Barcelona School of Economics

WORKING PAPERS

Minimal-Access Rights in School Choice and the Deferred Acceptance Mechanism

BSE Working Paper 1264
June 2021 (Revised: July 2023)

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Minimal-Access Rights in School Choice and the Deferred Acceptance Mechanism*

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July 4, 2023

Abstract

A classical school choice problem consists of a set of schools with priorities over students and a set of students with preferences over schools. Schools' priorities are often based on multiple criteria, e.g., merit-based test scores as well as minimal-access rights (siblings attending the school, students' proximity to the school, etc.). Traditionally, minimal-access rights are incorporated into priorities by always giving minimal-access students higher priority over non-minimal-access students. However, stability based on such adjusted priorities can be considered unfair because a minimal-access student may be admitted to a popular school while another student with higher merit-score but without minimal-access right is rejected, even though the former minimal-access student could easily attend another of her minimal-access schools.

We therefore weaken stability to minimal-access stability: minimal-access rights only promote access to at most one minimal-access school. Apart from minimal-access stability, we also would want a school choice mechanism to satisfy strategy-proofness and minimal-access monotonicity, i.e., additional minimal-access rights for a student do not harm her. Our main result is that the deferred acceptance mechanism is the only mechanism that satisfies minimal-access stability, strategy-proofness, and minimal-access monotonicity. Since this mechanism is in fact stable, our result can be interpreted as an impossibility result: fairer outcomes that are made possible by the weaker property of minimal-access stability are incompatible with strategy-proofness and minimal-access monotonicity.

Keywords: school choice, priorities, minimal-access rights, justified envy, stability, deferred acceptance.

JEL–Numbers: C78; D47; D63; D78.

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*We thank an associate editor, three anonymous reviewers, S. Bonkoungou, and participants of the Lausanne market design workshop for their valuable comments.

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1 Introduction

A classical school choice problem[1] consists of a set of schools that have priorities over students and a set of students who have preferences over schools. Priorities are often determined by various components such as a merit-based component (e.g., entrance exam scores or existing grade point averages) and a normative component (e.g., having a sibling already attending a school, living in walking distance to a school, or public transport accessibility). However, these components are fundamentally different since academic merit applies to all schools equally while aspects due to a sibling attending a school and easy logistics to get to school only apply to some schools. We therefore refer to the first component as "absolute priority" and the second (augmenting) component as "minimal-access rights." Traditionally, a school's final priority ranking over students is such that students who have minimal-access rights are ranked above those who do not have minimal-access rights, and within each of these two groups of students the absolute priority (i.e., the merit-based ranking) applies.

More specifically, if only one minimal-access criterion, e.g., walk-zone accessibility, is considered, then one way to adjust absolute priorities is, at each school, to always give walk-zone students higher priority over non-walk-zone students. However, stability based on such minimal-access adjusted priorities can be criticized as giving students with walk-zone rights at several schools advantages that go beyond granting a minimal-access to a walk-zone school: for example, a walk-zone student may be admitted to a popular school while another student with higher merit-based (absolute) priority but without walk-zone right is rejected, even though the former walk-zone student could easily attend another walk-zone school. Such an outcome, while stable with respect to minimal-access adjusted priorities, might be considered unfair.

This criticism is first mentioned and illustrated by Duddy (2019, page 362), who writes that the priority profile of schools

> "... can fail to capture important aspects of the information from which it is derived. In particular, important information is lost when a student satisfies a priority criterion across multiple schools. This loss of information means that matching mechanisms must treat situations that are substantially different from one another as though they were identical."

Duddy (2019) then offers various examples to illustrate his point of view and suggests a model extension that allows to treat additional priority criteria across multiple schools in a more differentiated way. In addition to multiple types of minimal-access rights (walk-zone rights, siblings-at-a-school rights, etc.), Duddy (2019) considers probabilistic matchings. In contrast, we consider only

[1] See Pathak (2011) and Abdulkadiroğlu (2013) for surveys on mechanism and market design in school choice.

one type of minimal-access criterion, e.g., walk-zone rights, and focus on deterministic matchings.[2] However, we do adopt Duddy's differential treatment of minimal-access rights and weaken the standard notion of stability with respect to minimal-access adjusted priorities to minimal-access stability: minimal-access rights only matter to guarantee access to one minimal-access school (if possible).

To be more precise, stability is classically based on (minimal-access) adjusted priorities, and it requires, in addition to non-wastefulness and individual rationality, the absence of justified envy: student i would justifiably envy student j if she would like to attend student j's school and she has a higher adjusted priority at that school than student j (Balinski and Sönmez, 1999). The interpretation is that minimal-access rights apply across *all* minimal-access schools, which is why we refer to the derived property in our model as no justified *max* envy. If minimal-access rights are interpreted as minimal guarantees, then a situation where student i is matched to a minimal-access school (or better) and envies student j only because of the minimal-access right (that is, student j is ranked higher in merit and has no minimal-access right for his school, while student i does) does no longer justify a complaint; we call the associated notion no justified *min* envy. Using no justified min envy instead of no justified max envy weakens stability to minimal-access stability.[3]

Apart from minimal-access stability, we would want a school choice mechanism to satisfy strategy-proofness, that is, no student can obtain a better match by misrepresenting her preferences. Apart from being a strategic robustness property, strategy-proofness in matching models represents a certain notion of fairness. Former Boston Public Schools superintendent Thomas Payzant (Payzant, 2005),[4] in a memo to the Boston School Committee on May 25, 2005, describes the rationale for switching away from a manipulable school choice mechanism as follows:

> "A strategy-proof algorithm levels the playing field by diminishing the harm done to parents who do not strategize or do not strategize well."

Finally, we introduce a natural monotonicity property for the school choice model with minimal-access rights: minimal-access monotonicity requires that additional minimal-access rights for a student do not harm her. An alternative motivation for minimal-access monotonicity would be that it should not be beneficial for students to hide or renounce some of their minimal-access rights

[2] Since we interpret our main result as an impossibility result to derive matching mechanisms that can in fact accommodate the differential treatment that Duddy (2019) calls for, it suffices to show that impossibility result for a less general model.

[3] Note that our weakening of the classical stability notion is closely linked to the presence of minimal-access rights. Other relaxations of classical stability and alternative justified envy notions have, for instance, recently been studied in Aygün and Bó (2021), Ehlers and Morrill (2020), Tang and Zhang (2021), and Troyan et al. (2020).

[4] A direct on-line reference for this quote does not seem available anymore but we refer, for instance, to Pathak and Sönmez (2008, page 1637).

(Aygün and Bó, 2021, show that in the selection mechanism used for federal universities in Brazil, students can get better assignments by not claiming all their "privileges.").

The deferred acceptance mechanism that is based on adjusted priorities satisfies all the desirable properties discussed above; in fact, it even satisfies the stronger stability property with respect to adjusted priorities. Based on Duddy's (2019) critique, however, a different mechanism, one that can treat minimal-access rights in a more differentiated way, could be desirable. To further explore this line of thought, we first need to answer the question:

> "Which mechanisms satisfy minimal-access stability, strategy-proofness, and minimal-access monotonicity?"

Our answer to this question is perhaps disappointing: apart from the deferred acceptance mechanism that is based on adjusted priorities, there exists no other mechanism that satisfies the three properties (Theorem 1). Hence, it is impossible for a school-choice mechanism to satisfy minimal-access stability, strategy-proofness, and minimal-access monotonicity while treating minimal-access rights in a differentiated way, as demanded by Duddy (2019).

2 Model and main result

A standard school choice problem consists of a population of students and a set of schools. Students are defined by their preferences over schools, and schools are defined by their capacities and priorities over students. Priorities are often determined by various components such as a merit-based component (e.g., entrance exam scores or existing grade point averages) and a normative component (e.g., having a sibling already attending a school, living in walking distance, or public transport accessibility). We refer to the first component as "absolute priority" and the second (augmenting) component as "minimal-access rights" (see our discussion at the beginning of the Introduction).

We define an **(extended school choice) problem** as a sextuple $(I, S, q, P, \succ, r)$ with

- a finite set $\boldsymbol{I}$ of **students**;
- a finite set $\boldsymbol{S}$ of **schools**;
- a list of **capacities** $\boldsymbol{q} \equiv (q_s)_{s \in S}$ where for each $s \in S$, $q_s \in \mathbb{N}$;
- a list of strict **preferences** $\boldsymbol{P} \equiv (P_i)_{i \in I}$ over $S \cup \{\emptyset\}$, where $\emptyset$ represents the "no-school option";

- a list of strict **(absolute) priority relations** $\succ \equiv (\succ_s)_{s \in S}$ over I; and

- a list of **minimal-access rights** $\boldsymbol{r} \equiv (r(i))_{i \in I}$ where for each $i \in I$, $r(i) \subseteq S$.

For each $i \in I$, we call $r(i)$ student i's **minimal-access schools**. For each $s \in S$, let $r(s) \equiv \{i \in I : s \in r(i)\}$. Let $\mathcal{P}_i$ denote the set of possible preferences of student i. Let $P_i \in \mathcal{P}_i$ and $s, s' \in S \cup \{\emptyset\}$. We write $s \, R_i \, s'$ if $s \, P_i \, s'$ or $s = s'$. A school $s \in S$ is **acceptable** for student i if $s \, P_i \, \emptyset$. In the sequel, since the set of students and schools and the schools' capacities remain fixed, a problem is more compactly denoted by $(P, \succ, r)$. Note that the only difference to a *"classical" school choice problem* $(P, \succ)$ is the separation of priorities into absolute priorities and minimal-access rights. Aygün and Bó (2021) present a model for Brazilian university admissions that is close to our extended school choice model with exam grades taking the role of absolute priorities and "privileges" that can be strategically used, or not, to gain admission.

