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ON THE HOMOLOGY AND COHOMOLOGY OF CONGRUENCE SUBGROUPS

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Introduction

For any prime p let $\Gamma_n(p)$ denote the congruence subgroup of $\mathrm{SL}_n(\mathbb{Z})$ of level $p: \Gamma_n(p)$ is the kernel of the surjective homomorphism $\mathrm{SL}_n(\mathbb{Z}) \twoheadrightarrow \mathrm{SL}_n(\mathbb{F}_p)$ induced by the reduction mod p ($\mathbb{F}_p$ is the field with p elements). The groups $\Gamma_n(p)$ are torsion-free for all odd primes p.

Recent results of Charney [4] and Suslin [14] provide examples of coefficient groups M such that the groups $\Gamma_n(p)$ are homology stable with M-coefficients and that $\mathrm{SL}(\mathbb{F}_p)$ acts trivially on $H_*(\Gamma(p); M): M = \mathbb{Q}$, $\mathbb{Z}[1/p]$ or $\mathbb{Z}/m$ (p not dividing m). For this choice of M we compare the stable homology groups of $\Gamma_n(p)$ with those of $\mathrm{SL}_n(\mathbb{Z})$ (Theorems 1.4 and 1.5). We then study the stable homology groups $H_i(\Gamma_n(p); \mathbb{Z}/q^d)$ ($p \neq q$ primes, $d \geq 1$) for $0 \leq i \leq 5$ (Section 2).

Finally we look at the cohomology with $\mathbb{Z}$-coefficients and prove that the restriction homomorphism $H^4(\mathrm{SL}_n(\mathbb{Z}); \mathbb{Z}) \to H^4(\Gamma_n(p); \mathbb{Z})$ is zero for all odd primes p and $n \geq 9$ (Corollary 3.3). This is an immediate consequence of the following result on the second Chern class of $\Gamma_n(p): c_2(\Gamma_n(p)) = 0$ for all odd primes p and $n \geq 2$.

1. The stable homology groups of the congruence subgroups

Let R be a commutative ring with unit and I an ideal in R. We assume throughout this section that $\mathrm{SL}(R/I)$ ($= \varinjlim \mathrm{SL}_n(R/I)$) is generated by elementary matrices (i.e., $\mathrm{SK}_1(R/I) = 0$). The projection $R \twoheadrightarrow R/I$ induces then a surjective homomorphism $\mathrm{SL}_n(R) \twoheadrightarrow \mathrm{SL}_n(R/I)$ for $n \geq 2$. We define the congruence subgroup $\Gamma_n(I)$ as the kernel of $\mathrm{SL}_n(R) \twoheadrightarrow \mathrm{SL}_n(R/I)$ and $\Gamma(I) := \varinjlim \Gamma_n(I)$. We get the short exact se-

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quence $\Gamma(I) \rightarrowtail \mathrm{SL}(R) \twoheadrightarrow \mathrm{SL}(R/I)$ and, in this section, we want to give cases in which $\mathrm{SL}(R/I)$ acts trivially on the homology of $\Gamma(I)$.

We recall the definition of the homology stability: a sequence of groups $G_1 \subset G_2 \subset G_3 \subset \cdots$ is homology stable with M-coefficients if for all $i \geq 0$ there exists an integer $n_0(i)$ such that the inclusions induce isomorphisms $H_i(G_n; M) \xrightarrow{\cong} H_i(G_{n+1}; M)$ for $n \geq n_0(i)$. It is easy to prove the following result due to Charney [4].

