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The first k-invariant of a double loop space is trivial

By

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Introduction. Let X be a connected simple CW-complex and let $X[n]$ denote the n-th Postnikov section of X: $X[n]$ is a CW-complex obtained from X by adjoining cells of dimension $\geqq n+2$, such that $\pi_i X[n] = 0$ for $i > n$ and $\pi_i X[n] \cong \pi_i X$ for $i \leqq n$ (for example $X[1] = K(\pi_1 X, 1)$). The k-invariants $k^{n+1}(X)$ of X are maps $X[n-1] \to K(\pi_n X, n+1)$ and therefore cohomology classes in $H^{n+1}(X[n-1]; \pi_n X)$, $n \geqq 2$. All spaces we consider in this paper are connected CW-complexes. Our main objective is to show the following

Theorem A. *If X is a connected double loop space, then its first k-invariant $k^3(X) \in H^3(K(\pi_1 X, 1); \pi_2 X)$ satisfies*

$$k^3(X) = 0.$$

We prove actually a more general result. Let us define, for each prime number p, L_p as the product of all primes $q < p$ ($L_2 = 1$).

Theorem B. *Let p be a prime number and $n := 2p - 2$. If X is a connected n-fold loop space ($X \cong \Omega^n Y$ for some CW-complex Y) such that $\pi_i X = 0$ for $1 < i < n$ (i.e., $X[n-1] = K(\pi_1 X, 1)$), then the first non-trivial k-invariant of X, $k^{n+1}(X) \in H^{n+1}(K(\pi_1 X, 1); \pi_n X)$, satisfies*

$$L_p k^{n+1}(X) = 0.$$

Theorem A corresponds to the case $p = 2$. Both theorems are true without any finiteness condition on the space X.

The paper is organized as follows. In Sections 1 and 2 we establish some preliminaries which will be used in the proof of the main theorems: in Section 1 we describe the homomorphism induced in homology by the diagonal map $K \to (K \times K, K \vee K)$, when K is an Eilenberg-MacLane space $K(G, 2)$ and G an arbitrary abelian group; in Section 2 we look at the corresponding homomorphism in cohomology and at the cohomology suspension $\sigma^*: H^*(K(G, 2); M) \to H^{*-1}(K(G, 1); M)$. Section 3 is then devoted to the proof of Theorems A and B. In Section 4 we give a second proof by showing that these theorems are actually equivalent to well-known results on classical cohomology operations. Finally, some consequences concerning the Hurewicz homomorphism are discussed in Section 5.

1. The integral homology of $K(G, 2)$ and the diagonal map. The purpose of this section is to study the homomorphism induced in homology by the diagonal map

$$d\colon K \to (K \times K, K \vee K),$$

when K is an Eilenberg-MacLane space $K(G,2)$ and G an arbitrary abelian group. This will be necessary in order to understand the (co)homology suspension, which we use in the proof of the main theorems. Note that d is actually the composition $\lambda \circ \Delta$, where Δ is the diagonal map $K \to K \times K = K(G \oplus G, 2)$ and λ the inclusion $K \times K \hookrightarrow (K \times K, K \vee K)$.

Throughout this section we use Cartan's description of $H_*(K(G,2);\mathbb{Z})$ [2, Exposé 11] and the following notation: $E(y, 2j+1)$ denotes the exterior algebra (over $\mathbb{Z}$) generated by y of degree $2j+1$ and $P(x, 2j)$ the divided power algebra with one generator x of degree $2j$ ($P(x,2j) = \mathbb{Z}[\gamma_1(x) = x, \gamma_2(x), \ldots]$, $\deg \gamma_m(x) = 2jm$, $\gamma_m(x)\gamma_n(x) = \frac{(m+n)!}{m!\,n!}\gamma_{m+n}(x)$). We start by assuming that G is a cyclic group.

Lemma 1.1. *Let K be $K(\mathbb{Z}, 2)$. Then the induced homomorphism*

$$d_*\colon H_i(K;\mathbb{Z}) \to H_i(K \times K, K \vee K;\mathbb{Z})$$

maps $H_i(K;\mathbb{Z})$ injectively onto a direct summand of $H_i(K \times K, K \vee K;\mathbb{Z})$ if $i > 2$ and $d_*(H_2(K;\mathbb{Z})) = 0$.