A **matching** is a mapping $\mu : I \cup S \to 2^I \cup S$ such that (i) for each $i \in I$, $\mu(i) \in S$ or $\mu(i) = \emptyset$, (ii) for each $s \in S$, $\mu(s) \subseteq I$ and $|\mu(s)| \leq q_s$, and (iii) for each $(i, s) \in I \times S$, $\mu(i) = s$ if and only if $i \in \mu(s)$. For each $i \in I$, $\mu(i)$ is student i's **match**, i.e., the school or no-school option to which the student is matched. Similarly, for each $s \in S$, $\mu(s)$ is school s's **match**, i.e., the students to which the school is matched.

Matching μ is **individually rational** if for all $i \in I$, $\mu(i) \, R_i \, \emptyset$.

Matching μ is **non-wasteful** if for all $i \in I$ and all $s \in S$, $s \, P_i \, \mu(i)$ implies $|\mu(s)| = q_s$.

Student $i \in I$ has **justified max envy** at matching μ if there is a student $j \in I$ and a school $s \in S$ such that $\mu(j) = s \, P_i \, \mu(i)$ and

(1) $s \notin r(i)$, $s \notin r(j)$, and $i \succ_s j$; or

(2) $s \in r(i)$, $s \in r(j)$, and $i \succ_s j$; or

(3) $s \in r(i)$ and $s \notin r(j)$.

Matching μ is **stable** if it is individually rational, non-wasteful, and no student has justified max envy.

Remark 1 **(Stability and adjusted priorities).**
A student has justified max envy at a matching μ with respect to $(P, \succ, r)$ if and only if she has justified envy (Abdulkadiroğlu and Sönmez, 2003) at μ with respect to $(P, \succ^r)$ where $\boldsymbol{\succ^r} \equiv (\succ_s^r)_{s \in S}$ are the **adjusted priorities**: for each $s \in S$, the priority relation $\succ_s^r$ is such that

(1) for all $i, j \notin r(s)$, $i \succ_s^r j$ if and only if $i \succ_s j$;

(2) for all $i, j \in r(s)$, $i \succ_s^r j$ if and only if $i \succ_s j$; and

(3) for all $i \in r(s)$ and all $j \notin r(s)$, $i \succ_s^r j$.

Therefore, a matching is stable with respect to $(P, \succ, r)$ if and only if it is "classically" stable with respect to $(P, \succ^r)$, i.e., as in college admissions (see, e.g., Balinski and Sönmez, 1999, page 79). $\diamond$

Remark 2 (Stability and schools' responsive adjusted priority preferences).
By assuming that schools have priorities over all students, together with our stability notion, we implicitly assume that each school finds all students acceptable and has responsive adjusted priority preferences over sets of students. More precisely, school $s \in S$ with capacity q_s and adjusted priority relation $\succ_s^r$ compares sets of students as follows. Let $2^I_{q_s}$ denote the set of all subsets of I that do not exceed the capacity q_s, i.e., $2^I_{q_s} \equiv \{I' \subseteq I : |I'| \leq q_s\}$. Let P_s^r denote an *adjusted priority preference relation* on $2^I_{q_s}$, i.e., P_s^r strictly orders all sets in $2^I_{q_s}$. Then, P_s^r is responsive to $\succ_s^r$ if the following two conditions hold:

(a) for all $I' \in 2^I_{q_s}$ with $|I'| < q_s$ and all $i \in I \backslash I'$, $(I' \cup \{i\})\; P_s^r\; I'$ and

(b) for all $I' \in 2^I_{q_s}$ with $|I'| < q_s$ and all $i, j \in I \backslash I'$, $(I' \cup \{i\})\; P_s^r\; (I' \cup \{j\})$ if and only if $i \succ_s^r j$.

When formulating (a), we implicitly assume that each school finds all students acceptable. Note that a model extension by allowing schools to find some students unacceptable while still having responsive adjusted priority preference relations would not change our results (but require additional notation in order to adjust individual rationality and stability when unacceptable students are concerned). $\diamond$

The set of stable matchings is non-empty. A stable matching can be obtained by adapting Gale and Shapley's (1962) **(student-proposing) deferred acceptance algorithm** (see also Roth, 2008) to the context of extended school choice problems as follows. Let $(P, \succ, r)$ be a problem.

Step 0. Using $\succ$ and r, compute $\succ^r$.

Step 1. Each student i proposes to the acceptable school she most prefers or the no-school option (according to P_i). Among all proposers, up to its capacity, each school s tentatively assigns its seats to students who have highest priority according to $\succ_s^r$ and rejects all other proposers. The no-school option accepts all proposals it receives.

Steps 2, ... Each student i who was rejected at the previous step proposes to her next most preferred acceptable school or the no-school option (according to P_i). Each school s considers the proposers she tentatively accepted (if any) and all current proposers. Among these students, up to its capacity, school s tentatively assigns its seats to the students who have highest priority according to $\succ_s^r$ and rejects all other proposers. The no-school option accepts all proposals it receives.

The algorithm stops when each student is either tentatively matched or has been rejected by all her acceptable schools; we denote the resulting matching by $\gamma(P, \succ, r)$. Alternatively, when focusing on the deferred acceptance outcome for the classical school choice problem $(P, \succ^r)$, we use the equivalent notation $\gamma(P, \succ^r)$.

It follows from Gale and Shapley (1962) (see also Abdulkadiroğlu and Sönmez, 2003, Proposition 1) that the resulting matching is stable with respect to $(P, \succ^r)$. Hence, by Remark 1, the (student-proposing) deferred acceptance algorithm yields a matching that is stable with respect to $(P, \succ, r)$. Moreover, the resulting matching is student-optimal in the sense that all students weakly prefer it to any other stable matching.

Stability and no justified max envy are key properties when allocating school seats to students and both notions crucially depend on how priorities of students are adjusted to minimal-access rights. In particular, when using adjusted priorities, a student who has minimal-access rights for several schools obtains higher priority for *all* these schools. Duddy (2019) points out that stability based on these adjusted priorities can be considered unfair because instead of just guaranteeing minimal-access rights to *one* of these schools, it could create unfair situations where a minimal-access student with low grades is admitted to a popular school while a student with higher grades is rejected in spite of the fact that the minimal-access student could easily have been admitted to another (but potentially less preferred) minimal-access school. In fact, as soon as a school (or even the no-school option) that is at least as good as a minimal-access school is offered to a student, one could consider a claim to be assigned to a better school based on minimal-access rights as unjustifiable. In other words, when using adjusted priorities, a student may receive more minimal-access rights than needed to guarantee a minimal-access school welfare level.

In order to take the above criticism into account, we introduce the following stricter envy concept that declares envy due to minimal-access rights unjustified if the student is already matched to a minimal-access school or one that is at least as good as a minimal-access school. Student $i \in I$ has **justified min envy**[5] at matching μ if there is a student $j \in I$ and a school $s \in S$ such that $\mu(j) = s\ P_i\ \mu(i)$ and

[5]Our definition of justified min envy is based on Duddy's (2019) notion of strongly justified envy.

(1) $s \notin r(i)$, $s \notin r(j)$, and $i \succ_s j$; or

(2) $s \in r(i)$, $s \in r(j)$, and $i \succ_s j$; or

(3a) $s \in r(i)$, $s \notin r(j)$, and $i \succ_s j$; or

(3b) $s \in r(i)$, $s \notin r(j)$, $j \succ_s i$, and there is no school $s' \in S$ with $s' \in r(i)$ and $\mu(i)\ R_i\ s'$.

The only (but important) difference with justified max envy lies in condition (3b); i.e., the only situation of justified envy of agent i for school s that is assigned to agent j when agent i has lower priority than agent j happens when agent i has a minimal-access right for s, agent j does not, and all of agent i's minimal-access right schools are better than her matched school. A matching is **minimal-access stable** if it is individually rational, non-wasteful, and no student has justified min envy. Since it is harder to achieve justified min envy than justified max envy, the set of minimal-access stable matchings contains the set of stable matchings. We illustrate the difference between stable and minimal-access stable matchings in the following example. In particular, the example demonstrates that two classical results for the set of stable matchings cannot be extended to the set of minimal-access stable matchings: neither does the set of minimal-access stable matchings form a distributive lattice under the classical definitions of meet and join given by Conway (Blair, 1988; Knuth, 1976) nor does it permit a "Rural Hospitals Theorem."[6]

Example 1 (Stability versus minimal-access stability: cardinality of matched students). Consider the extended school choice problem $(I, S, q, P, \succ, r)$ where $I = \{1, 2, 3\}$ and $S = \{A, B, C\}$ such that for each $s \in S$, $q_s = 1$ and where preferences P and priorities $\succ$ are given in Table 1. More precisely, student 1 finds all schools acceptable with $A\ P_1\ B\ P_1\ C\ P_1\ \emptyset$; for student 2 only school A is acceptable; and for student 3 only school C is acceptable. The minimal-access rights are given by $r(1) = \{A, C\}$, $r(2) = \emptyset$, and $r(3) = \{C\}$ and the resulting adjusted priorities $\succ^r$ are also given in Table 1.

[6]A basic version of the Rural Hospitals Theorem (Roth, 1984, Theorem 9) in the school choice context states that the set of filled school seats is the same across all stable matchings, as is the set of students who are assigned seats. Thus, the number of students assigned to the no-school option does not vary across stable matchings. The first versions of the theorem appear in Gale and Sotomayor (1985), Roth (1984), and Roth (1986).

P_1	P_2	P_3		$\succ_A$	$\succ_B$	$\succ_C$		$\succ^r_A$	$\succ^r_B$	$\succ^r_C$
$\boldsymbol{A}$	$\boxed{A}$	$\boxed{\boldsymbol{C}}$		3	3	3		1	3	3
$\boxed{B}$				2	2	2		3	2	1
C				1	1	1		2	1	2

Table 1: Students' preferences P, priorities $\succ$, and adjusted priorities $\succ^r$ where $r(1) = \{A, C\}$, $r(2) = \emptyset$, and $r(3) = \{C\}$ (Example 1).

