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ON THE ALGEBRAIC K-THEORY OF $\mathbb{Z}$

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Let $\mathrm{GL}(\mathbb{Z})$ (respectively $\mathrm{SL}(\mathbb{Z})$) be the infinite general (respectively special) linear group and $\mathrm{St}(\mathbb{Z})$ the infinite Steinberg group of $\mathbb{Z}$. This paper studies the relationships between $K_i\mathbb{Z} := \pi_i B\mathrm{GL}(\mathbb{Z})^+$, $\mathrm{H}_i(\mathrm{SL}(\mathbb{Z}); \mathbb{Z})$ and $H_i(\mathrm{St}(\mathbb{Z}); \mathbb{Z})$ for $i = 4$ and 5 (they are well understood for $i \leq 3$). The main results describe the Hurewicz homomorphism $K_i\mathbb{Z} \to H_i(\mathrm{St}(\mathbb{Z}); \mathbb{Z})$: it is an isomorphism if $i = 4$ and its cokernel is cyclic of order 2 if $i = 5$ (more precisely, the induced homomorphism $K_5\mathbb{Z}/\text{torsion} \to H_5(\mathrm{St}(\mathbb{Z}); \mathbb{Z})/\text{torsion}$ is multiplication by 2). The relations between the integral homology of $\mathrm{St}(\mathbb{Z})$ and that of $\mathrm{SL}(\mathbb{Z})$ in dimensions 4 and 5 are also explained.

Introduction

Let $\mathrm{GL}(\mathbb{Z})$ be the infinite general linear group, $\mathrm{SL}(\mathbb{Z})$ the infinite special linear group and $\mathrm{St}(\mathbb{Z})$ the infinite Steinberg group of the ring of integers $\mathbb{Z}$. The relations between the groups $K_i\mathbb{Z} := \pi_i B\mathrm{GL}(\mathbb{Z})^+$, $H_i(\mathrm{SL}(\mathbb{Z}); \mathbb{Z})$ and $H_i(\mathrm{St}(\mathbb{Z}); \mathbb{Z})$ are well understood for $i \leq 3$: $K_1\mathbb{Z} \cong \mathbb{Z}/2$, $H_1(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}) = H_1(\mathrm{St}(\mathbb{Z}); \mathbb{Z}) = 0$; $K_2\mathbb{Z} \cong H_2(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}) \cong \mathbb{Z}/2$, $H_2(\mathrm{St}(\mathbb{Z}); \mathbb{Z}) = 0$; $K_3\mathbb{Z} \cong H_3(\mathrm{St}(\mathbb{Z}); \mathbb{Z}) \cong \mathbb{Z}/48$ and the Hurewicz homomorphism $K_3\mathbb{Z} \to H_3(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}) \cong \mathbb{Z}/24$ is surjective [1, Satz 1.5]. The purpose of this paper is to study these relations for $i = 4$ and 5.

1. Statement of the main results

In this section we present the main results of the paper.

Theorem 1.1. $H_4(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}) \cong H_4(\mathrm{St}(\mathbb{Z}); \mathbb{Z}) \oplus \mathbb{Z}/2$.

Theorem 1.2. *Let g denote the surjective canonical homomorphism* $\mathrm{St}(\mathbb{Z}) \twoheadrightarrow \mathrm{SL}(\mathbb{Z})$ *whose kernel is $K_2\mathbb{Z}$ and g_* the induced homomorphism* $H_5(\mathrm{St}(\mathbb{Z}); \mathbb{Z}) \to H_5(\mathrm{SL}(\mathbb{Z}); \mathbb{Z})$. *Then* $H_5(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}) \cong (\mathrm{Im}\, g_*) \oplus L$, *where* $L \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

Now let Hu denote the Hurewicz homomorphism $K_*\mathbb{Z} \to H_*(\mathrm{St}(\mathbb{Z}); \mathbb{Z})$.

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Theorem 1.3. Hu: $K_4\mathbb{Z} \to H_4(\mathrm{St}(\mathbb{Z}); \mathbb{Z})$ *is an isomorphism.*

Theorem 1.4. *There exists an exact sequence*

$$K_5\mathbb{Z} \xrightarrow{\mathrm{Hu}} H_5(\mathrm{St}(\mathbb{Z}); \mathbb{Z}) \to \mathbb{Z}/2 \to 0 .$$

Theorem 1.5. *The homomorphisms*

$$K_5\mathbb{Z}/\mathrm{torsion} \to H_5(\mathrm{St}(\mathbb{Z}); \mathbb{Z})/\mathrm{torsion}$$

and

$$K_5\mathbb{Z}/\mathrm{torsion} \to H_5(\mathrm{SL}(\mathbb{Z}); \mathbb{Z})/\mathrm{torsion},$$

induced by the Hurewicz homomorphism, are multiplications by 2. (*Recall that these three groups are infinite cyclic.*)

We shall prove Theorems 1.1 and 1.2 in Section 2, Theorems 1.3, 1.4 and 1.5 in Section 4.

2. The group extension $K_2\mathbb{Z} \rightarrowtail \mathrm{St}(\mathbb{Z}) \twoheadrightarrow \mathrm{SL}(\mathbb{Z})$

In this section we study the relationships between the (co)homology of $\mathrm{St}(\mathbb{Z})$ and that of $\mathrm{SL}(\mathbb{Z})$ using the Serre spectral sequence of the universal central extension

$$K_2\mathbb{Z} \cong \mathbb{Z}/2 \rightarrowtail \mathrm{St}(\mathbb{Z}) \overset{g}{\twoheadrightarrow} \mathrm{SL}(\mathbb{Z}) .$$

In particular we will prove Theorems 1.1 and 1.2.

We start by restricting our attention to cohomology with $\mathbb{Z}/2$-coefficients. Let us first recall the following lemma (cf. [7, p. 154, Corollary 8.12]) which describes the structure of $H^*(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}/2)$, since the cohomology of the group $\mathrm{SL}(\mathbb{Z})$ is the same as that of the H-space $B\mathrm{SL}(\mathbb{Z})^+$.

Lemma 2.1. *Let X be a connected H-space of finite type, then $H^*(X; \mathbb{Z}/2) = \bigotimes_{i=0}^{\infty} B_i$, where each*

$$B_i = \begin{cases} \mathbb{Z}/2[x_i] & \text{or} \\ \mathbb{Z}/2[x_i]/(x_i^{e_i} = 0) , & \text{where } e_i \text{ is a power of } 2 . \end{cases} \qquad \square$$

We want to look more precisely at the mod 2 cohomology classes of $\mathrm{SL}(\mathbb{Z})$. It is well known that all Stiefel–Whitney classes w_i ($\deg w_i = i$) are non-zero (except $w_1 = 0$) and algebraically independent in $H^*(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}/2)$ [6]. Because the space $B\mathrm{SL}(\mathbb{Z})^+$ is simply connected we have

$$H^1(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}/2) = 0$$

and

$$H^2(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}/2) \cong \mathrm{Hom}(K_2\mathbb{Z}, \mathbb{Z}/2) \cong \mathbb{Z}/2\,,$$

generated by w_2.

