Approximation of Passage Times of γ-reflected Processes with fBm Input

Enkelejd Hashorva and Lanpeng Ji*

Abstract: Define a γ-reflected process $W_\gamma(t) = Y_H(t) - \gamma \inf_{s\in[0,t]} Y_H(s)$, $t \ge 0$ with input process $\{Y_H(t), t \ge 0\}$ which is a fractional Brownian motion with Hurst index $H \in (0,1)$ and a negative linear trend. In risk theory $R_\gamma(u) = u - W_\gamma(t), t \ge 0$ is referred to as the risk process with tax payments of a loss-carry-forward type. For various risk processes numerous results are known for the approximation of the first and last passage times to 0 (ruin times) when the initial reserve u goes to infinity. In this paper we show that for the γ-reflected process the conditional (standardized) first and last passage times are jointly asymptotically Gaussian and completely dependent. An important contribution of this paper is that it links ruin problems with extremes of non-homogeneous Gaussian random fields defined by Y_H which are also investigated in this contribution.

Key Words: Gaussian approximation; passage times; γ-reflected process; workload process; risk process with tax; fractional Brownian motion; Piterbarg constant; Pickands constant.

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1 Introduction and Main Result

Let $\{X_H(t), t \ge 0\}$ be a standard fractional Brownian motion (fBm) with Hurst index $H \in (0,1)$ meaning that X_H is a centered Gaussian process with covariance function

$$Cov(X_H(t), X_H(s)) = \frac{1}{2}(t^{2H} + s^{2H} - \mid t - s \mid^{2H}), \quad t, s \ge 0.$$

We shall define the γ-reflected process with input process $Y_H(t) = X_H(t) - ct$ by

$$W_\gamma(t) = Y_H(t) - \gamma \inf_{s\in[0,t]} Y_H(s), \quad t \ge 0, \tag{1}$$

where $\gamma \in [0,1]$ and $c > 0$ are two fixed constants.

Motivations for studying W_γ come from both risk and queuing theory. For instance, in queuing theory W_1 is

*University of Lausanne, UNIL-Dorigny 1015 Lausanne, Switzerland, enkelejd.hashorva@unil.ch, lanpeng.ji@unil.ch

the so-called workload process (or queue length process), see e.g., Harrison (1985), Asmussen (1987), Zeevi and Glynn (2000), Whitt (2002) and Awad and Glynn (2009) among many others. In advanced risk theory the process $R_\gamma(t) = u - W_\gamma(t), t \ge 0, u \ge 0$ is referred to as the risk process with tax payments of a loss-carry-forward type, see e.g., Asmussen and Albrecher (2010).

Recently Hashorva et al. (2013) studed the asymptotics of the probability $\mathbb{P}\left\{\sup_{t\in[0,T]} W_\gamma(t) > u\right\}$ as $u \to \infty$ for both $T < \infty$ and $T = \infty$. Continuing the investigation of the aforementioned paper in this contribution we shall investigate the approximation of first and last passage times of W_γ. Specifically, define the first and last passage times of W_γ to a constant threshold $u > 0$ by

$$\tau_1(u) = \inf\{t \ge 0, W_\gamma(t) > u\} \quad \text{and} \quad \tau_2(u) = \sup\{t \ge 0, W_\gamma(t) > u\}, \tag{2}$$

respectively (here we use that $\inf\{\emptyset\} = \infty$). Further, define $\tau_1^*(u),\ \tau_2^*(u), u > 0$ in the same probability space such that

$$(\tau_1^*(u), \tau_2^*(u)) \stackrel{d}{=} (\tau_1(u), \tau_2(u)) \Big| (\tau_1(u) < \infty), \tag{3}$$

where $\stackrel{d}{=}$ stands for equality of distribution functions.

The first and last passage times of Gaussian processes conditioned on that $\tau_1(u) < \infty$ are analysed in Hüsler and Piterbarg (2008) and Hüsler and Zhang (2008) when $\gamma = 0$. Therein, the Guassian approximations of both $\tau_1^*(u)$ and $\tau_2^*(u)$ are derived as $u \to \infty$. The Gaussian approximation is not only of theoretical interest but also important for statistical estimation. First passage times (sometimes called ruin times) are also studied extensively in the framework of insurance risk processes, see the recent articles Griffin and Maller (2012), Griffin (2013), Griffin et al. (2013), Dębicki et al. (2013) and the monographs Embrechts et al. (1997), Asmussen and Albrecher (2010) for approximations of ruin times of various risk processes. In this framework, $\tau_1^*(u)$ can be interpreted as the conditional ruin time of the fBm risk process with tax payments of a loss-carry-forward type. With motivation from the aforementioned contributions, this paper is concerned with the Gaussian approximation of the random vector $(\tau_1^*(u), \tau_2^*(u))$, as $u \to \infty$. For the derivation of the tail asymptotics of $\sup_{t\in[0,T]} W_\gamma(t)$ Hashorva et al. (2013) showed that the investigation of the supremum of certain non-stationary Gaussian random fields is crucial. One key merit of our problem of approximating the joint distribution function of $(\tau_1^*(u), \tau_2^*(u))$ is that it leads, as in the case of the analysis of the tail asymptotics of $\sup_{t\in[0,T]} W_\gamma(t)$, to an interesting unsolved problem of asymptotic theory of Gaussian random fields. Although the latter investigation was not initially in the scope of this paper, the result derived in Theorem 2.1 is important for various theoretical questions. Next, set

$$A(u) = \frac{H^{H+1/2}}{(1-H)^{H+1/2} c^{H+1}} u^H, \quad \text{and} \quad \tilde{t}_0 = \frac{H}{c(1-H)}$$

and denote by $\stackrel{d}{\to}$ and $\stackrel{p}{\to}$ the convergence in distribution and in probability, respectively. Further, let $\mathcal{N}$ be a $N(0,1)$ random variable. Our principal result is the following theorem:

Theorem 1.1 *Let the γ-reflected process $\{W_\gamma(t), t \ge 0\}$ be given as in* (1) *with $\gamma \in (0,1)$, and let $\tau_1^*(u), \tau_2^*(u)$ be defined as in* (3). *Then, as $u \to \infty$*

$$\left(\frac{\tau_1^*(u) - \tilde{t}_0 u}{A(u)}, \frac{\tau_2^*(u) - \tilde{t}_0 u}{A(u)} \right) \stackrel{d}{\to} (\mathcal{N}, \mathcal{N}). \tag{4}$$

Remarks: a) The joint convergence in (4) implies $(\tau_2^*(u) - \tau_1^*(u))/A(u) \stackrel{p}{\to} 0$ as $u \to \infty$.

b) For any $u \ge 0$ $\mathbb{P}\{\tau_1(u) < \infty\} = 1$ when $\gamma = 1$ (cf. Duncan and Jin (2008)), which is the reason of considering only the case that $\gamma \in (0,1)$. Under the latter assumption on γ we have further that $\mathbb{P}\left\{\tau_2(u) < \infty \middle| \tau_1(u) < \infty\right\} = 1$, which follows from the fact that $\lim_{t\to\infty} W_\gamma(t) = -\infty$ almost surely since in view of Remark 5 in Kozachenko et al. (2011)

$$\lim_{t\to\infty} \frac{\sup_{s\in[0,t]} |X_H(s)|}{t} = 0, \quad \forall H \in (0,1).$$

c) It is surprising that the Gaussian approximation of the conditional first and last passage times does not involve the reflection constant γ.