By applying the two versions[7] of the deferred acceptance algorithm, one immediately verifies that the unique stable matching is

$$\mu^* : \begin{array}{ccc} 1 & 2 & 3 \\ | & | & | \\ A & \emptyset & C \end{array}$$

which is the boldfaced matching in Table 1. The stable matching μ^* is by definition also minimal-access stable. However, there is exactly one other minimal-access stable matching, namely

$$\mu : \begin{array}{ccc} 1 & 2 & 3 \\ | & | & | \\ B & A & C \end{array}$$

which is the boxed matching in Table 1. To see that μ is minimal-access stable, first note that only student 1 does not get her most preferred school. Second, the only school that student 1 prefers to her match is school A. Note that $2 \in \mu(A)$, $1 \in r(A)$, $2 \notin r(A)$, and $2 \succ_A 1$. However, student 1 does not have justified min envy because $\mu(1) = B \; P_1 \; C$ and $C \in r(1)$. One easily verifies that apart from μ^* and μ there is no other minimal-access stable matching.

Since students 1 and 2 both most prefer school A and $\mu^*(1) = A = \mu(2)$, there does not exist a minimal-access stable matching that is unanimously most preferred by all students. In particular, the set of minimal-access stable matchings does not form a distributive lattice under the classical definitions of meet and join given by Conway (Blair, 1988; Knuth, 1976).

Finally, note that different minimal-access stable matchings may have different numbers of students assigned to the no-school option. So, the set of minimal-access stable matchings does not permit a "Rural Hospitals Theorem." Interestingly, at the unique stable matching μ^* one student is assigned to the no-school option, while at the only other minimal-access stable matching, no student is assigned to the no-school option. $\diamond$

[7] The second version of the deferred acceptance algorithm, the school-proposing deferred acceptance algorithm (see Subsection 3.1), is obtained by switching the roles of students and schools (i.e., proposers and receivers) and yields the stable matching that is student-pessimal, i.e., all students weakly prefer any other stable matching.

Our next example shows that a minimal-access stable matching can Pareto dominate the unique stable matching.

Example 2 (Stability versus minimal-access stability: Pareto domination). Consider the extended school choice problem $(I, S, q, P, \succ, r)$ where $I = \{1, 2, 3\}$ and $S = \{A, B, C\}$ such that for each $s \in S$, $q_s = 1$ and where preferences P and priorities $\succ$ are given in Table 2. More precisely, students find all schools acceptable with preferences given in Table 2. The minimal-access rights are given by $r(1) = r(2) = \{A, C\}$ and $r(3) = \emptyset$ and the resulting adjusted priorities $\succ^r$ are also given in Table 2.

P_1	P_2	P_3	$\succ_A$	$\succ_B$	$\succ_C$	$\succ^r_A$	$\succ^r_B$	$\succ^r_C$
$\boxed{B}$	A	$\boxed{A}$	3	3	3	1	3	1
$\boldsymbol{A}$	B	$\boldsymbol{B}$	1	1	1	2	1	2
C	$\boxed{\boldsymbol{C}}$	C	2	2	2	3	2	3

Table 2: Students' preferences P, priorities $\succ$, and adjusted priorities $\succ^r$ where $r(1) = r(2) = \{A, C\}$ and $r(3) = \emptyset$ (Example 2).

By applying the two versions of the deferred acceptance algorithm (see Footnote 7), one immediately verifies that the unique stable matching is

$$\mu^* : \begin{array}{ccc} 1 & 2 & 3 \\ | & | & | \\ A & C & B \end{array}$$

which is the boldfaced matching in Table 2. The stable matching μ^* is by definition also minimal-access stable. However, there is exactly one other minimal-access stable matching, namely

$$\mu : \begin{array}{ccc} 1 & 2 & 3 \\ | & | & | \\ B & C & A \end{array}$$

which is the boxed matching in Table 2. To see that μ is minimal-access stable, first note that only student 2 does not get her most preferred school. Second, student 2 prefers school B to her match and $1 \in \mu(B)$, $2 \notin r(B)$, $1 \notin r(B)$, but $1 \succ_B 2$. Third, student 2 prefers school A to her match and $3 \in \mu(A)$, $2 \in r(A)$, $3 \notin r(A)$, and $3 \succ_A 2$, but student 2 does not have justified min envy because $\mu(2) = C$ and $C \in r(2)$. One easily verifies that apart from μ^* and μ there is no other minimal-access stable matching.

Since students 1 and 3 both get their most preferred schools at μ and student 2 gets the same school at both μ and μ^*, the minimal-access stable matching μ Pareto dominates the stable matching μ^*. $\diamond$

Examples 1 and 2 illustrate how choosing a minimal-access stable matching that is different from a stable matching could be attractive because, in some situations, more students could be matched or a Pareto improvement could be implemented.

A mechanism φ is a function that selects for each problem $(P, \succ, r)$ a matching $\varphi(P, \succ, r)$. For each student i, $\varphi_i(P, \succ, r)$ denotes the school to which the student is assigned by φ. Mechanism $\boldsymbol{\varphi}$ is **individually rational / non-wasteful / (minimal-access) stable** if for each problem $(P, \succ, r)$, $\varphi(P, \succ, r)$ is individually rational / non-wasteful / (minimal-access) stable.

A mechanism is minimal-access monotonic if for each student an expansion of her minimal-access rights induces the mechanism to assign her to a weakly more preferred school. Formally, mechanism $\boldsymbol{\varphi}$ is **minimal-access monotonic** if for each student i and for each pair of problems $(P, \succ, r)$ and $(P, \succ, r')$ with $r'(i) \subseteq r(i)$ and for each student $j \neq i$, $r'(j) = r(j)$, we have $\varphi_i(P, \succ, r) \; R_i \; \varphi_i(P, \succ, r')$. Assuming students can renounce / hide minimal-access rights, a mechanism is minimal-access monotonic if whenever a student renounces / hides some of her minimal-access rights, the mechanism assigns her to a weakly less preferred school. Put differently, it is always optimal for a student to not hide any of her minimal-access rights (Aygün and Bó, 2021, demonstrate for Brazilian federal university admissions that students often have a better chance at admission if they hide privileges). Thus, a minimal-access monotonic mechanism is strategically simple and hence levels the playing field. In the context of classical exchange economies, Postlewaite (1979) is the first to introduce and study hiding-proofness and destruction-proofness with respect to individual endowments; minimal-access monotonicity is a natural version of these properties in our model.

The well-known non-manipulability property strategy-proofness requires that no student can ever benefit from misrepresenting her preferences. Formally, mechanism $\boldsymbol{\varphi}$ is **strategy-proof** if for each problem $(P, \succ, r)$, for each student i, and for all preferences $P'_i \in \mathcal{P}_i$, $\varphi_i(P, \succ, r) \; R_i \; \varphi_i(P'_i, P_{-i}, \succ, r)$ where $P_{-i} \equiv (P_j)_{j \neq i}$.

The mechanism that always assigns the matching obtained by the (student-proposing) deferred acceptance algorithm based on adjusted priorities is called **(minimal-access adjusted) deferred acceptance mechanism** and denoted by $\boldsymbol{\gamma}$. As mentioned earlier, γ is stable and hence also minimal-access stable. The following lemma also shows that γ inherits strategy-proofness from the deferred acceptance mechanism in the standard setting. Finally, the lemma shows that γ is minimal-access monotonic; this follows from the deferred acceptance mechanism in the standard setting "respecting improvements" (Balinski and Sönmez, 1999)— the intuition being that since more minimal-access rights for a student improve her position in the priority ranking of some schools, chances to be matched to a more desirable school throughout the deferred acceptance algorithm increase.

Lemma 1 (**Properties of the deferred acceptance mechanism**).
The deferred acceptance mechanism γ is stable, strategy-proof, and minimal-access monotonic.

Proof. We only have to prove strategy-proofness and minimal-access monotonicity. Let $(P, \succ, r)$ be a problem. Let $i \in I$ and $P'_i \in \mathcal{P}_i$. Recall that, with a slight abuse of notation, γ also denotes the deferred acceptance mechanism in the standard setting (defined for problems of the form $(P, \succ^r)$ instead of $(P, \succ, r)$). Then, it follows from Dubins and Freedman (1981) and Roth (1982) that[8] $\gamma_i(P, \succ, r) = \gamma_i(P, \succ^r)\, R_i\, \gamma_i(P'_i, P_{-i}, \succ^r) = \gamma_i(P'_i, P_{-i}, \succ, r)$. Hence, γ is strategy-proof.

Let $i \in I$ and let $(P, \succ, r)$ and $(P, \succ, r')$ be two problems with $r'(i) \subseteq r(i)$ and for each student $j \neq i$, $r'(j) = r(j)$. By Remark 1, r and r' induce adjusted priorities $\succ^r$ and $\succ^{r'}$ such that for each $j \in I$ and each $s \in S$, $i \succ_s^{r'} j$ implies $i \succ_s^r j$. Hence, $\succ^r$ is a so-called improvement of $\succ^{r'}$ for student i and, from Balinski and Sönmez (1999, Theorem 5), it follows that $\gamma_i(P, \succ, r) = \gamma_i(P, \succ^r)\ R_i\ \gamma_i(P, \succ^{r'}) = \gamma_i(P, \succ, r')$. □

The following example demonstrates that picking another minimal-access stable matching than the matching obtained by γ can lead to a violation of both strategy-proofness and minimal-access monotonicity.