Lemma 1.1. *If the groups $\Gamma_n(I)$ $(n = 2, 3, 4, \ldots,)$ are homology stable with M-coefficients, then the action (induced by conjugation) of the group* $\mathrm{SL}(R/I)$ *on $H_*(\Gamma(I); M)$ is trivial.*

The short exact sequence $\Gamma(I) \rightarrowtail \mathrm{SL}(R) \twoheadrightarrow \mathrm{SL}(R/I)$ induces the fibration $B\Gamma(I) \to B\mathrm{SL}(R) \to B\mathrm{SL}(R/I)$. By performing the $+$ construction we get the following commutative diagram where $F(I)$ denotes the fiber of the map $B\mathrm{SL}(R)^+ \to B\mathrm{SL}(R/I)^+$:

$$\begin{array}{ccccc} B\Gamma(I) & \longrightarrow & B\mathrm{SL}(R) & \longrightarrow & B\mathrm{SL}(R/I) \\ \downarrow f & & \downarrow + & & \downarrow + \\ F(I) & \longrightarrow & B\mathrm{SL}(R)^+ & \longrightarrow & B\mathrm{SL}(R/I)^+ \end{array}$$

Because $\mathrm{SK}_1(R/I) = 0$ the group $\mathrm{SL}(R/I)$ is perfect and the space $B\mathrm{SL}(R/I)^+$ is simply connected; therefore $F(I)$ is connected.

Lemma 1.2. *The two following conditions are equivalent:*

(a) *the action of* $\mathrm{SL}(R/I)$ *on $H_*(\Gamma(I); M)$ is trivial;*

(b) *the map f induces an isomorphism $f_*: H_*(B\Gamma(I); M) \xrightarrow{\cong} H_*(F(I); M)$.*

Proof. If the condition (a) is satisfied, then we can apply the comparison theorem for spectral sequences on the above diagram and we obtain (b). Conversely, if (b) holds, then the action of $\mathrm{SL}(R/I)$ on $H_*(B\Gamma(I); M)$ is trivial since $B\mathrm{SL}(R/I)^+$ is simply connected.

Remark. If the conditions of Lemma 1.2 are satisfied for $M = \mathbb{Z}$ and if the commutator subgroup $[\Gamma(I), \Gamma(I)]$ is perfect, then we have a homotopy equivalence $F(I) \simeq B\Gamma(I)^+_{[\Gamma(I), \Gamma(I)]}$.

We now restrict our attention to the case $R = \mathbb{Z}$, $I = p\mathbb{Z}$ where p is a prime number and denote the kernel of $\mathrm{SL}_n(\mathbb{Z}) \twoheadrightarrow \mathrm{SL}_n(\mathbb{F}_p)$ by $\Gamma_n(p)$ $(n \geq 2)$. Recall that the groups $\mathrm{SL}_n(\mathbb{Z})$ are homology stable with $\mathbb{Z}$-coefficients (cf. [15] or [10]): $H_i(\mathrm{SL}_n(\mathbb{Z}); \mathbb{Z}) \cong H_i(\mathrm{SL}_{n+1}(\mathbb{Z}); \mathbb{Z})$ for $n \geq 2i+1$. The congruence subgroups $\Gamma_n(p)$ are also homology stable with some coefficient groups. More precisely, if $M = \mathbb{Q}$, $\mathbb{Z}[1/p]$ or $\mathbb{Z}/m$ for all integers m with $(m, p) = 1$, then $H_i(\Gamma_n(p); M) \cong H_i(\Gamma_{n+1}(p); M)$ for $n \geq 2i+5$

(cf. [4, Theorem 5.2 and Example 5.4]). By Lemma 1.1 this stability result provides conditions on M for the triviality of the action of $\mathrm{SL}(\mathbb{F}_p)$ on $H_*(\Gamma(p); M)$.

Corollary 1.3. *Let p be a prime and* $M=\mathbb{Q}$, $\mathbb{Z}[1/p]$ *or* $\mathbb{Z}/m$ *for all integers m with* $(m,p)=1$. *Then the action of the group* $\mathrm{SL}(\mathbb{F}_p)$ *on* $H_*(\Gamma(p); M)$ *is trivial.*

Remark 1. The statement of Corollary 1.3 is also proved by Suslin (cf. [14, Proposition 1.3]) for $\mathbb{Z}/m$ and we can modify this proof to get the same assertion for $M=\mathbb{Q}$ or $\mathbb{Z}[1/p]$.

