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The Hurewicz homomorphism in algebraic K-theory

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Abstract

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This paper investigates the Hurewicz homomorphism $h_n : K_nA \to H_n(E(A); \mathbb{Z})$ between the algebraic K-theory of a ring A and the homology of the linear group $E(A)$ generated by elementary matrices over A. The main theorem asserts that for any $n \geq 2$, the kernel of h_n is a torsion group of finite exponent, and provides an upper bound, independent of A, for its exponent. The proof of this uses the fact that $BE(A)^+$ is an infinite loop space, because its basic idea is to observe that the result follows from the study of the kernel of the Hurewicz homomorphism in the range of stability. The discussion of the problem involves then the description of the relationship between the Hurewicz map and the k-invariants of the space $BE(A)^+$. Finally, some partial information on the cokernel of h_n is also obtained.

Introduction

Let GL(A) be the infinite general linear group (considered as a discrete group) over the ring with identity A, $E(A)$ its subgroup generated by elementary matrices, and $B\text{GL}(A)^+$ and $BE(A)^+$ the infinite loop spaces obtained by performing the plus construction on the classifying spaces of GL(A) and $E(A)$ respectively. The purpose of this paper is to investigate the Hurewicz homomorphism between algebraic K-theory and linear group homology. Since $BE(A)^+$ is the universal cover of $B\text{GL}(A)^+$, we concentrate actually our attention to the *Hurewicz homomorphism*

$$h_n : K_nA = \pi_n BE(A)^+ \to H_n(BE(A)^+; \mathbb{Z}) \cong H_n(E(A); \mathbb{Z})$$

for $n \geq 2$. Obviously, h_2 is an isomorphism and h_3 is surjective since the space $BE(A)^+$ is simply-connected.

In [10, Proposition 3] Soulé has shown that the kernel of h_n is a torsion group

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that involves at most the prime numbers p satisfying $p \le (n+1)/2$, but his argument does not imply that this group has finite exponent. The special case $n=3$ was first considered by Suslin [11, p. 370] who has obtained the result $2 \ker h_3 = 0$ by exhibiting $\ker h_3$ as a quotient of $K_2A \otimes \mathbb{Z}/2$. Recently, Sah [9, Proposition 2.5] has also established that $2 \ker h_3 = 0$ for any ring A (unfortunately, there is a gap in his proof: see Remark 1.9).

In the first section of this paper, we prove that for any n, $\ker h_n$ is effectively a torsion group of *finite exponent* and we give a *universal bound* for its exponent. More precisely, we define integers R_j ($j \ge 1$) which are independent of A and such that for any $n \ge 2$,

$$R_{n-1} \ker h_n = 0$$

for *any ring A* (in particular, we do not assume any finiteness condition). Observe that this implies Soulé's assertion because the integers R_{n-1} have the property that a prime p divides R_{n-1} if and only if $p \le (n+1)/2$. Notice also that our result is a generalization of Suslin and Sah's computation in relation to *all* dimensions since $R_2 = 2$. In order to get our theorem, we show that $\ker h_n$ is a quotient of a homology group of a space having only finitely many nontrivial homotopy groups, and then, the delooping of the space $BE(A)^+$ enables us to compare $\ker h_n$ with a finite number of stable homology groups of Eilenberg–Mac Lane spaces.

Our method works similarly for the Hurewicz homomorphism

$$h_n : K_nA = \pi_n B\mathrm{St}(A)^+ \to H_n(B\mathrm{St}(A)^+; \mathbb{Z}) \cong H_n(\mathrm{St}(A); \mathbb{Z})$$

for $n \ge 3$, where $\mathrm{St}(A)$ is the infinite Steinberg group of A. Remember that $B\mathrm{St}(A)^+$ is a 2-connected infinite loop space: therefore, h_3 is an isomorphism and h_4 is surjective. Here, our result is:

$$R_{n-2} \ker h_n = 0$$

for *any ring A* and for any integer $n \ge 3$.

The remainder of Section 1 is devoted to the study of the kernel of the Hurewicz homomorphism in low dimensions: for $n \le 5$, we are able to describe its exponent more precisely.

In Section 2, we look at the same problem from another point of view: we examine the *k-invariants* of the spaces $BE(A)^+$ and $B\mathrm{St}(A)^+$, and exhibit a relationship between their order and the knowledge of the Hurewicz homomorphism. We also prove the following statement: if F is an algebraically closed field and n an even integer, the k-invariant $k^{n+1}(B\mathrm{SL}(F)^+)$ is trivial and the Hurewicz homomorphism $h_n : K_nF \to H_n(\mathrm{SL}(F); \mathbb{Z})$ is split injective.