Example 3 (**A minimal-access stable mechanism that is neither strategy-proof nor minimal-access monotonic**). Consider again the extended school choice problem of Example 1. Let φ be a minimal-access stable mechanism such that $\varphi(P, \succ, r) = \mu$. We show first that φ is not strategy-proof. Let P'_1 be the preference relation with $C\ P'_1\ A\ P'_1\ B\ P'_1\ \emptyset$. The unique minimal-access stable matching at profile $P' \equiv (P'_1, P_2, P_3)$ is μ^*. To see this, note first that at any minimal-access stable matching, student 3 is assigned to school C, and second that μ is not minimal-access stable because student 1 has justified min envy with respect to student 2 and school A. Hence, $\varphi_1(P', \succ, r) = A\, P_1\, B = \varphi_1(P, \succ, r)$ and φ is not strategy-proof.

Next, we show that φ is not minimal-access monotonic. Consider the minimal-access rights r' where $r'(1) = \{A\} \subsetneq \{A, C\} = r(1)$, $r'(2) = r(2) = \emptyset$, and $r'(3) = r(3) = \{C\}$. Then, by the same arguments as before, $\varphi(P, \succ, r') = \mu^*$. Hence, $\varphi_1(P, \succ, r') = \mu^*(1) = A\, P_1\, B = \varphi_1(P, \succ, r)$ and φ is not minimal-access monotonic. ◇

[8]See also Abdulkadiroğlu and Sönmez (2003, Proposition 2).

The following theorem shows that the fact that minimal-access stable mechanism φ in Example 3 fails to satisfy strategy-proofness and walk-zone monotonicity is not a coincidence: the only mechanism satisfying all three properties is the deferred acceptance mechanism γ.

Theorem 1 (Characterization).
A mechanism φ is minimal-access stable, strategy-proof, and minimal-access monotonic if and only if $\varphi = \gamma$.

Theorem 1 demonstrates that apart from the deferred acceptance mechanism that is based on adjusted priorities, there exists no other mechanism that satisfies the three normatively appealing properties we have considered. Hence, it is impossible for a school-choice mechanism to satisfy minimal-access stability, strategy-proofness, and minimal-access monotonicity while treating minimal-access rights in a differentiated way, as demanded by Duddy (2019).

Proof sketch of Theorem 1. From Lemma 1 it follows that the deferred acceptance mechanism satisfies minimal-access stability, strategy-proofness, and minimal-access monotonicity. Next, we explain the uniqueness part of the proof; the full proof that there is no other mechanism that satisfies the three properties is delegated to Appendix A.

Let φ satisfy minimal-access stability, strategy-proofness, and minimal-access monotonicity. Suppose $\varphi \neq \gamma$. Then, there exists a problem $(P, \succ, r)$ such that $\varphi(P, \succ, r) \neq \gamma(P, \succ, r)$.

First, we show that it is without loss of generality to assume that problem $(P, \succ, r)$ is such that (t.a) φ and γ assign different matchings and (t.b) each student who receives a different match, finds only one school acceptable (see Transformation Claim, Claim 1, in Appendix A).

Second, since φ and γ are individually rational, we can partition the set of students at $(P, \succ, r)$ by using the following types (see Type Claim, Claim 2, in Appendix A). Types 1 and 2 are students with different matches under φ and γ who find exactly one school acceptable: type 1 students find only the school they are matched to under γ acceptable and are matched to $\emptyset$ under φ; type 2 students find only the school they are matched to under φ acceptable and are matched to $\emptyset$ under γ. Type 3 students are matched equally under φ and γ.

Third, we prove that while $\varphi(P, \succ, r)$ is minimal-access stable, it cannot be stable due to the Rural Hospitals Theorem and the assumption that $\varphi(P, \succ, r) \neq \gamma(P, \succ, r)$. Hence,

(e.1.) there exists some student with justified max envy at $\varphi(P, \succ, r)$ and

(e.2.) any justified max envy at $\varphi(P, \succ, r)$ is not justified min envy.

Fourth, we show that students with justified max envy cannot be of Type 2. Hence, from (e.1) there exists a student i of Type 1 or Type 3 that has justified max envy.

Fifth, if student i is of Type 1, then we can strictly reduce his minimal-access rights (possibly followed by a transformation of preferences à la Transformation Claim, Claim 1) to obtain a problem that satisfies again (t.a) and (t.b). If, on the other hand, student i is of Type 3, then we can identify a student of Type 1, say k, with justified max envy and carry out a reduction of student k's minimal-access rights (possibly followed by a transformation of preferences à la Transformation Claim, Claim 1) to obtain a problem that satisfies again (t.a) and (t.b).

Thus, starting from a problem that satisfies (t.a) and (t.b) we obtain each time a new problem that satisfies again (t.a) and (t.b) but with a strictly smaller number of minimal-access rights for some Type 1 student (while maintaining the minimal-access rights of the other students). However, since the number of students is finite, the total (finite) number of minimal-access rights cannot be strictly decreased perpetually. Thus, we obtain a contradiction and conclude that $\varphi = \gamma$. □

In the standard setting, the unique stable mechanism that satisfies strategy-proofness is the deferred acceptance mechanism (see, e.g., Roth and Sotomayor, 1990, Theorem 4.6). Together with Lemma 1, this implies that in our setting the unique stable mechanism that satisfies strategy-proofness is the deferred acceptance mechanism. However, the deferred acceptance mechanism is not the unique stable mechanism that satisfies minimal-access monotonicity. A mechanism we present as an independence example for strategy-proofness (Example 4) satisfies minimal-access monotonicity and minimal-access stability (in fact, it satisfies stability).

Before discussing the independence of the properties that characterize the deferred acceptance mechanism in Theorem 1 (see Section 4), we would like to explore what happens to other well-known mechanisms in the presence of minimal-access rights.

3 Other well-known mechanisms and minimal-access rights

Using the adjusted priorities $\succ^r$ we can adapt three more well-known mechanisms: the school-proposing deferred acceptance mechanism, the immediate acceptance mechanism, and the top trading cycles mechanism.

3.1 The school-proposing deferred acceptance mechanism

We adapt the classical **school-proposing deferred acceptance algorithm** to our model with minimal-access rights. Let $(P, \succ, r)$ be a problem.

Step 0. Using $\succ$ and r, compute $\succ^r$.

Step 1. Each school s proposes to the students with highest priority (under $\succ_s^r$), up to its capacity. The no-school option proposes to all students. Among all proposers, each student i tentatively accepts the most preferred acceptable school or the no-school option (according to P_i) and rejects all other proposers.

Steps 2, ... For each student who rejected school s at the previous step, school s proposes to the next highest priority student (according to $\succ_s^r$) to whom it has not yet made a proposal. The no-school option proposes to all students. Each student i considers the proposer she tentatively accepted (if any) and all current proposers. Among these proposers, student i tentatively accepts the most preferred acceptable school or the no-school option (according to P_i) and rejects all other proposers.

The algorithm stops when students no longer reject proposals. The mechanism that always assigns the matching obtained by the school-proposing deferred acceptance algorithm based on adjusted priorities is called **(minimal-access adjusted) school-proposing deferred acceptance mechanism** and we denote it by γ^S. It is well-known that γ^S is stable but not strategy-proof. We now show that γ^S is not minimal-access monotonic. Consider the extended school choice problem $(I, S, q, P, \succ, r)$ where $I = \{1, 2\}$ and $S = \{A, B\}$ such that for each $s \in S$, $q_s = 1$ and $2 \succ_s 1$. Students' preferences are given by Table 3. The minimal-access rights are given by $r(1) = \{A\}$ and $r(2) = \emptyset$.

P_1	P_2	$\succ_A^r$	$\succ_B^r$	$\succ_A^{r'}$	$\succ_B^{r'}$
$\boldsymbol{B}$	$\boldsymbol{A}$	1	2	2	2
$\boxed{A}$	$\boxed{B}$	2	1	1	1

Table 3: Students' preferences P and adjusted priorities $\succ^r$ and $\succ^{r'}$ (Subsection 3.1).

One immediately verifies that $\gamma^S(P, \succ, r)$ is the boxed matching in Table 3. Let r' be the minimal-access rights defined by $r'(1) = r'(2) = \emptyset$. Then, $\gamma^S(P, \succ, r')$ is the bold-faced matching in Table 3. Since $\gamma_1^S(P, \succ, r') = B \, P_1 \, A = \gamma_1^S(P, \succ, r)$, γ^S is not minimal-access monotonic.

3.2 The immediate acceptance mechanism

We adapt the classical **immediate acceptance algorithm** to our model with minimal-access rights. Let $(P, \succ, r)$ be a problem.

Step 0. Using $\succ$ and r, compute $\succ^r$.

Step 1. Each student i proposes to the acceptable school she most prefers or the no-school option (according to P_i). Among all proposers, up to its capacity, each school s assigns its seats to the students who have highest priority according to $\succ_s^r$ and rejects all other proposals. The no-school option accepts all proposals it receives.

Steps 2, ... Each student i who was rejected at the previous step proposes to her next most preferred acceptable school or the no-school option (according to P_i). Among all proposers, up to its capacity, each school s assigns its remaining seats (if any) to the students who have highest priority according to $\succ_s^r$ and rejects all other proposals. The no-school option accepts all proposals it receives.

The algorithm stops when each student is either matched or has been rejected by all her acceptable schools. The mechanism that always assigns the matching obtained by the immediate acceptance algorithm based on adjusted priorities is called **(minimal-access adjusted) immediate acceptance mechanism**. It follows from Abdulkadiroğlu and Sönmez (2003) that the immediate acceptance mechanism is neither strategy-proof nor stable. Note that the immediate acceptance mechanism is not stable "even if" students have no minimal-access rights. Since in this case, stability equals minimal-access stability, the immediate acceptance mechanism is also not minimal-access stable. However, since more minimal-access rights for a student improve her position in the priority ranking of some schools, chances to be matched to a more desirable school earlier in the immediate acceptance algorithm increase and thus the immediate acceptance mechanism satisfies minimal-access monotonicity.