Remark 2. Let $F(p)$ denote the fiber of $B\mathrm{SL}(\mathbb{Z})^+ \to B\mathrm{SL}(\mathbb{F}_p)^+$; since $K_2\mathbb{F}_p=0$, $F(p)$ is simply connected and therefore $H_1(F(p); \mathbb{Z})=0$. On the other hand $H_1(\Gamma_n(p); \mathbb{Z})$ is a (non-trivial) p-group for all $n\geq 3$ (cf. [7, Theorem 1.1]). We then deduce from Lemma 1.2 that for $M=\mathbb{Z}$ or $\mathbb{Z}/p$, the action of $\mathrm{SL}(\mathbb{F}_p)$ on $H_*(\Gamma(p); M)$ cannot be trivial, and from Lemma 1.1 that the groups $\Gamma_n(p)$ are not homology stable with M-coefficients.

Our next objective is to compare the stable homology groups of $\Gamma_n(p)$ with those of $\mathrm{SL}_n(\mathbb{Z})$; we consider coefficients in $\mathbb{Q}$ and in $\mathbb{Z}/m$ ($(m,p)=1$).

Theorem 1.4. *Let p be a prime. Then for* $i\geq 0$ *the inclusion* $\Gamma_n(p)\hookrightarrow \mathrm{SL}_n(\mathbb{Z})$ *induces an isomorphism*

$$H_i(\Gamma_n(p); \mathbb{Q}) \xrightarrow{\cong} H_i(\mathrm{SL}_n(\mathbb{Z}); \mathbb{Q})$$

for $n\geq 2i+5$.

Proof. By Corollary 1.3, $\mathrm{SL}(\mathbb{F}_p)$ acts trivially on $H_*(\Gamma(p); \mathbb{Q})$. Since $H_i(\mathrm{SL}(\mathbb{F}_p); \mathbb{Q})=0$ for $i>0$ the theorem follows from a spectral sequence argument and homology stability.

Remark. Theorem 1.4 is also proved in [3] where the stable rational homology of these groups is computed: $H_*(\Gamma(p); \mathbb{Q}) \cong H_*(\mathrm{SL}(\mathbb{Z}); \mathbb{Q}) = \Lambda(x_5, x_9, \ldots, x_{4k+1}, \ldots)$ with $\deg x_{4k+1}=4k+1$ for $k\geq 1$.

Theorem 1.5. *Let p and q be primes* $(p\neq q)$ *and s the least integer* ≥ 2 *such that* $p^s\equiv 1 \bmod q$.

(a) *For* $0\leq i\leq 2s-3$ *the inclusion* $\Gamma_n(p)\hookrightarrow \mathrm{SL}_n(\mathbb{Z})$ *induces an isomorphism*

$$H_i(\Gamma_n(p); \mathbb{Z}/q^d) \xrightarrow{\cong} H_i(\mathrm{SL}_n(\mathbb{Z}); \mathbb{Z}/q^d)$$

for $n\geq 2i+5$ *and* $d\geq 1$.

(b) *The following sequence is exact for* $d\geq 1$:

$$H_{2s-1}(\Gamma(p); \mathbb{Z}/q^d) \to H_{2s-1}(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}/q^d) \to \mathbb{Z}/(p^s-1)\otimes \mathbb{Z}/q^d \to$$

$$\to H_{2s-2}(\Gamma(p); \mathbb{Z}/q^d) \to H_{2s-2}(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}/q^d) \to 0.$$