Finally, the *cokernel* of the Hurewicz homomorphism is the purpose of Section 3, but our method provides an upper bound for its exponent only in small dimensions, for instance for $n \le 6$ in the case of $h_n : K_nA \to H_n(\mathrm{St}(A); \mathbb{Z})$.

Throughout the paper, we first formulate our results in general, i.e., for any m-connected CW-complex, and then apply them to algebraic K-theory. Note that all spaces considered here are connected CW-complexes, and that if the coefficients of homology (or homotopy) groups are not written explictly, integral coefficients must be understood.

1. The kernel of the Hurewicz homomorphism

The main objective of this section is to show the existence of upper bounds for the exponent of the kernel of the Hurewicz homomorphisms $h_n : K_nA \to H_n(E(A); \mathbb{Z})$, $n \geq 2$, and $h_n : K_nA \to H_n(\mathrm{St}(A); \mathbb{Z})$, $n \geq 3$. We start by looking at the general situation. If Y is a connected CW-complex and i a positive integer, let us denote by $Y \to Y[i]$ its ith Postnikov section: $Y[i]$ is a CW-complex obtained from Y by adjoining cells of dimension $\geq i+2$ such that $\pi_k Y[i] = 0$ for $k > i$ and $\pi_k Y \xrightarrow{\cong} \pi_k Y[i]$ for $k \leq i$.

Theorem 1.1. *Let Y be a connected simple CW-complex, X the r-fold loop space $\Omega^r Y$ $(r \geq 0)$, and n a positive integer. The kernel of the Hurewicz homomorphism $h_n : \pi_n X \to H_n(X; \mathbb{Z})$ is a subgroup of*

$$H_{n+r+1}(Y[n+r-1]; \mathbb{Z}) / \Theta(H_{n+r+1}(Y[n+r]; \mathbb{Z}))\,,$$

where Θ is the homomorphism induced by the inclusion $Y[n+r] \hookrightarrow Y[n+r-1]$.

Proof. Consider the exact integral homology sequence of the pair $(X[n-1], X[n])$:

$$\begin{aligned} \cdots &\to H_{n+1}X[n] \to H_{n+1}X[n-1] \\ &\to H_{n+1}(X[n-1], X[n]) \xrightarrow{\partial} H_nX[n] \to \cdots \end{aligned}$$

It is clear that $H_nX[n] \cong H_nX$; on the other hand, according to [15, p. 422], there is a natural isomorphism $H_{n+1}(X[n-1], X[n]) \cong \pi_n X$ and ∂ corresponds under this isomorphism to the Hurewicz homomorphism $h_n : \pi_n X \to H_n X$. Thus, the analogous exact sequence for the pair $(Y[n+r-1], Y[n+r])$ produces the commutative diagram

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H_{n+1}X[n] & \longrightarrow & H_{n+1}X[n-1] & \longrightarrow & \pi_n X & \xrightarrow{h_n} & H_n X & \to \cdots \\ & & & & & \cong\downarrow & & \downarrow & \\ \cdots \to & H_{n+r+1}Y[n+r] & \xrightarrow{\Theta} & H_{n+r+1}Y[n+r-1] & \to & \pi_{n+r}Y & \xrightarrow{h_{n+r}} & H_{n+r}Y & \to \cdots \end{array}$$

where the vertical arrows are the r-fold suspensions, the left one being an isomorphism. Consequently, the kernel of h_n is a subgroup of the kernel of h_{n+r}, which is isomorphic to $H_{n+r+1}Y[n+r-1]/\Theta(H_{n+r+1}Y[n+r])$. □

Definition 1.2. Let $L_1 := 1$ and for $k \geq 2$ let L_k be the product of all prime numbers p for which there exists a sequence of nonnegative integers $(a_1, a_2, a_3, \ldots)$ satisfying:

(a) $a_1 \equiv 0 \mod 2p-2$, $a_i \equiv 0$ or $1 \mod 2p-2$ for $i \geq 2$,

(b) $a_i \geq p\, a_{i+1}$ for $i \geq 1$,

(c) $\sum_{i=1}^{\infty} a_i = k$.

Observe that L_k divides the product of all primes $p \leq (k/2)+1$. These integers occur in the determination of the stable homology groups of Eilenberg–Mac Lane spaces [3, Théorème 7]: for any abelian group G and for any integer $s \geq 2$, one has $L_{i-s}H_i(K(G,s);\mathbb{Z}) = 0$ if $s < i < 2s$.