3.3 The top trading cycles mechanism

Inspired by David Gale's **top trading cycle algorithm**, Abdulkadiroğlu and Sönmez (2003) introduced the so-called top trading cycles mechanism, which we adapt to our model with minimal-access rights. Let $(P, \succ, r)$ be a problem.

Step 0. Using $\succ$ and r, compute $\succ^r$.

Step 1. Each student i points to the acceptable school she most prefers or the no-school option (according to P_i). The no-school option points to all students and each school s points to the student who has highest priority according to $\succ_s^r$. There is at least one cycle. Each student in a cycle is assigned to the school (or the no-school option) she points to and she is removed. The capacity of each school (but not the no-school option) that is in a cycle is reduced by 1. If the capacity of a school is now 0, then the school is removed (the no-school option is not removed).

Steps 2, ... Each remaining student i points to the school she most prefers among the remaining schools or the no-school option (according to P_i). The no-school option points to all students and each remaining school s points to the student who has highest priority according to $\succ_s^r$ among all remaining students. There is at least one cycle. Each student in a cycle is assigned to the school (or the no-school option) she points to and she is removed. The capacity of each school (but not the no-school option) that is in a cycle is reduced by 1. If the capacity of a school is now 0, then the school is removed (the no-school option is not removed).

The algorithm stops when each student has been removed and matched to a school or the no-school option. The mechanism that always assigns the matching obtained by the top trading cycles algorithm based on adjusted priorities is called **(minimal-access adjusted) top trading cycles mechanism**. It follows from Abdulkadiroğlu and Sönmez (2003) that the top trading cycles mechanism is strategy-proof but not stable. Note that the top trading cycles mechanism is not stable "even if" students have no minimal-access rights. Since in this case, stability equals minimal-access stability, the top trading cycles mechanism is also not minimal-access stable. Furthermore, since more minimal-access rights for a student improve her position in the priority ranking of some schools, chances to form a trading cycle that leads to matching with a more desirable school earlier in the top trading cycles algorithm increase and thus the top trading cycles mechanism satisfies minimal-access monotonicity (the formal proof is relegated to Appendix B).

4 Independence of properties in Theorem 1

The following three mechanisms show that the three properties in Theorem 1 are logically unrelated. We label the following independence examples by the property that is *not* satisfied.

Example 4 (Strategy-proofness). We define a mechanism $\bar{\gamma}$ as follows. For each $(P, \succ, r)$,

$$\bar{\gamma}(P, \succ, r) \equiv \begin{cases} \gamma(P, \succ, r) & \text{if for some } k \in I,\ r(k) \neq \emptyset; \\ \text{any}^9 \text{ stable matching at } (P, \succ, r) & \text{if for each } k \in I,\ r(k) = \emptyset. \end{cases}$$

By definition, $\bar{\gamma}$ is stable and $\bar{\gamma} \neq \gamma$. We now show that $\bar{\gamma}$ is also minimal-access monotonic. Let $i \in I$ and $(P, \succ, r)$ and let $(P, \succ, r')$ be two problems with $r'(i) \subsetneq r(i)$ and for each student $j \neq i$, $r'(j) = r(j)$. Then,

$$\bar{\gamma}_i(P, \succ, r) = \gamma_i(P, \succ, r)\ R_i\ \gamma_i(P, \succ, r')\ R_i\ \bar{\gamma}_i(P, \succ, r').$$

[9] To guarantee $\bar{\gamma} \neq \gamma$, one has to pick some stable matching different from $\gamma(P, \succ, r)$ for some problem $(P, \succ, r)$.

The first equality follows from the definition of $\bar{\gamma}$. The first R_i-comparison follows from minimal-access monotonicity of γ (Lemma 1). The second R_i-comparison follows from the student-optimality of the stable matching $\gamma(P, \succ, r')$ at problem $(P, \succ, r')$, i.e., all students weakly prefer stable matching $\gamma(P, \succ, r')$ to any other stable matching at problem $(P, \succ, r')$. Hence, $\bar{\gamma}$ is minimal-access monotonic.

To see that $\bar{\gamma}$ is not strategy-proof, let $(P, \succ, r)$ be a problem with $\bar{\gamma}(P, \succ, r) \neq \gamma(P, \succ, r)$. Then, there is a student $i \in I$ who can state (truncation) preferences P_i' that are obtained from her preferences P_i by declaring all schools less preferred than $\gamma_i(P, \succ, r)$ as unacceptable while keeping all other acceptable schools in the same order. Thus, $\bar{\gamma}_i(P_i', P_{-i}, \succ, r) = \gamma_i(P, \succ, r)\ P_i\ \bar{\gamma}_i(P, \succ, r)$. Therefore, student i is better off by misrepresenting her preferences. Hence, $\bar{\gamma}$ is not strategy-proof. $\diamond$

The above example shows, in particular, that, apart from the deferred acceptance mechanism, there do exist other stable mechanisms that satisfy minimal-access monotonicity.

Example 5 (Minimal-access stability). The top trading cycles mechanism (Subsection 3.3) satisfies strategy-proofness, minimal-access monotonicity (see Appendix B), but not minimal-access stability (see Subsection 3.3). $\diamond$

Example 6 (Minimal-access monotonicity). We define a mechanism $\tilde{\gamma}$ as follows. In the particular situation where all schools have the same (particular) priority order, all students but the lowest priority student have no minimal-access rights, and the lowest priority student has at least 2 minimal-access rights, we apply the associated serial dictatorship with a small twist: as soon as there is only one minimal-access seat left, the (student-proposing) deferred acceptance algorithm is applied. In all other situations, the deferred acceptance mechanism is applied directly. Formally, let $(I, S, q, P, \succ, r)$ be a problem. Let $I = \{1, 2, \ldots, n\}$. We distinguish between two cases.

Case 1: For each $s \in S$, $1 \succ_s 2 \succ_s \cdots \succ_s n$ and for each $i \in I \backslash \{n\}$, $r(i) = \emptyset$ and $|r(n)| > 1$.

For each $s \in S$, let $q_s(0) \equiv q_s$. The following procedure outputs a matching.

Begin Procedure

Step 1. Student 1 is assigned to her most preferred school or the no-school option (according to P_1), say s_1^*. If $s_1^* \in S$, then $q_{s^*}(1) \equiv q_{s^*}(0) - 1$ and for each $s \in S \backslash \{s_1^*\}$, $q_s(1) \equiv q_s(0)$. If $s_1^* = \emptyset$, then for each $s \in S$, $q_s(1) \equiv q_s(0)$. Go to Step 2.

Step $i > 1$.

(a) If $i < n$ and $\sum_{\tilde{s}\in r(n)} q_{\tilde{s}}(i-1) > 1$, then student i is assigned to her most preferred school or the no-school option (according to P_i), say s_i^*, among the schools in the set $\{s \in S : q_s(i-1) > 0\}$. If $s_i^* \in S$, then $q_{s^*}(i) \equiv q_{s^*}(i-1)-1$ and for each $s \in S\backslash\{s_i^*\}$, $q_s(i) \equiv q_s(i-1)$. If $s_i^* = \emptyset$, then for each $s \in S$, $q_s(i) \equiv q_s(i-1)$. Go to the next step.

(b) If $i < n$ and $\sum_{\tilde{s}\in r(n)} q_{\tilde{s}}(i-1) = 1$, then students $\{i, \ldots, n\}$ are matched to the remaining seats of the schools in $\{s \in S : q_s(i-1) > 0\}$ and the no-school option by applying the (school-proposing) deferred acceptance algorithm (with adjusted priorities based on agent n's last remaining minimal-access school). The procedure ends.

(c) If $i = n$, then student n is assigned to her most preferred school or the no-school option (according to P_n), say s_n^*, among the schools in the set $\{s \in S : q_s(n-1) > 0\}$. The procedure ends.

End Procedure

Let $\tilde{\gamma}(P, \succ, r)$ be the matching that is obtained by the above procedure. For later convenience, we refer to steps i(a) and i(c) in the procedure ($i = 1$, [$i < n$ and $\sum_{\tilde{s}\in r(n)} q_{\tilde{s}}(i-1) > 1$], and $i = n$) as the "serial dictatorship (SD) steps." Step i(b) in the procedure is referred to as the DA step.

Case 2: Otherwise. In this case, the mechanism coincides with the deferred acceptance mechanism, i.e., $\tilde{\gamma}(P, \succ, r) \equiv \gamma(P, \succ, r)$.

It is easy to see that mechanism $\tilde{\gamma}$ is strategy-proof. In Case 2 this follows immediately from strategy-proofness of the deferred acceptance mechanism. In Case 1 this is due to the SD steps (in particular, by misstating her preferences, no student can change the set of schools that is available to her) and strategy-proofness of the deferred acceptance mechanism.

Mechanism $\tilde{\gamma}$ is also minimal-access stable. This is obvious in Case 2 since the deferred acceptance mechanism always yields a stable matching. We now show that in Case 1 mechanism $\tilde{\gamma}$ always yields a minimal-access stable matching. Let $\mu = \tilde{\gamma}(P, \succ, r)$. First, since the no-school option is available at each SD and DA step, μ is individually rational. Second, μ is non-wasteful because (i) at the SD steps, unoccupied seats are always available and (ii) the deferred acceptance mechanism is non-wasteful. Third, there is no justified min envy:

(a) At each SD step $i < n$, student i does not have justified min envy with respect to any student $j \in I\backslash\{i\}$ because $j \succ_{\mu(j)} i$ or $\mu(i)\ R_i\ \mu(j)$.

