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Inverse Applications of the Generalized Littlewood Theorem Concerning Integrals of the Logarithm of Analytic Functions

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Abstract: Recently, we established and used the generalized Littlewood theorem concerning contour integrals of the logarithm of analytical function to obtain new criteria equivalent to the Riemann hypothesis. The same theorem was subsequently applied to calculate certain infinite sums and study the properties of zeroes of a few analytical functions. In this article, we discuss what are, in a sense, inverse applications of this theorem. We first prove a Lemma that if two meromorphic on the whole complex plane functions *f(z)* and *g(z)* have the same zeroes and poles, taking into account their orders, and have appropriate asymptotic for large |*z*|, then for some integer *n*, $\frac{d^n \ln(f(z))}{dz^n} = \frac{d^n \ln(g(z))}{dz^n}$. The use of this Lemma enables proofs of many identities between elliptic functions, their transformation and *n*-tuple product rules. In particular, we show how exactly for any complex number *a*, ℘(z)-a, where ℘(z) is the Weierstrass ℘ function, can be presented as a product and ratio of three elliptic θ_1 functions of certain arguments. We also establish *n*-tuple rules for some elliptic theta functions.

Keywords: logarithm of an analytical function; generalized Littlewood theorem; elliptic functions; zeroes and poles of analytical function

MSC: 30E20; 30C15; 33B20; 33B99

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1. Introduction

The generalized Littlewood theorem concerning contour integrals of the logarithm of analytical function is stated as follows [1,2]:

Theorem 1 (The Generalized Littlewood theorem). *Let C denote the rectangle bounded by the lines $x = X_1, x = X_2, y = Y_1, y = Y_2$ where $X_1 < X_2, Y_1 < Y_2$ and let f(z) be analytic and non-zero on C and meromorphic inside it, and let also g(z) be analytic on C and meromorphic inside it. Let F(z) = ln(f(z)) be the logarithm defined as follows: we start with a particular determination on $x = X_2$ and obtain the value at other points by continuous variation along y = const from $\ln(X_2 + iy)$. If, however, this path would cross a zero or pole of f(z), we take F(z) to be $F(z \pm i0)$ according as to whether we approach the path from above or below. Let also $\widetilde{F}(z) = \ln(f(z))$ be the logarithm defined by continuous variation along any smooth curve fully lying inside the contour which avoids all poles and zeroes of f(z) and starts from the same particular determination on $x = X_2$. Suppose also that the poles and zeroes of the functions f(z), g(z) do not coincide.*

Then

$$\int_C F(z)g(z)dz = 2\pi i \left(\sum_{\rho_g} res(g(\rho_g) \cdot \widetilde{F}(\rho_g)) - \sum_{\rho_f^0} \int_{X_1+iY_\rho^0}^{X_\rho^0+iY_\rho^0} g(z)dz + \sum_{\rho_f^{pol}} \int_{X_1+iY_\rho^{pole}}^{X_\rho^{pole}+iY_\rho^{pole}} g(z)dz \right) \quad (1)$$

where the sum is over all ρ_g which are poles of the function g(z) lying inside C, all $\rho_f^0 = X_\rho^0 + iY_\rho^0$ which are zeroes of the function f(z) both counted taking into account their multiplicities (that is *the corresponding term is multiplied by 485 m for a zero of the order m) and which lie inside C, and all* $\rho_f^{pole} = X_\rho^{pole} + iY_\rho^{pole}$ *which are poles of the function f(z) counted taking into account their multiplicities and which lie inside C. The assumption is that all relevant integrals on the right-hand side of the equality exist.*

The proof of this theorem [2] is very close to the proof of the standard Littlewood theorem corresponding to the case *g(z)* = 1, see e.g., [3]. Especially interesting are some particular cases when the contour integral $\int\limits_C F(z)g(z)dz$ disappears (tends to zero) if the contour tends to infinity; that is when $X_1, Y_1 \to -\infty, X_2, Y_2 \to +\infty$. This means that Equation (1) takes the form

$$\sum_{\rho_f^0} \int\limits_{-\infty+iY_\rho^0}^{X_\rho^0+iY_\rho^0} g(z)dz - \sum_{\rho_f^{pol}} \int\limits_{-\infty+iY_\rho^{pole}}^{X_\rho^{pole}+iY_\rho^{pole}} g(z)dz = \sum_{\rho_g} res(g(\rho_g)\cdot F(\rho_g)) \tag{2}$$

If the integrals here can be calculated explicitly, in this way one obtains equalities involving finite or infinite sums (this last case is the most interesting one).

Previously, this approach was used by us to analyze some properties of the zeroes of the Riemann zeta-function (see, e.g., [4] for a general discussion of this function); in particular, to establish a number of theorems equivalent to the Riemann hypothesis, see, e.g., [1,2,5]. (Some of these results were recently included in the corresponding chapter of the Encyclopedia of Mathematics and its Applications [6]). Later on, we used the generalized Littlewood theorem to calculate many infinite sums over integers and to study the properties of zeroes of some analytical functions, including the elliptic functions [7]; the latter is especially close to the subject of the current paper.

In the present paper, we discuss what, in a sense, is the inverse application of this generalized Littlewood theorem, which were not considered before. Namely, we demonstrate how from the circumstance that certain analytical functions have the same poles and zeroes, together with some additional information, the formulae connecting these functions can be established. These applications are first illustrated by certain trigonometrical functions and gamma functions and then applied to prove numerous equalities and transformation rules between different elliptic functions. Certainly, these equalities and rules are known for the most part, but we believe that in the frame of our approach, they are proven in a quite clearer and more transparent way. Moreover, to the best knowledge of the present author, the essential part of the content of the Section 4.3 "n-tuple products" is unknown in its generality; see also the remarks below.

2. The Main Lemma

Let us state the following Lemma 1, which is the tool of proof of our main results; see especially Sections 3 and 4.

Lemma 1. *Let the functions f(z) and g(z) be analytic and meromorphic on the complex plane, and let for some integer n the existence of a sequence of contours Ci tending to infinity, such as defined in the conditions of the generalized Littlewood theorem, and such that* $\int\limits_{C_i} \frac{1}{(z-a)^{n+1}} \ln(f(z))dz$ *and* $\int\limits_{C_i} \frac{1}{(z-a)^{n+1}} \ln(g(z))dz$ *tend to zero. Here a is an arbitrary complex number not coinciding with any zero or pole of the functions f(z) and g(z). Let also the poles and zeroes of these functions, taking into account their multiplicities, coincide. Then* $\frac{d^n \ln(f(z))}{dz^n} = \frac{d^n \ln(g(z))}{dz^n}$.

Proof. By the Lemma conditions, the values of the contour integrals $\int\limits_{C_i} \frac{1}{(z-a)^{n+1}} \ln(f(z))dz$ and $\int\limits_{C_i} \frac{1}{(z-a)^{n+1}} \ln(g(z))dz$ for contours Ci and points a considered above tend to zero, and the identity of poles and zeroes of the functions f(z), g(z) take place. Thus, according to Equation (2), we have $\frac{d^n \ln(f(z))}{dz^n}|_{z=a} = \frac{d^n \ln(g(z))}{dz^n}|_{z=a}$ for all a not coinciding with any zero or pole of the functions *f(z)*, *g(z)*. □

Certainly, this Lemma has close connections with the famous Cauchy–Liouville's theorem stating that any bounded entire analytical function is a constant; for its generalizations by Hadamard and others, see, e.g., [3]. Nevertheless, it turns out to be rather useful in applications, and thus, in our opinion, deserves a special attention.

3. Illustrative Applications

Let us first consider some simple illustrations.

3.1. Gamma Function

First, we apply Lemma 1 with $n = 2$ to the functions $\Gamma(2z)$ and $\Gamma(z)\Gamma(z+1/2)$. Both these functions have the same poles at $z = 0, -1/2, -1, -3/2, -2\ldots$ and no zeroes, and for large $|z|$ the asymptotic of their logarithms is $O(z \ln z)$; see, e.g., [8,9] for the general discussion of gamma function and its derivatives. Thus, Lemma 1 gives $\psi'(z) + \psi'(z+1/2) = 4\psi'(2z)$. Integration gives $\psi(z) + \psi(z+1/2) = 2\psi(2z) + C_1$, and one more integration gives $\Gamma(z)\Gamma(z+1/2) = C_2 e^{C_1 z}\Gamma(2z)$, where C_1 and C_2 are integration constants to be determined.

Now the elementary $\Gamma(1) = \Gamma(2) = 1$, $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(3/2) = \Gamma(1+1/2) = \frac{1}{2}\sqrt{\pi}$ enable, after the substitution of $z = 1/2$ and $z = 1$ into above equality, to recover the Legendre duplication rule:

$$\Gamma(z)\Gamma(z+1/2) = 2^{1-2z}\sqrt{\pi}\Gamma(2z) \quad (3)$$

The reflection formula

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)} \quad (4)$$

Is also trivially obtained by applying Lemma 1 with $n = 2$ to $\Gamma(1-z)\Gamma(z)$ and $\frac{1}{\sin(\pi z)}$. Thus $\psi'(1-z) + \psi'(z) = -\frac{\pi^2}{\cos(\pi z)}$. The first integration gives $-\psi(1-z) + \psi(z) = -\pi\cot(\pi z) + C_1$; substitution of $z = 1/2$, without even the knowledge of $\psi(1/2)$, gives $C_1 = 0$. Second integration gives $\Gamma(1-z)\Gamma(z) = \frac{C_2}{\sin(\pi z)}$, substitution of $z = 1/2$ gives $C_2 = \pi$.