Organisation of the rest of the paper: In the next section we present a key result on the supremum of some Gaussian random fields defined by Y_H and then display the proof of Theorem 1.1. Section 3 is dedicated to the proof of Theorem 2.1. A variant of Piterbarg Lemma suitable for Gaussian random fields is presented in Appendix.

2 Further Results and Proof of Theorem 1.1

Following the idea of Hüsler and Piterbarg (1999, 2008), and as discussed in Hashorva et al. (2013) it is convenient to introduce the following family of Gaussian random fields:

$$Y_u(s,t) := \frac{X_H(ut) - \gamma X_H(us)}{(1 + ct - c\gamma s)u^H}, \quad s, t \ge 0.$$

The variance function of $\{Y_u(s,t), s, t \ge 0\}$ is given by

$$V_Y^2(s,t) = \frac{(1-\gamma)t^{2H} + (\gamma^2 - \gamma)s^{2H} + \gamma(t-s)^{2H}}{(1 + ct - c\gamma s)^2}, \quad s, t \ge 0. \tag{5}$$

Moreover, on the set $\{(s,t) : 0 \le s \le t < \infty\}$ it attains its maximum at the unique point $(0, \tilde{t}_0)$ with $\tilde{t}_0 = \frac{H}{c(1-H)}$ and further

$$V_Y(0, \tilde{t}_0) = \frac{H^H (1-H)^{1-H}}{c^H}.$$

By changing time $t = t'u, s = s'u$ and noting that the distribution of Y_u does not depend on u, we obtain

$$\begin{aligned} \mathbb{P}\{\tau_1(u) < \infty\} &= \mathbb{P}\{\exists t \in [0,\infty) \text{ such that } W_\gamma(t) > u\} \\ &= \mathbb{P}\left\{\exists t' \in [0,\infty) \text{ such that } Y_u(s',t') > u^{1-H} \text{ for some } s' \in [0,t']\right\} \end{aligned}$$

$$= \mathbb{P}\left\{\exists t \in [0,\infty) \text{ such that } Y(s,t) > u^{1-H} \text{ for some } s \in [0,t]\right\},$$

where

$$Y(s,t) := \frac{X_H(t) - \gamma X_H(s)}{1 + c(t-\gamma s)}, \quad s,t \geq 0. \tag{6}$$

In order to complete the proof of Theorem 1.1 we need to know the tail asymptotic behaviour of the supremum of the Gaussian random field Y over a region which might depend on u. Therefore, we shall investigate first the tail asymptotic behaviour of the supremum of certain non-stationary Gaussian random fields (including Y as a special case) over a region depending on u in Theorem 2.1 followed then by the proof of Theorem 1.1.

Hereafter, we assume that all considered Gaussian random fields (or processes) have almost surely continuous sample paths. We need to introduce some more notation starting with the well-known Pickands constant $\mathcal{H}_\alpha$ given by

$$\mathcal{H}_\alpha := \lim_{T\to\infty} \frac{1}{T}\mathcal{H}_\alpha[0,T], \quad \alpha \in (0,2],$$

where

$$\mathcal{H}_\alpha[0,T] = \mathbb{E}\left(\exp\left(\sup_{t\in[0,T]}\left(\sqrt{2}B_\alpha(t) - t^\alpha\right)\right)\right) \in (0,\infty), \qquad T \in (0,\infty),$$

with $\{B_\alpha(t), t \geq 0\}$ a fBm with Hurst index $\alpha/2 \in (0,1]$. It is known that $\mathcal{H}_1 = 1$ and $\mathcal{H}_2 = 1/\sqrt{\pi}$, see Pickands (1969), Albin (1990), Piterbarg (1996), Dębicki (2002), Debicki et al. (2004), Mandjes (2007), Dębicki and Mandjes (2011), Dieker and Yakir (2013) for various properties of Pickands constant and its generalizations.

Next we introduce another constant, usually referred to as Piterbarg constant, given by

$$\mathcal{P}_\alpha^a := \lim_{S\to\infty} \mathcal{P}_\alpha^a[0,S], \quad \alpha \in (0,2],\ a > 0,$$

where

$$\mathcal{P}_\alpha^a[S,T] = \mathbb{E}\left(\exp\left(\sup_{t\in[S,T]}\left(\sqrt{2}B_\alpha(t) - (1+a)|t|^\alpha\right)\right)\right) \in (0,\infty), \quad S < T.$$

It is also known that

$$\mathcal{P}_1^a = 1 + \frac{1}{a} \quad \text{and} \quad \mathcal{P}_2^a = \frac{1}{2}\left(1 + \sqrt{1 + \frac{1}{a}}\right) \tag{7}$$

see e.g., Dębicki and Mandjes (2003) and Dębicki and Tabiś (2011). As it will be seen in Theorem 2.1 below both Pickands and Piterbarg constants are important for our study. We denote by $\Phi(\cdot)$ the standard normal distribution (of a $N(0,1)$ random variable), and further set $\Psi(\cdot) := 1 - \Phi(\cdot)$.

In the following we investigate the tail asymptotic behaviour of the supremum of non-stationary Gaussian random fields over a region which is depend on u. Our next result is of interest on its own, and furthermore is the key to the proof of Theorem 1.1.