Proof. Our argument is based on the calculation of the E^2-term of the Hochschild–Serre spectral sequence of the extension $\Gamma(p) \rightarrowtail \mathrm{SL}(\mathbb{Z}) \twoheadrightarrow \mathrm{SL}(\mathbb{F}_p)$: $E^2_{i,j} = H_i(\mathrm{SL}(\mathbb{F}_p); H_j(\Gamma(p); \mathbb{Z}/q^d))$ (by Corollary 1.3, $H_j(\Gamma(p); \mathbb{Z}/q^d)$ is a trivial $\mathrm{SL}(\mathbb{F}_p)$-module). If r denotes the least positive integer such that $p^r \equiv 1 \bmod q$ then, according to [11], $H_i(\mathrm{SL}(\mathbb{F}_p); \mathbb{Z}/q^d) = 0$ for $0 < i \leq 2r-2$ and $H_{2r-1}(\mathrm{SL}(\mathbb{F}_p); \mathbb{Z}/q^d) \cong \mathbb{Z}/(p^r-1) \otimes \mathbb{Z}/q^d$. But $H_1(\mathrm{SL}(\mathbb{F}_p); \mathbb{Z}) = H_2(\mathrm{SL}(\mathbb{F}_p); \mathbb{Z}) = 0$ since $\mathrm{SL}(\mathbb{F}_p)$ is perfect and $K_2\mathbb{F}_p = 0$. Therefore we may conclude that $H_i(\mathrm{SL}(\mathbb{F}_p); \mathbb{Z}/q^d) = 0$ for $0 < i \leq 2s-2$ where $s = \max\{r, 2\}$.

Thus we have $E^2_{i,j} = 0$ for $0 < i \leq 2s-2$, which implies (a) by homology stability, and the exact sequence

$$\begin{aligned} &H_{2s-1}(\Gamma(p); \mathbb{Z}/q^d) \to H_{2s-1}(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}/q^d) \to H_{2s-1}(\mathrm{SL}(\mathbb{F}_p); \mathbb{Z}/q^d) \\ &\quad \to H_{2s-2}(\Gamma(p); \mathbb{Z}/q^d) \to H_{2s-2}(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}/q^d) \to 0. \end{aligned}$$

The assertion (b) follows from

$$H_{2s-1}(\mathrm{SL}(\mathbb{F}_p); \mathbb{Z}/q^d) \cong \begin{cases} \mathbb{Z}/(p^s-1) \otimes \mathbb{Z}/q^d, & \text{if } s = r, \\ K_3\mathbb{F}_p \otimes \mathbb{Z}/q^d \cong \mathbb{Z}/(p^2-1) \otimes \mathbb{Z}/q^d, & \text{if } s = 2. \end{cases}$$

Finally we mention the following result on the homology of $\Gamma(p^t)$ for $t \geq 1$.

Lemma 1.6. *Let p be a prime and m an integer such that $(m, p) = 1$. Then*

$$H_*(\Gamma(p^t); \mathbb{Z}/m) \cong H_*(\Gamma(p); \mathbb{Z}/m) \quad \textit{for all } t \geq 1.$$

Proof. Let t be a given integer ≥ 1. The projection $\mathbb{Z}/p^t \twoheadrightarrow \mathbb{Z}/p$ induces a surjective homomorphism $\mathrm{SL}(\mathbb{Z}/p^t) \twoheadrightarrow \mathrm{SL}(\mathbb{Z}/p)$ whose kernel will be called K. It follows from [14, Lemma 1.7] that $\tilde{H}_*(K; \mathbb{Z}/m) = 0$ and therefore the induced homomorphism $H_*(\mathrm{SL}(\mathbb{Z}/p^t); \mathbb{Z}/m) \to H_*(\mathrm{SL}(\mathbb{Z}/p); \mathbb{Z}/m)$ is an isomorphism. By Corollary 1.3 we may apply the comparison theorem for spectral sequences to the commutative diagram

$$\begin{array}{ccccc} \Gamma(p^t) & \rightarrowtail & \mathrm{SL}(\mathbb{Z}) & \twoheadrightarrow & \mathrm{SL}(\mathbb{Z}/p^t) \\ \downarrow & & \| & & \downarrow \\ \Gamma(p) & \rightarrowtail & \mathrm{SL}(\mathbb{Z}) & \twoheadrightarrow & \mathrm{SL}(\mathbb{Z}/p) \end{array}$$

which implies the desired isomorphism.

2. Calculations in low dimensions

In order to compute homology groups of $\Gamma_n(p)$ using the previous section let us recall the following results on the homology of $\mathrm{SL}(\mathbb{Z})$.