Definition 1.3. $R_j := \prod_{k=1}^{j} L_k$ for $j \geq 1$. For example, $R_1 = 1$, $R_2 = 2$, $R_3 = 4$, $R_4 = 24$, $R_5 = 144$, $R_6 = 288, \ldots$ It turns out that a prime p divides R_j if and only if $p \leq (j/2)+1$.

Lemma 1.4. *Let Z be a space with finitely many nontrivial homotopy groups: $\pi_q Z = 0$ if $q \leq m$ or $q > t$, where m and t are integers satisfying $t > m \geq 1$. If $i = t+1$ or $t+2$ and if $i < 2m+2$, then $R_{i-m-1}H_i(Z;\mathbb{Z}) = 0$.*

Proof. Let us consider the fibrations $K(\pi_{s+1}Z, s+1) \to Z[s+1] \to Z[s]$ for $s = m+1, m+2, \ldots, t-1$, and apply the Serre spectral sequence: we get the exact sequences

$$H_i K(\pi_{s+1}Z, s+1) \to H_i Z[s+1] \to H_i Z[s]$$

which give inductively the exponent $L_{i-m-1} \cdot L_{i-m-2} \cdot \cdots \cdot L_{i-t+1} \cdot L_{i-t} = R_{i-m-1}$ (since $i-t = 1$ or 2 according to $i = t+1$ or $t+2$) for $H_i Z[t]$. This is exactly the assertion because $Z[t] = Z$. □

Theorem 1.5. *Let X be an m-connected r-fold loop space (with $m, r \geq 0$ but $m+r \geq 1$) and n an integer such that $m+1 \leq n \leq 2m+r$. Then the Hurewicz homomorphism $h_n : \pi_n X \to H_n(X;\mathbb{Z})$ has the following property:*

$$R_{n-m} \ker h_n = 0\,.$$

Proof. Since the statement of the theorem is obvious for $n = m+1$, assume that $n \geq m+2$ and that $X \simeq \Omega^r Y$, where Y is an $(m+r)$-connected space. It follows from Theorem 1.1 that $\ker h_n$ is a subquotient of $H_{n+r+1}Y[n+r-1]$; it is then sufficient to show that this group is annihilated by R_{n-m}. But this is true by the previous lemma, because the space $Y[n+r-1]$ is $(m+r)$-connected and its homotopy groups are trivial in dimensions $> n+r-1$. □

Notice that the upper bound R_{n-m} for the exponent of ker h_n does not depend on the space X (in particular, there is no finiteness condition on X). Now, let us apply this result to the infinite loop spaces $BE(A)^+$ and $B\mathrm{St}(A)^+$.

Corollary 1.6. (a) *For any ring A and for any integer $n \geq 2$, the Hurewicz homomorphism $h_n : K_nA \to H_n(E(A); \mathbb{Z})$ fulfills R_{n-1} ker $h_n = 0$.*

(b) *For any ring A and for any integer $n \geq 3$, the Hurewicz homomorphism $h_n : K_nA \to H_n(\mathrm{St}(A); \mathbb{Z})$ fulfills R_{n-2} ker $h_n = 0$.* □

It is actually possible to get more details on the exponent of the kernel of h_n in low dimensions. For an abelian group G, we shall write $\exp(G)$ for the exponent of G.

Theorem 1.7. *Let X be an m-connected r-fold loop space. The kernel of the Hurewicz homomophism satisfies:*

(a) *if $m + r \geq 2$, then* $\exp(\ker h_{m+2})$ *divides* $\exp(\pi_{m+1}X \otimes \mathbb{Z}/2)$.

(b) *if $m + r \geq 3$, then* $\exp(\ker h_{m+3})$ *divides the product*

$$\exp(\pi_{m+2}X \otimes \mathbb{Z}/2) \cdot \exp(\mathrm{Tor}(\pi_{m+1}X, \mathbb{Z}/2)) .$$

Proof. Suppose that $X \simeq \Omega^r Y$, where Y is an $(m + r)$-connected space. In order to establish (a), we must determine $\exp(H_{m+r+3}Y[m + r + 1])$. But the equality $Y[m + r + 1] = K(\pi_{m+1}X, m + r + 1)$ implies that $H_{m+r+3}Y[m + r + 1] \cong \pi_{m+1}X \otimes \mathbb{Z}/2$.