Corollary 2.2. *$H^*(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}/2) = \mathbb{Z}/2[w_2] \otimes (\bigotimes_{i=1}^{\infty} B_i)$, where each B_i is as in Lemma* 2.1 *and generated by one element of degree* ≥ 3. □

We know that

$$H^3(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

(since $H_3(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}) \cong \mathbb{Z}/24$ [1, Satz 1.5]); one copy of $\mathbb{Z}/2$ is generated by w_3 and we define the generator α of the other copy as follows: let h be the homomorphism $\mathrm{SL}(\mathbb{Z}) \twoheadrightarrow \mathrm{SL}(\mathbb{F}_3)$ induced by the reduction mod 3 ($\mathbb{F}_3$ is the field of three elements) and $h^*: H^3(\mathrm{SL}(\mathbb{F}_3); \mathbb{Z}/2) \cong \mathbb{Z}/2 \to H^3(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}/2)$ the induced homomorphism which is injective because of the surjectivity of $h_*: H_3(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}) \cong \mathbb{Z}/24 \to H_3(\mathrm{SL}(\mathbb{F}_3); \mathbb{Z}) \cong \mathbb{Z}/8$ (cf. [1, §3]).

Definition 2.3. Let us call α the image of the generator of $H^3(\mathrm{SL}(\mathbb{F}_3); \mathbb{Z}/2)$ under the homomorphism h^* ($\alpha \in H^3(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}/2)$).

We shall use the following notation: $\beta: H^*(-; \mathbb{Z}/2) \to H^{*+1}(-; \mathbb{Z})$ denotes the Bockstein homomorphism and $\mathrm{red}_2: H^*(-; \mathbb{Z}) \to H^*(-; \mathbb{Z}/2)$ the reduction mod 2 associated with the short exact sequence $\mathbb{Z} \rightarrowtail \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2$; we also need the Steenrod square operations $\mathrm{Sq}^i: H^*(-; \mathbb{Z}/2) \to H^{*+i}(-; \mathbb{Z}/2)$ ($\mathrm{Sq}^1 = \mathrm{red}_2 \circ \beta$).

Remark 2.4. (a) We deduce from the definition of α and from the commutative diagram

$$\begin{array}{ccc} H^3(\mathrm{SL}(\mathbb{F}_3); \mathbb{Z}/2) & \xrightarrow{\beta} & H^4(\mathrm{SL}(\mathbb{F}_3); \mathbb{Z}) \\ \downarrow{\scriptstyle h^*} & & \downarrow{\scriptstyle h^*} \\ H^3(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}/2) & \xrightarrow{\beta} & H^4(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}) \end{array}$$

that $\beta(\alpha) \neq 0$. On the other hand since $w_3 = \mathrm{Sq}^1 w_2$ (by Wu's formula) one has $\beta(w_3) = 0$. Consequently $\alpha \neq w_3$.

(b) $\mathrm{Sq}^1\alpha = \mathrm{red}_2(\beta(\alpha)) = 0$ because $\beta(\alpha)$ is the element of order 2 in $H^4(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}) \cong \mathbb{Z}/24$.

Definition 2.5. Let z be a generator of the infinite cyclic group $\mathrm{Hom}(H_5(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}), \mathbb{Z})$ and ζ an element of $H^5(\mathrm{SL}(\mathbb{Z}); \mathbb{Z})$ such that $\rho(\zeta) = z$, where ρ is the homomorphism $H^5(\mathrm{SL}(\mathbb{Z});\mathbb{Z}) \twoheadrightarrow \mathrm{Hom}(H_5(\mathrm{SL}(\mathbb{Z});\mathbb{Z}),\mathbb{Z})$ given by the universal coefficient theorem. Finally we define $\eta := \mathrm{red}_2(\zeta) \in H^5(\mathrm{SL}(\mathbb{Z});\mathbb{Z}/2)$.

Remark 2.6. (a) $\mathrm{Sq}^1\eta = 0$ since $\beta(\eta) = 0$.

(b) Let ρ denote now the homomorphism $H^5(\mathrm{SL}(\mathbb{Z});\mathbb{Z}/2) \twoheadrightarrow \mathrm{Hom}(H_5(\mathrm{SL}(\mathbb{Z});\mathbb{Z}),\mathbb{Z}/2)$, then $\rho(\eta) \neq 0$ (because $\rho(\eta) = \mathrm{red}_2(z) \neq 0$).

Lemma 2.7. *The cohomology classes w_5, w_2w_3, $w_2\alpha$ and η are linearly independent in $H^5(\mathrm{SL}(\mathbb{Z});\mathbb{Z}/2)$.*

Proof. Let $r, s, t, u \in \{0,1\}$ such that $rw_5 + sw_2w_3 + tw_2\alpha + u\eta = 0$. Using Wu's formula we obtain $\mathrm{Sq}^1(rw_5 + sw_2w_3 + tw_2\alpha + u\eta) = sw_3^2 + tw_3\alpha = 0$ and we may conclude, according to Lemma 2.1, that $s = t = 0$. We apply the homomorphism ρ to the remaining equation $rw_5 + u\eta = 0$: since $w_5 = \mathrm{Sq}^1 w_4$ we get $\rho(w_5) = 0$ and $u\rho(\eta) = 0$; Remark 2.6(b) then implies that $u = 0$ and $r = 0$. $\square$

We are now able to work with the Serre spectral sequence of $\mathbb{Z}/2 \rightarrowtail \mathrm{St}(\mathbb{Z}) \overset{g}{\twoheadrightarrow} \mathrm{SL}(\mathbb{Z})$, whose E_2-term is $H^*(\mathrm{SL}(\mathbb{Z}); H^*(\mathbb{Z}/2;\mathbb{Z}/2)) \cong H^*(\mathrm{SL}(\mathbb{Z});\mathbb{Z}/2) \otimes \mathbb{Z}/2[x]$ (x denotes the generator of $H^*(\mathbb{Z}/2;\mathbb{Z}/2) = \mathbb{Z}/2[x]$, $\deg x = 1$); in particular $E_2^{1,j} = 0$ $\forall j \geq 0$. Because $H^1(\mathrm{St}(\mathbb{Z});\mathbb{Z}/2) = 0$ we get $d_2(x) = w_2$ and, $\forall n \geq 0$ and $\forall y \in H^*(\mathrm{SL}(\mathbb{Z});\mathbb{Z}/2)$, $d_2(x^{2n}y) = 0$, $d_2(x^{2n+1}y) = x^{2n}w_2y$; it follows from Corollary 2.2 that $d_2(x^{2n+1}y) \neq 0$ if $y \neq 0$. Consequently the E_3-term has the following properties: $E_3^{1,j} = E_3^{2,j} = 0$ $\forall j \geq 0$, $E_3^{i,2n+1} = 0$ $\forall i, n \geq 0$, $E_3^{3,2n} = E_2^{3,2n}$ $\forall n \geq 0$. In order to understand the action of d_3 we use the fact that the Sq^i operations commute with the transgression: $d_3(x^2) = d_3(\mathrm{Sq}^1 x) = \mathrm{Sq}^1(d_2(x)) = \mathrm{Sq}^1 w_2 = w_3$ and, $\forall y \in E_3^{*,0}$, $d_3(x^2y) = w_3y$; again the structure of $H^*(\mathrm{SL}(\mathbb{Z});\mathbb{Z}/2)$ implies that $d_3(x^2y) \neq 0$ if $y \neq 0$. Obviously $d_3(x^4) = 0$, $d_4(x^4) = 0$ but, as above, $d_5(x^4) = d_5(\mathrm{Sq}^2x^2) = \mathrm{Sq}^2(d_3(x^2)) = \mathrm{Sq}^2 w_3$; by Wu's formula $\mathrm{Sq}^2 w_3 = w_5 + w_2w_3$ and we conclude that $d_5(x^4) = w_5$ since $w_2w_3 = 0$ in $E_3^{5,0}$.