Let us now prove the same formula in a slightly different way relying on the Laurent expansions of the corresponding logarithms. We know $\Gamma^{-1}(1+z) = 1 + \gamma z + O(z^2)$ so that from the functional equation $\Gamma(1+z) = z\Gamma(z)$, we have $\Gamma^{-1}(z) = z + \gamma z^2 + O(z^3)$, $\Gamma^{-1}(z)\Gamma(1-z) = z + O(z^3)$, and $\ln(\Gamma^{-1}(z)\Gamma(1-z)/z) = O(z^2)$. We also know that $\ln(\frac{\sin \pi z}{\pi z}) = O(z^2)$. The generalized Littlewood theorem (its inverse application, Lemma 1) shows that $O(z^2)$ terms in both these formulae are identical. Thus, $\ln(\Gamma^{-1}(z)\Gamma^{-1}(1-z)/z) = \ln\frac{\sin \pi z}{\pi z}$ and Equation (4) follows.

In exactly the same way we obtain an easy proof of $\sum\limits_{k=0}^{n-1} \psi'(z+\frac{k}{n}) = n^2\psi'(nz)$ and, by integration, $\sum\limits_{k=0}^{n-1} \psi(z+\frac{k}{n}) = n\psi(nz) + C_1(n)$. Substituting $z = 1$ we obtain $\psi(1) + \sum\limits_{k=1}^{n-1} \psi(1+\frac{k}{n}) = n\psi(n) + C_1(n)$. The use of $\psi(1) = -\gamma$ and $\psi(1+z) = \psi(z) + \frac{1}{z}$, whence $\psi(n) = \sum\limits_{k=1}^{n-1}\frac{1}{k} - \gamma$, gives $-\gamma + \sum\limits_{k=1}^{n-1}\psi(\frac{k}{n}) + n\sum\limits_{k=1}^{n-1}\frac{1}{k} = n(\sum\limits_{k=1}^{n-1}\frac{1}{k} - \gamma) + C_1(n)$, so that $C_1(n) = \sum\limits_{k=1}^{n-1}\psi(\frac{k}{n}) + (n-1)\gamma$. Applying Gauss' identity $\sum\limits_{k=1}^{n}\psi(\frac{k}{n}) = -n(\gamma + \ln n)$ [8,9], we obtain $C_1(n) = -n\ln n$ whence $\sum\limits_{k=0}^{n-1}\psi(z+\frac{k}{n}) = n\psi(nz) - n\ln n$. One more integration gives $\prod\limits_{k=0}^{n-1}\Gamma(z+\frac{k}{n}) = C_2(n)e^{-nz\ln n}\Gamma(nz)$ and $C_2(n) = n\prod\limits_{k=1}^{n-1}\Gamma(\frac{k}{n})$, but unfortunately we do not see an obvious way to find that $C_2(n) = (2\pi)^{\frac{n-1}{2}} n^{\frac{1}{2}}$ for the case, and thus, fully restore the Gauss multiplication theorem [8,9]:

$$\prod_{k=0}^{n-1} \Gamma(z + \frac{k}{n}) = (2\pi)^{\frac{n-1}{2}} n^{\frac{1}{2}} e^{-nz \ln n} \Gamma(nz) \tag{5}$$

3.2. Trigonometrical Functions

The corresponding relations for trigonometrical functions are very well known, so we limit ourselves with quite a short exposition (it also will be useful when similar transformation rules for elliptical functions are discussed; see below). The equivalence of the sums over poles and zeroes expressed via $\int_C \frac{1}{(z-a)^3} \ln(\sin(nz))dz$ and $\int_C \frac{1}{(z-a)^3} \ln(\prod_{k=0}^{n-1} \sin(z + \frac{\pi k}{n}))dz$ (Lemma 1) readily furnishes

$$\sum_{k=0}^{n-1} \frac{1}{\sin^2(z + \frac{\pi k}{n})} = \frac{n^2}{\sin^2(nz)} \tag{6}$$

The integration of Equation (6) gives $\sum_{k=0}^{n-1} \cot(z + \frac{\pi k}{n}) = n \cot(nz) + C(n)$, the substitution $z = \frac{\pi}{2n}$ shows that $C(n) = 0$ whence

$$\sum_{k=0}^{n-1} \cot(z + \frac{\pi k}{n}) = \cot(nz) \tag{7}$$

The next integration gives

$$\prod_{k=0}^{n-1} \sin(z + \frac{k\pi}{n}) = C_1(n) \sin(nz) \tag{8}$$

This is a well-known (and very good) student exercise to establish the Euler relation $\prod_{k=1}^{n-1} \sin(\frac{k\pi}{n}) = \frac{n}{2^{n-1}}$. This relation corresponds to the case when z tends to zero in Equation (8). Thus,

$$\prod_{k=0}^{n-1} \sin(z + \frac{k\pi}{n}) = 2^{1-n} \sin(nz) \tag{9}$$

Finally, let us note the following rarely mentioned equality between the products and sums of squares for the sine function. From Equations (9) and (6), it immediately follows that

$$\prod_{k=0}^{n-1} \sin^2(z + \frac{k\pi}{n}) = n^2 2^{2-2n} \frac{1}{\sum_{k=0}^{n-1} \sin^{-2}(z + \frac{k\pi}{n})} \tag{10}$$

Attention should be paid to the sign if extracting the square root of Equation (10).

4. Applications to Elliptic Functions

4.1. Definitions and Main Properties of Elliptic Functions

Elliptic functions, which are much studied because of their high importance in mathematics and physics (see, e.g., [10–16]; whenever possible, below we will mostly cite "encyclopedia-like" Ref. [10]), are a fertile ground for the approach based on the generalized Littlewood theorem. First, we need to give necessary definitions and specify a notation used, especially because, unfortunately, different notations and conventions still co-exist in the research field of elliptic functions. We define four theta functions as follows:

$$\theta_1(z,q) = \theta_1(z|\tau) = 2\sum_{n=0}^{\infty}(-1)^n q^{(n+1/2)^2}\sin((2n+1)z) \tag{11a}$$

$$\theta_2(z,q) = \theta_2(z|\tau) = 2\sum_{n=0}^{\infty} q^{(n+1/2)^2}\cos((2n+1)z) \tag{11b}$$

$$\theta_3(z,q) = \theta_3(z|\tau) = 1 + 2\sum_{n=0}^{\infty} q^{n^2}\cos(2nz) \tag{11c}$$

$$\theta_4(z,q) = \theta_4(z|\tau) = 1 + 2\sum_{n=0}^{\infty}(-1)^n q^{n^2}\cos(2nz) \tag{11d}$$

Here, $q = e^{i\pi\tau}$ (it is named a nome), and $\mathrm{Im}\tau > 0$. As functions of z for any fixed τ, they are entire and 2π periodic, and they are quasiperiodic on the lattice formed by the points $z_{m,n} = (m + n\tau)\pi$, in the sense that the following relation holds:

$$\theta_1(z + (m + n\tau)\pi|\tau) = (-1)^{m+n} q^{-n^2} e^{-2inz}\theta_1(z|\tau) \tag{12}$$

And similar relations exist for other theta functions. Here and below, unless specifically stated to the contrary, $n, m \in Z$, but we will not always repeat this statement. The notations $\theta_j(z,q)$ and $\theta_j(z|\tau)$ are used on equal footing.

These properties guaranty that for large $|z|$, the theta function is at most $O(exp(C|z|^2))$ with some constant C; hence, in the limit of infinitely large contours, we have the disappearance of the contour integrals $\int_{C_i} \frac{1}{(z-a)^{n+1}} \ln(\theta_j(z))dz$ for j = 1, 2, 3, 4 and n = 3, 4... The location of zeroes ρ_i, which are all simple for these functions, is also well known. Namely, the functions $\theta_j(z|\tau)$ for j = 1, 2, 3, 4 have zeroes at the points $(m + n\tau)\pi$, $(m + \frac{1}{2} + n\tau)\pi$, $(m + \frac{1}{2} + (n + \frac{1}{2})\tau)\pi$ and $(m + (n + \frac{1}{2})\tau)\pi$, respectively [10]. Taylor expansions of the theta functions are the following [10]: $\theta_1(\pi z|\tau) = \pi z \theta_1{}'(0|\tau)\exp(-\sum_{j=1}^{\infty}\frac{1}{2j}\delta_{2j}z^{2j})$, $\theta_2(\pi z|\tau) = \theta_2(0|\tau)\exp(-\sum_{j=1}^{\infty}\frac{1}{2j}\alpha_{2j}z^{2j})$, $\theta_3(\pi z|\tau) = \theta_3(0|\tau)\exp(-\sum_{j=1}^{\infty}\frac{1}{2j}\beta_{2j}z^{2j})$, and $\theta_4(\pi z|\tau) = \theta_4(0|\tau)\exp(-\sum_{j=1}^{\infty}\frac{1}{2j}\gamma_{2j}z^{2j})$ (that is, $\theta_1(\pi z|\tau) = \pi z\theta_1{}'(0|\tau)(1 - \frac{\delta_2}{2}z^2 - (\frac{\delta_4}{4} - \frac{\delta_2^2}{8})z^4 + O(z^6))$), where:

$$\delta_{2j}(\tau) = \sum_{n=-\infty}^{\infty}\sum_{\substack{m=-\infty \\ |m|+|n|\neq 0}}^{\infty}\frac{1}{(m+n\tau)^{2j}} \tag{13a}$$

$$\alpha_{2j}(\tau) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{1}{(m+\frac{1}{2}+n\tau)^{2j}} \tag{13b}$$

$$\beta_{2j}(\tau) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{1}{(m+\frac{1}{2}+(n+\frac{1}{2})\tau)^{2j}} \tag{13c}$$

$$\gamma_{2j}(\tau) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{1}{(m+(n+\frac{1}{2})\tau)^{2j}} \tag{13d}$$

The order of summation is important for these sums if j = 1; see additional discussion in our previous paper [7].