Similarly, (b) follows from the computation of $H_{m+r+4}Y[m + r + 2]$. The Serre spectral sequence of the fibration

$$K(\pi_{m+2}X, m + r + 2) \to Y[m + r + 2] \to K(\pi_{m+1}X, m + r + 1)$$

gives the exact sequence

$$H_{m+r+4}K(\pi_{m+2}X, m + r + 2) \to H_{m+r+4}Y[m + r + 2]$$
$$\to H_{m+r+4}K(\pi_{m+1}X, m + r + 1) .$$

Remember first the isomorphism $H_{m+r+4}K(\pi_{m+2}X, m + r + 2) \cong \pi_{m+2}X \otimes \mathbb{Z}/2$. Observe then that the third group in the exact sequence may be calculated by using the isomorphism [3, Corollaire of Théorème 7]

$$H_{m+r+4}K(\pi_{m+1}X, m + r + 1) \cong H_{m+r+4}(K(\mathbb{Z}, m + r + 1); \pi_{m+1}X)$$

(since $m + r \geq 3$), the universal coefficient theorem

$$H_{m+r+4}(K(\mathbb{Z}, m + r + 1); \pi_{m+1}X)$$
$$\cong \mathrm{Hom}(H_{m+r+4}K(\mathbb{Z}, m + r + 1), \pi_{m+1}X)$$
$$\oplus \mathrm{Tor}(H_{m+r+3}K(\mathbb{Z}, m + r + 1), \pi_{m+1}X) ,$$

and the facts that $H_{m+r+4}K(\mathbb{Z}, m+r+1)=0$ and $H_{m+r+3}K(\mathbb{Z}, m+r+1) \cong \mathbb{Z}/2$. □

Corollary 1.8. *Let A be any ring with identity.*

(a) *If* h_n *denotes the Hurewicz homomorphism* $K_nA \to H_n(E(A);\mathbb{Z})$, *then* $\exp(\ker h_3)$ *divides* $\exp(K_2A \otimes \mathbb{Z}/2)$ *and* $\exp(\ker h_4)$ *divides the product* $\exp(K_3A \otimes \mathbb{Z}/2) \cdot \exp(\mathrm{Tor}(K_2A, \mathbb{Z}/2))$.

(b) *If* h_n *denotes the Hurewicz homomorphism* $K_nA \to H_n(\mathrm{St}(A);\mathbb{Z})$, *then* $\exp(\ker h_4)$ *divides* $\exp(K_3A \otimes \mathbb{Z}/2)$ *and* $\exp(\ker h_5)$ *divides the product* $\exp(K_4A \otimes \mathbb{Z}/2) \cdot \exp(\mathrm{Tor}(K_3A, \mathbb{Z}/2))$. □

Remark 1.9. In the case of the space $BE(A)^+$ and $n=3$, our argument uses the fact that $BE(A)^+$ is a loop space ($r=1$): consequently, $\ker h_3$ is a subquotient of $H_5K(K_2A, 3)$, and the result follows from the isomorphism $H_5K(K_2A, 3) \cong K_2A \otimes \mathbb{Z}/2$ (this is a stable homology group). The proof of the assertion $2\ker h_3 = 0$ given by Sah in [9, Proposition 2.5] is not complete because he tries to deduce it from the observation that $\ker h_3$ is a quotient of $H_4K(K_2A, 2)$, instead of going to the range of stability and comparing $\ker h_3$ with $H_5K(K_2A, 3)$.

Example 1.10. Let F be a field of characteristic 2. For $n=2$ and 3, K_nF is 2-torsion-free (cf. [14, Theorem 1.10] for $n=2$ and [7, Corollary 4.13] for $n=3$), and the groups $K_nF \otimes \mathbb{Z}/2$ are well understood: $K_2F \otimes \mathbb{Z}/2 \cong {}_2\mathrm{Br}(F)$ (the 2-torsion subgroup of the Brauer group of F), and $K_3F \otimes \mathbb{Z}/2 \cong K_3F/2 \cong K_3^{\mathrm{M}}F/2 \cong I^3/I^4$, where $K_3^{\mathrm{M}}F$ is the third Milnor K-group of F and I the unique maximal ideal of the Witt ring $W(F)$ of nondegenerate symmetric bilinear forms over F (cf. [6] and [7, Corollary 4.13]). Consequently, we deduce from the previous corollary the following assertions:

(a) in the case of the Hurewicz homomorphism $h_n : K_nF \to H_n(\mathrm{SL}(F);\mathbb{Z})$, $\exp(\ker h_3)$ is a divisor of $\exp({}_2\mathrm{Br}(F))$ and $\exp(\ker h_4)$ a divisor of $\exp(I^3/I^4)$,

(b) in the case of the Hurewicz homomorphism $h_n : K_nF \to H_n(\mathrm{St}(F);\mathbb{Z})$, $\exp(\ker h_4)$ is a divisor of $\exp(I^3/I^4)$ and $\exp(\ker h_5)$ a divisor of $\exp(K_4F \otimes \mathbb{Z}/2)$.