Theorem 2.1 *Let S, T be two positive constants, and let $\{X(s,t), (s,t) \in [0,S] \times [0,T]\}$ be a centered Gaussian random field, with standard deviation function $\sigma(\cdot,\cdot)$ and correlation function $r(\cdot,\cdot,\cdot,\cdot)$. Assume that $\sigma(\cdot,\cdot)$ attains its maximum on $[0,S] \times [0,T]$ at the unique point $(0,t_0)$, with $t_0 \in (0,T)$, and further*

$$\sigma(s,t) = 1 - b_1 s^\beta (1+o(1)) - b_2|t-t_0|^2(1+o(1)) - b_3 s|t-t_0|(1+o(1)) \tag{8}$$

as $(s,t) \to (0,t_0)$ for some constants $\beta \in (1,2)$, and $b_i > 0, i = 1,2$, $b_3 \in \mathbb{R}$ satisfying $b_2 + b_3/2 > 0$. Suppose further that

$$r(s,s',t,t') = 1 - (a_1|s-s'|^\beta + a_2|t-t'|^\beta)(1+o(1)) \quad \text{as } (s,t),(s',t') \to (0,t_0) \tag{9}$$

for some constants $a_i > 0$, $i = 1,2$. Then, for any $x \in \mathbb{R}$

$$\mathbb{P}\left\{\sup_{(s,t)\in\widetilde{\Delta_x^1}(u)} X(s,t) > u\right\} = \sqrt{\frac{\pi}{b_2}} a_2^{\frac{1}{\beta}} \mathcal{P}_\beta^{b_1/a_1} \mathcal{H}_\beta u^{\frac{2}{\beta}-1} \Psi(u) \Phi(\sqrt{2b_2}x)(1+o(1)) \tag{10}$$

$$\mathbb{P}\left\{\sup_{(s,t)\in\widetilde{\Delta_x^2}(u)} X(s,t) > u\right\} = \sqrt{\frac{\pi}{b_2}} a_2^{\frac{1}{\beta}} \mathcal{P}_\beta^{b_1/a_1} \mathcal{H}_\beta u^{\frac{2}{\beta}-1} \Psi(u) \Psi(\sqrt{2b_2}x)(1+o(1)) \tag{11}$$

as $u \to \infty$, where $\delta_1(u) = (\ln u/u)^{\frac{2}{\beta}}$, $\delta_2(u) = \ln u/u$ and

$$\widetilde{\Delta_x^1}(u) = [0,\delta_1(u)] \times [t_0 - \delta_2(u), t_0 + xu^{-1}], \quad \widetilde{\Delta_x^2}(u) = [0,\delta_1(u)] \times [t_0 + xu^{-1}, t_0 + \delta_2(u)]. \tag{12}$$

Remarks 2.2 *a) If $\beta \in (0,1)$, then (8) becomes*

$$\sigma(s,t) = 1 - b_1 s^\beta (1+o(1)) - b_2|t-t_0|^2(1+o(1)) \quad \text{as } (s,t) \to (0,t_0). \tag{13}$$

We mention that in this case both (10) and (11) are still valid.

b) It can be shown along the proof of Theorem 2.1 that if $x = x(u)$ satisfies the following two conditions

$$\lim_{u\to\infty} x(u) = \infty, \qquad x(u) = o(u^\epsilon) \quad \text{as } u \to \infty, \text{ for any } \epsilon > 0, \tag{14}$$

then (10) still holds with $\Phi(\sqrt{2b_2}x)$ replaced by 1. Similarly, if $x = -x(u)$ with $x(u)$ satisfying (14), then (11) holds with $\Psi(\sqrt{2b_2}x)$ replaced by 1.

PROOF OF THEOREM 1.1 Define

$$T_1(u) = \inf\{t \ge 0 : \ Y(s,t) > u^{1-H} \text{ for some } s \in [0,t]\}$$

and

$$T_2(u) = \sup\{t \ge 0 : Y(s,t) > u^{1-H} \text{ for some } s \in [0,t]\}.$$

Clearly $\tau_i(u) \overset{d}{=} uT_i(u), i = 1,2$, with $\overset{d}{=}$ denoting equivalence in distribution. Consider first the approximation of $\tau_1(u)$. For any $x \in \mathbb{R}$ and $u > 0$ we have

$$\begin{aligned}\mathbb{P}\left\{\frac{\tau_1(u) - \tilde{t}_0 u}{A(u)} \le x \Big| \tau_1(u) < \infty\right\} &= \mathbb{P}\left\{T_1(u) \le \tilde{t}_0 + xA(u)u^{-1} \Big| T_1(u) < \infty\right\} \\ &= \frac{\mathbb{P}\left\{\sup_{0 \le s \le t \le \tilde{t}_0 + xA(u)u^{-1}} Y(s,t) > u^{1-H}\right\}}{\mathbb{P}\{\tau_1(u) < \infty\}}.\end{aligned}$$

In view of Hashorva et al. (2013) for any $H, \gamma \in (0,1)$

$$\mathbb{P}\{\tau_1(u) < \infty\} = \mathbb{P}\left\{\sup_{t \ge 0} W_\gamma(t) > u\right\} = \mathcal{W}_H(u)\Psi\left(\frac{c^H u^{1-H}}{H^H(1-H)^{1-H}}\right)(1+o(1)) \quad \text{as } u \to \infty, \tag{15}$$

where

$$\mathcal{W}_H(u) = 2^{\frac{1}{2} - \frac{1}{2H}} \frac{\sqrt{\pi}}{\sqrt{H(1-H)}} \mathcal{H}_{2H} \mathcal{P}_{2H}^{\frac{1-\gamma}{\gamma}} \left(\frac{c^H u^{1-H}}{H^H(1-H)^{1-H}}\right)^{(1/H-1)}.$$

Next, we focus on the analysis of $\mathbb{P}\left\{\sup_{0 \le s \le t \le \tilde{t}_0 + xA(u)u^{-1}} Y(s,t) > u^{1-H}\right\}$. By Bonforroni's inequality

$$p_3(u) \le \mathbb{P}\left\{\sup_{0 \le s \le t \le \tilde{t}_0 + xA(u)u^{-1}} Y(s,t) > u^{1-H}\right\} \le p_1(u) + p_2(u) + p_3(u), \tag{16}$$

where $p_i(u), i = 1,2,3$, are defined in (17), (21) and (22) below. In the following, we shall give the asymptotics of $p_3(u)$ as $u \to \infty$, and give bounds for both $p_1(u)$ and $p_2(u)$ for u large, assuring that they are relatively negligible.