From these formulae, we easily obtain the Taylor expansions of the logarithms of theta functions in some vicinity of zero, like

$$\ln[\theta_1(\pi z|\tau)/(\pi z\theta_1{}'(0|\tau))] = -\sum_{j=1}^{\infty}\frac{1}{2j}\delta_{2j}z^{2j} \tag{14}$$

or $\ln[\theta_2(\pi z|\tau)/\theta_2(0|\tau)] = -\sum_{j=1}^{\infty}\frac{1}{2j}\alpha_{2j}z^{2j}$. Note, also useful for the future relation is

$$\delta_2 = -\frac{\pi^2\theta_1'''(0|\tau)}{3\theta_1{}'(0|\tau)} \tag{15}$$

which immediately follows from the Taylor series of $\theta_1(\pi z|\tau)$ given above.

The Weierstrass σ function is defined as [10]

$$\sigma(z|\Lambda) = z\prod_{\omega\in\Lambda/\{0\}}^{\infty}(1-\frac{z}{\omega})\exp(\frac{z}{\omega}+\frac{z^2}{2\omega^2}) \tag{16}$$

where Λ is a lattice formed by points $2m\omega_1 + 2n\omega_2$, with again $n, m \in Z$, and $\mathrm{Im}\frac{\omega_2}{\omega_1} > 0$. We have by definition $\zeta(z|\Lambda) = \frac{d}{dz}(\ln\sigma(z|\Lambda))$, $\wp(z|\Lambda) = -\frac{d\zeta(z|\Lambda)}{dz}$, $\wp_z(z|\Lambda) = \frac{d\wp(z|\Lambda)}{dz}$. Below we will use the notation $\sigma(z|\Lambda)$, $\zeta(z|\Lambda)$, etc. if an arbitrary lattice is used; if $\omega_1 = 1/2$, which is usually the case, we shall write $\sigma(z,\tau)$, $\zeta(z,\tau)$, etc.

The following Laurent expansion is well known (and evident from the above definition) [10]:

$$\zeta(z,\tau) = \frac{1}{z} - \sum_{k=2}^{\infty}\delta_{2k}(\tau)z^{2k-1} \tag{17}$$

where $\delta_{2k}(\tau)$ are the Eisenstein series defined by Equation (13a). To simplify the notation, below we may omit τ, writing $\delta_{2k}(\tau)$ simply as δ_{2k}. Thus, $z\zeta(z,\tau) = 1 - \sum_{k=2}^{\infty}\delta_{2k}z^{2k}$ and $\ln(z\zeta(z,\tau)) = -\delta_4 z^4 - \delta_6 z^6 - (\delta_8 + \frac{1}{2}\delta_4^2)z^8 + O(z^{10})$. Of course, we have

$$\wp(z,\tau) := -\frac{d}{dz}\zeta(z,\tau) = \frac{1}{z^2} + \sum_{k=2}^{\infty}(2k-1)\delta_{2k}(\tau)z^{2k-2} \tag{18}$$

which is often written as

$$\wp(z,\tau) = \frac{1}{z^2} + z^2\sum_{k=0}^{\infty}\frac{d_k}{k!}z^{2k} \tag{18a}$$

where evidently $d_k = (2k+3)k!\delta_{2k+4}$.

Jacobi elliptical functions $sn(z,k)$, $cn(z,k)$, $dn(z,k)$ and other similar, are properly scaled ratios of the appropriate theta functions [10]. They will not be considered here.

4.2. Equalities between Different Elliptic Functions

It is well known that all double-periodic meromorphic functions can be expressed via Weierstrass elliptical functions and their derivatives, and numerous concrete examples of the corresponding expressions/representations are commonly found in the literature [10–16]. For this reason, here, we do not aim to consider many corresponding examples. Our purpose is to illustrate the principle and to concentrate on the cases which, for some reasons, are seemingly not widely indicated or even not known in all generality.

Our first example is the Weierstrass sigma function $\sigma(z|\Lambda)$. From its definition, it is clear that this is an entire function having simple zeroes at the lattice points. Thus, the Weierstrass sigma function $\sigma(z,\tau)$ and the elliptic theta function $\theta_1(\pi z|\tau)$ are both entire functions that have simple zeroes at $m + n\tau$. We know the Taylor expansion of the $\ln[\theta_1(\pi z|\tau)/(\pi z\theta_1{}'(0|\tau))]$; see Equation (14). Due to the Lemma 1, the Taylor expansion of the $\ln(\sigma(z,\tau)/z)$ must be exactly the same starting from the $O(z^4)$ term: $\ln(\sigma(z,\tau)/z) = a + bz^2 - \sum_{j=2}^{\infty} \frac{1}{2j}\delta_{2j}z^{2j}$. Directly from Equation (16), we see that $a = b = 0$; hence

$$\sigma(z,\tau) = \exp\left(\frac{z^2}{2}\delta_2(\tau)\right)\frac{\theta_1(\pi z|\tau)}{\pi\theta_1{}'(0|\tau)} \tag{19}$$

Scaling for an arbitrary $\sigma(z|\Lambda)$ immediately gives the "standard" formula [10]:

$$\sigma(z|\Lambda) = 2\omega_1 \exp\left(\frac{\eta_1 z^2}{2\omega_1}\right)\frac{\theta_1(\pi z/(2\omega_1), q)}{\pi\theta_1{}'(0,q)} \tag{20}$$

where $\eta_1 = -\frac{\pi^2}{12\omega_1}\frac{\theta_1{}'''(0,q)}{\theta_1{}'(0,q)}$. Here and below, it might be useful to exploit [10]

$$\theta_1{}'(0,q) = 2q^{1/4}\prod_{j=1}^{\infty}\left(1-q^{2j}\right)^3 \tag{21}$$

Remark 1. *In light of the abovesaid, the Taylor expansion of the function* $\frac{\sigma(z,\tau)}{z}$ *is given by* $\frac{\sigma(z,\tau)}{z} = \exp(-\frac{1}{4}\delta_4 z^4 - \frac{1}{6}\delta_6 z^6 - \frac{1}{8}\delta_8 z^8 + \ldots)$*, sf. [10,13]. To fully restore the known results, we need to add the formulae expressing* δ_{2j} *for j = 4, 5, 6... via* δ_4 *and* δ_6*, which readily follow from, e.g., the differential equation [10]*

$$\frac{d^2\wp(z,\tau)}{dz^2} = 6\wp^2(z,\tau) - 30\delta_4 \tag{22}$$

by equating the Taylor series coefficients obtained by differentiating of Equation (18). Usually, these formulae are written in the form including the coefficients d_k *of Equation (18a):*

$$\sum_{k=0}^{n} C_n^k d_k d_{n-k} = \frac{2n+9}{3n+6}d_{n+2} \tag{23}$$

(Differentiating Equation (18a), we see that the coefficient in front of the z^{2n+4} *term in the Taylor expansion of l.h.s of Equation (22) is* $\frac{(2n+5)(2n+6)d_{n+2}}{(n+2)!}$*, while in the r.h.s it is* $6\sum_{k=0}^{n}\frac{d_k}{k!}\frac{d_{n-k}}{(n-k)!} + 12\frac{d_{n+2}}{(n+2)!}$*).*

Our next example of the inverse application of the generalized Littlewood theorem to elliptic functions is the Weierstrass $\wp(z|\Lambda)$ functions. These functions have second-order poles at the same points where simple zeroes of $\theta_1(\frac{\pi z}{2\omega_1}, q)$ function are located, which means that the function $\theta_1{}^{-2}(\frac{\pi z}{2\omega_1}, q)$ has the same poles as the Weierstrass $\wp(z|\Lambda)$ function. Here, $\tau = \omega_2/\omega_1$ and $q = e^{i\pi\tau}$; again, we use this notation to be consistent with the data presented in [10].