Remark 1.11. If the 2-adic Quillen–Lichtenbaum conjecture (in the sense of Dwyer and Friedlander) is true for the ring $\mathbb{Z}$, then it follows from [4, Corollary 4.3] that $H^*(\mathrm{SL}(\mathbb{Z});\mathbb{Z}/2) \cong \mathbb{Z}/2[w_2, w_3, w_4, \ldots] \otimes \Lambda(u_3, u_5, u_7, \ldots)$, with $\deg w_i = i$ (w_i is the ith Stiefel–Whitney class), $\deg u_i = i$, and consequently that $H^4(\mathrm{St}(\mathbb{Z});\mathbb{Z}/2) \cong \mathbb{Z}/2$, $H^5(\mathrm{St}(\mathbb{Z});\mathbb{Z}/2) \cong \mathbb{Z}/2$ because of [1, Lemma 2.8]. Since $H_3(\mathrm{St}(\mathbb{Z});\mathbb{Z}) \cong K_3\mathbb{Z} \cong \mathbb{Z}/48$, the 2-torsion subgroup of $H_4(\mathrm{St}(\mathbb{Z});\mathbb{Z})$ would then vanish. But we have proved in [1, Theorem 1.3] that the Hurewicz homomorphism $h_4 : K_4\mathbb{Z} \to H_4(\mathrm{St}(\mathbb{Z});\mathbb{Z})$ is an isomorphism. Therefore, we obtain the following:

Conjecture A. The 2-torsion subgroup of $K_4\mathbb{Z}$ is trivial.

In dimension 5, the 2-adic Quillen–Lichtenbaum conjecture would imply that $\mathrm{Hom}(H_5(\mathrm{St}(\mathbb{Z});\mathbb{Z}),\mathbb{Z}/2)\cong\mathbb{Z}/2$, in other words that the 2-torsion subgroup of $H_5(\mathrm{St}(\mathbb{Z});\mathbb{Z})$ is trivial, because it is known that $H_5(\mathrm{St}(\mathbb{Z});\mathbb{Z})\cong\mathbb{Z}\oplus$torsion group, and we would conclude that the kernel of the Hurewicz homomorphism $h_5: K_5\mathbb{Z}\to H_5(\mathrm{St}(\mathbb{Z});\mathbb{Z})$ would be annihilated by 2, because of Corollary 1.8(b):

Conjecture B. The exponent of the 2-torsion subgroup of $K_5\mathbb{Z}$ is at most 2.

As mentioned in the Introduction, Sah has examined the kernel of $h_3: K_3A\to H_3(E(A);\mathbb{Z})$: he has shown in particular that h_3 is injective if K_2A is 2-divisible [9, Proposition 2.5]. The end of this section is devoted to a generalization of this result in relation to all dimensions.

Theorem 1.12. (a) *Let A be a ring, n an integer ≥ 3, and p a prime number. Assume that K_jA is uniquely p-divisible if $2\leq j<n-2p+3$, and p-divisible if $j=n-2p+3\geq 2$. Then the kernel of $h_n: K_nA\to H_n(E(A);\mathbb{Z})$ contains no p-torsion.*

(b) *The same result holds for $h_n: K_nA\to H_n(\mathrm{St}(A);\mathbb{Z})$, $n\geq 4$, given the condition that K_jA is uniquely p-divisible if $3\leq j<n-2p+3$ and p-divisible if $j=n-2p+3\geq 3$.*

In order to prove this theorem we first need to recall some information on the torsion in the stable homology of Eilenberg–Mac Lane spaces.