We first consider bounds for $p_1(u)$ and $p_2(u)$. Since on the set $\{(s,t) : 0 \le s \le t < \infty\}$ the maximum of the variance function $V_Y^2(s,t)$ is attained uniquely at $(0, \tilde{t}_0)$, we obtain from the Borell-TIS inequality (e.g., Adler and Taylor (2007)) that for any constant $K \ge 2\tilde{t}_0$, there exist constants $\rho > 0$ small enough and $\theta \in (0,1)$ such that, for u sufficiently large

$$p_1(u) := \mathbb{P}\left\{\sup_{\substack{0 \le s \le t \le K \\ s \in [\rho, K] \text{ or } t \in [0, \tilde{t}_0 - \rho]}} Y(s,t) > u^{1-H}\right\} \le \exp\left(-\frac{(u^{1-H} - d)^2}{2\theta V_Y^2(0, \tilde{t}_0)}\right), \tag{17}$$

with $d = \mathbb{E}\left(\sup_{0 \le s \le t \le K} Y(s,t)\right) < \infty$. If follows that

$$1 - \frac{V_Y(s,t)}{V_Y(0,\tilde{t}_0)} = \begin{cases} \frac{c^2(1-H)^3}{2H}(\tilde{t}_0 - t)^2(1+o(1)) + \frac{(\gamma-\gamma^2)(1-H)^{2H}c^{2H}}{2H^{2H}} s^{2H}(1+o(1)), & H \le 1/2, \\ \frac{c^2(1-H)^3}{2H}(\tilde{t}_0 - t + \gamma s)^2(1+o(1)) + \frac{(\gamma-\gamma^2)(1-H)^{2H}c^{2H}}{2H^{2H}} s^{2H}(1+o(1)), & H > 1/2 \end{cases} \tag{18}$$

as $(s,t) \to (0, \tilde{t}_0)$ and further the correlation function of Y satisfies

$$1 - Cov\left(\frac{Y(s,t)}{V_Y(s,t)}, \frac{Y(s',t')}{V_Y(s',t')}\right) = \frac{1}{2\tilde{t}_0^{2H}}\left(|\, t - t' \,|^{2H} + \gamma^2 \,|\, s - s' \,|^{2H}\right)(1+o(1)) \tag{19}$$

as $(s,t), (s',t') \to (0, \tilde{t}_0)$. In addition, for the chosen $\rho > 0$ small enough there exists some $\mathbb{C} > 0$ such that for any $(s,t), (s',t') \in [0,\rho] \times [\tilde{t}_0 - \rho, \tilde{t}_0 + \rho]$

$$\mathbb{E}\left(Y(s,t) - Y(s',t')\right)^2 \le \mathbb{C}(|t - t'|^{2H} + |s - s'|^{2H}). \tag{20}$$

Next, let

$$A=\frac{H^{1/2}}{c(1-H)^{3/2}},\quad \tilde{u}=\frac{u^{1-H}}{V_Y(0,\tilde{t}_0)}.$$

In the light of (18) and (20), by the Piterbarg inequality (see Theorem 8.1 in Piterbarg (1996) or Theorem 8.1 in Piterbarg (2001)) for all u sufficiently large

$$p_2(u):=\mathbb{P}\left\{\sup_{\substack{(s,t)\in[0,\rho]\times[\tilde{t}_0-\rho,\tilde{t}_0+xA(u)u^{-1}]\\ s\in[\tilde{\delta}_1(\tilde{u}),\rho]\text{ or } t\in[\tilde{t}_0-\rho,\tilde{t}_0-\tilde{\delta}_2(\tilde{u})]}} Y(s,t)>u^{1-H}\right\}\le C_1\, u^{\frac{2(1-H)}{H}}\exp\left(-\frac{u^{2(1-H)}}{2V_Y^2(0,\tilde{t}_0)}-C_2(\ln u)^2\right) \tag{21}$$

for some positive constants $C_i, i=1,2$, where $\tilde{\delta}_1(\tilde{u})=(\ln\tilde{u}/\tilde{u})^{1/H}, \tilde{\delta}_2(\tilde{u})=\ln\tilde{u}/\tilde{u}$. Further, we have

$$p_3(u):=\mathbb{P}\left\{\sup_{(s,t)\in[0,\tilde{\delta}_1(\tilde{u})]\times[\tilde{t}_0-\tilde{\delta}_2(\tilde{u}),\tilde{t}_0+xA(u)u^{-1}]} Y(s,t)>u^{1-H}\right\}=\mathbb{P}\left\{\sup_{(s,t)\in\widehat{\Delta^1_{Ax}}(\tilde{u})}\frac{Y(s,t)}{V_Y(0,\tilde{t}_0)}>\tilde{u}\right\}, \tag{22}$$

where $\widehat{\Delta^1_{Ax}}(\tilde{u})=[0,\tilde{\delta}_1(\tilde{u})]\times[\tilde{t}_0-\tilde{\delta}_2(\tilde{u}),\tilde{t}_0+Ax\tilde{u}^{-1}]$. Utilizing (18) and (19) we obtain from Theorem 2.1 that

$$\mathbb{P}\left\{\sup_{(s,t)\in\widehat{\Delta^1_{Ax}}(\tilde{u})}\frac{Y(s,t)}{V_Y(0,\tilde{t}_0)}>\tilde{u}\right\} = \mathcal{H}_{2H}\mathcal{P}_{2H}^{\frac{1-\gamma}{\gamma}}2^{-\frac{1}{2H}}\sqrt{2\pi}A\frac{c(1-H)}{H}\Psi(\tilde{u})\tilde{u}^{\frac{1}{H}-1}\Phi(x)(1+o(1)) \tag{23}$$

as $u\to\infty$. Consequently, we conclude from (16)-(17), (21)-(23) that

$$\mathbb{P}\left\{\sup_{0\le s\le t\le\tilde{t}_0+xA(u)u^{-1}} Y(s,t)>u^{1-H}\right\}=\mathcal{H}_{2H}\mathcal{P}_{2H}^{\frac{1-\gamma}{\gamma}}2^{-\frac{1}{2H}}\sqrt{2\pi}A\frac{c(1-H)}{H}\Psi(\tilde{u})\tilde{u}^{\frac{1}{H}-1}\Phi(x)(1+o(1))$$

as $u\to\infty$, and thus in the light of (15)

$$\lim_{u\to\infty}\sup_{x\in\mathbb{R}}\left|\mathbb{P}\left\{\frac{\tau_1(u)-\tilde{t}_0u}{A(u)}\le x\middle|\tau_1(u)<\infty\right\}-\Phi(x)\right|=0.$$