Proof. It is known that $H_*(K;\mathbb{Z}) \cong P(x,2)$; the induced homomorphism $\Delta_*\colon H_*(K;\mathbb{Z}) \to H_*(K \times K;\mathbb{Z})$ satisfies clearly $\Delta_*(x) = x \otimes 1 + 1 \otimes x$ and for $m \geqq 1$, according to [2, Exposé 7],

$$\begin{aligned}\Delta_*(\gamma_m(x)) = \gamma_m(\Delta_*(x)) = \gamma_m(x \otimes 1 + 1 \otimes x) &= \sum_{k=0}^{m} \gamma_k(x \otimes 1)\,\gamma_{m-k}(1 \otimes x)\\ &= \sum_{k=0}^{m} \gamma_k(x) \otimes \gamma_{m-k}(x).\end{aligned}$$

Since the homomorphism $\lambda_*\colon H_{2m}(K \times K;\mathbb{Z}) \to H_{2m}(K \times K, K \vee K;\mathbb{Z})$ is surjective with kernel $H_{2m}(K;\mathbb{Z}) \otimes H_0(K;\mathbb{Z}) \oplus H_0(K;\mathbb{Z}) \otimes H_{2m}(K;\mathbb{Z})$, we get $d_*(\gamma_m(x)) = \lambda_*\Delta_*(\gamma_m(x)) = \sum_{k=1}^{m-1} \gamma_k(x) \otimes \gamma_{m-k}(x)$. Consequently, $d_*(H_2(K;\mathbb{Z})) = 0$ but, if $m > 1$, $d_*\colon H_{2m}(K;\mathbb{Z}) \to H_{2m}(K \times K, K \vee K;\mathbb{Z})$ is split injective.

Lemma 1.2. *Let p be a prime number, r a positive integer, $K = K(\mathbb{Z}/p^r, 2)$ and let us consider the homomorphism $d_*\colon H_i(K;\mathbb{Z}) \to H_i(K \times K, K \vee K;\mathbb{Z})$.*

(a) *d_* is injective for $2 < i < 2p$; $d_*(H_2(K;\mathbb{Z})) = 0$; if $i = 2p$, the kernel of d_* is the cyclic subgroup of order p of $H_{2p}(K;\mathbb{Z}) \cong \mathbb{Z}/p^{r+1}$;*

(b) *d_* maps $H_i(K;\mathbb{Z})$ onto a direct summand of $H_i(K \times K, K \vee K;\mathbb{Z})$ for $0 \leqq i \leqq 2p$.*

Proof. For $i \geqq 2$, $H_i(K;\mathbb{Z})$ is the p-torsion subgroup of $H_i(C \otimes D)$, where C is the complex $P(x_0, 2) \otimes E(y_0, 3)$ with the differential $\delta(x_0) = 0$, $\delta(y_0) = -p^r x_0$, $D = \bigotimes_{j=1}^{\infty} D_j$ and for $j \geqq 1$, D_j is the complex $E(x_j, 2p^j + 1) \otimes P(y_j, 2p^j + 2)$ with $\delta(x_j) = 0$,

$\delta(y_j) = px_j$ (cf. [2, Exposé 11]). Thus, for $2 \leqq i \leqq 2p$, $H_i(K;\mathbb{Z})$ is the p-torsion subgroup of $H_i(C)$. Observe that

$$H_i(C) \cong \begin{cases} 0, \text{ if } i \text{ is odd,} \\ \mathbb{Z}/mp^r, \text{ generated by } \gamma_m(x_0), \text{ if } i = 2m. \end{cases}$$

In particular, $H_{2m}(K;\mathbb{Z})$ is cyclic of order p^r for $1 \leqq m < p$ and of order p^{r+1} if $m = p$. Because $d_*(\gamma_m(x_0)) = \sum_{k=1}^{m-1} \gamma_k(x_0) \otimes \gamma_{m-k}(x_0)$ as in the previous proof, we conclude that $d_*(H_2(K;\mathbb{Z})) = 0$ and that $d_*(H_{2m}(K;\mathbb{Z}))$ is a cyclic direct summand of order p^r of $H_{2m}(K \times K, K \vee K;\mathbb{Z})$ for $1 < m \leqq p$.

We are now able to consider arbitrary abelian groups.