Lemma 2.1 [2]. $H_1(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}) = 0$, $H_2(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}) \cong \mathbb{Z}/2$, $H_3(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}) \cong \mathbb{Z}/24$.

Lemma 2.2. *Let C denote the Serre class of all finite abelian groups containing only 2- and 3-torsion.*

(a) $H_4(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}) \in C$,

(b) $H_5(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}) \cong \mathbb{Z} \oplus H$, *where* $H \in C$.

Proof. Let X be the simply connected space $B\mathrm{SL}(\mathbb{Z})^+$. Because $K_2\mathbb{Z} \cong \mathbb{Z}/2$, $K_3\mathbb{Z} \cong \mathbb{Z}/48$ [8] and $K_4\mathbb{Z} \in C$ (cf. [9] and [13]) the homology groups $H_i(X; \mathbb{Z}) \cong H_i(\mathrm{SL}(\mathbb{Z}); \mathbb{Z})$ are also elements of C for $0 < i \leq 4$ and the Hurewicz homomorphism $\pi_5 X \to H_5(X; \mathbb{Z})$ is a C-isomorphism. This implies (b) because $\pi_5 X = K_5\mathbb{Z} \cong \mathbb{Z} \oplus (\text{finite group} \in C)$ (cf. [9] and [13]).

Throughout this section we assume that p and q are prime numbers such that $p \neq q$. We are now able to compute the stable homology groups $H_i(\Gamma(p); \mathbb{Z}/q^d)$, $d \geq 1$, for $0 \leq i \leq 3$ and to obtain partial results for $i = 4$ and 5.

A direct consequence of Theorem 1.5 is that $H_1(\Gamma(p); \mathbb{Z}/q) = 0$. Note that this result is obvious by the computation of $H_1(\Gamma_n(p); \mathbb{Z})$ for $n \geq 3$ by Lee and Szczarba [7, Theorem 1.1]: $H_1(\Gamma_n(p); \mathbb{Z})$ is isomorphic to the additive group of all $n \times n$ matrices with entries in $\mathbb{F}_p$ and trace zero.

For $i = 2$ and 3 we can again use Theorem 1.5 but we get more information by looking at the homotopy exact sequence of the fibration $F(p) \to B\mathrm{SL}(\mathbb{Z})^+ \xrightarrow{\pi} B\mathrm{SL}(\mathbb{F}_p)^+$ (cf. Section 1). Since $K_2\mathbb{F}_p = K_4\mathbb{F}_p = 0$ the fiber $F(p)$ is simply connected and for all primes q we obtain the exact sequence

$$0 \to (\pi_3 F(p))_q \to (K_3\mathbb{Z})_q \to (K_3\mathbb{F}_p)_q \to (\pi_2 F(p))_q \to (K_2\mathbb{Z})_q \to 0 \qquad (*)$$

where $(\ )_q$ denotes the q-primary component. We have proved (cf. [1, Sätze 2.5, 2.6, 2.8]) that the induced homomorphism $\pi_* : K_3\mathbb{Z} \cong \mathbb{Z}/48 \to K_3\mathbb{F}_p \cong \mathbb{Z}/(p^2-1)$ is surjective for $p = 2$ and $p = 3$ and that its image is a cyclic group of order 24 if $p \geq 5$.

Theorem 2.3. *Let p and q be primes such that $p \neq q$ and $q \neq 2$.*

(a) *For* $n \geq 9$ *and* $d \geq 1$: $H_2(\Gamma_n(p); \mathbb{Z}/q^d) \cong \mathbb{Z}/((p^2-1)/3, q^d)$, *if* $p \neq 3$ *and* $H_2(\Gamma_n(3); \mathbb{Z}/q^d) = 0$.

(b) *For* $n \geq 11$ *and* $d \geq 1$: $H_3(\Gamma_n(p); \mathbb{Z}/q^d) \cong \mathbb{Z}/((p^2-1)/3, q^d)$, *if* $p \neq 3$ *and* $H_3(\Gamma_n(3); \mathbb{Z}/q^d) = 0$.