Using similar arguments, we conclude by the properties of the random field Y and (11) that

$$\mathbb{P}\left\{\sup_{t\ge\tilde{t}_0+xA(u)u^{-1},s\in[0,t]} Y(s,t)>u^{1-H}\right\}=\mathcal{H}_{2H}\mathcal{P}_{2H}^{\frac{1-\gamma}{\gamma}}2^{-\frac{1}{2H}}\sqrt{2\pi}A\frac{c(1-H)}{H}\Psi(\tilde{u})\tilde{u}^{\frac{1}{H}-1}\Psi(x)(1+o(1))$$

as $u\to\infty$, where we used the fact that for any large enough integer $K>\tilde{t}_0$

$$\mathbb{P}\left\{\sup_{0\le s\le t<\infty} Y(s,t)>u^{1-H}\right\}=\mathbb{P}\left\{\sup_{0\le s\le t<K} Y(s,t)>u^{1-H}\right\}(1+o(1))\qquad\text{as } u\to\infty,$$

see Hashorva et al. (2013). Therefore

$$\begin{aligned}\mathbb{P}\left\{\frac{\tau_2(u)-\tilde{t}_0u}{A(u)}\le x\middle|\tau_1(u)<\infty\right\} &= 1-\mathbb{P}\left\{\frac{\tau_2(u)-\tilde{t}_0u}{A(u)}\ge x\middle|\tau_1(u)<\infty\right\}\\ &= 1-\mathbb{P}\left\{T_2(u)\ge\tilde{t}_0+xA(u)u^{-1}\middle|T_1(u)<\infty\right\}\\ &= 1-\frac{\mathbb{P}\left\{\sup_{t\ge\tilde{t}_0+xA(u)u^{-1},s\in[0,t]}Y(s,t)>u^{1-H}\right\}}{\mathbb{P}\left\{\tau_1(u)<\infty\right\}}\\ &\to \Phi(x)\qquad\text{as } u\to\infty\end{aligned}$$

for any $x\in\mathbb{R}$. Hence the proof follows by a direct application of Lemma 2.3 below. ☐

Lemma 2.3 *Let $(Z_{u1}, Z_{u2}), u > 0$ be a bivariate random sequence such that $Z_{u2} \ge Z_{u1}$ almost surely for all large u. If the following convergence in distribution*

$$Z_{ui} \stackrel{d}{\to} \mathcal{Z} \qquad as\ u \to \infty$$

holds for $i = 1, 2$ with $\mathcal{Z}$ a non-degenerate random variable, then we have the joint convergence in distribution

$$(Z_{u1}, Z_{u2}) \stackrel{d}{\to} (\mathcal{Z}, \mathcal{Z}) \qquad as\ u \to \infty. \tag{24}$$

Proof: Let x, y be any two continuous points of the distribution function $\mathbb{P}\{\mathcal{Z} \le t\}, t \in \mathbb{R}$. It is sufficient to show that

$$\lim_{u\to\infty} \mathbb{P}\{Z_{u1} \le x, Z_{u2} \le y\} = \mathbb{P}\{\mathcal{Z} \le \min(x, y)\}.$$

In fact, if $x \ge y$ by the assumption that $Z_{u2} \ge Z_{u1}$ holds for all large u we have

$$\mathbb{P}\{Z_{u1} \le x, Z_{u2} \le y\} = \mathbb{P}\{Z_{u2} \le y\} \to \mathbb{P}\{\mathcal{Z} \le y\} \qquad \text{as } u \to \infty.$$

Further, if $x \le y$

$$\begin{aligned}
\mathbb{P}\{Z_{u1} \le x, Z_{u2} \le y\} &= \mathbb{P}\{Z_{u1} \le x\} - \mathbb{P}\{Z_{u1} \le x, Z_{u2} > y\} \\
&\ge \mathbb{P}\{Z_{u1} \le x\} - \mathbb{P}\{Z_{u1} \le y, Z_{u2} > y\} \\
&= \mathbb{P}\{Z_{u1} \le x\} - \Big(\mathbb{P}\{Z_{u2} > y\} - \mathbb{P}\{Z_{u1} > y\}\Big) \\
&\to \mathbb{P}\{\mathcal{Z} \le x\} \qquad \text{as } u \to \infty
\end{aligned}$$

and

$$\mathbb{P}\{Z_{u1} \le x, Z_{u2} \le y\} \le \mathbb{P}\{Z_{u1} \le x\} \to \mathbb{P}\{\mathcal{Z} \le x\} \qquad \text{as } u \to \infty$$

hold, hence the claim follows. □

3 Proof of Theorem 2.1

Proof of Theorem 2.1 We present only the proof of (10) with $x \ge 0$, since the other cases can be dealt with using the same argumentations. For simplicity we shall assume that $a_1 = a_2 = 1$; the general case follows by a time scaling.

Since our approach is asymptotic in natural and that $\delta_1(u)$ and $\delta_2(u)$ both converge to 0 as u tends to infinity, the properties (8) and (9) are the only necessary properties of the Gaussian random field X needed for the asymptotics (which can be seen from the proof below). Therefore, we conclude that

$$\mathbb{P}\left\{\sup_{(s,t)\in\widetilde{\Delta_x^1}(u)} X(s,t) > u\right\} = \mathbb{P}\left\{\sup_{(s,t)\in\widetilde{\Delta_x^1}(u)} \tilde{\xi}(s,t) > u\right\}(1+o(1)) =: \pi(u)(1+o(1)) \qquad \text{as } u \to \infty,$$

with $\{\tilde{\xi}(s,t), s,t\ge 0\}$ any Gaussian random field possessing the properties (8) and (9). Particularly, we set

$$\tilde{\xi}(s,t)=\frac{\xi(s,t)}{(1+b_1s^\beta)(1+b_2|t-t_0|^2+b_3|t-t_0|s)},\quad s,t\ge 0,$$

with $\{\xi(s,t), s,t\ge 0\}$ a centered Gaussian random field with covariance function

$$r_\xi(s,t)=\exp(-s^\beta-t^\beta),\quad s,t\ge 0.$$

Since $\beta<2$, for any positive constants S_1,S_2, we can divide the intervals $[0,\delta_1(u)]$ and $[t_0-\delta_2(u),t_0+xu^{-1}]$ into several sub-intervals of length $S_1u^{-2/\beta}$ and $S_2u^{-2/\beta}$, respectively. Specifically, let for $S_1,S_2>0$

$$\triangle_0^i=u^{-\frac{2}{\beta}}[0,S_i],\quad \triangle_k^i=u^{-\frac{2}{\beta}}[kS_i,(k+1)S_i],\quad k\in\mathbb{Z},\ i=1,2.$$