Lemma 1.3. *Let G be an abelian group, $K = K(G, 2)$, p a prime and i an integer satisfying $0 \leqq i \leqq 2p$. Let us call T (respectively S) the subgroup of all torsion elements of $H_i(K;\mathbb{Z})$ (resp. of $H_i(K \times K, K \vee K;\mathbb{Z})$) whose order is not divisible by p. Then the image of the homomorphism $H_i(K;\mathbb{Z})/T \to H_i(K \times K, K \vee K;\mathbb{Z})/S$ induced by d_* is a direct summand of $H_i(K \times K, K \vee K;\mathbb{Z})/S$.*

Proof. For finitely generated abelian groups G, the inclusion of the image of $H_i(K;\mathbb{Z})/T$ into $H_i(K \times K, K \vee K;\mathbb{Z})/S$ splits naturally by Lemmas 1.1 and 1.2. Now let G be an arbitrary abelian group and $\{G_\alpha\}_{\alpha \in A}$ the set of its finitely generated subgroups: $G = \varinjlim G_\alpha$ and $K = \varinjlim K(G_\alpha, 2)$. The lemma follows from the fact that the diagonal map and homology commute with direct limits.

Lemma 1.4. *Let G be an abelian group, $K = K(G, 2)$ and p a prime number.*

(a) *If z belongs to the kernel of $d_*\colon H_{2p-1}(K;\mathbb{Z}) \to H_{2p-1}(K \times K, K \vee K;\mathbb{Z})$, then z is a torsion element whose order is not divisible by p.*
(b) *If z belongs to the kernel of $d_*\colon H_{2p}(K;\mathbb{Z}) \to H_{2p}(K \times K, K \vee K;\mathbb{Z})$, then there exists an element $y \in H_{2p}(K;\mathbb{Z})$ such that $z = py$.*

Proof. (a) We first assume that G is finitely generated. Lemma 1.1 implies that z is a torsion element; let mp^r be its order (with $(m, p) = 1$, $r \geqq 0$). Since mz is a p-torsion element of the kernel of d_*, we deduce from Lemma 1.2(a) that $mz = 0$, i.e., z is of order m. Now, if G is an arbitrary abelian group and $\{G_\alpha\}_{\alpha \in A}$ the set of its finitely generated subgroups, let K_α denote $K(G_\alpha, 2)$ and $\theta_\alpha\colon H_*(K_\alpha;\mathbb{Z}) \to H_*(K;\mathbb{Z})$ the homomorphism induced by the inclusion $G_\alpha \hookrightarrow G$. If $z \in H_{2p-1}(K;\mathbb{Z})$ satisfies $d_*(z) = 0$, there exists an $\alpha \in A$ and a $z_\alpha \in H_{2p-1}(K_\alpha;\mathbb{Z})$ such that $d_*(z_\alpha) = 0$ and $\theta_\alpha(z_\alpha) = z$, because $H_{2p-1}(K;\mathbb{Z}) = \varinjlim H_{2p-1}(K_\alpha;\mathbb{Z})$. We have seen that the order of z_α is not divisible by p; of course the same is true for the order of z.

(b) We start again by considering a finitely generated group G; let $H \cong \mathbb{Z}/p^{r_1} \oplus \cdots \oplus \mathbb{Z}/p^{r_n}$ be its p-torsion subgroup. As above, it is clear that z is a torsion element, of order mp^r (with $(m, p) = 1$, $r \geqq 0$). Assertion (b) is trivial when $r = 0$. If r is positive, it follows from Lemma 1.2(a) that mz is an element of $H_{2p}(K(\mathbb{Z}/p^{r_1}, 2);\mathbb{Z}) \oplus \cdots \oplus H_{2p}(K(\mathbb{Z}/p^{r_n}, 2);\mathbb{Z}) \cong \mathbb{Z}/p^{r_1+1} \oplus \cdots \oplus \mathbb{Z}/p^{r_n+1}$ and that $pmz = 0$. Therefore $mz = pw$ for some $w \in H_{2p}(K;\mathbb{Z})$; this proves (b) since $(m, p) = 1$. If G is an arbitrary abelian group, we proceed as above. Let z be an element of $H_{2p}(K;\mathbb{Z})$ with $d_*(z) = 0$. There is an $\alpha \in A$ and a $z_\alpha \in H_{2p}(K_\alpha;\mathbb{Z})$ such that $d_*(z_\alpha) = 0$ and $\theta_\alpha(z_\alpha) = z$. But we have just established the existence of a $y_\alpha \in H_{2p}(K_\alpha;\mathbb{Z})$ satisfying $z_\alpha = p y_\alpha$; consequently, $z = py$ where $y := \theta_\alpha(y_\alpha)$.