But contrary to the entire theta function in the power of -1, the Weierstrass $\wp$ functions have zeroes. The situation is easiest with the functions $\wp(z|\Lambda) - e_i$, where $e_1 = \wp(\omega_1)$, $e_2 = \wp(\omega_2)$ and $e_3 = \wp(\omega_1 + \omega_2)$. By construction, these functions have zeroes at points, coinciding with the "demi-lattice" points $m + \frac{1}{2} + n\tau$, $m + \frac{1}{2} + (n + \frac{1}{2})\tau$ or $m + (n + \frac{1}{2})\tau$, respectively, and it is clear that these are second-order zeroes. It is also known that there are no other zeroes for functions $\wp(z|\Lambda) - e_i$. Thus, in particular, for the function $\wp(z|\Lambda) - e_1$, the positions of second-order zeroes coincide with those of the entire function $\theta_2(\frac{\pi z}{2\omega_1}, q)$, and Lemma 1 gives $\frac{d^2}{dz^2}\ln(\wp(z|\Lambda) - e_1) = \frac{d^2}{dz^2}\ln\frac{\theta_2^2(\pi z/(2\omega_1),q)}{\theta_1^2(\pi z/(2\omega_1),q)}$, so that $\wp(z|\Lambda) - e_1 = C_2(\Lambda)\exp(C_1(\Lambda)z + C(\Lambda)z^2)\frac{\theta_2^2(\pi z/(2\omega_1),q)}{\theta_1^2(\pi z/(2\omega_1),q)}$. Coefficients $C(\Lambda)$ and $C_1(\Lambda)$ must be equal to zero since other coefficients are incompatible with the double periodicity of both sides of the equality. (The ratio of the squares of theta functions in the r.h.s. here is proportional to the square of Jacobi elliptic function $cs^2(\varsigma, k)$; see, e.g., [10] for definitions). Analysis for the value of $z = 0$ equating the coefficients in front of the $1/z^2$ term in the Taylor expansions shows $C_2 = \left(\frac{\pi\theta'^2{}_1(0,q)}{2\omega_1\theta_2^2(0,q)}\right)^2$. Then, we can use the known equality [10]:

$$\theta_1{}'(0,q) = \theta_2(0,q)\theta_3(0,q)\theta_4(0,q) \tag{24}$$

to write the "standard" form:

$$\wp(z|\Lambda) - e_1 = \left(\frac{\pi\theta_3(0,q)\theta_4(0,q)}{2\omega_1}\right)^2 \frac{\theta_2^2(\pi z/(2\omega_1),q)}{\theta_1^2(\pi z/(2\omega_1),q)} \tag{25}$$

Similar equalities can be in the same fashion established for other $\wp(z|\Lambda) - e_j$ functions, which we will not do here; see, e.g., [10] for the list of formulae. We also will not exercise ourselves on the expression of the ratio of theta functions occurring in these formulae via appropriate Jacobi elliptic functions; see again [10]. Instead (see our motivation discussed above), we want to underline again that there is nothing special with the functions $\wp(z|\Lambda) - e_j$. The following theorem, which seems to almost never appear in the monographs devoted to the elliptic functions, holds true.

Theorem 2. *Let a be an arbitrary complex number not equal to e_1, e_2, e_3 defined above, and numbers $\alpha_{1,2}$ be such that the equality $\wp(\alpha_j|\Lambda) = a$, j = 1, 2, holds true. Let also the difference $\alpha_1 - \alpha_2 \neq 2n\omega_1 + 2m\omega_2$. Then,*

$$\wp(z|\Lambda) - a = C\frac{\theta_1(\pi z/(2\omega_1) - \pi\alpha_1/(2\omega_1))\theta_1(\pi z/(2\omega_1) - \pi\alpha_2/(2\omega_1)), q)}{\theta_1^2(\pi z/(2\omega_1), q)} \tag{26}$$

where $\tau = \omega_2/\omega_1$, $q = e^{i\pi\tau}$,

$$C = \frac{\pi^2}{4\omega_1^2}\frac{\theta_1{}'^2(0,q)}{\theta_1(\pi\alpha_1/(2\omega_1), q)\theta_1(\pi\alpha_2/(2\omega_1), q)} \tag{27}$$

Proof. First, we note that due to the argument principle, double-periodic function $\wp(z|\Lambda) - a$, having in each fundamental parallelogram one pole of the second order, has in this parallelogram exactly two zeroes taking into account their multiplicity. (Exactly for $a = e_j$, $j = 1, 2, 3$, excluded by the theorem conditions, this function has one double zero inside the fundamental parallelogram). Thus, for any a, we can find the numbers $\alpha_{1,2}$ requested by the theorem. Clearly, the functions $\wp(z|\Lambda) - a$ and $\theta_1{}^{-2}(\pi z/(2\omega_1), q)$ have the same poles of the second order at the lattice points. Function $\wp(z|\Lambda) - a$ additionally has simple zeroes at the points $\alpha_{1,2} + 2n\omega_1 + 2m\omega_2$, and the product $\theta_1(\pi z/(2\omega_1) - \pi\alpha_1/(2\omega_1), q) \times \theta_1(\pi z/(2\omega_1) - \pi\alpha_2/(2\omega_1), q)$ is an entire function also having simple zeroes exactly at these points. The functions involved are both double periodic; hence, the Lemma 1 is applicable already for n = 2. The double integration of the obtained equality of the second derivatives of the corresponding logarithms gives $\wp(z|\Lambda) - a = C(\omega_1, q)\frac{\theta_1(\pi z/(2\omega_1) - \pi\alpha_1/(2\omega_1), q)\theta_1(\pi z/(2\omega_1) - \pi\alpha_2/(2\omega_1)), q)}{\theta_1^2(\pi z/(2\omega_1), q)}$. (Again, the possible factor $\exp(Bz) \equiv 1$ due to the double periodicity of the functions involved). The coefficient C is established by analyzing the case when z tends to zero by equating the terms proportional to 1/z2 in Laurent expansions. We have $C = \frac{\pi^2}{4\omega_1^2}\frac{\theta_1{}'^2(0,q)}{\theta_1(\pi\alpha_1/(2\omega_1), q)\theta_1(\pi\alpha_2/(2\omega_1), q)}$, and the theorem is proven. □

Corollary 1. *Applying (26), (27) for a = 0, we have the following presentation of the Werierstrass $\wp(z)$ function via θ_1 elliptical functions:*

$$\wp(z|\Lambda) = \frac{\pi^2}{4\omega_1^2}\frac{{\theta_1{}'}^2(0,q)}{\theta_1(\pi\alpha_1/(2\omega_1),q)\theta_1(\pi\alpha_2/(2\omega_1),q)} \times \frac{\theta_1(\pi z/(2\omega_1)-\pi\alpha_1/(2\omega_1),q)\theta_1(\pi z/(2\omega_1)-\pi\alpha_2/(2\omega_1)),q)}{\theta_1^2(\pi z/(2\omega_1),q)} \quad (28)$$

where $\alpha_{1,2}$ are some solutions of the equation $\wp(z|\Lambda)=0$ such that $\alpha_1-\alpha_2 \neq 2n\omega_1+2m\omega_2$. In particular, we can take these solutions as discussed (for $\omega_1=1/2$) in Refs. [17,18] to be equal to $\pm\alpha_1$; here, α_1 belongs to the first fundamental parallelogram. Then, $\wp(z|\Lambda)=-\pi^2\frac{{\theta_1{}'}^2(0,q)}{\theta_1^2(\pi\alpha_1,q)}\times\frac{\theta_1(\pi z-\pi\alpha_1,q)\theta_1(\pi z+\pi\alpha_1,q)}{\theta_1^2(\pi z,q)}$. See also our previous paper [10] for some additional considerations of zeroes $\alpha_{1,2}$ and sums over inverse powers of the zeroes of the Weierstrass $\wp(z)$ function.

Remark 2. *It is instructive to see what happens in the limit $\tau\to i\infty$ when $\wp(z,\tau)$ tends to $\wp(z)=\frac{\pi^2}{\sin^2\pi z}-\frac{\pi^2}{3}$. The smallest in the module solutions of $\wp(z)=0$ are then given by the complex numbers $\alpha_{1,2}=\pm(\frac{1}{2}+\frac{i}{2\pi}\ln(5+2\sqrt{6}))$—the values which are still not easy to work with. Thus, for even greater simplicity, let us consider the more transparent case of, e.g., $\wp(z)-\pi^2=\frac{\pi^2}{\sin^2\pi z}-\frac{4\pi^2}{3}$. We have: $\wp(z)-\pi^2=\frac{\pi^2}{\sin^2\pi z}-\frac{4\pi^2}{3}=-\frac{4\pi^2}{3\sin^2\pi z}\sin(\pi z-\frac{\pi}{3})\sin(\pi z+\frac{\pi}{3})$, and this gives the aforementioned presentation for the case.*

The derivative of the Weierstrass $\wp$ function $\wp_z(z,\tau)$ has third-order poles at $k+j\tau$ and zeroes at $k+1/2+j\tau$, $k+(j+1/2)\tau$, $k+1/2+(j+1/2)\tau$ [10]. Thus, the lattice, formed by the irregular points of the logarithm (i.e., poles and zeroes) of the $\wp_z(z,\tau)$, coincides with the zeroes of the function $\sigma(2z,\tau)$. We can convert the third-order poles of the $\wp_z(z,\tau)$ function into simple zeroes by multiplying it by the $\sigma^4(z,\tau)$ function: thus, both functions $\sigma(2z,\tau)$ and $\wp_z(z,\tau)\sigma^4(z,\tau)$ are entire and have the same simple zeroes. Using our method, we can then establish the well-known equality

$$\sigma(2z,\tau) = -\wp_z(z,\tau)\sigma^4(z,\tau) \quad (29)$$

Of course, there are also infinitely many other possibilities of such types. For instance, we see that the entire function $\sigma(z+\frac{1}{2},\tau)$ has simple zeroes at $m+\frac{1}{2}+n\tau$ points, thus at the same points as the entire function $\theta_2(\pi z|\tau)$, so that $\sigma(z+\frac{1}{2},\tau)=C\exp(C_1 z+C_2 z^2)\theta_2(\pi z|\tau)$. Omitting clearly from the abovesaid details, we arrive at

$$\sigma(z+\frac{1}{2},\tau) = \frac{\sigma(1/2,\tau)}{\theta_2(0|\tau)}\exp(\zeta(1/2,\tau)z+\frac{1}{2}\delta_2(\tau)z^2)\theta_2(\pi z|\tau) \quad (30)$$

Similar equalities hold for the functions $\sigma(z+\frac{\tau}{2},\tau)$, $\sigma(z+\frac{1}{2}+\frac{\tau}{2},\tau)$; see, e.g., [10].