Let further for any $u>0$

$$h_1(u)=\lfloor S_1^{-1}(\ln u)^{\frac{2}{\beta}}\rfloor+1,\ h_2(u)=\lfloor S_2^{-1}(\ln u)u^{\frac{2}{\beta}-1}\rfloor+1,\ i=1,2,\ h_{2,x}(u)=\lfloor S_2^{-1}xu^{\frac{2}{\beta}-1}\rfloor+1.$$

Here $\lfloor\cdot\rfloor$ denotes the ceiling function. Applying Bonferroni's inequality we obtain

$$\begin{aligned}\pi(u)&\le\sum_{k_1=0}^{h_1(u)}\sum_{k_2=-h_2(u)}^{h_{2,x}(u)}\mathbb{P}\left\{\sup_{(s,t)\in\triangle_{k_1}^1\times(t_0+\triangle_{k_2}^2)}\tilde{\xi}(s,t)>u\right\}\\&=\sum_{k_2=-h_2(u)}^{h_{2,x}(u)}\mathbb{P}\left\{\sup_{(s,t)\in\triangle_0^1\times(t_0+\triangle_{k_2}^2)}\tilde{\xi}(s,t)>u\right\}+\sum_{k_1=1}^{h_1(u)}\sum_{k_2=-h_2(u)}^{h_{2,x}(u)}\mathbb{P}\left\{\sup_{(s,t)\in\triangle_{k_1}^1\times(t_0+\triangle_{k_2}^2)}\tilde{\xi}(s,t)>u\right\}\\&=:I_{1,x}(u)+I_{2,x}(u)\end{aligned}$$

and

$$\begin{aligned}\pi(u)&\ge\sum_{k_2=-h_2(u)+1}^{h_{2,x}(u)-1}\mathbb{P}\left\{\sup_{(s,t)\in\triangle_0^1\times(t_0+\triangle_{k_2}^2)}\tilde{\xi}(s,t)>u\right\}\\&\quad-\sum_{-h_2(u)+1\le i<j\le h_{2,x}(u)-1}\mathbb{P}\left\{\sup_{(s,t)\in\triangle_0^1\times(t_0+\triangle_i^2)}\tilde{\xi}(s,t)>u,\sup_{(s,t)\in\triangle_0^1\times(t_0+\triangle_j^2)}\tilde{\xi}(s,t)>u\right\}\\&=:J_{1,x}(u)-J_{2,x}(u).\end{aligned}$$

Next we derive the required asymptotic bounds of $I_{1,x}(u)$ and $J_{1,x}(u)$, and show that

$$I_{2,x}(u)=J_{2,x}(u)(1+o(1))=o(I_{1,x}(u))=o(J_{1,x}(u))\quad\text{as } u\to\infty,\ S_i\to\infty, i=1,2.\tag{25}$$

Assuming further that $b_3>0$, we have

$$\begin{aligned}J_{1,x}(u)&\ge\sum_{k_2=0}^{h_{2,x}(u)-1}\mathbb{P}\left\{\sup_{(s,t)\in\triangle_0^1\times\triangle_{k_2}^2}\frac{\xi(s,t)}{1+b_1s^\beta}>u(1+b_2((k_2+1)S_2u^{-\frac{2}{\beta}})^2+b_3((k_2+1)S_2u^{-\frac{2}{\beta}})(S_1u^{-\frac{2}{\beta}}))\right\}\\&\quad+\sum_{k_2=-h_2(u)+1}^{-1}\mathbb{P}\left\{\sup_{(s,t)\in\triangle_0^1\times\triangle_{k_2}^2}\frac{\xi(s,t)}{1+b_1s^\beta}>u(1+b_2(-k_2S_2u^{-\frac{2}{\beta}})^2+b_3(-k_2S_2u^{-\frac{2}{\beta}})(S_1u^{-\frac{2}{\beta}}))\right\}\end{aligned}$$

$$=: J_{1,1,x}(u) + J_{1,2,x}(u).$$

In view of Lemma 4.1 in Appendix

$$\begin{aligned}
J_{1,1,x}(u) &= \mathcal{P}_\beta^{b_1}[0,S_1]\mathcal{H}_\beta[0,S_2]\frac{1}{\sqrt{2\pi}u}\sum_{k_2=0}^{h_{2,x}(u)-1}\frac{1}{1+b_2((k_2+1)S_2u^{-\frac{2}{\beta}})^2+b_3((k_2+1)S_2u^{-\frac{2}{\beta}})(S_1u^{-\frac{2}{\beta}})}\\
&\quad\times\exp\left(-\frac{u^2(1+b_2((k_2+1)S_2u^{-\frac{2}{\beta}})^2+b_3((k_2+1)S_2u^{-\frac{2}{\beta}})(S_1u^{-\frac{2}{\beta}}))^2}{2}\right)(1+o(1))\\
&= \mathcal{P}_\beta^{b_1}[0,S_1]\mathcal{H}_\beta[0,S_2]\Psi(u)\\
&\quad\times\sum_{k_2=0}^{h_{2,x}(u)-1}\exp\left(-b_2((k_2+1)S_2u^{1-\frac{2}{\beta}})^2-b_3u^2((k_2+1)S_2u^{-\frac{2}{\beta}})(S_1u^{-\frac{2}{\beta}})\right)(1+o(1))\\
&= \mathcal{P}_\beta^{b_1}[0,S_1]\frac{\mathcal{H}_\beta[0,S_2]}{S_2}\Psi(u)u^{\frac{2}{\beta}-1}\int_0^x e^{-b_2y^2}dy(1+o(1)) \qquad (26)
\end{aligned}$$

as $u\to\infty$, where in the last equation we utilised the facts that

$$h_{2,x}(u)\to\infty,\quad h_{2,x}(u)S_2u^{1-\frac{2}{\beta}}\to x,\quad u^2(h_{2,x}(u)S_2u^{-\frac{2}{\beta}})(S_1u^{-\frac{2}{\beta}})\to 0$$

as $u\to\infty$. Similarly

$$J_{1,2,x}(u) = \mathcal{P}_\beta^{b_1}[0,S_1]\frac{\mathcal{H}_\beta[0,S_2]}{S_2}\Psi(u)u^{\frac{2}{\beta}-1}\int_{-\infty}^0 e^{-b_2y^2}dy(1+o(1)) \qquad (27)$$

as $u\to\infty$. Therefore we conclude that

$$J_{1,x}(u) \geq \mathcal{P}_\beta^{b_1}[0,S_1]\frac{\mathcal{H}_\beta[0,S_2]}{S_2}\Psi(u)u^{\frac{2}{\beta}-1}\int_{-\infty}^x e^{-b_2y^2}dy(1+o(1)) \quad \text{as } u\to\infty. \qquad (28)$$