Let us recall the following result on the stable homology of Eilenberg-MacLane spaces (cf. [2, Exposé 11, Thm. 2]).

Lemma 1.5. *Let M_j be the product of all prime numbers $q \leqq \frac{j}{2} + 1$, for $j \geqq 1$ ($M_1 = 1$). Then for any abelian group G and for each integer $n \geqq 2$, one has*

$$M_{i-n} H_i(K(G,n);\mathbb{Z}) = 0$$

if $n < i < 2n$.

Corollary 1.6. *For a prime p, let us call L_p the product of all prime numbers $q < p$ ($L_2 = 1$). Let G be an abelian group, p a prime number and $n := 2p - 2$. Then the iterated homology suspension $(\sigma_*)^n$: $H_{n+1}(K(G,1);\mathbb{Z}) \to H_{2n+1}(K(G,n+1);\mathbb{Z})$ satisfies*

$$L_p(\sigma_*)^n(H_{n+1}(K(G,1);\mathbb{Z})) = 0.$$

Proof. We use again the notation $K = K(G,2)$ and consider the sequence

$$H_{n+1}(K(G,1);\mathbb{Z}) \xrightarrow{\sigma_*} H_{n+2}(K;\mathbb{Z}) \xrightarrow{d_*} H_{n+2}(K \times K, K \vee K;\mathbb{Z})$$

where $d_* \circ \sigma_* = 0$ by [3, p. 382]. If $x \in H_{n+1}(K(G,1);\mathbb{Z})$, Lemma 1.4(b) implies that $\sigma_*(x) = py$ for some $y \in H_{n+2}(K;\mathbb{Z})$. Consequently, $\frac{M_n}{p}(\sigma_*)^n(x) = \frac{M_n}{p}(\sigma_*)^{n-1}(py) = M_n(\sigma_*)^{n-1}(y) = 0$ in $H_{2n+1}(K(G,n+1);\mathbb{Z})$ by the previous lemma. The corollary is then proved since $L_p = \frac{M_n}{p}$.

2. The cohomology suspension. In this section we are interested in the homomorphism induced in cohomology by the diagonal map d, in order to study the cohomology suspension σ^*: $H^*(K(G,2);M) \to H^{*-1}(K(G,1);M)$.

Lemma 2.1. *Let p be a prime number, G and M abelian groups, $K = K(G,2)$ and let $d^\natural$: $\mathrm{Hom}(H_{2p}(K \times K, K \vee K;\mathbb{Z}), M) \to \mathrm{Hom}(H_{2p}(K;\mathbb{Z}), M)$ denote the homomorphism induced by d. If $x \in \mathrm{Hom}(H_{2p}(K;\mathbb{Z}), M)$ fulfills $px = 0$, then there exists a $w \in \mathrm{Hom}(H_{2p}(K \times K, K \vee K;\mathbb{Z}), M)$ such that $d^\natural(w) = x$ and $pw = 0$.*

Proof. We consider d_*: $H_{2p}(K;\mathbb{Z}) \to H_{2p}(K \times K, K \vee K;\mathbb{Z})$ and call N its kernel, I its image, and ψ and ϕ the inclusions $N \hookrightarrow H_{2p}(K;\mathbb{Z})$ and $I \hookrightarrow H_{2p}(K \times K, K \vee K;\mathbb{Z})$ respectively. We get the exact sequence

$$0 \longrightarrow \mathrm{Hom}(I, M) \xrightarrow{d^\natural} \mathrm{Hom}(H_{2p}(K;\mathbb{Z}), M) \xrightarrow{\psi^\natural} \mathrm{Hom}(N, M) \longrightarrow \cdots$$