We summarize the information we have on $E_\infty^{i,j}$ for $i + j \leq 5$: $E_\infty^{i,j} = 0$ for $i + j \leq 5$, $j > 0$; $E_\infty^{1,0} = E_\infty^{2,0} = 0$; $E_\infty^{3,0} \cong H^3(\mathrm{SL}(\mathbb{Z});\mathbb{Z}/2)/(w_3 = 0)$; $E_\infty^{4,0} \cong H^4(\mathrm{SL}(\mathbb{Z});\mathbb{Z}/2)/(w_2^2 = 0)$; $E_\infty^{5,0} \cong H^5(\mathrm{SL}(\mathbb{Z});\mathbb{Z}/2)/(w_5 = w_2w_3 = w_2\alpha = 0)$. This implies the following result:

Lemma 2.8. *The homomorphism $g: \mathrm{St}(\mathbb{Z}) \to \mathrm{SL}(\mathbb{Z})$ induces a surjective homomorphism $g^*: H^i(\mathrm{SL}(\mathbb{Z});\mathbb{Z}/2) \to H^i(\mathrm{St}(\mathbb{Z});\mathbb{Z}/2)$ for $i \leq 5$.*

For $i = 2$ $\ker g^* \cong \mathbb{Z}/2$, generated by w_2; for $i = 3$ $\ker g^* \cong \mathbb{Z}/2$, generated by w_3; for $i = 4$ $\ker g^* \cong \mathbb{Z}/2$, generated by w_2^2; for $i = 5$ $\ker g^* \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$, generated by w_5, w_2w_3 and $w_2\alpha$. $\square$

For the next step of our argument we look at the Serre spectral sequence for integral cohomology of the central extension $\mathbb{Z}/2 \rightarrowtail \mathrm{St}(\mathbb{Z}) \overset{g}{\twoheadrightarrow} \mathrm{SL}(\mathbb{Z})$. Since $E_2^{i,j} \cong H^i(\mathrm{SL}(\mathbb{Z}); H^j(\mathbb{Z}/2;\mathbb{Z}))$ fulfils $E_2^{i,j} = 0$ for j odd, we have $E_2 \equiv E_3$.

Lemma 2.9. $H^5(\mathrm{St}(\mathbb{Z}); \mathbb{Z}) \cong H^5(\mathrm{SL}(\mathbb{Z}); \mathbb{Z})/(\mathbb{Z}/2)$.

Proof. The unique non-zero terms $E_3^{i,j}$ of the line $i+j=5$ are $E_3^{3,2} \cong H^3(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and $E_3^{5,0} \cong H^5(\mathrm{SL}(\mathbb{Z}); \mathbb{Z})$. We can show that the differential $d_3: E_3^{3,2} \to E_3^{6,0}$ is injective by reducing the spectral sequence mod 2 and looking at the corresponding differential of the mod 2 spectral sequence, which is injective; consequently $E_\infty^{3,2} = 0$. For the same reason $d_5: E_5^{0,4} = E_2^{0,4} \cong \mathbb{Z}/2 \to E_5^{5,0}$ is injective and $E_\infty^{5,0} \cong E_5^{5,0}/(\mathbb{Z}/2)$. It is possible to deduce from $|H^4(\mathrm{St}(\mathbb{Z}); \mathbb{Z})| = 2|E_\infty^{4,0}|$ ($|\;|$ denotes the order of the group) that $E_5^{5,0} = E_2^{5,0}$; therefore $E_\infty^{5,0} \cong H^5(\mathrm{SL}(\mathbb{Z}); \mathbb{Z})/(\mathbb{Z}/2)$ and the lemma is proved. □

We finally consider the Serre spectral sequence for integral homology of the central extension $\mathbb{Z}/2 \rightarrowtail \mathrm{St}(\mathbb{Z}) \overset{g}{\twoheadrightarrow} \mathrm{SL}(\mathbb{Z})$: $E_{i,j}^2 \cong H_i(\mathrm{SL}(\mathbb{Z}); H_j(\mathbb{Z}/2; \mathbb{Z}))$ satisfies $E_{i,j}^2 = 0$ for $i = 1$ or j even, $j \geq 2$. We will need the following information:

Lemma 2.10. *The differential* $d^3: E_{3,1}^3 \to E_{0,3}^3 \cong \mathbb{Z}/2$ *is zero.*

Proof. Suppose $d^3: E_{3,1}^3 \to E_{0,3}^3$ is surjective. Then $E_{0,3}^\infty = 0$ and, because $E_{1,2}^\infty = 0$, $|H_3(\mathrm{St}(\mathbb{Z}); \mathbb{Z})| = 2|E_{3,0}^\infty|$, we must have $E_{2,1}^\infty \cong \mathbb{Z}/2$ which implies that $d^2: E_{4,0}^2 \to E_{2,1}^2 \cong \mathbb{Z}/2$ is zero and that $E_{4,0}^\infty = E_{4,0}^2 \cong H_4(\mathrm{SL}(\mathbb{Z}); \mathbb{Z})$. Thus the induced homomorphism $g_*: H_4(\mathrm{St}(\mathbb{Z}); \mathbb{Z}) \to H_4(\mathrm{SL}(\mathbb{Z}); \mathbb{Z})$ is surjective.