The next example is apparently also not quite common. We observe that $\sigma(2z,\tau)$ and $\theta_1(\pi z|\tau)\theta_2(\pi z|\tau)\theta_3(\pi z|\tau)\theta_4(\pi z|\tau)$ are both entire and have the same simple zeroes. Thus, we have $8\frac{d^3}{dz^3}\ln(\sigma(2z,\tau))=\pi^3\frac{d^3}{dz^3}\ln[\theta_1(\pi z|\tau)\theta_2(\pi z|\tau)\theta_3(\pi z|\tau)\theta_4(\pi z|\tau)]$ and $4\frac{d^2}{dz^2}\ln(\sigma(2z,\tau))=\pi^2\frac{d^2}{dz^2}\ln[\theta_1(\pi z|\tau)\theta_2(\pi z|\tau)\theta_3(\pi z|\tau)\theta_4(\pi z|\tau)]+C_1$. Equating when z tends to zero, we obtain $0=C_1-\delta_2-\alpha_2-\beta_2-\gamma_2$; see Equation (13) for definitions. In Ref. [7], we used the generalized Littlewood theorem in the direct way showing that $3\delta_2=\alpha_2+\beta_2+\gamma_2$, thus $C_1=4\delta_2$. For the first derivatives $2\frac{d}{dz}\ln(\sigma(2z,\tau))=\pi\frac{d}{dz}\ln[\theta_1(\pi z|\tau)\theta_2(\pi z|\tau)\theta_3(\pi z|\tau)\theta_4(\pi z|\tau)]+C_1 z+C_2$, so that trivially $C_2=0$. Thus $\sigma(2z,\tau)=C\exp(2\delta_2 z^2)\theta_1(\pi z|\tau)\theta_2(\pi z|\tau)\theta_3(\pi z|\tau)\theta_4(\pi z|\tau)$. Equating functions at zero, we have $2=C\pi\theta_1'(0|\tau)\theta_2(0|\tau)\theta_3(0|\tau)\theta_4(0|\tau)$ and with Equation (25), $C=2/(\pi[\theta_2(0)\theta_3(0)\theta_4(0)]^2)$; hence,

$$\sigma(2z,\tau) = \frac{2}{\pi\theta_2^2(0)\theta_3^2(0)\theta_4^2(0)}\exp(2\delta_2 z^2)\theta_1(\pi z|\tau)\theta_2(\pi z|\tau)\theta_3(\pi z|\tau)\theta_4(\pi z|\tau) \quad (31)$$

As usual, $\delta_2(\tau)$ can be expressed via the derivatives ratio using Equation (15). Now note that by Equation (19) $\sigma(2z,\tau) = \exp\left(2z^2\delta_2(\tau)\right)\frac{\theta_1(2\pi z|\tau)}{\pi\theta_1{}'(0|\tau)}$, so that we have the following expression

$$\theta_1(2z|\tau) = \frac{2}{\theta_2(0|\tau)\theta_3(0|\tau)\theta_4(0|\tau)}\theta_1(z|\tau)\theta_2(z|\tau)\theta_3(z|\tau)\theta_4(z|\tau) \tag{32}$$

This, of course, can also be established by our approach in the "direct way". See the discussion of the transformation rules for the elliptical functions in the next subsection.

Analogously—and this is our final example in this sub-section—we see that $\wp_z(z,\tau)$ and $\theta_1^{-3}(\pi z|\tau)\theta_2(\pi z|\tau)\theta_3(\pi z|\tau)\theta_4(\pi z|\tau)$ have the same poles and zeroes. Omitting standard clearly from the abovesaid details, we obtain $\wp_z(z,\tau) = C\theta_1^{-3}(\pi z|\tau)\theta_2(\pi z|\tau)\theta_3(\pi z|\tau)\theta_4(\pi z|\tau)$, and equating at zero, we have $-2 = C\pi^{-3}[\theta_1'(0|\tau)]^{-3}\theta_2(0|\tau)\theta_3(0|\tau)\theta_4(0|\tau) = C\pi^{-3}[\theta_2(0|\tau)\theta_3(0|\tau)\theta_4(0|\tau)]^{-2}$. Hence, $C = -2\pi^3[\theta_2(0|\tau)\theta_3(0|\tau)\theta_4(0|\tau)]^2$ and

$$\wp_z(z) = -\frac{2\pi^3}{[\theta_2(0)\theta_3(0)\theta_4(0)]^2}\theta_1^{-3}(\pi z|\tau)\theta_2(\pi z,\tau)\theta_3(\pi z,\tau)\theta_4(\pi z,\tau) \tag{33}$$

Here in the r.h.s. the expressions of $\frac{\theta_2(\pi z|\tau)}{\theta_1(\pi z|\tau)}$, $\frac{\theta_3(\pi z|\tau)}{\theta_1(\pi z|\tau)}$ and $\frac{\theta_4(\pi z|\tau)}{\theta_1(\pi z|\tau)}$ via Jacobi elliptical functions $cs(\varsigma,k), ds(\varsigma,k), ns(\varsigma,k)$, respectively (see [10] for details), can be used.

4.3. n-Tuple Products

Let us consider the entire theta function $\theta_1(nz|n\tau)$ with the integer n = 2, 3, 4... It has zeroes at the points $nz = k\pi + jn\pi\tau$, i.e., $z = \frac{k}{n}\pi + j\pi\tau$ (as usual, $k, j \in Z$), and thus, has the same zeroes as the product $\prod_{k=0}^{n-1}\theta_1(z + \frac{k\pi}{n}|\tau)$. We illustrate this coincidence in Figure 1 below.

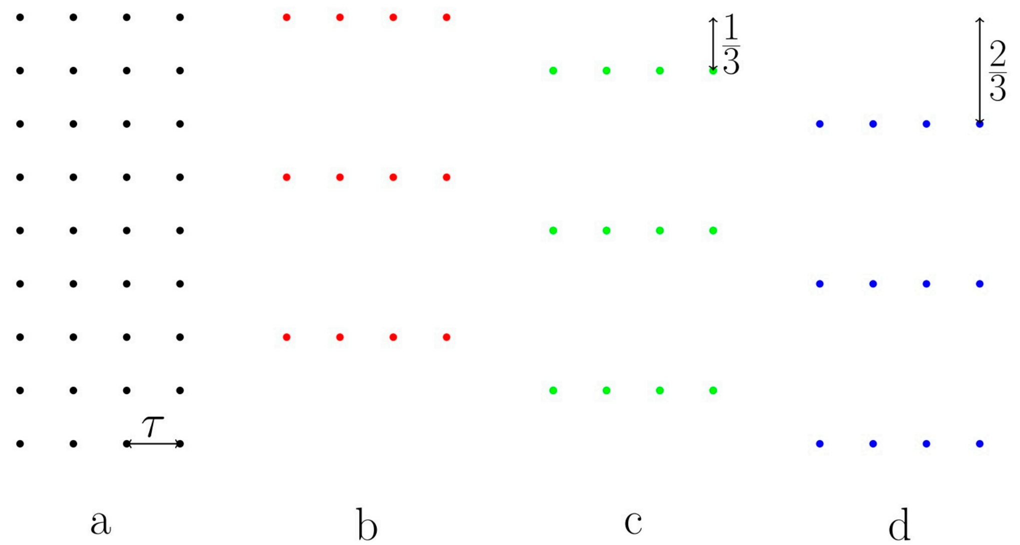

Figure 1. Illustrating the coincidence of the location of zeroes for n = 3. The purely imaginary value of τ is taken for ease of presentation and understanding. In (**a**), the zeroes of the function $\theta_1(3\pi z|3\tau)$ are shown; in (**b**–**d**), the zeroes for the functions $\theta_1(\pi z|\tau)$, $\theta_1(\pi z + \frac{\pi}{3}|\tau)$ and $\theta_1(\pi z + \frac{2\pi}{3}|\tau)$, respectively, are shown.