If z is an element of N, then $\psi(z) = py$ for some $y \in H_{2p}(K;\mathbb{Z})$ by Lemma 1.4(b); therefore $\psi^{\natural}(x)(z) = x(\psi(z)) = x(py) = px(y) = 0$. We obtain $\psi^{\natural}(x) = 0$ and the existence of an element $v \in \mathrm{Hom}(I, M)$ with $d^{\natural}(v) = x$ and $pv = 0$. Now let R (respectively S) be the subgroup of all torsion elements of I (resp. of $H_{2p}(K \times K, K \vee K;\mathbb{Z})$) whose order is not divisible by p and let us consider the diagram

$$\begin{array}{ccccccc}
 & & \mathrm{Hom}(H_{2p}(K \times K, K \vee K;\mathbb{Z})/S, M) & \longrightarrow & \mathrm{Hom}(H_{2p}(K \times K, K \vee K;\mathbb{Z}), M) & & \\
 & & \downarrow{\scriptstyle \phi^{\sim}} & & \downarrow{\scriptstyle \phi^{\natural}} & & \\
0 & \longrightarrow & \mathrm{Hom}(I/R, M) & \xrightarrow{\ \mu\ } & \mathrm{Hom}(I, M) & \xrightarrow{\ v\ } & \mathrm{Hom}(R, M),
\end{array}$$

where $\phi^{\natural}$ and $\phi^{\sim}$ are induced by ϕ. Since multiplication by p: $R \to R$ is an isomorphism, multiplication by p: $\mathrm{Hom}(R, M) \to \mathrm{Hom}(R, M)$ is also an isomorphism: consequently, $v(v) = 0$ because $pv(v) = v(pv) = 0$, and v belongs to the image of μ by exactness of the bottom sequence. Lemma 1.3 and the commutativity of the diagram complete the proof.

Lemma 2.2. *Let p be a prime number, G and M abelian groups, $K = K(G, 2)$ and let $d^{\#}$: $\mathrm{Ext}(H_{2p-1}(K \times K, K \vee K;\mathbb{Z}), M) \to \mathrm{Ext}(H_{2p-1}(K;\mathbb{Z}), M)$ denote the homomorphism induced by d. If $x \in \mathrm{Ext}(H_{2p-1}(K;\mathbb{Z}), M)$ is such that $px = 0$, then x belongs to the image of $d^{\#}$.*

P r o o f. Let T (respectively S) be the subgroup of all torsion elements of $H_{2p-1}(K;\mathbb{Z})$ (resp. of $H_{2p-1}(K \times K, K \vee K;\mathbb{Z})$) whose order is not divisible by p. Since $px = 0$ a similar argument shows that x is in the image of the homomorphism μ: $\mathrm{Ext}(H_{2p-1}(K;\mathbb{Z})/T, M) \to \mathrm{Ext}(H_{2p-1}(K;\mathbb{Z}), M)$ induced by the canonical surjection. According to Lemmas 1.3 and 1.4(a), the homomorphism $d^{\sim}$: $\mathrm{Ext}(H_{2p-1}(K \times K, K \vee K;\mathbb{Z})/S, M) \to \mathrm{Ext}(H_{2p-1}(K;\mathbb{Z})/T, M)$ induced by d is surjective. Therefore x is an element of the image of $\mu \circ d^{\sim}$ and thus, also of the image of $d^{\#}$.

Corollary 2.3. *Let p be a prime number, G and M abelian groups and let σ^*: $H^{2p}(K(G,2); M) \to H^{2p-1}(K(G,1); M)$ denote the cohomology suspension. If $x \in H^{2p}(K(G,2); M)$ satisfies $px = 0$, then $\sigma^*(x) = 0$.*

P r o o f. Let d^*: $H^{2p}(K \times K, K \vee K; M) \to H^{2p}(K; M)$ be the homomorphism induced by d, where $K = K(G, 2)$, and let us look at the short exact sequence given by the universal coefficient theorem:

$$\mathrm{Ext}(H_{2p-1}(K;\mathbb{Z}), M) \overset{\tau}{\rightarrowtail} H^{2p}(K; M) \overset{\varrho}{\twoheadrightarrow} \mathrm{Hom}(H_{2p}(K;\mathbb{Z}), M).$$

By Lemma 2.1, there exists an element $w \in H^{2p}(K \times K, K \vee K; M)$ such that $\varrho(d^*(w)) = \varrho(x)$ and $pw = 0$. Consequently, $x - d^*(w)$ belongs to the image of τ and $p(x - d^*(w)) = 0$. But Lemma 2.2 then implies that $x - d^*(w)$ (and, of course, also x) belongs to the image of d^*. Finally the sequence

$$H^{2p}(K \times K, K \vee K; M) \xrightarrow{d^*} H^{2p}(K; M) \xrightarrow{\sigma^*} H^{2p-1}(K(G,1); M),$$

where $\sigma^* \circ d^* = 0$ (cf. [3, p. 383]), produces the assertion.