On the other hand we consider the following commutative diagram:

$$\begin{array}{ccccccccc}
K_4\mathbb{Z} & \xrightarrow{\mathrm{Hu}} & H_4(B\mathrm{St}(\mathbb{Z})^+; \mathbb{Z}) & & & & & & \\
\| & & \downarrow{\scriptstyle g_*} & & & & & & \\
K_4\mathbb{Z} & \xrightarrow{\mathrm{Hu}} & H_4(B\mathrm{SL}(\mathbb{Z})^+; \mathbb{Z}) & \longrightarrow & \mathbb{Z}/4 & \longrightarrow & K_3\mathbb{Z} & \overset{\mathrm{Hu}}{\twoheadrightarrow} & H_3(B\mathrm{SL}(\mathbb{Z})^+; \mathbb{Z})
\end{array}$$

where Hu denotes the Hurewicz homomorphism. The bottom sequence is the Whitehead exact sequence of $B\mathrm{SL}(\mathbb{Z})^+$ and the kernel of $\mathrm{Hu}: K_3\mathbb{Z} \to H_3(B\mathrm{SL}(\mathbb{Z})^+; \mathbb{Z})$ is cyclic of order 2 (cf. [8] and [1, Satz 1.5]). Since $B\mathrm{St}(\mathbb{Z})^+$ is 2-connected, $\mathrm{Hu}: K_4\mathbb{Z} \to H_4(B\mathrm{St}(\mathbb{Z})^+; \mathbb{Z})$ is surjective and g_* cannot be surjective: that gives us a contradiction. □

Lemma 2.11. *There exist the following exact sequences*:

(a) $\quad 0 \to H_4(\mathrm{St}(\mathbb{Z}); \mathbb{Z}) \xrightarrow{g_*} H_4(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}) \to \mathbb{Z}/2 \to 0$,

(b) $\quad H_5(\mathrm{St}(\mathbb{Z}); \mathbb{Z}) \xrightarrow{g_*} H_5(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}) \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to 0$.

Proof. It follows from the previous lemma that $E_{3,1}^\infty = E_{3,1}^3$ and, since $|H_3(\mathrm{St}(\mathbb{Z}); \mathbb{Z})| = 2|E_{3,0}^\infty|$, that $E_{4,0}^\infty$ is the kernel of a surjective homomorphism $H_4(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}) \twoheadrightarrow \mathbb{Z}/2$. Obviously $E_{5,0}^\infty = E_{5,0}^3 = \ker d^2: E_{5,0}^2 \to E_{3,1}^2$. (Note that $E_{5,0}^2 \cong H_5(\mathrm{SL}(\mathbb{Z}); \mathbb{Z})$ and $E_{3,1}^2 \cong H_3(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$.) We obtain the exact sequence

$$H_5(\mathrm{St}(\mathbb{Z});\mathbb{Z}) \xrightarrow{g_*} E^2_{5,0} \xrightarrow{d^2} E^2_{3,1} \to H_4(\mathrm{St}(\mathbb{Z});\mathbb{Z})$$
$$\xrightarrow{g_*} H_4(\mathrm{SL}(\mathbb{Z});\mathbb{Z}) \to \mathbb{Z}/2 \to 0\,.$$

We deduce from the universal coefficient theorem and Borel's theorem that $H^5(\mathrm{SL}(\mathbb{Z});\mathbb{Z}) \cong \mathbb{Z} \oplus \mathrm{Ext}(H_4(\mathrm{SL}(\mathbb{Z});\mathbb{Z}),\mathbb{Z})$ and $H^5(\mathrm{St}(\mathbb{Z});\mathbb{Z}) \cong \mathbb{Z} \oplus \mathrm{Ext}(H_4(\mathrm{St}(\mathbb{Z});\mathbb{Z}),\mathbb{Z})$. Lemma 2.9 then implies the injectivity of $g_*: H_4(\mathrm{St}(\mathbb{Z});\mathbb{Z}) \to H_4(\mathrm{SL}(\mathbb{Z});\mathbb{Z})$. □

Proof of Theorem 1.1. Since we know from Lemma 2.8 that $H^4(\mathrm{St}(\mathbb{Z});\ \mathbb{Z}/2) \cong H^4(\mathrm{SL}(\mathbb{Z});\ \mathbb{Z}/2)/(\mathbb{Z}/2)$, the universal coefficient theorem gives us: $\mathrm{Hom}(H_4(\mathrm{St}(\mathbb{Z});\mathbb{Z}),\ \mathbb{Z}/2) \cong \mathrm{Hom}(H_4(\mathrm{SL}(\mathbb{Z});\ \mathbb{Z}),\ \mathbb{Z}/2)/(\mathbb{Z}/2)$. Therefore we may conclude that the short exact sequence of Lemma 2.11(a) splits. □

Proof of Theorem 1.2. It follows again from Lemma 2.8, the universal coefficient theorem and Theorem 1.1 that $\mathrm{Hom}(H_5(\mathrm{St}(\mathbb{Z});\mathbb{Z}),\ \mathbb{Z}/2) \cong \mathrm{Hom}(H_5(\mathrm{SL}(\mathbb{Z});\ \mathbb{Z}),\ \mathbb{Z}/2)/(\mathbb{Z}/2 \oplus \mathbb{Z}/2)$. The assertion of Theorem 1.2 is then a consequence of Lemma 2.11(b). □

Remark 2.12. Recall that by Borel's theorem $H_5(\mathrm{St}(\mathbb{Z});\mathbb{Z})/\mathrm{torsion}$ and $H_5(\mathrm{SL}(\mathbb{Z});\mathbb{Z})/\mathrm{torsion}$ are infinite cyclic groups. According to Theorem 1.2, $g_*: H_5(\mathrm{St}(\mathbb{Z});\ \mathbb{Z}) \to H_5(\mathrm{SL}(\mathbb{Z});\mathbb{Z})$ induces an isomorphism

$$H_5(\mathrm{St}(\mathbb{Z});\mathbb{Z})/\mathrm{torsion} \xrightarrow{\cong} H_5(\mathrm{SL}(\mathbb{Z});\mathbb{Z})/\mathrm{torsion}\,.$$

Remark 2.13. It is possible to show that the order of the kernel of $g_*: H_5(\mathrm{St}(\mathbb{Z});\ \mathbb{Z}) \to H_5(\mathrm{SL}(\mathbb{Z});\mathbb{Z})$ divides 4.

3. Homological relations between the spaces $B\mathrm{St}(\mathbb{Z})^+$ and $B\mathrm{SL}(\mathbb{F}_3)^+$

We consider now the commutative triangle

$$\begin{tikzcd} B\mathrm{St}(\mathbb{Z})^+ \arrow[rr, "f"] \arrow[dr, "g"'] & & B\mathrm{SL}(\mathbb{F}_3)^+ \\ & B\mathrm{SL}(\mathbb{Z})^+ \arrow[ur, "h"'] & \end{tikzcd}$$

where f and h are induced by the reduction mod 3 and g is the map induced by the canonical homomorphism of Section 2 (recall that $\mathrm{St}(\mathbb{F}_3) = \mathrm{SL}(\mathbb{F}_3)$ and that $B\mathrm{SL}(\mathbb{F}_3)^+$ is 2-connected since $K_2\mathbb{F}_3 = 0$). In order to prove our main results in Section 4 we need to examine the image of $f_*: H_5(B\mathrm{St}(\mathbb{Z})^+;\mathbb{Z}) \to H_5(B\mathrm{SL}(\mathbb{F}_3)^+;\mathbb{Z})$.