Lemma 1.13. *Let G be an abelian group, s and i two integers with $2\leq s<i<2s$, and p a prime. Then $H_i(K(G,s);\mathbb{Z})$ is p-torsion-free if one of the following conditions is satisfied:*

(a) $i-s<2p-2$,

(b) $i-s=2p-2$ *and G is p-divisible,*

(c) $i-s>2p-2$ *and G is uniquely p-divisible.*

Proof. Let us suppose that $s+2\leq i<2s$ since the result is trivial for $i=s+1$. We use the isomorphism $H_i(K(G,s);\mathbb{Z})\cong H_i(K(\mathbb{Z},s);G)$ [3, Corollaire of Théorème 7] and the universal coefficient theorem

$$H_i(K(\mathbb{Z},s);G)\cong H_iK(\mathbb{Z},s)\otimes G\oplus\mathrm{Tor}(H_{i-1}K(\mathbb{Z},s),G)\,.$$

It is known by [3, Théorème 7] that the stable homology groups $H_{s+k}K(\mathbb{Z},s)$ are finite cyclic groups whose order divides L_k (for $1\leq k<s$). In particular, $H_{s+k}K(\mathbb{Z},s)$ contains no p-torsion if $k<2p-2$: this implies (a). In order to get (b), we then only have to look at $H_{s+(2p-2)}K(\mathbb{Z},s)\otimes G$: $H_{s+(2p-2)}K(\mathbb{Z},s)$ is cyclic of order $p \cdot t$, with t not divisible by p, but $H_{s+(2p-2)}K(\mathbb{Z}, s) \otimes G$ is p-torsion-free since G is p-divisible. Similarly, if $i - s > 2p - 2$ there is p-torsion neither in $H_i K(\mathbb{Z}, s) \otimes G$ nor in $\mathrm{Tor}(H_{i-1}K(\mathbb{Z}, s), G)$, because G is p-divisible, respectively uniquely p-divisible. □

Proof of Theorem 1.12. Because $BE(A)^+$ is a simply-connected infinite loop space, let us assume that $BE(A)^+ \simeq \Omega^{n-2}Y$, where Y is an $(n-1)$-connected space. According to Theorem 1.1, it is sufficient to show that the p-torsion subgroup of $H_{2n-1}Y[2n-3]$ is trivial. But it turns out that this is equivalent to the fact that $H_{2n-1}K(\pi_s Y, s) \cong H_{2n-1}K(K_{s-n+2}A, s)$ contains no p-torsion for $s = n$, $n+1, n+2, \ldots, 2n-3$. We may deduce this from assertion (a) of the lemma for $2n - 2p + 1 < s \le 2n - 3$, from (b) for $s = 2n - 2p + 1$, because of the p-divisibility of $K_{n-2p+3}A$, and finally from (c) for $n \le s < 2n - 2p + 1$, since $K_{s-n+2}A$ is uniquely p-divisible. We proceed similarly for the space $B\mathrm{St}(A)^+$. □

2. Postnikov-invariants

It is also possible to discuss the Hurewicz homomorphism $h_n : \pi_n X \to H_n(X, \mathbb{Z})$ by looking at the Postnikov k-invariants $k^{n+1}(X) \in H^{n+1}(X[n-1]; \pi_n X)$. The method of the proofs of Theorems 1.1 and 1.5 provides in fact the following result: if X is an m-connected r-fold loop space, then $R_{n-m}k^{n+1}(X) = 0$ for $m + 1 \le n \le 2m + r$ [2]. On the other hand, the finiteness of the order of the k-invriant $k^{n+1}(X)$ produces a map $f_n : X \to K(\pi_n X, n)$ inducing multiplication by this order, i.e., by a divisor of R_{n-m}, on $\pi_n X$, and the commutative diagram

$$\begin{array}{ccc} \pi_n X & \xrightarrow{(f_n)_*} & \pi_n K(\pi_n X, n) \cong \pi_n X \\ {\scriptstyle h_n}\downarrow & & \downarrow{\scriptstyle \cong\, h_n} \\ H_n(X; \mathbb{Z}) & \xrightarrow{(f_n)_*} & H_n(K(\pi_n X, n); \mathbb{Z}) . \end{array}$$

Remark that this implies in particular that if $x \in \ker h_n$, then $R_{n-m}x = 0$: this is another way to formulate the proof of Theorem 1.5.

In the case of the algebraic K-theory, the upper bounds for the order of the k-invariants are given by

$$R_{n-1}k^{n+1}(BE(A)^+) = 0 \quad \text{for } n \ge 2$$

and

$$R_{n-2}k^{n+1}(B\mathrm{St}(A)^+) = 0 \quad \text{for } n \ge 3 ,$$

for any ring A. Therefore, the next assertion is a direct consequence of the above diagram.