3. Proof of Theorems A and B.

Proposition 3.1. *Let G and M be abelian groups, p a prime number, $n := 2p - 2$ and L_p the product of all primes $q < p$. Then the n-fold iterated cohomology suspension $(\sigma^*)^n$: $H^{2n+1}(K(G, n+1); M) \to H^{n+1}(K(G, 1); M)$ satisfies:*

$$L_p(\sigma^*)^n(y) = 0$$

for all $y \in H^{2n+1}(K(G, n+1); M)$.

Proof. Let us begin by noting that Lemma 1.5 and the universal coefficient theorem imply that $M_n y = 0$ and, consequently, that $M_n(\sigma^*)^n(y) = 0$ (in the special case $p = 2$, that $2(\sigma^*)^2(y) = 0$). Now we want to show that, in fact, $L_p(\sigma^*)^n(y) = 0$, where $L_p = \frac{M_n}{p}$ (in particular if $p = 2$, that $(\sigma^*)^2(y) = 0$). Let us call x the element $(\sigma^*)^{n-1}(L_p y) \in H^{n+2}(K(G, 2); M)$; clearly, x fulfills $px = 0$. Thus we may apply Corollary 2.3: $L_p(\sigma^*)^n(y) = \sigma^*(x) = 0$.

This proposition enables us to prove Theorems A and B. Let p be a prime, $n := 2p - 2$, Y an n-connected CW-complex and $X = \Omega^n Y$. We assume that $\pi_i X = 0$ for $1 < i < n$ and define $G := \pi_1 X \cong \pi_{n+1} Y$ and $M := \pi_n X \cong \pi_{2n} Y$. The first non-trivial k-invariant of X, $k^{n+1}(X) \in H^{n+1}(K(G, 1); M)$, and that of Y, $k^{2n+1}(Y) \in H^{2n+1}(K(G, n+1); M)$, are related by the formula

$$(\sigma^*)^n(k^{2n+1}(Y)) = k^{n+1}(X),$$

where $(\sigma^*)^n$ is the n-fold iterated cohomology suspension [3, p. 438]. Therefore, Proposition 3.1 provides the assertion

$$L_p k^{n+1}(X) = 0.$$

For instance if $p = 2$, the conclusion is: $k^3(X) = 0$.

4. Classical cohomology operations. We have just established the equivalence of the assertions of Proposition 3.1 and Theorem B (respectively Theorem A if $p = 2$). The purpose of this section is to notice that these assertions are actually equivalent to known results on classical cohomology operations; this provides a second proof of our main theorems.

Let G be an abelian group, p a prime number, $n := 2p - 2$, and as usual let $P_p^1 \in H^{2n+1}(K(G, n+1); G/pG)$ denote the Steenrod operation of degree n (cf. [2, Exposé 15]); if $p = 2$, we interpret P_2^1 as the Steenrod square $Sq^2 \in H^5(K(G, 3); G/2G)$. This is a stable cohomology operation and it is well-known that

$$(*) \qquad (\sigma^*)^n(P_p^1) = 0,$$

where $(\sigma^*)^n$: $H^{2n+1}(K(G, n+1); G/pG) \to H^{n+1}(K(G, 1); G/pG)$ is the n-fold iterated cohomology suspension. The vanishing of $(\sigma^*)^n(P_p^1)$ is equivalent to the following

Proposition 4.1. *Let G and M be abelian groups, p a prime and $n := 2p - 2$. If an element $u \in H^{2n+1}(K(G, n+1); M)$ is such that $pu = 0$, then the n-fold iterated cohomology suspension $(\sigma^*)^n$: $H^{2n+1}(K(G, n+1); M) \to H^{n+1}(K(G, 1); M)$ satisfies*

$$(\sigma^*)^n(u) = 0.$$

Proof. Consider the isomorphism given by the universal coefficient theorem

$$H^{2n+1}(K(G, n+1); M) \cong \mathrm{Hom}(H_{2n+1}(K(G, n+1); \mathbb{Z}), M) \oplus \mathrm{Ext}(H_{2n}(K(G, n+1); \mathbb{Z}), M),$$

and note that the p-primary component of $H_{2n+1}(K(G, n+1); \mathbb{Z})$, respectively $H_{2n}(K(G, n+1); \mathbb{Z})$, is G/pG, respectively 0 (cf. [2, Exposé 11]). Since $pu = 0$, u corresponds to an element $\bar{u}$ of $\mathrm{Hom}(G/pG, M)$ and we write $\bar{u}_*$ for the induced homomorphism $H^*(-; G/pG) \to H^*(-; M)$. For instance, the element $\overline{P_p^1} \in \mathrm{Hom}(G/pG, G/pG)$, corresponding to P_p^1, is the identity. Therefore, it turns out that $u = \bar{u}_*(P_p^1)$. It is then easy to complete the proof: $(\sigma^*)^n(u) = (\sigma^*)^n(\bar{u}_*(P_p^1)) = \bar{u}_*((\sigma^*)^n(P_p^1)) = 0$.