We shall use throughout this section the following notation. We define F (respectively $\bar{F}$) as the fibre of h (respectively f) and get the commutative diagram

$$\begin{array}{ccccc} \bar{F} & \xrightarrow{j} & B\mathrm{St}(\mathbb{Z})^+ & \xrightarrow{f} & B\mathrm{SL}(\mathbb{F}_3)^+ \\ \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle g} & & \| \\ F & \xrightarrow{i} & B\mathrm{SL}(\mathbb{Z})^+ & \xrightarrow{h} & B\mathrm{SL}(\mathbb{F}_3)^+ \end{array}$$

where both rows are fibrations. As usual we shall denote by $j^*, f^*, i^*, h^*, k^*, g^*$ the induced homomorphisms in cohomology. If y is an element of $H^*(B\mathrm{SL}(\mathbb{Z})^+;\mathbb{Z}/2)$ let us define $\tilde{y} := i^*(y) \in H^*(F;\mathbb{Z}/2)$ and $\bar{y} := g^*(y) \in H^*(B\mathrm{St}(\mathbb{Z})^+;\mathbb{Z}/2)$. According to [5] the ring $H^*(B\mathrm{SL}(\mathbb{F}_3)^+;\mathbb{Z}/2)$ is generated by cohomology classes e_i and c_i, $i \geq 2$, where $\deg e_i = 2i-1$ and $\deg c_i = 2i$.

Remark 3.1. (a) The space F is simply connected and $\bar{F}$ is 2-connected. The groups $H_i(F;\mathbb{Z})$ are finite for $i = 2, 3, 4$ and $H_5(F;\mathbb{Z}) \cong \mathbb{Z} \oplus (\text{finite group})$, because the same results hold for $B\mathrm{SL}(\mathbb{Z})^+$ and all homology groups of $B\mathrm{SL}(\mathbb{F}_3)^+$ are finite (this is also true for $\bar{F}$). Note that a Serre spectral sequence argument shows that $H_2(F;\mathbb{Z}) \cong \mathbb{Z}/2$ and $H_3(F;\mathbb{Z}) \cong \mathbb{Z}/3$.

(b) The relation between F and the classifying space of the congruence subgroup of $\mathrm{SL}(\mathbb{Z})$ of level 3 is explained in [2, Section 1].

We start by looking at mod 2 cohomology. Obviously $\tilde{w}_2 \neq 0$ in $H^2(F;\mathbb{Z}/2)$ and, since $h^*(e_2) = \alpha \in H^3(B\mathrm{SL}(\mathbb{Z})^+;\mathbb{Z}/2)$ (cf. Definition 2.3), $\tilde{\alpha} = 0$ but $\tilde{w}_3 \neq 0$ in $H^3(F;\mathbb{Z}/2)$. Because h is an H-map, F is an H-space and, by Lemma 2.1, $\tilde{w}_2\tilde{w}_3 \neq 0$ in $H^5(F;\mathbb{Z}/2)$.

Lemma 3.2. *Let γ_m denote the homomorphism $H^*(F;\mathbb{Z}/2) \to H^*(F;\mathbb{Z}/2^m)$ induced by the inclusion $\mathbb{Z}/2 \hookrightarrow \mathbb{Z}/2^m$. Then $\gamma_m(\tilde{w}_2\tilde{w}_3) \neq 0$ for all $m \geq 1$.*

Proof. Let $m \geq 1$ be a given integer and θ_m the homomorphism $H^*(F;\mathbb{Z}/2^m) \to H^*(F;\mathbb{Z}/2)$ induced by the surjection $\mathbb{Z}/2^m \twoheadrightarrow \mathbb{Z}/2$. We call a (respectively b) the generator of $H^2(F;\mathbb{Z}/2^m) \cong \mathrm{Hom}(H_2(F;\mathbb{Z}), \mathbb{Z}/2^m) \cong \mathbb{Z}/2$ (respectively of $H^3(F;\mathbb{Z}/2^m) \cong \mathrm{Ext}(H_2(F;\mathbb{Z}), \mathbb{Z}/2^m) \cong \mathbb{Z}/2$). It is clear that $\theta_m(b) = \tilde{w}_3$ which implies actually the equality $\tilde{w}_2 b = \tilde{w}_2\tilde{w}_3$ in $H^5(F;\mathbb{Z}/2)$. On the other hand we deduce from $\gamma_m(\tilde{w}_2) = a$ that $\gamma_m(\tilde{w}_2 b) = ab$; therefore $\gamma_m(\tilde{w}_2\tilde{w}_3) = ab$.

We complete the proof by showing that $ab \neq 0$ in $H^5(F;\mathbb{Z}/2^m)$. Let $\mu^*: H^*(F;\mathbb{Z}/2^m) \to H^*(F \times F;\mathbb{Z}/2^m)$ denote the homomorphism induced by the H-space structure of F. Since F is simply connected we have obviously $\mu^*(a) = a \otimes 1 + 1 \otimes a$ and $\mu^*(b) = b \otimes 1 + 1 \otimes b$. If $ab = 0$, then $0 = \mu^*(ab) = \mu^*(a)\mu^*(b) = a \otimes b + b \otimes a$ in $H^5(F \times F;\mathbb{Z}/2^m)$, which is not the case. $\square$

Corollary 3.3. *Let $\rho : H^5(F;\mathbb{Z}/2) \twoheadrightarrow \mathrm{Hom}(H_5(F;\mathbb{Z}), \mathbb{Z}/2)$ be the homomorphism given by the universal coefficient theorem. Then $\rho(\tilde{w}_2\tilde{w}_3) \neq 0$.*

Proof. Suppose $\rho(\tilde{w}_2\tilde{w}_3)=0$; then the exactness of the sequence $\mathrm{Ext}(H_4(F;\mathbb{Z}), \mathbb{Z}/2) \overset{\nu}{\rightarrowtail} H^5(F;\mathbb{Z}/2) \overset{\rho}{\twoheadrightarrow} \mathrm{Hom}(H_5(F;\mathbb{Z}), \mathbb{Z}/2)$ implies the existence of an element $\sigma \in \mathrm{Ext}(H_4(F;\mathbb{Z}), \mathbb{Z}/2)$ such that $\nu(\sigma) = \tilde{w}_2\tilde{w}_3$. Let 2^{m-1} be the exponent of the 2-torsion subgroup of $H_4(F;\mathbb{Z})$ and let us consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}(H_4(F;\mathbb{Z}),\mathbb{Z}/2) \arrow[r, "\gamma'_m"] \arrow[d, tail, "\nu"] & \mathrm{Ext}(H_4(F;\mathbb{Z}),\mathbb{Z}/2^m) \arrow[d, tail] \\
H^5(F;\mathbb{Z}/2) \arrow[r, "\gamma_m"] & H^5(F;\mathbb{Z}/2^m)
\end{tikzcd}$$

where γ_m and γ'_m are induced by the inclusion $\mathbb{Z}/2 \hookrightarrow \mathbb{Z}/2^m$. It follows from the Hom-Ext-sequence that $\gamma'_m=0$; therefore $\gamma_m(\tilde{w}_2\tilde{w}_3)=\gamma_m(\nu(\sigma))=0$, which contradicts the previous lemma. □

Lemma 3.4. *The element $\rho(\tilde{w}_2\tilde{w}_3)$ belongs to the image of the reduction* mod 2 $\mathrm{Hom}(H_5(F;\mathbb{Z}),\mathbb{Z}) \to \mathrm{Hom}(H_5(F;\mathbb{Z}),\mathbb{Z}/2)$.