(b) At DA step i, no student in $k \in \{i, \ldots, n-1\}$ has justified min envy with respect to any student $j \in \{1, \ldots, i-1\}$ because $j \succ_{\mu(j)} k$. Student n does not have min envy with respect to any student $j \in \{1, \ldots, i-1\}$ since the deferred acceptance mechanism is minimal-access stable and $\sum_{\tilde{s} \in r(n)} q_{\tilde{s}}(i-1) = 1$. Finally, since the deferred acceptance mechanism is minimal-access stable, no student in $k \in \{i, \ldots, n\}$ has justified min envy with respect to any other student $j \in \{i, \ldots, n\} \backslash \{k\}$.

(c) At SD step n, student n does not have justified min envy with respect to any other student. To see this, note that at this step, student n is assigned to her most preferred school or the no-school option (according to P_n) among the schools in the set $\{s \in S : q_s(n-1) > 0\}$. Since $\sum_{\tilde{s} \in r(n)} q_{\tilde{s}}(n-1) \geq 1$, there is a minimal-access school $\tilde{s} \in r(n)$ with $\mu(n) \, R_n \, \tilde{s}$.

Finally, mechanism $\tilde{\gamma}$ is not minimal-access monotonic. For an illustration, consider the extended school choice problem $(I, S, q, P, \succ, r)$ where $I = \{1, 2\}$ and $S = \{A, B\}$ such that for each $s \in S$, $q_s = 1$ and $1 \succ_s 2$. Students' preferences are given by Table 4. The minimal-access rights are given by $r(1) = \emptyset$ and $r(2) = \{A, B\}$.

P_1	P_2
$\boxed{A}$	A
B	$\boxed{B}$

Table 4: Students' preferences P (Example 6).

One immediately verifies that $\tilde{\gamma}(P, \succ, r)$ is the boxed matching in Table 4. Let r' be the minimal-access rights defined by $r'(1) = r(1) = \emptyset$ and $r'(2) = \{A\} \subsetneq \{A, B\} = r(2)$. Then, $\tilde{\gamma}(P, \succ, r')$ is the encircled matching in Table 4. Since $\tilde{\gamma}_2(P, \succ, r') = A \, P_2 \, B = \tilde{\gamma}_2(P, \succ, r)$, $\tilde{\gamma}$ is not minimal-access monotonic. $\diamond$

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A Appendix: Proof of the uniqueness part of Theorem 1

Proof of Theorem 1: uniqueness of $\boldsymbol{\gamma}$. We prove that γ is the only mechanism that satisfies minimal-access stability, strategy-proofness, and minimal-access monotonicity.

Let φ satisfy minimal-access stability, strategy-proofness, and minimal-access monotonicity and suppose $\varphi \neq \gamma$. Then, there exists a problem $(P, \succ, r)$ such that $\varphi(P, \succ, r) \neq \gamma(P, \succ, r)$. We first show that problem $(P, \succ, r)$ can be transformed into a problem $(\bar{P}, \succ, r)$ such that (t.a) φ and γ assign different matchings and (t.b) each student who receives a different match, finds only one school acceptable.

Claim 1 (**Transformation Claim**). *Let $(P, \succ, r)$ such that $\varphi(P, \succ, r) \neq \gamma(P, \succ, r)$. Then, there is a preference profile $\bar{P}$ such that*

(t.a.) $\varphi(\bar{P}, \succ, r) \neq \gamma(\bar{P}, \succ, r)$ *and*

(t.b.) *for each student ℓ with $\varphi_\ell(\bar{P}, \succ, r) \neq \gamma_\ell(\bar{P}, \succ, r)$, only one school is acceptable under $\bar{P}_\ell$.*

Proof of the Transformation Claim. Let $\mu \equiv \varphi(P, \succ, r)$ and $\mu^* \equiv \gamma(P, \succ, r)$.

Let ℓ be a student with $\mu(\ell) \neq \mu^*(\ell)$. Since φ and γ are individually rational, $\mu(\ell)\ R_\ell\ \emptyset$ and $\mu^*(\ell)\ R_\ell\ \emptyset$. Therefore, at least one school is acceptable under P_ℓ.

Next, assume that ℓ is a student such that $\mu(\ell) \neq \mu^*(\ell)$ and who finds at least two different schools acceptable under P_ℓ. We distinguish between two cases.

Case 1. $\mu^*(\ell)\, P_\ell\, \mu(\ell)$. Let $\bar{P}_\ell$ be a preference relation where $\mu^*(\ell)$ is the only acceptable school. Let $\bar{P} \equiv (\bar{P}_\ell, P_{-\ell})$, $\bar{\mu} \equiv \varphi(\bar{P}, \succ, r)$, and $\bar{\mu}^* \equiv \gamma(\bar{P}, \succ, r)$. Since γ is strategy-proof, $\bar{\mu}^*(\ell) = \mu^*(\ell)$. Since φ is individually rational, $\bar{\mu}(\ell) \in \{\emptyset, \mu^*(\ell)\}$. By strategy-proofness of φ, $\bar{\mu}(\ell) \neq \mu^*(\ell)$. Hence, $\bar{\mu}(\ell) = \emptyset$. Moreover, since $\mu^*(\ell)\, P_\ell\, \mu(\ell)\, R_\ell\, \emptyset$, $\bar{\mu}^*(\ell) \neq \emptyset = \bar{\mu}(\ell)$.

Case 2. $\mu(\ell)\, P_\ell\, \mu^*(\ell)$. Let $\bar{P}_\ell$ be a preference relation where $\mu(\ell)$ is the only acceptable school. Let $\bar{P} \equiv (\bar{P}_\ell, P_{-\ell})$, $\bar{\mu} \equiv \varphi(\bar{P}, \succ, r)$, and $\bar{\mu}^* \equiv \gamma(\bar{P}, \succ, r)$. Since φ is strategy-proof, $\bar{\mu}(\ell) = \mu(\ell)$. Since γ is individually rational, $\bar{\mu}^*(\ell) \in \{\emptyset, \mu(\ell)\}$. By strategy-proofness of γ, $\bar{\mu}^*(\ell) \neq \mu(\ell)$. Hence, $\bar{\mu}^*(\ell) = \emptyset$. Moreover, since $\mu(\ell)\, P_\ell\, \mu^*(\ell)\, R_\ell\, \emptyset$, $\bar{\mu}(\ell) \neq \emptyset = \bar{\mu}^*(\ell)$.

Note that relative to P, $\bar{P} = (\bar{P}_\ell, P_{-\ell})$ only involves a transformation of student ℓ's preferences. After the transformation, student ℓ finds only one school acceptable. Hence, we reduced the number of students who find more than one school acceptable by one student. Moreover, $\varphi_\ell(\bar{P}, \succ, r) \neq \gamma_\ell(\bar{P}, \succ, r)$. In particular, $\varphi(\bar{P}, \succ, r) \neq \gamma(\bar{P}, \succ, r)$.

Thus, if at problem $(P, \succ, r)$ there are students who have different matches at φ and γ and who find at least two different schools acceptable, then we can, student by student, transform $(P, \succ, r)$ until we obtain a preference profile such that (t.a) and (t.b) are satisfied. □

Based on the Transformation Claim (Claim 1), we now can assume without loss of generality that

$$\text{problem } (P, \succ, r) \text{ satisfies (t.a) and (t.b).} \tag{1}$$

Since φ and γ are individual rational, we can assign the following types to students. Types 1 and 2 are students with different matches under φ and γ who find exactly one school acceptable: Type 1 students find only the school they are matched to under γ acceptable and are matched to $\emptyset$ under φ; Type 2 students find only the school they are matched to under φ acceptable and are matched to $\emptyset$ under γ. Type 3 students are matched equally under φ and γ.

Claim 2 (Type Claim). *Each student ℓ is exactly of one of the following types:*

Type 1: *$\gamma_\ell(P, \succ, r) \in S$ is the only acceptable school under P_ℓ and $\varphi_\ell(P, \succ, r) = \emptyset$.*

Type 2: *$\varphi_\ell(P, \succ, r) \in S$ is the only acceptable school under P_ℓ and $\gamma_\ell(P, \succ, r) = \emptyset$.*

Type 3: *$\varphi_\ell(P, \succ, r) = \gamma_\ell(P, \succ, r)\, R_\ell\, \emptyset$.*

To summarize, we consider $(P, \succ, r)$ satisfying (t.1) and (t.2) such that $\varphi(P, \succ, r) \neq \gamma(P, \succ, r)$ and the Transformation Claim (Claim 1) and Type Claim (Claim 2) hold. We now derive a contradiction.

Start. Let $\mu \equiv \varphi(P, \succ, r)$ and $\mu^* \equiv \gamma(P, \succ, r)$.

Suppose μ is stable at $(P, \succ, r)$. Then, since μ^* is also stable at $(P, \succ, r)$, it follows from Remark 1 and the Rural Hospitals Theorem (see Footnote 6) that any student who is matched (to a school) at μ is also matched (to a school) at μ^*, and vice versa. Hence, all students are of Type 3. However, this contradicts (t.a). So, μ is not stable at $(P, \succ, r)$.

Since φ is minimal-access stable, μ is minimal-access stable at $(P, \succ, r)$. Since μ is not stable at $(P, \succ, r)$ this means that

(e.1.) there exists some student i with justified max envy at μ with respect to some student $j \in I \backslash \{i\}$ and some school $s \in S$ and

(e.2.) any justified max envy at μ is not justified min envy.

Then, $\mu(j) = s \, P_i \, \mu(i)$, $s \in r(i)$, $s \notin r(j)$, $j \succ_s i$, and there is a school $s' \in S$ with $s' \in r(i)$ and $\mu(i) \, R_i \, s'$. Thus,

$$|r(i)| > 1. \tag{2}$$

Since $s \, P_i \, \mu(i)$, student i is not of Type 2. Thus, by the Type Claim (Claim 2), it suffices to distinguish between the two cases where student i is of Type 1 or 3.