Theorem 2.1. *For any ring A, there exist homomorphisms* $(f_n)_*: H_n(E(A);\mathbb{Z}) \to K_nA$, *respectively* $(f_n)_*: H_n(\mathrm{St}(A);\mathbb{Z}) \to K_nA$ *such that*

(a) *the composition* $(f_n)_* \circ h_n : K_nA \to H_n(E(A);\mathbb{Z}) \to K_nA$ *is multiplication by a divisor of* R_{n-1} *for* $n \geq 2$,

(b) *the composition* $(f_n)_* \circ h_n : K_nA \to H_n(\mathrm{St}(A);\mathbb{Z}) \to K_nA$ *is multiplication by a divisor of* R_{n-2} *for* $n \geq 3$. □

Remark 2.2. Consequently, for any ring A the Hurewicz homomorphism with coefficients

$$h_n : K_n(A;\mathbb{Z}/p) \to H_n(E(A);\mathbb{Z}/p)$$

is split injective if p is a prime number $>(n+1)/2$ $(n \geq 2)$, because R_{n-1} is not divisible by p. Let us also recall that, if $A = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, the Friedlander–Milnor conjecture holds for $B\mathrm{SL}(A) \to B\mathrm{SL}(A)^{\mathrm{top}}$, where $B\mathrm{SL}(A)^{\mathrm{top}}$ denotes the classifying space of $\mathrm{SL}(A)$ with the given topology as a Lie group (cf. [8], [12, Corollary 4.6], [13, Proposition 3.5]): thus, we may conclude that $K_n(A;\mathbb{Z}/p)$ is a direct summand of $H_n(B\mathrm{SL}(A)^{\mathrm{top}};\mathbb{Z}/p)$ for $n \geq 2$ and $p > (n+1)/2$.

Remark 2.3. The results of this paper may also be formulated for algebraic L-theory. In particular, the kernels of the Hurewicz homomorphisms $h_n : {}_1L_nA \to H_n(O(A);\mathbb{Z})$ and $h_n : {}_{-1}L_nA \to H_n(\mathrm{Sp}(A);\mathbb{Z})$ are annihilated by R_n for any ring A and any integer $n \geq 1$. It is actually possible to replace the integers R_n by smaller integers $\bar{R}_n$ (cf. [2, Definition 2.3 and Theorem 2.4]). Remark 2.2 holds also for L-theory since the Friedlander–Milnor conjecture is true in this situation for $A = \mathbb{R}$ or $\mathbb{C}$ [5].

The main result of this section is the following theorem on the k-invariants of the space $B\mathrm{SL}(F)^+$ for algebraically closed fields F.

Theorem 2.4. *Let F be an algebraically closed field and n a positive even integer. Then,*

(a) $k^{n+1}(B\mathrm{SL}(F)^+) = 0$ *in* $H^{n+1}(B\mathrm{SL}(F)^+[n-1]; K_nF)$,

(b) *the Hurewicz homomorphism* $h_n : K_nF \to H_n(\mathrm{SL}(F);\mathbb{Z})$ *is split injective.*

Proof. Since $B\mathrm{SL}(F)^+$ is a simply-connected infinite loop space, we may consider an $(n-1)$-connected space Y with $B\mathrm{SL}(F)^+ \simeq \Omega^{n-2}Y$. The k-invariant $k^{n+1}(B\mathrm{SL}(F)^+)$ is then the image of $k^{2n-1}(Y)$ under the $(n-2)$-fold iterated cohomology suspension

$$H^{2n-1}(Y[2n-3]; K_nF) \to H^{n+1}(B\mathrm{SL}(F)^+[n-1]; K_nF)$$

[15, p. 438]. Now, look at the universal coefficient theorem

$$H^{2n-1}(Y[2n-3]; K_nF)$$
$$\cong \mathrm{Hom}(H_{2n-1}Y[2n-3], K_nF) \oplus \mathrm{Ext}(H_{2n-2}Y[2n-3], K_nF),$$

and observe that the group $\mathrm{Ext}(H_{2n-2}Y[2n-3], K_nF)$ vanishes because Suslin has proved that K_nF is divisible for algebraically closed fields [13, Section 2]. Moreover, he has shown that K_nF is torsion-free if n is an even integer: this and the fact that $H_{2n-1}Y[2n-3]$ is a torsion group (see Lemma 1.4) imply that $\mathrm{Hom}(H_{2n-1}Y[2n-3], K_nF)$ is trivial. Consequently, $H^{2n-1}(Y[2n-3]; K_nF)=0$ and $k^{n+1}(B\mathrm{SL}(F)^+)=0$. Assertion (b) follows then from the commutative diagram introduced at the beginning of this section. □

3. The cokernel of the Hurewicz homomorphism

Finally, we try to get some information on the cokernel of the Hurewicz homomorphism. We start again by looking at the general situation.