Proof. We look at the commutative diagram

$$\begin{tikzcd}
H^5(B\mathrm{SL}(\mathbb{Z})^+;\mathbb{Z}/2) \arrow[r, "\beta"] \arrow[d, "i^*"] & H^6(B\mathrm{SL}(\mathbb{Z})^+;\mathbb{Z}) \arrow[d, "i^*"] \\
H^5(F;\mathbb{Z}/2) \arrow[r, "\beta"] & H^6(F;\mathbb{Z})
\end{tikzcd}$$

where β denotes again the Bockstein homomorphism. It is easy to check that $\beta(w_2w_3)=c_3(\mathrm{SL}(\mathbb{Z}))$, i.e., the third Chern class of the inclusion $\mathrm{SL}(\mathbb{Z}) \hookrightarrow \mathrm{GL}(\mathbb{C})$ (cf. [1]), because this equality holds in the cohomology of $B\mathrm{SO}$. We then deduce from $i^*(c_3(\mathrm{SL}(\mathbb{Z})))=0$ [3] that $\beta(\tilde{w}_2\tilde{w}_3)=\beta(i^*(w_2w_3))=i^*(\beta(w_2w_3))=0$. Consequently $\tilde{w}_2\tilde{w}_3$ belongs to the image of the reduction mod 2 and the same is true for $\rho(\tilde{w}_2\tilde{w}_3)$. □

Lemma 3.5. $k^*(\tilde{w}_2\tilde{w}_3)=0$ *in* $H^5(\bar{F};\mathbb{Z}/2)$.

Proof. This follows from $g^*(w_2w_3)=0$ (cf. Lemma 2.8) since $k^*(\tilde{w}_2\tilde{w}_3)=j^*(g^*(w_2w_3))$. □

We are now able to prove the main result of this section. Recall that $H_5(B\mathrm{SL}(\mathbb{F}_3)^+;\mathbb{Z}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/13$ [4].

Proposition 3.6. *The 2-torsion subgroup of the image of the homomorphism $f_*: H_5(B\mathrm{St}(\mathbb{Z})^+;\mathbb{Z}) \to H_5(B\mathrm{SL}(\mathbb{F}_3)^+;\mathbb{Z})$ is cyclic of order 2.*

Proof. The homomorphisms g, i, j, k induce the commutative diagram

$$\begin{array}{ccc} H_5(\bar{F};\mathbb{Z})/\text{torsion} \cong \mathbb{Z} & \xrightarrow{j_*} & H_5(B\mathrm{St}(\mathbb{Z})^+;\mathbb{Z})/\text{torsion} \cong \mathbb{Z} \\ \downarrow{\scriptstyle k_*} & & \downarrow{\scriptstyle g_*} \\ H_5(F;\mathbb{Z})/\text{torsion} \cong \mathbb{Z} & \xrightarrow{i_*} & H_5(B\mathrm{SL}(\mathbb{Z})^+;\mathbb{Z})/\text{torsion} \cong \mathbb{Z} \end{array}$$

and we know that g_* is an isomorphism (Remark 2.12). It follows from Corollary 3.3. and Lemmas 3.4 and 3.5 that k_* is multiplication by an even number (or 0); thus j_* is also multiplication by an even number (or 0).

On the other hand the Serre spectral sequence of the fibration $\bar{F} \xrightarrow{j} B\mathrm{St}(\mathbb{Z})^+ \xrightarrow{f} B\mathrm{SL}(\mathbb{F}_3)^+$ produces an exact sequence $H_5(\bar{F};\mathbb{Z}) \xrightarrow{j_*} H_5(B\mathrm{St}(\mathbb{Z})^+;\mathbb{Z}) \xrightarrow{f_*} H_5(B\mathrm{SL}(\mathbb{F}_3)^+;\mathbb{Z})$. Therefore the 2-torsion subgroup of the cokernel of j_* is cyclic of order 2 and the proof is complete. $\square$

Our next objective is to examine the image of the torsion subgroup of $H_5(B\mathrm{St}(\mathbb{Z})^+;\mathbb{Z})$ under the homomorphism f_*. We first consider the homomorphism $h^*: H^*(B\mathrm{SL}(\mathbb{F}_3)^+;\mathbb{Z}/2) \to H^*(B\mathrm{SL}(\mathbb{Z})^+;\mathbb{Z}/2)$; recall that by Definition 2.3 $h^*(e_2) = \alpha \in H^3(B\mathrm{SL}(\mathbb{Z})^+;\mathbb{Z}/2)$.

Lemma 3.7. $h^*(c_2) = w_2^2$, $h^*(e_3) = \mathrm{Sq}^2\alpha$, $h^*(c_3) = w_3^2$.

Proof. Since $e_2^2 = c_3$ [5, p. 565] (and consequently $\mathrm{Sq}^2 e_2 = e_3$) we get $h^*(c_3) = \alpha^2$ and $h^*(e_3) = \mathrm{Sq}^2\alpha$.

We use the Eilenberg–Moore spectral sequence of the fibration $F \xrightarrow{i} B\mathrm{SL}(\mathbb{Z})^+ \xrightarrow{h} B\mathrm{SL}(\mathbb{F}_3)^+$ which converges to $H^*(F;\mathbb{Z}/2)$. Let R be the polynomial ring $H^*(B\mathrm{SL}(\mathbb{F}_3)^+;\mathbb{Z}/2)$. In order to get the E_1-term of this (second quadrant) spectral sequence we choose an R-free resolution of the field of two elements $\mathbb{F}_2$:

$$\cdots \to \bigoplus_{k=1}^{\infty} Ru_k \to R \to \mathbb{F}_2$$

where $u_k^2 = 0$ $\forall k \geq 1$ and $\operatorname{bideg} u_1 = (-1,3)$, $\operatorname{bideg} u_2 = (-1,4)$, $\operatorname{bideg} u_3 = (-1,5)$, $\operatorname{bideg} u_4 = (-1,7), \ldots$. We obtain E_1 by tensoring this resolution with $H^*(B\mathrm{SL}(\mathbb{Z})^+;\mathbb{Z}/2)$ over R; in particular $E_1^{0,*} \cong H^*(B\mathrm{SL}(\mathbb{Z})^+;\mathbb{Z}/2)$.