Remark 4.2. Proposition 3.1 is an immediate consequence of Proposition 4.1 because $pL_p y = 0$ for all $y \in H^{2n+1}(K(G, n+1); M)$. On the other hand, if u is an element of $H^{2n+1}(K(G, n+1); M)$ such that $pu = 0$, then $p(\sigma^*)^n(u) = 0$ and Proposition 3.1 (i.e., $L_p(\sigma^*)^n(u) = 0$) implies that $(\sigma^*)^n(u) = 0$ since p doesn't divide L_p. Thus, the statements of Propositions 3.1 and 4.1 (and consequently also Theorem B and assertion (*)) are equivalent.

In the case $p = 2$, Theorem A corresponds to the vanishing of $(\sigma^*)^2(Sq^2)$, where $Sq^2 \in H^5(K(G,3); G/2G)$ is the Steenrod square and $(\sigma^*)^2$ is the double suspension $H^5(K(G,3); G/2G) \to H^4(K(G,2); G/2G) \to H^3(K(G,1); G/2G)$ (observe that $\sigma^*(Sq^2)$ is the cup-square). We close this section by mentioning related results on Postnikov and Pontryagin squares.

Let $P_0 \in H^3(K(G,1); \Gamma(G))$ be the Postnikov square and $P_1 \in H^4(K(G,2); \Gamma(G))$ the Pontryagin square; these cohomology operations are defined in [5, §5]. Here G is an abelian group and $\Gamma(G)$ is the group introduced in [4, Chapter II].

Lemma 4.3. *Let σ^*: $H^4(K(G,2); \Gamma(G)) \to H^3(K(G,1); \Gamma(G))$ be the cohomology suspension. Then $\sigma^*(P_1) = P_0$.*

Proof. The path fibration $K(G,1) \to PK(G,2) \xrightarrow{h} K(G,2)$ over $K(G,2)$ induces the diagram

$$\begin{array}{ccccc} H^2(K(G,2); G) & \xrightarrow{h^*} & H^2(PK(G,2), K(G,1); G) & \xleftarrow[\cong]{\delta} & H^1(K(G,1); G) \\ \downarrow{\scriptstyle P_1} & & \downarrow{\scriptstyle P_1} & & \downarrow{\scriptstyle P_0} \\ H^4(K(G,2); \Gamma(G)) & \xrightarrow{h^*} & H^4(PK(G,2), K(G,1); \Gamma(G)) & \xleftarrow[\cong]{\delta} & H^3(K(G,1); \Gamma(G)) \end{array}$$

where δ is the connecting homomorphism of the cohomology sequence of the pair $(PK(G,2), K(G,1))$; δ is an isomorphism since $PK(G,2)$ is a contractible space. The diagram commutes because P_1 is natural and $P_1 \circ \delta = \delta \circ P_0$ by [5, (5.5)]. Recall that the suspension σ^* is $\delta^{-1} \circ h^*$ by definition [3, p. 373], and that σ^* maps the characteristic class of $H^2(K(G,2); G)$ onto the characteristic class of $H^1(K(G,1); G)$. It is then obvious that $\sigma^*(P_1) = P_0$.

The Whitehead exact sequences (cf. [4] and [3, p. 555, Thm. 3.12]) of the Eilenberg-MacLane spaces $K(G,2)$ and $K(G,3)$,

$$\cdots \to \underbrace{\pi_4 K(G,2)}_{=0} \to H_4(K(G,2);\mathbb{Z}) \to \Gamma(G) \to \underbrace{\pi_3 K(G,2)}_{=0} \to \cdots$$

and

$$\cdots \to \underbrace{\pi_5 K(G,3)}_{=0} \to H_5(K(G,3);\mathbb{Z}) \to G/2G \to \underbrace{\pi_4 K(G,3)}_{=0} \to \cdots,$$

yield the isomorphisms $H_4(K(G,2);\mathbb{Z}) \cong \Gamma(G)$ and $H_5(K(G,3);\mathbb{Z}) \cong G/2G$. Thus, we may consider the homology suspension $\sigma_*\colon \Gamma(G) \to G/2G$ which induces the homomorphism $\sigma_\bullet\colon H^*(-;\Gamma(G)) \to H^*(-;G/2G)$.

