0 = \mathrm{e}^{-rT}\left(\mathrm{e}^{\hat{\mu}+\hat{\sigma}^2/2}\,\Phi(d_1) - K\Phi(d_2)\right) \quad \text{with} \quad d_1 = \frac{\hat{\mu}+\hat{\sigma}^2 - \ln K}{\hat{\sigma}},\ d_2 = d_1 - \hat{\sigma},$$

where

$$\hat{\mu} = \ln S_0 + \frac{T+\frac{T}{n}}{2}\left(r - \frac{\sigma^2}{2}\right), \quad \hat{\sigma}^2 = \sigma^2 T\frac{(2n+1)(n+1)}{6n^2}.$$

Here σ^2 denotes the variance of the log-returns, which can be estimated from historical data. The arithmetic average option price $^{(BS)}\mathrm{AA}_0$ in the Black-Scholes model can not be obtained by an explicit formula. Thus we use a Quasi-Monte Carlo simulated price (cf. HARTINGER AND PREDOTA [22]).

In Table 4 the Black-Scholes prices are compared with the NIG resp. VG Esscher prices $\mathrm{AA}_0^{\mathrm{MC}}$ and the relative difference is given. Note that in the Black-Scholes setting the Esscher pricing principle also yields the correct (unique) option prices. Whereas the NIG prices are very similar to the VG prices, the Black-Scholes prices slightly differ from the VG Esscher prices and tend to be significantly lower than the latter if the options are out of the money.

Figures 4 and 5 depict the dependence of the price difference of Asian options in the Black Scholes model and in the NIG model on strike K and maturity T. The qualitative behavior of these differences is similar to the corresponding differences for European option prices depicted in Figure 6.

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Stock	model	T	K	AA_0^{MC}	${}^{(BS)}AA_0$	r.d.(%)	GA_0^{MC}	${}^{(BS)}GA_0$	r.d.(%)
OMV	VG	10	94	6.2674	6.2528	0.23	6.2407	6.2262	0.23
($S_0 = 100$)			97	3.5803	3.5760	0.12	3.5584	3.5542	0.12
			98.5	2.4538	2.4652	0.46	2.4348	2.4463	0.47
			100	1.5538	1.5752	1.36	1.5378	1.5594	1.39
			101.5	0.9063	0.9249	2.01	0.8933	0.9122	2.07
			103	0.4901	0.4958	1.15	0.4802	0.4862	1.23
			106	0.1192	0.1066	11.82	0.1148	0.1026	11.89
		20	94	6.6063	6.5788	0.42	6.5564	6.5295	0.41
			97	4.1180	4.1039	0.34	4.0760	4.0627	0.33
			98.5	3.0781	3.0715	0.22	3.0404	3.0345	0.19
			100	2.2092	2.2073	0.09	2.1758	2.1747	0.05
			101.5	1.5209	1.5190	0.13	1.4917	1.4907	0.07
			103	1.0058	0.9989	0.69	0.9811	0.9751	0.62
			106	0.3892	0.3749	3.81	0.3735	0.3602	3.69
	NIG	10	94	6.2642	6.2528	0.18	6.2380	6.2262	0.19
			97	3.5864	3.5760	0.29	3.5649	3.5542	0.30
			98.5	2.4554	2.4652	0.40	2.4369	2.4463	0.38
			100	1.5508	1.5752	1.55	1.5352	1.5594	1.55
			101.5	0.9031	0.9249	2.36	0.8904	0.9122	2.39
			103	0.4854	0.4958	2.10	0.4759	0.4862	2.12
			106	0.1160	0.1066	8.82	0.1117	0.1026	8.87
		20	94	6.5989	6.5788	0.31	6.5497	6.5295	0.31
			97	4.1090	4.1039	0.12	4.0680	4.0627	0.13
			98.5	3.0832	3.0715	0.38	3.0462	3.0345	0.39
			100	2.1998	2.2073	0.34	2.1672	2.1747	0.35
			101.5	1.5178	1.5190	0.08	1.4894	1.4907	0.09
			103	1.0004	0.9989	0.15	0.9761	0.9751	0.10
			106	0.3805	0.3749	1.49	0.3653	0.3602	1.42

Table 4: Comparison of simulated Asian option prices in the NIG model and the Black-Scholes model