We know that $i^*(\alpha) = 0$ and we deduce from [3] that $i^*(w_2^2) = 0$, $i^*(w_3^2) = 0$ (because w_2^2 (respectively w_3^2) is the reduction mod 2 of the second (respectively the third) Chern class of the inclusion $\mathrm{SL}(\mathbb{Z}) \hookrightarrow \mathrm{GL}(\mathbb{C})$). This implies that $\alpha \in E_1^{0,3}$, $w_2^2 \in E_1^{0,4}$ and $w_3^2 \in E_1^{0,6}$ have to be killed by some differential: for placement reasons these three classes belong to the image of the differential d_1 of bidegree $(1,0)$. Since u_1 and u_2 generate $E_1^{-1,3}$ and $E_1^{-1,4}$ respectively, we have $d_1(u_1) = \alpha$ and $d_1(u_2) = w_2^2$ (that gives us $h^*(c_2) = w_2^2$). $E_1^{-1,6}$ is generated by

w_3u_1, αu_1, w_2u_2 and therefore $\operatorname{Im} d_1 : E_1^{-1,6} \to E_1^{0,6}$ is generated by $w_3\alpha$, α^2 and w_2^3. We then may conclude that $w_3^2 = rw_3\alpha + s\alpha^2 + tw_2^3$ for some $r, s, t \in \{0,1\}$. But it follows from Lemma 2.1 that $r = t = 0$, $s = 1$: $w_3^2 = \alpha^2$; consequently $h^*(c_3) = w_3^2$. □

Definition 3.8. $\xi := w_3 + \alpha \in H^3(B\mathrm{SL}(\mathbb{Z})^+; \mathbb{Z}/2)$.

Remark 3.9. (a) $\xi^2 = 0$ since $w_3^2 = \alpha^2$.

(b) $h^*(e_3) = \mathrm{Sq}^2\alpha = \mathrm{Sq}^2(w_3 + \xi) = w_2w_3 + w_5 + \mathrm{Sq}^2\xi$ by Wu's formula.

(c) Since $f^* = g^* \circ h^*$ it follows from Lemmas 2.8 and 3.7 that $f^*(e_2) = \bar{\alpha} = \bar{\xi}$ $(\neq 0)$ generates $H^3(B\mathrm{St}(\mathbb{Z})^+; \mathbb{Z}/2)$ and that $f^*(c_2) = 0$, $f^*(e_3) = \mathrm{Sq}^2\bar{\xi}$, $f^*(c_3) = 0$.

Lemma 3.10. *Let us call* $\hat{c}_3$ *a generator of* $H^6(B\mathrm{SL}(\mathbb{F}_3)^+;\ \mathbb{Z}) \cong \mathbb{Z}/26$, *then* $h^*(\hat{c}_3) = c_3(\mathrm{SL}(\mathbb{Z})) \in H^6(B\mathrm{SL}(\mathbb{Z})^+; \mathbb{Z})$ ($c_3(\mathrm{SL}(\mathbb{Z}))$ *is the third Chern class of the inclusion* $\mathrm{SL}(\mathbb{Z}) \hookrightarrow \mathrm{GL}(\mathbb{C})$).

Proof. Let β denote again the Bockstein homomorphism $H^*(-;\mathbb{Z}/2) \to H^{*+1}(-;\mathbb{Z})$. The element $w_2\alpha$ of $H^5(B\mathrm{SL}(\mathbb{Z})^+; \mathbb{Z}/2)$ satisfies $i^*(w_2\alpha) = 0$ since $i^*(\alpha) = 0$. We define $\tau := \beta(w_2\alpha)$; of course $i^*(\tau) = 0$ in $H^6(F; \mathbb{Z})$. Moreover we know from [3] that $i^*(c_3(\mathrm{SL}(\mathbb{Z}))) = 0$; note that $\tau \neq c_3(\mathrm{SL}(\mathbb{Z}))$ because $\mathrm{red}_2(\tau) = \mathrm{Sq}^1(w_2\alpha) = w_3\alpha \neq w_3^2 = \mathrm{red}_2(c_3(\mathrm{SL}(\mathbb{Z})))$. We can conclude by looking at the Serre spectral sequence of $F \xrightarrow{i} B\mathrm{SL}(\mathbb{Z})^+ \xrightarrow{h} B\mathrm{SL}(\mathbb{F}_3)^+$ that the kernel of $i^* : H^6(B\mathrm{SL}(\mathbb{Z})^+;\ \mathbb{Z}) \to H^6(F; \mathbb{Z})$ is generated by τ and $c_3(\mathrm{SL}(\mathbb{Z}))$. But $h^*(\hat{c}_3)$ belongs to this kernel, i.e., $h^*(\hat{c}_3) = r\tau + sc_3(\mathrm{SL}(\mathbb{Z}))$ where $r, s \in \{0,1\}$. We apply red_2 to this equation and obtain $h^*(c_3) = rw_3\alpha + sw_3^2$. On the other hand, since $h^*(c_3) = w_3^2$ by Lemma 3.7, we get $r = 0$, $s = 1$, so $h^*(\hat{c}_3) = c_3(\mathrm{SL}(\mathbb{Z}))$. □

Corollary 3.11. *Let* $\rho : H^5(B\mathrm{SL}(\mathbb{Z})^+; \mathbb{Z}/2) \twoheadrightarrow \mathrm{Hom}(H_5(B\mathrm{SL}(\mathbb{Z})^+; \mathbb{Z}),\ \mathbb{Z}/2)$ *be the homomorphism given by the universal coefficient theorem and* η *the cohomology class introduced in Definition* 2.5. *Then* $\rho(\mathrm{Sq}^2\xi) = \rho(\eta)$.

Proof. The commutative diagram

$$\begin{array}{ccc} H^5(B\mathrm{SL}(\mathbb{F}_3)^+; \mathbb{Z}/2) & \xrightarrow{\quad f^* \quad} & H^5(B\mathrm{St}(\mathbb{Z})^+; \mathbb{Z}/2) \\ {\scriptstyle\rho}\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\rho} \\ \mathrm{Hom}(H_5(B\mathrm{SL}(\mathbb{F}_3)^+; \mathbb{Z}), \mathbb{Z}/2) & \xrightarrow{f^\natural} & \mathrm{Hom}(H_5(B\mathrm{St}(\mathbb{Z})^+; \mathbb{Z}), \mathbb{Z}/2) \end{array}$$

where $f^\natural$ is induced by f_*, and the injectivity of $f^\natural$ (consequence of Proposition 3.6) give us: $\rho(\mathrm{Sq}^2\bar{\xi}) = \rho(f^*(e_3)) = f^\natural(\rho(e_3)) \neq 0$; since $\mathrm{Sq}^2\bar{\xi} = g^*(\mathrm{Sq}^2\xi)$ we get $\rho(\mathrm{Sq}^2\xi) \neq 0$ in $\mathrm{Hom}(H_5(B\mathrm{SL}(\mathbb{Z})^+;\ \mathbb{Z}),\ \mathbb{Z}/2)$.