Using similar arguments we further obtain that

$$\begin{aligned}
I_{1,x}(u) &\leq \sum_{k_2=0}^{h_{2,x}(u)-1}\mathbb{P}\left\{\sup_{(s,t)\in\triangle_0^1\times\triangle_{k_2}^2}\frac{\xi(s,t)}{1+b_1s^\beta}>u(1+b_2(k_2S_2u^{-\frac{2}{\beta}})^2)\right\}\\
&\quad+\sum_{k_2=-h_2(u)}^{-1}\mathbb{P}\left\{\sup_{(s,t)\in\triangle_0^1\times\triangle_{k_2}^2}\frac{\xi(s,t)}{1+b_1s^\beta}>u(1+b_2(-(k_2+1)S_2u^{-\frac{2}{\beta}})^2)\right\}\\
&= \mathcal{P}_\beta^{b_1}[0,S_1]\frac{\mathcal{H}_\beta[0,S_2]}{S_2}\Psi(u)u^{\frac{2}{\beta}-1}\int_{-\infty}^x e^{-b_2y^2}dy(1+o(1)) \qquad (29)
\end{aligned}$$

as $u\to\infty$. Next we verify (25). Specifically

$$\begin{aligned}
I_{2,x}(u) &\leq \sum_{k_1=1}^{h_1(u)}\sum_{k_2=0}^{h_{2,x}(u)}\mathbb{P}\left\{\sup_{(s,t)\in\triangle_{k_1}^1\times\triangle_{k_2}^2}\xi(s,t)>u(1+b_1(k_1S_1u^{-\frac{2}{\beta}})^\beta+b_2(k_2S_2u^{-\frac{2}{\beta}})^2)\right\}\\
&\quad+\sum_{k_1=1}^{h_1(u)}\sum_{k_2=-h_2(u)}^{-1}\mathbb{P}\left\{\sup_{(s,t)\in\triangle_{k_1}^1\times\triangle_{k_2}^2}\xi(s,t)>u(1+b_1(k_1S_1u^{-\frac{2}{\beta}})^\beta+b_2(-(k_2+1)S_2u^{-\frac{2}{\beta}})^2)\right\}.
\end{aligned}$$

Similar argumentations as in (28) yield

$$I_{2,x}(u)\leq\mathcal{H}_\beta[0,S_1]\mathcal{H}_\beta[0,S_2]\Psi(u)(S_2^{-1}u^{\frac{2}{\beta}-1})\int_{-\infty}^x e^{-b_2y^2}dy\sum_{k_1=1}^{h_1(u)}\exp\left(-b_1(k_1S_1)^\beta\right)(1+o(1)) \qquad (30)$$

as $u\to\infty$. Further, we write

$$\begin{aligned} J_{2,x}(u) &= \sum_{-h_2(u)+1\le i<j\le h_{2,x}(u)-1} \mathbb{P}\left\{\sup_{(s,t)\in\triangle_0^1\times(t_0+\triangle_i^2)}\tilde{\xi}(s,t)>u,\ \sup_{(s,t)\in\triangle_0^1\times(t_0+\triangle_j^2)}\tilde{\xi}(s,t)>u\right\}\\ &=: \Sigma_{1,x}(u)+\Sigma_{2,x}(u), \end{aligned}$$

where $\Sigma_{1,x}(u)$ is the sum over indexes $j=i+1$, and $\Sigma_{2,x}(u)$ is the sum over indexes $j>i+1$. Let

$$B(i,S_2,u)=u(1+b_2(|i|S_2u^{-\frac{2}{\beta}})^2),\quad i\in\mathbb{Z},\ S_2>0,\ u>0.$$

It follows that

$$\begin{aligned} \Sigma_{1,x}(u) &\le \sum_{i=-1}^{h_{2,x}(u)-1}\mathbb{P}\left\{\sup_{(s,t)\in\triangle_0^1\times\triangle_i^2}\frac{\xi(s,t)}{1+b_1s^\beta}>B(0,S_2,u),\ \sup_{(s,t)\in\triangle_0^1\times\triangle_{i+1}^2}\frac{\xi(s,t)}{1+b_1s^\beta}>B(0,S_2,u)\right\}\\ &\quad+\sum_{i=-h_2(u)+1}^{-2}\mathbb{P}\left\{\sup_{(s,t)\in\triangle_0^1\times\triangle_i^2}\frac{\xi(s,t)}{1+b_1s^\beta}>B(i+2,S_2,u),\ \sup_{(s,t)\in\triangle_0^1\times\triangle_{i+1}^2}\frac{\xi(s,t)}{1+b_1s^\beta}>B(i+2,S_2,u)\right\} \end{aligned}$$

and, for any $i,j\in\mathbb{Z}$

$$\begin{aligned} &\mathbb{P}\left\{\sup_{(s,t)\in\triangle_0^1\times\triangle_i^2}\frac{\xi(s,t)}{1+b_1s^\beta}>B(j,S_2,u),\ \sup_{(s,t)\in\triangle_0^1\times\triangle_{i+1}^2}\frac{\xi(s,t)}{1+b_1s^\beta}>B(j,S_2,u)\right\}\\ &=\mathbb{P}\left\{\sup_{(s,t)\in\triangle_0^1\times\triangle_0^2}\frac{\xi(s,t)}{1+b_1s^\beta}>B(j,S_2,u)\right\}+\mathbb{P}\left\{\sup_{(s,t)\in\triangle_0^1\times\triangle_1^2}\frac{\xi(s,t)}{1+b_1s^\beta}>B(j,S_2,u)\right\}\\ &-\mathbb{P}\left\{\sup_{(s,t)\in\triangle_0^1\times(\triangle_0^2\cup\triangle_1^2)}\frac{\xi(s,t)}{1+b_1s^\beta}>B(j,S_2,u)\right\}. \end{aligned}$$

Therefore, analogous to the derivation of (28), we obtain

$$\limsup_{u\to\infty}\frac{\Sigma_{1,x}(u)}{\Psi(u)u^{\frac{2}{\beta}-1}} \le \mathcal{P}_\beta^{b_1}[0,S_1]\frac{2\mathcal{H}_\beta[0,S_2]-\mathcal{H}_\beta[0,2S_2]}{S_2}\left(x+\int_{-\infty}^0 e^{-b_2y^2}\,dy\right). \tag{31}$$