Proof. We first consider the case $q = 3$ ($p \neq 3$). We have the exact sequence $(*)$

$$(\pi_3 F(p))_3 \rightarrowtail \mathbb{Z}/3 \xrightarrow{\pi_*} (\mathbb{Z}/(p^2-1))_3 \twoheadrightarrow (\pi_2 F(p))_3$$

and π_* is injective since $p \neq 3$. Consequently $(\pi_3 F(p))_3 = 0$ and $(\pi_2 F(p))_3 \cong$

$(\mathbb{Z}/(p^2-1)/3)_3$. Because $F(p)$ is simply connected the Hurewicz homomorphism $\pi_i F(p) \to H_i(F(p); \mathbb{Z})$ is surjective for $i=3$ and an isomorphism for $i=2$. It follows that $H_2(F(p); \mathbb{Z})_3 \cong (\mathbb{Z}/(p^2-1)/3)_3$ and $H_3(F(p); \mathbb{Z})_3 = 0$. We deduce from Lemma 1.2 and the universal coefficient theorem (for $d \geq 1$):

$$H_2(\Gamma(p); \mathbb{Z}/3^d) \cong H_2(F(p); \mathbb{Z}/3^d) \cong \mathbb{Z}/((p^2-1)/3, 3^d),$$

$$H_3(\Gamma(p); \mathbb{Z}/3^d) \cong H_3(F(p); \mathbb{Z}/3^d) \cong \mathbb{Z}/((p^2-1)/3, 3^d).$$

Now for $q \geq 5$ $(K_3\mathbb{Z})_q = (K_2\mathbb{Z})_q = 0$ and the exact sequence (*) gives us:

$$(\pi_3 F(p))_q = 0 \quad \text{and}$$

$$(\pi_2 F(p))_q \cong (\mathbb{Z}/(p^2-1))_q \cong \begin{cases} (\mathbb{Z}/(p^2-1)/3)_q & \text{for } p \neq 3, \text{ since } q \neq 3, \\ 0 & \text{for } p = 3, \text{ since } q \neq 2. \end{cases}$$

As above we get:

$$H_2(\Gamma(p); \mathbb{Z}/q^d) \cong H_3(\Gamma(p); \mathbb{Z}/q^d) \cong \mathbb{Z}/((p^2-1)/3, q^d) \quad \text{for } p \neq 3$$

and

$$H_2(\Gamma(3); \mathbb{Z}/q^d) = H_3(\Gamma(3); \mathbb{Z}/q^d) = 0.$$

The proof is then complete by homology stability.

We also discuss the case $q=2$ which is more complicated.

Theorem 2.4. *Let p be a prime* $\neq 2$.

(a) *If* $p^2-1 \not\equiv 0 \bmod 16$, *then* $H_2(\Gamma_n(p); \mathbb{Z}/2^d) \cong \mathbb{Z}/2$ *for* $n \geq 9$ *and* $d \geq 1$.

(b) *If* $p^2-1 \equiv 0 \bmod 16$, *then one has the following short exact sequence for* $n \geq 9$:

$$(\mathbb{Z}/(p^2-1)/8)_2 \rightarrowtail H_2(\Gamma_n(p); \mathbb{Z})_2 \twoheadrightarrow \mathbb{Z}/2.$$

(c) *For* $n \geq 11$ *and* $d \geq 1$: $H_3(\Gamma_n(p); \mathbb{Z}/2^d) \cong H_2(\Gamma_n(p); \mathbb{Z}/2^d) \oplus l\mathbb{Z}/2$ *where* $l=0$ *or* 1.