It follows from Lemma 3.10 and from $\beta(e_3) = 13\hat{c}_3$ that $\beta(h^*(e_3)) = h^*(\beta(e_3)) = 13c_3(\mathrm{SL}(\mathbb{Z})) = c_3(\mathrm{SL}(\mathbb{Z}))$ because $c_3(\mathrm{SL}(\mathbb{Z}))$ is an element of order 2

[1]. On the other hand, according to Remark 3.9(b), $\beta(h^*(e_3)) = \beta(w_2w_3) + \beta(w_5) + \beta(\mathrm{Sq}^2\xi) = c_3(\mathrm{SL}(\mathbb{Z})) + \beta(\mathrm{Sq}^2\xi)$ ($\beta(w_5) = 0$ since $w_5 = \mathrm{Sq}^1 w_4$). Thus we get $\beta(\mathrm{Sq}^2\xi) = 0$; therefore $\rho(\mathrm{Sq}^2\xi)$ belongs to the image of the reduction mod 2 $\mathrm{Hom}(H_5(B\mathrm{SL}(\mathbb{Z})^+;\mathbb{Z}),\mathbb{Z}) \to \mathrm{Hom}(H_5(B\mathrm{SL}(\mathbb{Z})^+;\mathbb{Z}),\mathbb{Z}/2)$. We then deduce from Definition 2.5 that $\rho(\mathrm{Sq}^2\xi) = \rho(\eta)$. □

Remark 3.12. The previous corollary implies that $\rho(\mathrm{Sq}^2\bar{\xi}) = \rho(\bar{\eta})$ in $\mathrm{Hom}(H_5(B\mathrm{St}(\mathbb{Z})^+;\mathbb{Z}),\mathbb{Z}/2)$.

We consider again the homomorphism $f_*: H_5(B\mathrm{St}(\mathbb{Z})^+;\mathbb{Z}) \to H_5(B\mathrm{SL}(\mathbb{F}_3)^+;\mathbb{Z})$.

Proposition 3.13. *Let T denote the torsion subgroup of $H_5(B\mathrm{St}(\mathbb{Z})^+;\mathbb{Z})$. Then the 2-torsion subgroup of $f_*(T)$ is trivial.*

Proof. Because $\rho(\bar{\eta})$ is by definition an element of the image of the reduction mod 2 $\mathrm{Hom}(H_5(B\mathrm{St}(\mathbb{Z})^+;\mathbb{Z}),\mathbb{Z}) \to \mathrm{Hom}(H_5(B\mathrm{St}(\mathbb{Z})^+;\mathbb{Z}),\mathbb{Z}/2)$, one has $\rho(\bar{\eta})(T) = 0$. Observe that, by commutativity of the diagram introduced in the proof of Corollary 3.11, $f^{\square}(\rho(e_3)) = \rho(\mathrm{Sq}^2\bar{\xi}) = \rho(\bar{\eta})$. Consequently $f^{\square}(\rho(e_3))(T) = 0$ and, since $f^{\square}$ is induced by f_*, $\rho(e_3)(f_*(T)) = 0$, which implies that there is no 2-torsion in $f_*(T)$. □

4. The Whitehead sequence of the space $B\mathrm{St}(\mathbb{Z})^+$

Proof of Theorems 1.3 and 1.4. We use the map $f: B\mathrm{St}(\mathbb{Z})^+ \to B\mathrm{SL}(\mathbb{F}_3)^+$ in order to compare the Whitehead exact sequence (cf. [7, p. 555, Theorem 3.12]) of $B\mathrm{St}(\mathbb{Z})^+$ with that of $B\mathrm{SL}(\mathbb{F}_3)^+$ (both spaces are 2-connected). We get the following commutative diagram where both rows are exact (Hu denotes the Hurewicz homomorphism):

$$\begin{tikzcd}
K_5\mathbb{Z} \arrow[r, "\mathrm{Hu}"] & H_5(B\mathrm{St}(\mathbb{Z})^+;\mathbb{Z}) \arrow[r, "\varphi"] \arrow[d, "f_*"] & \underbrace{K_3\mathbb{Z}\otimes\mathbb{Z}/2}_{\cong\mathbb{Z}/2} \arrow[r] \arrow[d, "\psi"] & K_4\mathbb{Z} \arrow[r, "\mathrm{Hu}"] & H_4(B\mathrm{St}(\mathbb{Z})^+;\mathbb{Z}) \arrow[r] & 0 \\
K_5\mathbb{F}_3 \arrow[r, "\mathrm{Hu}"] & \underbrace{H_5(B\mathrm{SL}(\mathbb{F}_3)^+;\mathbb{Z})}_{\cong\mathbb{Z}/2\oplus\mathbb{Z}/13} \arrow[r, "\chi"] & \underbrace{K_3\mathbb{F}_3\otimes\mathbb{Z}/2}_{\cong\mathbb{Z}/2} \arrow[r] & K_4\mathbb{F}_3 = 0
\end{tikzcd}$$

Note that the Whitehead exact sequence can also be obtained from the Serre spectral sequence of the fibration $A(\mathbb{Z}) \to B\mathrm{St}(\mathbb{Z})^+ \xrightarrow{p} K(K_3\mathbb{Z}, 3)$ (respectively $A(\mathbb{F}_3) \to B\mathrm{SL}(\mathbb{F}_3)^+ \to K(K_3\mathbb{F}_3, 3)$), where p is the Postnikov approximation map and $A(\mathbb{Z})$ the fibre of p.

The homomorphism ψ is actually $f_* \otimes 1$ and, since $f_* : K_3\mathbb{Z} \cong \mathbb{Z}/48 \to K_3\mathbb{F}_3 \cong \mathbb{Z}/8$ is surjective (cf. [1, §3]), ψ is an isomorphism. Proposition 3.6 says that $\chi \circ f_*$ is surjective and, by commutativity of the diagram, that φ is surjective. The proof is then complete because the group $\mathrm{St}(\mathbb{Z})$ and the space $B\mathrm{St}(\mathbb{Z})^+$ have the same homology. □

Proof of Theorem 1.5. The above diagram and Proposition 3.13 show that $\varphi(T) = 0$ (T denotes the torsion subgroup of $H_5(B\mathrm{St}(\mathbb{Z})^+; \mathbb{Z})$). Therefore we have an exact sequence

$$K_5\mathbb{Z}/\mathrm{torsion} \xrightarrow{\mathrm{Hu}'} H_5(B\mathrm{St}(\mathbb{Z})^+; \mathbb{Z})/\mathrm{torsion} \xrightarrow{\varphi'} \mathbb{Z}/2$$

where Hu′ (respectively φ') is induced by Hu (respectively φ). It follows from the surjectivity of φ that φ' is also surjective. Consequently Hu′ is multiplication by 2. The analogous statement for the space $B\mathrm{SL}(\mathbb{Z})^+$ is then a consequence of Remark 2.12. □

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