Case T1. Student i is of Type 1. Thus, $\mu^*(i)$ is the only acceptable school under P_i and $\mu(i) = \emptyset$. Then, it follows from $s \, P_i \, \mu(i)$ that s is the only acceptable school under P_i, i.e., $\mu^*(i) = s$.

Minimal-access rights reduction step. Let r' be the minimal-access rights defined by $r'(i) \equiv \{s\}$ and for each $\ell \in I \backslash \{i\}$, $r'(\ell) \equiv r(\ell)$. So for school s, the set of students with minimal-access right did not change, i.e., $r'(s) = r(s)$, and for all other schools $S \backslash \{s\}$, the only possible change is that student i lost her minimal-access right, i.e., for each school $\tilde{s} \in S \backslash \{s\}$, $r'(\tilde{s}) = r(\tilde{s}) \backslash \{i\}$.

Let $\nu \equiv \varphi(P, \succ, r')$ and $\nu^* \equiv \gamma(P, \succ, r')$. Since φ is minimal-access monotonic, $\emptyset = \mu(i) \, R_i \, \nu(i)$. Hence, by individual rationality, $\nu(i) = \emptyset$. Recall that $\mu^*(i) = s$ is the only acceptable school under P_i and that under $\succ^{r'}$, student i retained his minimal-access right at school $\mu^*(i) = s$ while other students' minimal-access rights did not change. Hence, for problems $(P, \succ, r)$ and $(P, \succ, r')$, student i applies to school s at Step 1 of the deferred acceptance algorithm. Since at school s, students' priorities did not change from $\succ^r$ to $\succ^{r'}$ and all students $I \setminus \{i\}$ apply to the same schools in the consecutive steps of the deferred acceptance algorithm, the resulting matchings for problems $(P, \succ, r)$ and $(P, \succ, r')$ are the same; particularly, $\nu^*(i) = \mu^*(i) = s$. In particular, $\nu^*(i) \neq \nu(i)$.

Note that relative to $(P, \succ, r)$, at $(P, \succ, r')$ agent i's matches under φ and γ did not change but other agents' matches might have changed. However, since $\varphi(P, \succ, r') = \nu \neq \nu^* = \gamma(P, \succ, r')$, we can again apply the Transformation Claim (Claim 1) to obtain a problem $(\bar{P}, \succ, r')$ that satisfies (t.a) and (t.b) (possibly no transformation is required) and, in addition, has **reduced minimal-access rights** such that

$$1 = |r'(i)| < |r(i)|.$$

Then, return to the **Start** (first line after the Type Claim, Claim 2). Note that inequality (2) cannot apply to an agent with only one minimal-access school, so agent i, who featured as justified max envy agent in this case, cannot feature as a justified max envy agent later on.

Case T3. Student i is of Type 3. Thus, $\mu(i) = \mu^*(i)\ R_i\ \emptyset$. Since $\mu(j) = s \neq \emptyset$, student j is not of Type 1. Suppose student j is of Type 3. Then, $\mu^*(j) = \mu(j) = s$. But then, since student i has justified max envy with respect to student j and school s at matching μ, student i has justified max envy with respect to student j and school s at matching μ^* as well. Since this contradicts the stability of μ^* at $(P, \succ, r)$, student j is not of Type 3.

So, by the Type Claim (Claim 2), student j is of Type 2. Then, $s = \mu(j)$ is the only acceptable school for student j at P_j and $\mu^*(j) = \emptyset$. Since γ is non-wasteful, there is a student $k \in I$ such that $\mu^*(k) = s$ and $\mu(k) \neq s$. Obviously, student k is of Type 1. In particular, $\mu(k) = \emptyset$. Since μ^* is stable at $(P, \succ, r)$, it follows from $\mu^*(k) = s\, P_j\, \emptyset = \mu^*(j)$ and Remark 1 that $k \succ_s^r j$. Hence, at μ, student k has justified max envy with respect to student j and school s. So, as in Case T1, there is a student of Type 1 that has justified max envy at μ. Hence, we can apply the **minimal-access rights reduction step** of Case T1 (with student k in the role of student i), followed again by a transformation of the preferences (possibly applying the Transformation Claim, Claim 1) to obtain a problem $(\bar{P}, \succ, r')$ that satisfies (t.a) and (t.b) and, in addition, has **reduced minimal-access rights** such that

$$1 = |r'(k)| < |r(k)|.$$

Then, return to the **Start** (first line after the Type Claim, Claim 2). Recall that inequality (2) cannot apply to an agent with only one minimal-access school, so agent k, who featured as justified max envy agent in this case, cannot feature as a justified max envy agent later on.

Starting from problem $(P, \succ, r)$, Cases T1 and T3 explain how to obtain a new problem $(\bar{P}, \succ, r')$ with unequal matchings under φ and γ by strictly reducing the minimal-access rights of a Type 1 student (and possibly the transformation of preferences). Problem $(\bar{P}, \succ, r')$ is then used as new input at the **Start** (first line after the Type Claim, Claim 2). Hence, for $(\bar{P}, \succ, r')$ there exists an agent with justified max envy and, by inequality (2), minimal access rights for more than one school. Thus, the strict reduction step of the minimal-access rights of a Type 1 student is repeated. However, since the number of students is finite, the total (finite) number of minimal-access rights cannot be strictly decreased perpetually. Thus, we obtain a contradiction. Therefore, $\varphi = \gamma$. $\square$

B Appendix: Proof that the top trading cycles mechanism is minimal-access monotonic

For classical school choice problems, Proposition 9 in the supplementary material of Hatfield et al. (2016) implies that the top trading cycles mechanism is minimal-access monotonic in our setting of extended school choice problems. However, for completeness, below we provide a direct proof.

We first consider the "unit setting" where each school has 1 seat, i.e., for each $s \in S$, $q_s = 1$. Let τ denote the top trading cycles mechanism.

Let $i \in I$. Let $(P, \succ, r)$ and $(P, \succ, r')$ be two problems such that $r'(i) \subseteq r(i)$ and for each $j \in I \backslash \{i\}$, $r'(j) = r(j)$. With a slight abuse of notation we write $\tau(\succ)$ for $\tau(P, \succ, r)$ and $\tau(\succ')$ for $\tau(P, \succ, r')$. Step $t \geq 1$ in the top trading cycles algorithm applied to $(P, \succ, r)$ and $(P, \succ, r')$ is referred to as step t of $\tau(\succ)$ and $\tau(\succ')$, respectively. In addition, let t_i and t_i' denote the step of $\tau(\succ)$ and $\tau(\succ')$ at which student i is assigned to a school (or the no-school option $\emptyset$), respectively.

For each $t \in \{1, \ldots, t_i\}$, let $A(\succ, t)$ denote the set of agents (students and schools)[10] that are present at step t of $\tau(\succ)$. Similarly, for each $t' \in \{1, \ldots, t_i'\}$, let $A(\succ', t')$ denote the set of agents (students and schools) that are present at step t' of $\tau(\succ')$. Finally, for each $t \in \{1, \ldots, t_i\}$, let $P(i, \succ, t)$ denote the set of predecessors of student i at step t of $\tau(\succ)$, i.e., the agents (students and schools) from which there is a path (that does not involve $\emptyset$) to student i.[11]. For convenience, we always exclude student i and (obviously also) the no-school option $\emptyset$ from the set of predecessors, i.e., $i, \emptyset \notin P(i, \succ, t)$. In particular, it is possible that $P(i, \succ, t) = \emptyset$, i.e., no school points to student i at step t of $\tau(\succ)$. Finally, by a cycle C we here refer to the set of agents (students and schools) *and the no-school option* $\emptyset$ involved in a "pointing" (top trading) cycle.[12]

The following proposition shows that the top trading cycles mechanism is minimal-access monotonic.

Proposition 1.

$$\tau_i(\succ) \; R_i \; \tau_i(\succ'). \tag{3}$$

[10] The no-school option $\emptyset$ is not considered an agent.

[11] Here, each directed edge in the path refers to the "pointing" as described in the top trading cycles algorithm (see Subsection 3.3).

[12] Note that whenever the no-school option $\emptyset$ is in a top trading cycle the cycle is trivial in the sense that it only contains one student (and no school).

Proof of Proposition 1. If $t_i = 1$, then $\tau_i(\succ)$ is student i's most preferred school (or, if all schools are unacceptable, $\emptyset$), in which case (3) holds trivially. Let $t_i > 1$. Assume that (3) does not hold, i.e.,

$$\tau_i(\succ') \, P_i \, \tau_i(\succ). \tag{4}$$

We first prove the following lemma by induction.

Lemma 2. *For each step $t \in \{1, \ldots, t_i - 1\}$,*

(A) *$t < t'_i$;*

(B) *if C is a cycle at step t of $\tau(\succ')$, then*

 (1) *C is a cycle at step t of $\tau(\succ)$ or*

 (2) *$C \subseteq P(i, \succ, t) \cup \{\emptyset\}$;*

(C) *if C is a cycle at step t of $\tau(\succ)$, then C is a cycle at step t of $\tau(\succ')$.*

Proof of Lemma 2.

Induction basis. Let $t = 1$. We first prove (A). Suppose $t \geq t'_i$. Then, $t'_i = 1$. So, i is in a cycle at step 1 of $\tau(\succ')$. But then i is also in a cycle at step 1 of $\tau(\succ)$, which contradicts $t_i > 1$. Hence, $t < t'_i$. This proves (A).