Theorem 3.1. *If X is an m-connected space and n an integer such that $m+2 \le n \le 2m+1$ ($m \ge 1$), then the cokernel of the Hurewicz homomorphism $h_n : \pi_n X \to H_n(X; \mathbb{Z})$ satisfies:*
(a) $\operatorname{coker} h_n \cong H_n(X[n-1]; \mathbb{Z})$,
(b) $R_{n-m-1} \operatorname{coker} h_n = 0$.

Proof. The exact integral homology sequence of the pair $(X[n-1], X[n])$ gives the following exact sequence (cf. Proof of Theorem 1.1):

$$\cdots \to \pi_n X \xrightarrow{h_n} H_n X \to H_n X[n-1] \to \cdots,$$

where the homomorphism $H_nX \to H_nX[n-1]$ is surjective by Whitehead's theorem. Thus, $\operatorname{coker} h_n \cong H_nX[n-1]$. But the proof is then complete since the exponent of the group $H_nX[n-1]$ is finite and bounded by R_{n-m-1} as indicated in Lemma 1.4. □

Example 3.2. The first interesting case is $n = m+3$ (assuming $m \ge 2$). It follows from the theorem that $\operatorname{coker} h_{m+3} \cong H_{m+3}X[m+2]$ and the Serre spectral sequence of the fibration $K(\pi_{m+2}X, m+2) \to X[m+2] \to K(\pi_{m+1}X, m+1)$ produces the exact sequence

$$\underbrace{H_{m+3}K(\pi_{m+2}X, m+2)}_{0} \to H_{m+3}X[m+2] \to \underbrace{H_{m+3}K(\pi_{m+1}X, m+1)}_{\cong\, \pi_{m+1}X \otimes \mathbb{Z}/2}.$$

Consequently, the cokernel of h_{m+3} is isomorphic to a subgroup of $\pi_{m+1}X \otimes \mathbb{Z}/2$.

Corollary 3.3. *For any ring A, the cokernel of $h_5: K_5A \to H_5(\mathrm{St}(A);\mathbb{Z})$ is a subgroup of $K_3A \otimes \mathbb{Z}/2$.* □

In general, our argument succeeds only up to dimension $2m+1$. However, it also enables us to describe partially the case $n = 2m+2$ if the space we are looking at is a loop space.

Theorem 3.4. *Let X be an m-connected loop space ($m \geq 0$) and $QH_*(X;\mathbb{Z})$ the indecomposables of $H_*(X;\mathbb{Z})$. The cokernel of the composition*

$$\pi_{2m+2}X \xrightarrow{h_{2m+2}} H_{2m+2}(X;\mathbb{Z}) \twoheadrightarrow QH_{2m+2}(X;\mathbb{Z})$$

is annihilated by R_{m+1}.

Proof. Let X be $\simeq \Omega Y$ with Y an $(m+1)$-connected space and $n = 2m+2$. The homology suspension σ provides the commutative diagram

$$\begin{array}{ccc} \pi_n X & \xrightarrow{h_n} & H_n X \\ \cong\downarrow & & \downarrow\sigma \\ \pi_{n+1} Y & \xrightarrow{h_{n+1}} & H_{n+1} Y . \end{array}$$

If α is an element of H_nX, then $R_{m+1}\sigma(\alpha)$ belongs to the image of h_{n+1} by Theorem 3.1. Therefore, there exists an element $\beta \in \pi_nX$ such that $(h_n(\beta) - R_{m+1}\alpha) \in \ker\sigma$. But every element of $\ker\sigma$ is reductive [15, p. 383], and hence decomposable since it is in H_nX and $H_tX = 0$ for $t \leq (n/2) - 1$. It is then easy to conclude, because the images of $h_n(\beta)$ and $R_{m+1}\alpha$, under the surjection $H_nX \twoheadrightarrow QH_nX$, coincide. □

Corollary 3.5. *For any ring A, the cokernel of the composition*

$$K_4A \xrightarrow{h_4} H_4(E(A);\mathbb{Z}) \twoheadrightarrow QH_4(E(A);\mathbb{Z}),$$

respectively

$$K_6A \xrightarrow{h_6} H_6(\mathrm{St}(A);\mathbb{Z}) \twoheadrightarrow QH_6(\mathrm{St}(A);\mathbb{Z}),$$

is annihilated by 2, respectively by 4. □

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