This coincidence of zeroes, together with the asymptotic of the functions at hand for large values of $|z|$, immediately enable to write, by applying Lemma 1: $\theta_1(nz|n\tau) = C(n,\tau)\exp(C_1(n,\tau)z + C_2(n,\tau)z^2)\prod_{k=0}^{n-1}\theta_1(z + \frac{k\pi}{n}|\tau)$. $C_1(n,\tau)$ and $C_2(n,\tau)$ are clearly zero because other values are incompatible with the z-periodicity of the functions involved. (Purely imaginary $C(n,\tau) = ik$ with integer k might be compatible (for we do not have *double* periodicity now), but such a value cannot be obtained by comparison of the real Taylor expansions at $z = 0$). The constant $C(n, \tau)$ should be determined from this equality applied at any particular value of z. Using for z tending to zero $\theta_1(nz|n\tau) = nz\theta'_1(0|n\tau) + O(z^2)$, we have $nz\theta'_1(0|n\tau) = C(n,\tau)\theta'_1(0|\tau)z\prod_{k=1}^{n-1}\theta_1(\frac{k\pi}{n}|\tau)$; hence,

$$\theta_1(nz|n\tau) = \frac{n\theta_1{}'(0|n\tau)}{\theta_1{}'(0|\tau)\prod_{k=1}^{n-1}\theta_1(\frac{k\pi}{n}|\tau)}\prod_{k=0}^{n-1}\theta_1(z+\frac{k\pi}{n}|\tau) \tag{34}$$

In particular, $\theta_1(2z|2\tau) = \frac{2\theta_1{}'(0|2\tau)\theta_1(z|\tau)\theta_1(z+\frac{\pi}{2}|\tau)}{\theta_1{}'(0|\tau)\theta_1(\frac{\pi}{2}|\tau)} = \frac{2\theta_1{}'(0|2\tau)\theta_1(z|\tau)\theta_2(z|\tau)}{\theta_1{}'(0|\tau)\theta_2(0|\tau)}$ and $\theta_1(4z|4\tau) = \frac{4\theta_1{}'(0|4\tau)\theta_1(z|\tau)\theta_1(z+\frac{\pi}{4}|\tau)\theta_1(z+\frac{\pi}{2}|\tau)\theta_1(z+\frac{3\pi}{4}|\tau)}{\theta_1{}'(0|\tau)\theta_1(\frac{\pi}{4}|\tau)\theta_1(\frac{\pi}{2}|\tau)\theta_1(\frac{3\pi}{4}|\tau)} = \frac{4\theta_1{}'(0|4\tau)\theta_1(z|\tau)\theta_1(z+\frac{\pi}{4}|\tau)\theta_2(z|\tau)\theta_1(z+\frac{3\pi}{4}|\tau)}{\theta_1{}'(0|\tau)\theta_1(\frac{\pi}{4}|\tau)\theta_2(0|\tau)\theta_1(\frac{3\pi}{4}|\tau)}$.

Remark 3. *For unclear reasons, this relation is seemingly never presented in such a (general) form and, if presented for some particular case, is given in somewhat "artificial" form like, for example, the following Landen transformations [10]:* $\theta_1(4z|4\tau) = \frac{\theta_1(z|\tau)\theta_1(\frac{\pi}{4}-z|\tau)\theta_1(\frac{\pi}{4}+z|\tau)\theta_2(z|\tau)}{\theta_3(0|\tau)\theta_4(0|\tau)\theta_3(\frac{\pi}{4}|\tau)}$, *or* $\theta_1(2z|2\tau) = \frac{\theta_1(z|\tau)\theta_2(z|\tau)}{\theta_4(0|2\tau)}$. *(This looks understandable, because Landen transformations historically came from the manipulations of the elliptical integrals achieved mainly with variable changes [10,12–15]). Of course, the equivalence of the transformations can be shown using formulae connecting the values of the elliptical functions and their derivatives at certain values (see [10–16], especially [14]), but this is actually not easy. Alternatively, the comparison of these formulae can be seen as a proof of certain relations like, e.g.,* $\frac{2\theta_1{}'(0|2\tau)}{\theta_1{}'(0|\tau)\theta_1(\frac{\pi}{2}|\tau)} = \frac{1}{\theta_4(0|2\tau)}$.

Further, let us note the following. From Equation (11a) and the definition of nome, it follows that $lim_{\tau\to i\infty}\theta_1(z|\tau) = lim_{\tau\to i\infty}(2e^{i\pi\tau/4}\sin z) = 0$ *for any finite z. From the same, we have* $lim_{\tau\to i\infty}e^{-i\pi\tau/4}\theta_1(z|\tau) = 2\sin z$ *Applying this (properly scaled) limit to both sides of Equation (34), we obtain Equation (8).*

There is also a quite different possibility. We can consider not $\theta_1(nz|n\tau)$ but $\theta_1(nz|\tau)$ instead and obtain the following theorem.

Theorem 3. *For odd 2l + 1 = 3, 5, 7. . ., we have:*

$$\theta_1((2l+1)z|\tau) = C\prod_{k=-l}^{l}\prod_{j=-l}^{l}\theta_1(z+\frac{k\pi}{2l+1}+\frac{j\pi\tau}{2l+1}|\tau) \tag{35}$$

with

$$C^{-1} = \frac{1}{2l+1}\prod_{k=-l}^{l}\prod_{\substack{j=-l,\\ |j|+|l|\neq 0}}^{l}\theta_1(\frac{k\pi}{2l+1}+\frac{j\pi\tau}{2l+1}|\tau) \tag{36}$$

For even 2l = 2, 4, 6. . ., we have

$$\theta_1(2lz,\tau) = \widetilde{C}\prod_{m=-(l-1)}^{l}\theta_4(z+\frac{m\pi}{2l})\times\prod_{k=-(l-1)}^{l}\prod_{j=-(l-1)}^{l-1}\theta_1(z+\frac{k\pi}{2l}+\frac{j\pi\tau}{2l}|\tau) \tag{37}$$

with

$$\widetilde{C}^{-1} = \frac{1}{2l}\prod_{m=-(l-1)}^{l}\theta_4(\frac{m\pi}{2l})\times\prod_{k=-(l-1)}^{l}\prod_{\substack{j=-(l-1),\\ |j|+|k|\neq 0}}^{l-1}\theta_1(\frac{k\pi}{2l}+\frac{j\pi\tau}{2l}|\tau) \tag{38}$$

Proof. We see that the functions $\theta_1(nz|\tau)$ and $\prod_{k=0}^{n-1}\prod_{j=0}^{n-1}\theta_1(z+\frac{k\pi}{n}+\frac{j\pi\tau}{n}|\tau)$ are both entire and have the same zeroes (a simple "two-dimensional" generalization of Figure 1 suffices), but to find an exponential factor we need to "symmetrize" this product.

Let first n be odd, $n = 2\,l + 1$. Then, $\theta_1((2l+1)z|\tau)$ has the same zeroes and (no) poles as the "symmetric" product $\prod_{k=-l}^{l}\prod_{j=-l}^{l}\theta_1(z+\frac{k\pi}{n}+\frac{j\pi\tau}{n}|\tau)$. Thus, by the Lemma 1 we obtain

$$\theta_1((2l+1)z|\tau) = C\exp(C_1 z + C_2 z^2)\prod_{k=-l}^{l}\prod_{j=-l}^{l}\theta_1(z+\frac{k\pi}{n}+\frac{j\pi\tau}{n}|\tau) \tag{39}$$

As usual, the coefficient C_2 should be equal to zero due to the z-periodicity. The coefficient C_1 is equal to zero due to the even nature of the $\theta_1(nz|\tau)/z$ and $\frac{1}{z}\prod_{k=-l}^{l}\prod_{j=-l}^{l}\theta_1(z+\frac{k\pi}{n}+\frac{j\pi\tau}{n}|\tau)$ functions. This is seen as follows: $\theta_1(nz|\tau)/z$ is even, and all factors in the $\prod_{k=-l}^{l}\prod_{j=-l}^{l}\theta_1(z+\frac{k\pi}{n}+\frac{j\pi\tau}{n}|\tau)$ come in pairs with $\pm k, \pm j$ if $k \neq 0, j \neq 0$. For such pairs, we have $\theta_1(z+\frac{k\pi}{n}+\frac{j\pi}{n}|\tau)\theta_1(z-\frac{k\pi}{n}-\frac{j\pi}{n}) = -\theta_1(z+\frac{k\pi}{n}+\frac{j\pi}{n}|\tau)\theta_1(-z+\frac{k\pi}{n}+\frac{j\pi}{n})$ so that this product is an even function. If k or j are equal to zero, this is also valid. If both $k = j = 0$ simultaneously, the factor $\theta_1(z|\tau)/z$ is even. Thus, comparing the $O(z)$ terms of the Taylor development of the logarithms of both sides of Equation (39), we obtain $C_1 = 0$. Coefficient C is obtained equating the values at zero: $C^{-1} = \frac{1}{2l+1}\prod_{k=-l}^{l}\prod_{j=-l,\ |j|+|l|\neq 0}^{l}\theta_1(\frac{k\pi}{n}+\frac{j\pi\tau}{n}|\tau)$.