Next, we prove (B). Let C be a cycle at step 1 of $\tau(\succ')$. From (A) it follows that $i \notin C$. Suppose C is not a cycle at step 1 of $\tau(\succ)$. Then, $C \subseteq I \cup S$ and there is a non-empty set of schools $S^* \subseteq S \cap C$ that point to student i at step 1 of $\tau(\succ)$, and each of the other agents in $C \backslash S^*$ points to the same agent at step 1 of $\tau(\succ)$ and $\tau(\succ')$. Hence, $C \subseteq P(i, \succ, t)$. This proves (B).

Finally, we prove (C). Let C be a cycle at step 1 of $\tau(\succ)$. Since $1 < t_i$, $i \notin C$. Hence, C is a cycle at step 1 of $\tau(\succ')$, which proves (C).

Induction hypothesis. Suppose (A), (B), and (C) hold for each step $1, \ldots, t-1$ with $t < t_i$ $(t - 1 < t_i - 1)$.

Induction step. We prove that (A), (B), and (C) also hold for step t. Note first that, at each step of the top trading cycles algorithm, the only agents that are removed from the problem are the agents that are part of a cycle. Hence, by the induction hypothesis ((B) and (C) for steps $1, \ldots, t-1$) it follows that

(a) $A(\succ, t) \supseteq A(\succ', t)$ and

(b) $A(\succ, t) \backslash A(\succ', t) \subseteq P(i, \succ, t)$.

We first prove (A) for step t. Since $t - 1 < t < t_i$, it follows from (A) for step $t-1$ that $t - 1 < t'_i$. So, $t \leq t'_i$. Suppose $t = t'_i$. Then, i is in a cycle, say C, at step t of $\tau(\succ')$.

We claim that $C \subseteq P(i, \succ, t) \cup \{i, \emptyset\}$. This is obviously true if C is a trivial cycle (consisting of i and $\emptyset$ only).[13] Suppose C is a non-trivial cycle. Then, $\emptyset \notin C$. Since $C \subseteq A(\succ', t)$, by (a), $C \subseteq A(\succ, t)$. However, since $t < t_i$, C is not a cycle at step t of $\tau(\succ)$. Note that if agent i is the only agent in C that points to different objects in cycle C and at step t of $\tau(\succ)$, then $C \subseteq P(i, \succ, t) \cup \{i, \emptyset\}$ follows immediately. Now let $a^* \neq i$ be an agent in C that points to different objects in cycle C and at step t of $\tau(\succ)$, say b' in cycle C and $b \neq b'$ at step t of $\tau(\succ)$. Since $\emptyset \notin C$, $b' \neq \emptyset$. Since $b' \in C \subseteq A(\succ, t)$, agent a^* points to b either because of a minimal-access right at step t of $\tau(\succ)$ ($b = i$) or because b has a higher priority than / is preferred to b' but b is not present at step t of $\tau(\succ')$, i.e., it follows that $b \in \{i\} \cup [A(\succ, t) \backslash A(\succ', t)]$. If $b = i$, then $a^* \in P(i, \succ, t)$. Suppose $b \neq i$. Since by (b), $A(\succ, t) \backslash A(\succ', t) \subseteq P(i, \succ, t)$, we have $b \in P(i, \succ, t)$. Hence, by definition of $P(i, \succ, t)$, $a^* \in P(i, \succ, t)$ as well. This shows that $C \subseteq P(i, \succ, t) \cup \{i, \emptyset\}$.[14]

Now note that, since $t < t_i$, for each $s \in [S \cap P(i, \succ, t)] \cup \{\emptyset\}$, $\tau_i(\succ)\ R_i\ s$. Since $\tau_i(\succ') \in [S \cap C] \cup \{\emptyset\} \subseteq [S \cap P(i, \succ, t)] \cup \{\emptyset\}$, we have $\tau_i(\succ)\ R_i\ \tau_i(\succ')$, which contradicts assumption (4) we made at the beginning of the proof of Proposition 1. Hence, $t < t'_i$ and (A) for step t holds.

Next, we prove (B) for step t. Let C be a cycle at step t of $\tau(\succ')$. From (A) for step t it follows that $i \notin C$. Suppose C is not a cycle at step t of $\tau(\succ)$. Then, it suffices to show that $C \subseteq P(i, \succ, t) \cup \{\emptyset\}$.

Suppose C is a trivial cycle. Then, C consists of some student $j^* \neq i$ and $\emptyset$ only. Then, by (a), $j^* \in A(\succ', t) \subseteq A(\succ, t)$. However, C is not a cycle at step t of $\tau(\succ)$. Hence, j^* points to different objects in cycle C and at step t of $\tau(\succ)$. Then, j^* points to some school $s^* \in S$ at step t of $\tau(\succ)$. In particular, $s^* \in A(\succ, t) \backslash A(\succ', t)$. Since by (b), $A(\succ, t) \backslash A(\succ', t) \subseteq P(i, \succ, t)$, we have $s^* \in P(i, \succ, t)$. Hence, by definition of $P(i, \succ, t)$, $j^* \in P(i, \succ, t)$ as well. This shows that $C \subseteq P(i, \succ, t) \cup \{\emptyset\}$.

Now suppose C is a non-trivial cycle. Then, $\emptyset \notin C$. Since $C \subseteq A(\succ', t)$, by (a), $C \subseteq A(\succ, t)$. However, by assumption, C is not a cycle at step t of $\tau(\succ)$. Let a^* be an agent in C that points to different objects in cycle C and at step t of $\tau(\succ)$, say b' in cycle C and $b \neq b'$ at step t of $\tau(\succ)$. Since $\emptyset \notin C$, $b' \neq \emptyset$. Since $b' \in C \subseteq A(\succ, t)$, agent a^* points to b either because of a minimal-access right at step t of $\tau(\succ)$ ($b = i$) or because b has a higher priority than / is preferred to b' but b is not present at step t of $\tau(\succ')$, i.e., it follows that $b \in \{i\} \cup [A(\succ, t) \backslash A(\succ', t)]$. If $b = i$, then $a^* \in P(i, \succ, t)$. Suppose $b \neq i$. Since by (b), $A(\succ, t) \backslash A(\succ', t) \subseteq P(i, \succ, t)$, we have $b \in P(i, \succ, t)$.

[13] Here a cycle that contains multiple instances of $\emptyset$, i.e., $i_1 \to \emptyset \to i_2 \to \emptyset \cdots \to i_p \to \emptyset \to i_1$ is interpreted as p trivial cycles.

[14] It was sufficient to consider agents $a^* \neq i$ in C that point to different objects in cycle C and at step t of $\tau(\succ)$. To see this, let agent $a \neq i$ be an agent that points to the *same* agent in C and at step t of $\tau(\succ)$. Then, there is a path at step t of $\tau(\succ)$ from a to i or to an agent a^* in C that points to different objects in C and at step t of $\tau(\succ)$. Since we have shown that $a^* \in P(i, \succ, t)$, it immediately follows that $a \in P(i, \succ, t)$ as well.

Hence, by definition of $P(i, \succ, t)$, $a^* \in P(i, \succ, t)$ as well. This shows that $C \subseteq P(i, \succ, t)$.[15] This completes the proof of (B) for step t.

Finally, we prove (C). Let C be a cycle at step t of $\tau(\succ)$. Since $t < t_i$, $i \notin C$. So, $C \subseteq A(\succ, t) \cup \{\emptyset\}$ and $C \cap P(i, \succ, t) = \emptyset$. Since by (b), $A(\succ, t) \backslash A(\succ', t) \subseteq P(i, \succ, t)$, we have $C \subseteq A(\succ', t) \cup \{\emptyset\}$. Since by (a), $A(\succ', t) \subseteq A(\succ, t)$ and $i \notin C$, C is also a cycle at step t of $\tau(\succ')$. This proves (C) and completes the proof of Lemma 2. $\square$

With the result of Lemma 2 we can now complete the proof of Proposition 1.

From Lemma 2 (A) for $t = t_i - 1$ it follows that $t_i \leq t_i'$. Lemma 2 (B) and (C) for $t = 1, \ldots, t_i - 1$ implies that $A(\succ, t_i) \supseteq A(\succ', t_i)$. From the top trading cycles algorithm it follows that $A(\succ', t_i) \supseteq A(\succ', t_i')$. Hence, $A(\succ, t_i) \supseteq A(\succ', t_i')$. When student i is removed at step t_i / t_i' of $\tau(\succ)$ / $\tau(\succ')$, she is assigned to the school (or no-school option) that she most prefers in $A(\succ, t_i)$ / $A(\succ', t_i')$. Thus, $A(\succ, t_i) \supseteq A(\succ', t_i')$ implies $\tau_i(\succ) \; R_i \; \tau_i(\succ')$, i.e., equation (3) holds. $\square$

We now consider the general setting where schools can have multiple seats, i.e., for each $s \in S$, $q_s \geq 1$. The top trading cycles mechanism is also minimal-access monotonic in the general setting. This can be easily seen by applying minimal-access monotonicity from the unit setting as follows. First, make q_s copies of each school $s \in S$ and label them $1, 2, \ldots, q_s$. Second, let each copy of a school inherit the priority ordering of the school. Third, students' new preferences are obtained from their original preferences by replacing each school s by its q_s copies (in the strict order of increasing labels). Fourth, note that the top trading cycles matching for the original problem "coincides" with the top trading cycles matching for the new problem (by lumping together the students who are matched to copies of the same school). Fifth, by applying minimal-access monotonicity to problems in the unit setting we obtain minimal-access monotonicity in the general setting.

[15] It was sufficient to consider agents a^* in C that point to different objects in cycle C and at step t of $\tau(\succ)$. To see this, let agent a be an agent that points to the *same* agent in C and at step t of $\tau(\succ)$. Then, there is a path at step t of $\tau(\succ)$ from a to an agent a^* in C that points to different objects in C and at step t of $\tau(\succ)$. Since we have shown that $a^* \in P(i, \succ, t)$, it immediately follows that $a \in P(i, \succ, t)$ as well.