We are interested in $\sigma_\bullet(P_1) \in H^4(K(G,2); G/2G)$: we obtain the definition of this cohomology operation if we replace, in the definition of P_1 given in [5, § 5], $\gamma(g)$ and $[g,g'] \in \Gamma(G)$ by $\sigma_*(\gamma(g))$ and $\sigma_*([g,g'])$ respectively (cf. [4, § 5] for the notation $\gamma(-)$ and $[-,-]$). It turns out that $\sigma_*(\gamma(g))$ is the class of g in $G/2G$ and consequently that $\sigma_*([g,g']) = 0$ (for any $g, g' \in G$). Therefore, $\sigma_\bullet(P_1)$ is exactly the cup-square $\sigma^*(Sq^2) \in H^4(K(G,2); G/2G)$ and our result for $p = 2$ (Theorem A) is equivalent to the assertion $\sigma^*(\sigma_\bullet(P_1)) = 0$, where σ^* denotes the cohomology suspension $H^4(K(G,2); G/2G) \to H^3(K(G,1); G/2G)$. Since $\sigma_\bullet(P_0) = \sigma_\bullet(\sigma^*(P_1)) = \sigma^*(\sigma_\bullet(P_1))$, we get:

Proposition 4.4. $\sigma_\bullet(P_0) = \sigma^*(\sigma_\bullet(P_1)) = 0$.

5. The Hurewicz homomorphism. Let us close with some remarks on the Hurewicz homomorphism. The following conclusion follows immediately from Theorem A.

Corollary 5.1. *If X is a connected double loop space, then*

$$X[2] \cong K(\pi_1 X, 1) \times K(\pi_2 X, 2).$$

Corollary 5.2. *If X is a connected double loop space, then the Hurewicz homomorphism* Hu: $\pi_2 X \to H_2(X;\mathbb{Z})$ *is split injective.*

Proof. The previous corollary implies the existence of a map $f\colon X \to K(\pi_2 X, 2)$, which induces an isomorphism $f_*\colon \pi_2 X \xrightarrow{\cong} \pi_2 X$. Therefore, the composition

$$\pi_2 X \xrightarrow{\text{Hu}} H_2(X;\mathbb{Z}) \xrightarrow{f_*} H_2(K(\pi_2 X, 2);\mathbb{Z})$$

is also an isomorphism and Hu maps $\pi_2 X$ injectively onto a direct summand of $H_2(X;\mathbb{Z})$.

In order to generalize this result to the situation of Theorem B, let us recall the following

Lemma 5.3 [1, Lemma 4]. *Let X be a connected simple CW-complex and assume that the k-invariant $k^{n+1}(X)$ is a cohomology class of finite order s in $H^{n+1}(X[n-1]; \pi_n X)$. Then there exists a map $f\colon X \to K(\pi_n X, n)$ such that the induced homomorphism $f_*\colon \pi_n X \to \pi_n X$ is multiplication by s.*

Corollary 5.4. *Let p be a prime number, $n := 2p-2$ and L_p the product of all primes $q < p$. If X is a connected n-fold loop space such that $\pi_i X = 0$ for $1 < i < n$, then the Hurewicz homomorphism* Hu: $\pi_n\left(X; \mathbb{Z}\left[\frac{1}{L_p}\right]\right) \to H_n\left(X; \mathbb{Z}\left[\frac{1}{L_p}\right]\right)$ *is split injective.*

Proof. We know from Theorem B that the order of $k^{n+1}(X)$ divides L_p. Consequently, the map $f\colon X \to K(\pi_n X, n)$ given by Lemma 5.3 induces an isomorphism $f_*\colon \pi_n\left(X; \mathbb{Z}\left[\frac{1}{L_p}\right]\right) \xrightarrow{\cong} \pi_n\left(K(\pi_n X, n); \mathbb{Z}\left[\frac{1}{L_p}\right]\right)$. The argument used in the proof of Corollary 5.2 gives the desired assertion.

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