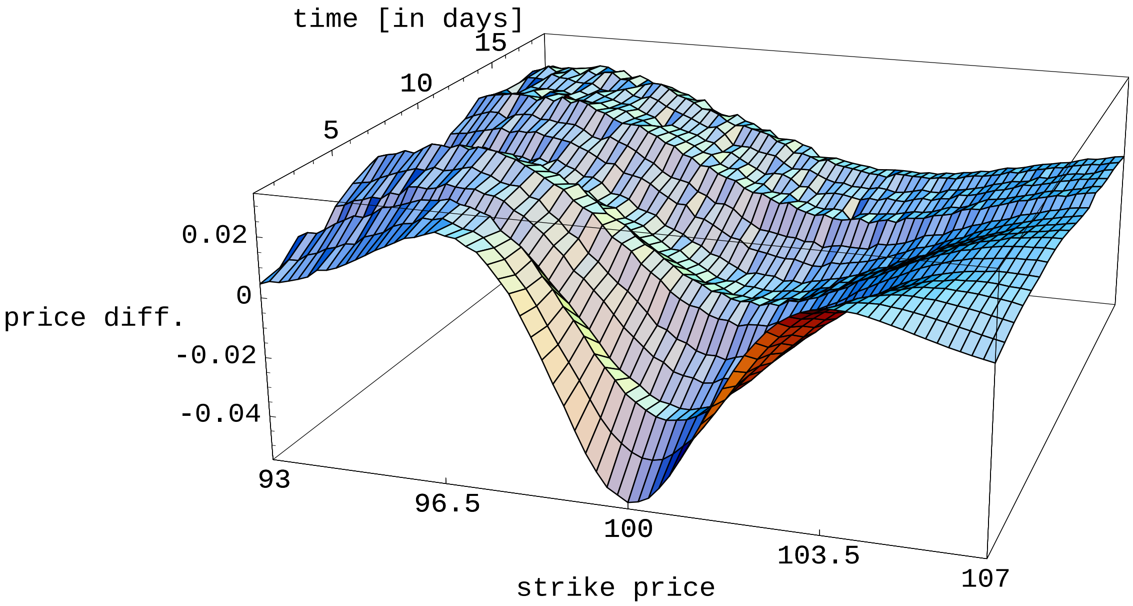

Figure 4: Price difference ${}^{(NIG)}GA_0 - {}^{(BS)}GA_0$ for the geometric average option in the BS and the NIG model

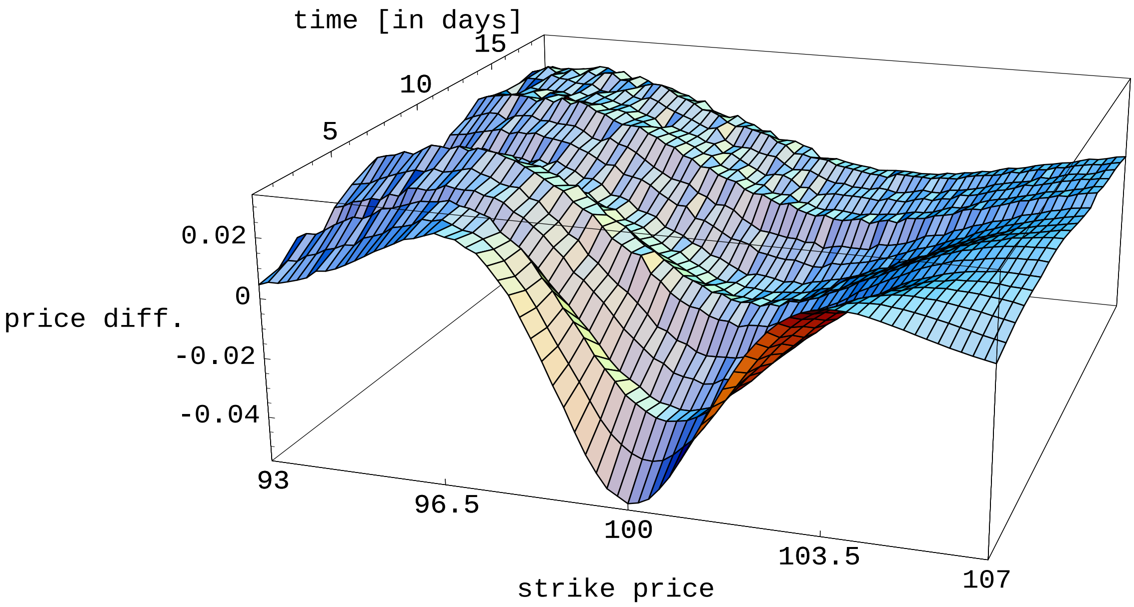

Figure 5: Price difference ${}^{(NIG)}AA_0 - {}^{(BS)}AA_0$ for the arithmetic average option in the BS and the NIG model

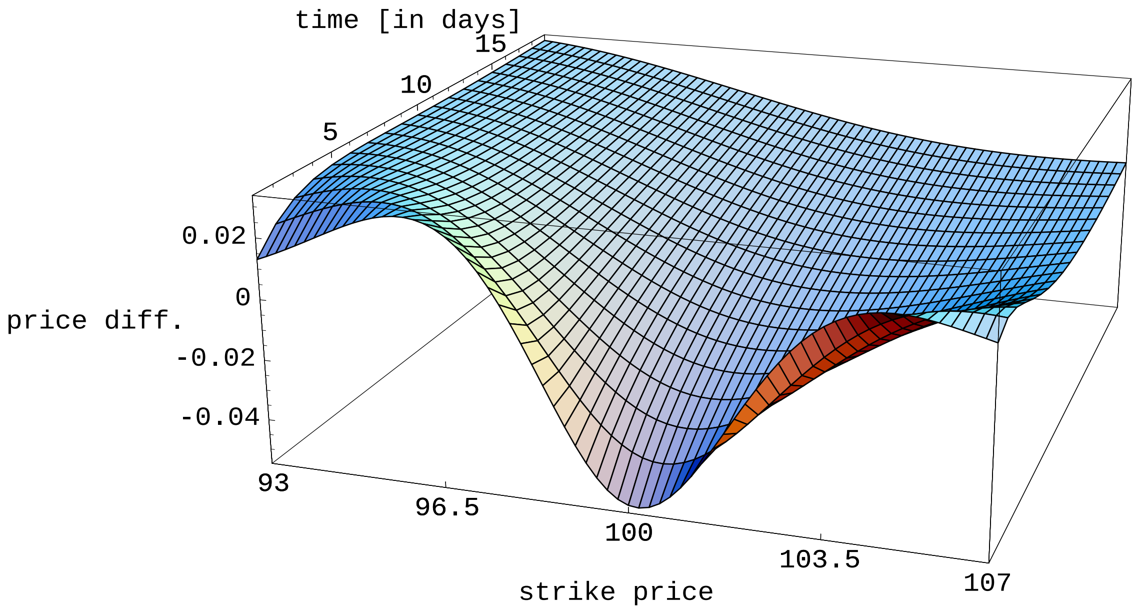

Figure 6: Price difference $^{(NIG)}EC_0 - {}^{(BS)}EC_0$ for the European option in the BS and the NIG model

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