What does happen if $n = 2l$ is even? Then, in passing from $\prod_{k=0}^{2l-1}\prod_{j=0}^{2l-1}\theta_1(z+\frac{k\pi}{n}+\frac{j\pi\tau}{n}|\tau)$ to the symmetrized form, we obtain not the fully symmetric product $\prod_{k=-(l-1)}^{l}\prod_{j=-(l-1)}^{l}\theta_1(z+\frac{k\pi}{n}+\frac{j\pi\tau}{n}|\tau)$. The situation with $\frac{1}{z}\prod_{k=-(l-1)}^{l-1}\prod_{j=-(l-1)}^{l-1}\theta_1(z+\frac{k\pi}{n}+\frac{j\pi\tau}{n}|\tau)$ is the same as before: this is an even function, but now we obtain also the "unpaired" factors at $k = l$ and $j = l$. These are the functions $\theta_1(z+\frac{\pi}{2}+\frac{\pi\tau j}{n}|\tau)$ and $\theta_1(z+\frac{\pi k}{n}+\frac{\pi\tau}{2}|\tau)$, respectively. They are respectively equal to $\theta_2(z+\frac{\pi\tau j}{n}|\tau)$ and $Az^{-iz}\theta_4(z+\frac{\pi k}{n}|\tau)$, where the constant A easily follows from [10]

$$\theta_1(z+\frac{\pi\tau}{2}|\tau) = -ie^{-iz}e^{-i\pi\tau/4}\theta_4(z|\tau) \tag{40}$$

but for our current purposes itis only important that it does not depend on z. The functions $\theta_2(z+\frac{\pi\tau j}{n}|\tau)$ with $1 \leq j \leq l-1$, when paired as $\theta_2(z+\frac{\pi\tau j}{n}|\tau)\theta_2(z-\frac{\pi\tau j}{n}) = \theta_2(z+\frac{\pi\tau j}{n}|\tau)\theta_2(-z+\frac{\pi\tau j}{n})$, again are even functions. The function $\theta_2(z|\tau)$, corresponding to $j = 0$, is also even. Thus, among the functions corresponding to $k = l$, only $\theta_1(z+\frac{\pi}{2}+\frac{\pi\tau}{2}|\tau)$ rests unpaired. It is equal to $Be^{-iz}\theta_3(z|\tau)$ [10], where B does not depend on z; the function $\theta_3(z|\tau)$ is even.

In the second group of functions (corresponding to $j = l$), the functions $Az^{-iz}\theta_4(z+\frac{\pi k}{n}|\tau)$ for $k \neq 0$ are also paired: $A_1e^{-iz}\theta_4(z+\frac{\pi k}{n}|\tau)A_2e^{-iz}\theta_4(z-\frac{\pi k}{2}) = A_1A_2e^{-2iz}\theta_4(z+\frac{\pi k}{n}|\tau)\theta_4(-z+\frac{\pi k}{2})$. For $k = 0$, we have $Ae^{-iz}\theta_4(z|\tau)$ with an even function $\theta_4(z|\tau)$; the case $k = j = l$ has already been considered just before.

Thus, we see that the product of all these unpaired factors, related with the theta functions at $k = l$ and $j = l$, is $e^{-2liz}\varphi(z)$, where $\varphi(z)$ is an even function. Thus, comparing the O(z) terms of Taylor expansions of the logarithms of both sides of the equation $\theta_1(2lz|\tau) = C\exp(C_1 z)\prod_{k=-(l-1)}^{l}\prod_{j=-(l-1)}^{l}\theta_1(z+\frac{k\pi}{n}+\frac{j\pi\tau}{n}|\tau)$, we obtain $C_1 = 2il$.

We might finish the consideration at this point, but it seems preferable to give the purely real form using Equation (40) again: $\theta_1(2lz,\tau) = \widetilde{C}\prod_{m=-(l-1)}^{l}\theta_4(z+\frac{m\pi}{2l})\times\prod_{k=-(l-1)}^{l}\prod_{j=-(l-1)}^{l-1}\theta_1(z+\frac{k\pi}{2l}+\frac{j\pi\tau}{2l}|\tau)$. (Note, that for $m = l$ in the first product, $\theta_4(z+\frac{\pi}{2}) = \theta_3(z)$ [12], we used this earlier). Coefficient $\widetilde{C}$ is obtained equating the values of both sides of the above equation at zero: $\widetilde{C}^{-1} = \frac{1}{2l}\prod_{m=-(l-1)}^{l}\theta_4(\frac{m\pi}{2l})\times\prod_{k=-(l-1)}^{l}\prod_{\substack{j=-(l-1),\\ |j|+|k|\neq 0}}^{l-1}\theta_1(\frac{k\pi}{2l}+\frac{j\pi\tau}{2l}|\tau)$. □

Remark 4. *1. The appearance of the factor e^{2ilz} during the derivation of the n-tuple relation for $\theta_1(nz|\tau)$ in the case of even n = 2l might look unexpected. However, let us see how we obtain it already for n = 2. Directly from the Lemma 1, we prove $\theta_1(2z|\tau) = \frac{2\theta_1(z|\tau)\theta_2(z|\tau)\theta_3(z|\tau)\theta_4(z|\tau)}{\theta_2(0|\tau)\theta_3(0|\tau)\theta_4(0|\tau)}$. From our Theorem 3, we have $\theta_1(2z|\tau) = Ce^{2iz}\theta_1(z|\tau)\theta_1(z+\frac{\pi}{2}|\tau)\theta_1(z+\frac{\pi\tau}{2}|\tau)\theta_1(z+\frac{\pi}{2}+\frac{\pi\tau}{2}|\tau)$, and this is consistent because $\theta_1(z+\frac{\pi}{2}|\tau) = \theta_2(z)$, $\theta_1(z+\frac{\pi\tau}{2}|\tau) = Ae^{-iz}\theta_4(z|\tau)$, $\theta_1(z+\frac{\pi}{2}+\frac{\pi\tau}{2}|\tau) = Be^{-iz}\theta_3(z|\tau)$.*

2. Along the same lines, relations similar to Equations (35)–(38) can be established for the Weierstrass sigma functions $\sigma(nz,n\tau)$, $\sigma(nz,\tau)$. These same relations can also be obtained from the above relations for the first theta function, supplemented with Equation (19) or Equation (20). The present author, however, was able to find only the relation analogous to Equation (35) for the Weierstrass $\sigma(nz,\tau)$ function, formulae 23.10.13–23.10.16 in [10], (with a misprint in 23.10.14: instead of $A_n = n\prod_{j=0}^{n-1}\prod_{\substack{l=0\\ |j|+|l|\neq 0}}^{n-1}\sigma^{-1}(\frac{2j\omega_1}{n}+\frac{2l\omega_3}{n})$ it is written $A_n = n\prod_{j=0}^{n-1}\prod_{\substack{l=0\\ l\neq j}}^{n-1}\sigma^{-1}(\frac{2j\omega_1}{n}+\frac{2l\omega_3}{n})$), but not for the theta functions.

4.4. Fundamental Modular Transformations and Jacobi's Triple Product

Let us briefly discuss the fundamental modular transformations. What is presented below are the known results, but now they are obtained in a very transparent and easy way.

Analogously to what has been done above, we establish $\theta_1(z|\tau+1) = C_1\theta_1(z|\tau)$: the possible exponential factor $\exp(Az^2+Bz) \equiv 1$. (Any $A \neq 0$ is incompatible with the z-periodicity, while $B = 0$ simply due to the evenness of the functions $\ln(\theta_1(z|\tau)/z)$).

Similarly, $\theta_1(z|-\frac{1}{\tau}) = C_2\exp(\frac{i\tau z^2}{\pi})\theta_1(\tau z|\tau)$. Here, in the possible factor $\exp(Az^2+Bz)$, $B = 0$ for the same reason as in the previous relation, while to establish the coefficient A we used the Taylor expansion (13a). But this calculation requires certain caution! Clearly,

$$\delta_2(-1/\tau) = \sum_{n=-\infty}^{\infty}\sum_{\substack{m=-\infty\\ |m|+|n|\neq 0}}^{\infty}\frac{1}{(m-n/\tau)^2} = \tau^2\sum_{n=-\infty}^{\infty}\sum_{\substack{m=-\infty\\ |m|+|n|\neq 0}}^{\infty}\frac{1}{(m\tau+n)^2} \quad (41)$$

where $\delta_2(\tau)$ is defined by Equation (13a), so that one might decide that $\frac{d^2}{dz^2}\ln(\theta_1(z|-\frac{1}{\tau})/z)|_{z=0} = \frac{d^2}{dz^2}\ln(\theta_1(\tau z|\tau)/z)|_{z=0}$ and $A = 0$ which is clearly impossible: $\theta_1(z|-\frac{1}{\tau})$ and $\theta_1(\tau z|\tau)$ have different incompatible periods. However, the *order* of the summation in the r.h.s of Equation (41) is different from that of Equation (13a), so we need to recall the

Eisenstein relation $\sum_{m=-\infty}^{\infty} \sum_{\substack{n=-\infty \\ |m|+|n|\neq 0}}^{\infty} \frac{1}{(m+n\tau)^2} = \delta_2(\tau) - \frac{2\pi i}{\tau}$ (see the discussion and further references in our recent paper [7]), and in such a way we obtain $A = \frac{i\tau}{\pi}$.