Further, for any $u>0$

$$\begin{aligned} \Sigma_{2,x}(u) &\le \sum_{i=-1}^{h_{2,x}(u)-1}\sum_{j\ge2}\mathbb{P}\left\{\sup_{(s,t)\in\triangle_0^1\times\triangle_0^2}\xi(s,t)>u,\ \sup_{(s,t)\in\triangle_0^1\times\triangle_j^2}\xi(s,t)>u\right\}\\ &\quad+\sum_{i=-h_2(u)+1}^{-2}\sum_{j\ge2}\mathbb{P}\left\{\sup_{(s,t)\in\triangle_0^1\times\triangle_0^2}\xi(s,t)>B(i+1,S_2,u),\ \sup_{(s,t)\in\triangle_0^1\times\triangle_j^2}\xi(s,t)>u\right\}\\ &\le \sum_{i=-1}^{h_{2,x}(u)-1}\sum_{j\ge2}\mathbb{P}\left\{\sup_{\substack{(s,t)\in\triangle_0^1\times\triangle_0^2\\(s',t')\in\triangle_0^1\times\triangle_j^2}}\zeta(s,t,s',t')>2u\right\}\\ &\quad+\sum_{i=-h_2(u)+1}^{-2}\sum_{j\ge2}\mathbb{P}\left\{\sup_{\substack{(s,t)\in\triangle_0^1\times\triangle_0^2\\(s',t')\in\triangle_0^1\times\triangle_j^2}}\zeta(s,t,s',t')>B(i+1,S_2,u)+u\right\}, \end{aligned}$$

where

$$\zeta(s,t,s',t')=\xi(s,t)+\xi(s',t'),\quad s,s',t,t'\ge0.$$

It is easy to check that, for u sufficiently large

$$2 \leq \mathbb{E}\left((\zeta(s,t,s',t'))^2\right) = 4 - 2(1 - r(|s-s'|, |t-t'|)) \leq 4 - ((j-1)S_2)^{\beta} u^{-2}$$

for any $(s,t) \in \triangle_0^1 \times \triangle_0^2, (s',t') \in \triangle_0^1 \times \triangle_j^2$. Borrowing the arguments of the proof of Lemma 6.3 in Piterbarg (1996) we conclude that

$$\limsup_{u\to\infty} \frac{\Sigma_{2,x}(u)}{\Psi(u) u^{\frac{2}{\beta}-1}} \quad \leq \quad \mathbb{C}\ x\ (\mathcal{H}_\beta[0,S_1])^2 S_2 \sum_{j\geq 1} \exp\left(-\frac{1}{8}(jS_2)^{\beta}\right) \tag{32}$$

for some positive constant $\mathbb{C}$. Hence the claim follows from (25–32) when $b_3 > 0$ by letting $S_2, S_1 \to \infty$. When $b_3 < 0$, the same results can be obtained using similar arguments as above and the fact that

$$1 - \sigma(s,t) \geq b_1 s^{\beta}(1+o(1)) + \left(b_2 + \frac{b_3}{2}\right) |t - t_0|^2 (1 + o(1))$$

as $(s,t) \to (0, t_0)$ which is utilised for verifying (25), and thus the proof is complete. □

4 Appendix: Piterbarg Lemma for Gaussian Random Fields

In order to find the asymptotics of supremum of centered non-smooth Gaussian processes two crucial results are important, namely the Pickands Lemma and the Piterbarg Lemma. Although for experts in this field the results are well-known, we would like to briefly mention them. Let $\{X(t), t \geq 0\}$ be a centered stationary Gaussian process with a.s. continuous sample paths and correlation function $r(t)$ which satisfies $r(t) = 1 - t^{\alpha}(1 + o(1))$ as $t \to 0$ with $\alpha \in (0,2]$ and $r(t) < 1$ for all $t > 0$. In the seminal paper Pickands (1969) it was shown that for any $T \in (0,\infty)$

$$\mathbb{P}\left\{\sup_{t\in[0,T]} X(t) > u\right\} = \mathcal{H}_\alpha T u^{\frac{2}{\alpha}} \Psi(u)(1 + o(1)) \quad \text{as} \quad u \to \infty. \tag{33}$$

The proof of (33) strongly relies on Pickands Lemma which says that

$$\mathbb{P}\left\{\sup_{t\in[0,u^{-\frac{2}{\alpha}}T]} X(t) > u\right\} = \mathcal{H}_\alpha[0,T]\Psi(u)(1 + o(1)) \quad \text{as} \quad u \to \infty. \tag{34}$$

In the seminal contribution Piterbarg (1972) V.I. Piterbarg rigorously proved (33) and then extended (34) to a result which we refer to as Piterbarg Lemma, namely for any constant $b > 0$

$$\mathbb{P}\left\{\sup_{t\in[0,u^{-\frac{2}{\alpha}}T]} \frac{X(t)}{1 + bt^{\alpha}} > u\right\} = \mathcal{P}_\alpha^b[0,T]\Psi(u)(1 + o(1)) \quad \text{as} \quad u \to \infty.$$

Our next result is a variant of Piterbarg Lemma for two-dimensional case. We omit its proof since it follows with exactly the same arguments as that of Lemma 6.1 in Piterbarg (1996).

Lemma 4.1 *Let $\{\xi(s,t), s,t \geq 0\}$ be a centered Gaussian random field with covariance function*

$$r_\xi(s,t) = \exp(-s^{\alpha_1} - t^{\alpha_2}), \quad s,t \geq 0, \quad \text{with } \alpha_1, \alpha_2 \in (0,2].$$

Let further S, T_1, T_2 be three constants such that $S > 0$ and $T_1 < T_2$. Then, for any constants $b_1 \geq 0, b_2 > 0$, and any positive function $g(u), u \geq 0$ satisfying $\lim_{u\to\infty} g(u)/u = 1$, we have

$$\mathbb{P}\left\{ \sup_{(s,t)\in[0,u^{-\frac{2}{\alpha_1}}S]\times[u^{-\frac{2}{\alpha_2}}T_1, u^{-\frac{2}{\alpha_2}}T_2]} \frac{\xi(s,t)}{(1+b_1 s^{\alpha_1})(1+b_2 t^{\alpha_2})} > g(u) \right\} = \mathcal{P}_{\alpha_1}^{b_1}[0,S]\mathcal{P}_{\alpha_2}^{b_2}[T_1,T_2]\Psi(g(u))(1+o(1)) \quad (35)$$

as $u \to \infty$.

Remark 4.2 *In the last formula we identify $\mathcal{P}_{\alpha_1}^{b_1}[0,S]$ to be $\mathcal{H}_{\alpha_1}[0,S]$ when $b_1 = 0$.*

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