Of course, $C_1 = \frac{\theta_1{}'(0|1+\tau)}{\theta_1{}'(0|\tau)}$ and $C_2 = \frac{\theta_1{}'(0|-1/\tau)}{\tau\theta_1{}'(0|\tau)}$. To move further, the use of the original definitions Equation (11) is necessary to obtain [10]:

$$\theta_1(z|1+\tau) = i^{1/2}\theta_1(z|\tau) \tag{42}$$

$$\theta_1(z|-\frac{1}{\tau}) = -i(-i\tau)^{1/2}\exp(\frac{i\tau z^2}{\pi})\theta_1(\tau z|\tau) \tag{43}$$

Finally, let us look at the Jacobi triple product. We note that a whole function, having zeroes at $m + j\tau$, that is coinciding with those of the function $\theta_1(\pi z|\tau)$, can be written as $\psi_1(z,\tau) = \prod_{j=0}^{\infty}(1 - e^{2\pi i((j+1)\tau - z)})(1 - e^{2\pi i(j\tau+z)})$, and similar expressions can be obtained for other theta functions. (This might not look too familiar because the more standard terminology/notation in the field is the use of $x = e^{2\pi iz}$, $q = e^{2\pi i\tau}$ (or $x = e^{\pi iz}$ and the nome $q = e^{\pi i\tau}$) and $(x;q) = \prod_{j=0}^{\infty}(1 - xq^j)$; then we write that $\psi_1(z,\tau) \equiv (x;q)(q/x;q)$, etc.). Application of Lemma 1 immediately enables us to write (in somewhat unusual notation): $\theta_1(\pi z|\tau) = C(\tau)\exp(C_1(\tau)z)(x;q)(q/x;q)$, where the possible quadratic in z $\exp(C_2(\tau)z^2)$ term is again excluded by periodicity. The exact value of C_1 is calculated comparing the values of the derivatives of the logarithms at $z = 0$. Trivially, $\frac{d}{dz}(\ln\frac{\theta_1(\pi z|\tau)}{z})|_{z=0} = 0$, while for the r.h.s we have $(x;q)(q/x;q)/z = \prod_{j=0}^{\infty}(1 - xq^j)(1 - q^{j+1}/x)/z = \frac{1-x}{z}\prod_{j=1}^{\infty}(1 - xq^j)(1 - q^j/x)$, and further $\lim_{x\to 1}\frac{d}{dx}\ln(\frac{1-x}{z}) = \lim_{x\to 1}\frac{d}{dx}\ln\frac{1-x}{2\pi i\ln x} = \lim_{\delta\to 0}\frac{d}{d\delta}\ln\frac{-\delta}{\ln(1+\delta)} = \frac{1}{2}$, together with $\lim_{x\to 1}(\frac{d}{dx}\ln(\prod_{j=1}^{\infty}(1 - xq^j)(1 - q^j/x))) = \lim_{x\to 1}(\sum_{j=1}^{\infty}(-\frac{q^j}{1-xq^j} + \frac{q^j/x^2}{1-q^j/x})) = 0$. Finally, $\lim_{z\to 0}\frac{d}{dz}\ln[(x;q)(q/x;q)/z] = \lim_{z\to 0}\frac{dx}{dz}\cdot\lim_{x\to 1}\frac{d}{dx}\ln[(x;q)(q/x;q)/z] = 2\pi i\cdot\lim_{x\to 1}\frac{d}{dx}\ln[(x;q)(q/x;q)/z] = \pi i$, so that $C_1(\tau) = -i\pi$. Using the well-known value $\theta_1{}'(0,q) = 2q^{1/8}\prod_{j=1}^{\infty}(1-q^j)^3$—see Equation (21) but reminding that now $q = e^{2\pi i\tau}$—we find $C(\tau) = ie^{\pi i\tau/4}(q;q)$ and thus restore (cf. e.g., [19])

$$\theta_1(\pi z|\tau) = ie^{\pi i(\tau/4 - z)}(x;q)(q/x;q)(q;q) \tag{44}$$

where $(q;q) = \prod_{j=1}^{\infty}(1-q^j)$. Similar relations can be established for other theta functions, including the best known [10]:

$$\theta_3(\pi z,q) = \prod_{n=1}^{\infty}(1-q^n)(1+q^{n-1/2}x)(1+q^{n-1/2}/x) \tag{45}$$

reminding that now $q = e^{2\pi i\tau}$ and $\theta_3(0,q) = \prod_{j=1}^{\infty}(1-q^j)(1+q^{j-1/2})$ [10].

5. Conclusions

We showed how the generalized Littlewood theorem concerning contour integrals of the logarithm of analytical function can be used in the inverse sense to readily enable the establishment of identities between different functions and their transformation rules. As of now, the most interesting applications are realized for “classic” elliptical functions, and our findings apparently include some earlier unknown cases. It is hardly doubtful that this line of research can be continued further. (In particular, the paper devoted to the considerations of n-tuple product rules for the derivative $\wp_z$ of the Weierstrass $\wp_z(z,\tau)$ function, having a very regular picture of poles and zeroes, is currently in preparation). More generally speaking, first of all, the continuation can probably be foreseen for such generalizations of the functions considered as, for example, q-gamma function (see, e.g., [20,21]; in preparation), elliptic gamma function [19], and the whole very reach area of the q-series. We believe this will be interesting and important not only in a purely mathematical sense but, given the broad importance of elliptic functions and q-series for physics and engineering (see, e.g., [16,22]), also far beyond.

We sincerely hope that numerous other applications of this approach, which is difficult to anticipate today, will be found as well.

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References

1. Sekatskii, S.K.; Beltraminelli, S.; Merlini, D. On equalities involving integrals of the logarithm of the Riemann ζ-function and equivalent to the Riemann hypothesis. *Ukr. Math. J.* **2012**, *64*, 218–228. [CrossRef]
2. Sekatskii, S.K.; Beltraminelli, S.; Merlini, D. On Equalities Involving Integrals of the Logarithm of the Riemann ζ-function with Exponential Weight which are Equivalent to the Riemann Hypothesis. *Int. J. Anal.* **2015**, *2015*, 980728. [CrossRef]
3. Titchmarsh, E.C. *The Theory of Functions*; Oxford Univ. Press: Oxford, UK, 1939.
4. Titchmarsh, E.C.; Heath-Brown, E.R. *The Theory of the Riemann Zeta-Function*; Clarendon Press: Oxford, UK, 1988.
5. Sekatskii, S.K. On the generalized Li's criterion equivalent to the Riemann hypothesis and its first applications. In *Schrödinger Operators, Spectral Analysis and Number Theory: In Memory of Erik Balslev*; Albeverio, S., Balslev, A., Weder, R., Eds.; Springer Proc. Math. Stat., V. 348; Springer Nature: Cham, Switzerland, 2021; pp. 241–254. [CrossRef]
6. Broughan, K.A. *Equivalents of the Riemann Hypothesis, Vol. 2 Encyclopedia of Mathematics and its Applications*; Cambridge Univ. Press: Cambridge, UK, 2017; p. 165.
7. Sekatskii, S.K. On the Use of the Generalized Littlewood Theorem Concerning Integrals of the Logarithm of Analytical Functions to Elliptic Functions. *Axioms* **2023**, *15*, 595. [CrossRef]
8. Erdélyi, A.; Magnus, W.; Oberhettinger, F.; Tricomi, F.G. *Higher Transcendental Functions*; McGraw-Hill Book Company Inc.: New York, NY, USA, 1953; Volume I.
9. NIST Digital Library of Mathematical Functions, DLMF: NIST Digital Library of Mathematical Functions, Chapter 5. Available online: https://dlmf.nist.gov/ (accessed on 10 July 2024).
10. NIST Digital Library of Mathematical Functions, DLMF: NIST Digital Library of Mathematical Functions, Chapters 20, 22 and 23. Available online: https://dlmf.nist.gov/ (accessed on 10 July 2024).
11. Lawden, D.F. *Elliptic Functions and Applications*; Applied Mathematical Sciences; Springer: New York, NY, USA, 1989; Volume 80.
12. Erdélyi, A.; Magnus, W.; Oberhettinger, F.; Tricomi, F.G. *Higher Transcendental Functions*; McGraw-Hill Book Company Inc.: New York, NY, USA, 1953; Volume II.
13. Abramowitz, M.; Stegun, I.A. (Eds.) *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*; National Bureau of Standards Applied Mathematics Series; U.S. Government Printing Office: Washington, DC, USA, 1964.
14. Whittaker, E.T.; Watson, G.N. *A Course of Modern Analysis*, 4th ed.; Cambridge University Press: Cambridge, UK, 1927.
15. Neville, E.H. *Jacobian Elliptic Functions*, 2nd ed.; Clarendon Press: Oxford, UK, 1951.
16. Pastras, G. *The Weierstrass Elliptic Function and Applications in Classical and Quantum Mechanics. A Primer for Advanced Undergraduates*; Springer: Cham, Switzerland, 2020.
17. Eichler, M.; Zagier, D. On the zeroes of the Weierstrass ℘-function. *Math. Ann.* **1982**, *258*, 399–407. [CrossRef]
18. Duke, W.; Imamoglu, O. The zeroes of the Weierstrass ℘-function and hypergeometric series. *Math. Ann.* **2008**, *340*, 897–905. [CrossRef]
19. Felder, G.; Varchenko, A. The elliptic Gamma function and SL(3,Z) × Z. *Adv. Math.* **2000**, *156*, 44–76. [CrossRef]
20. Jackson, F.H. The Basic Gamma-Function and the Elliptic Functions. *Proc. R. Soc. Lond. Ser. A Contain. Pap. A Math. Phys. Character* **1905**, *76*, 127–144.
21. Askey, R. The q-Gamma and q-Beta functions. *Appl. Anal.* **1978**, *8*, 125–141. [CrossRef]
22. Exton, H. *q-Hypergeometric Function and Applications*; Halstead Press: Ultimo, Australia, 1983.

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