Proof. We use again the exact sequence (*):

$$(\pi_3 F(p))_2 \rightarrowtail \mathbb{Z}/16 \xrightarrow{\pi_*} (\mathbb{Z}/(p^2-1))_2 \to (\pi_2 F(p))_2 \twoheadrightarrow \mathbb{Z}/2.$$

The image of π_* is cyclic of order 8 since $p \neq 2$. If $p^2-1 \not\equiv 0 \bmod 16$, then $(\pi_2 F(p))_2 \cong \mathbb{Z}/2$ and, as in the proof of the previous theorem,

$$H_2(\Gamma(p); \mathbb{Z}/2^d) \cong H_2(F(p); \mathbb{Z}/2^d) \cong \mathbb{Z}/2 \quad \text{for } d \geq 1.$$

If $p^2-1 \equiv 0 \bmod 16$, then we get the short exact sequence

$$(\mathbb{Z}/(p^2-1)/8)_2 \rightarrowtail H_2(F(p); \mathbb{Z})_2 \twoheadrightarrow \mathbb{Z}/2$$

and $H_2(F(p); \mathbb{Z})_2 \cong H_2(\Gamma(p); \mathbb{Z})_2$ since $p \neq 2$ (cf. Lemma 1.2). The exact sequence

(*) also implies that $(\pi_3 F(p))_2 \cong \mathbb{Z}/2$ and therefore $H_3(F(p);\mathbb{Z})_2 \cong 0$ or $\mathbb{Z}/2$ since the Hurewicz homomorphism is surjective. By the universal coefficient theorem we may conclude that $H_3(\Gamma(p);\mathbb{Z}/2^d) \cong H_2(\Gamma(p);\mathbb{Z}/2^d) \oplus l\mathbb{Z}/2$ with $l=0$ or 1. Again the homology stability completes the proof.

Remark. It is actually possible to show that $l=0$.

We finally look at the cases $i=4$ and $i=5$.

Theorem 2.5. *Let p and q be primes such that $p \neq q$, $(p^2-1,q)=(p^3-1,q)=1$. Then*

(a) $H_4(\Gamma_n(p);\mathbb{Z}/q^d)=0$ *for* $n \geq 13$ *and* $d \geq 1$,

(b) $H_5(\Gamma_n(p);\mathbb{Z}/q^d) \cong \mathbb{Z}/q^d$ *for* $n \geq 15$ *and* $d \geq 1$.

Proof. By Theorem 1.5 we have for $0 \leq i \leq 5$: $H_i(\Gamma_n(p);\mathbb{Z}/q^d) \cong H_i(\mathrm{SL}_n(\mathbb{Z});\mathbb{Z}/q^d)$ for $n \geq 2i+5$ and $d \geq 1$. The prime q must be ≥ 5 because $(p^2-1,q)=1$ and $p \neq q$. Consequently by Lemma 2.1 and Lemma 2.2, $H_4(\mathrm{SL}_n(\mathbb{Z});\mathbb{Z}/q^d)=0$ and $H_5(\mathrm{SL}_n(\mathbb{Z});\mathbb{Z}/q^d) \cong \mathbb{Z} \otimes \mathbb{Z}/q^d \cong \mathbb{Z}/q^d$ and the theorem is proved.

3. The restriction in cohomology with integral coefficients

In this section we study the restriction homomorphism

$$\mathrm{res}: H^i(\mathrm{SL}_n(\mathbb{Z});\mathbb{Z}) \to H^i(\Gamma_n(p);\mathbb{Z})$$

for $i \leq 4$, p an odd prime and n large. We do not consider the prime 2 which is the unique case where $\Gamma_n(p)$ is not torsion-free.

The problem is trivial for $i=1$ and 2 because $H^1(\mathrm{SL}(\mathbb{Z});\mathbb{Z}) = H^2(\mathrm{SL}(\mathbb{Z});\mathbb{Z})=0$ (cf. Lemma 2.1). Note that since $H_1(\Gamma_n(p);\mathbb{Z})$ is a finite p-group [7, Theorem 1.1] $H^1(\Gamma_n(p);\mathbb{Z})=0$ and $H^2(\Gamma_n(p);\mathbb{Z}) \cong H_1(\Gamma_n(p);\mathbb{Z})$ for large n.

The first nontrivial case occurs for $i=3$. Because $H^3(\mathrm{SL}_n(\mathbb{Z});\mathbb{Z}) \cong \mathbb{Z}/2$ (Lemma 2.1) the study of $\mathrm{res}: H^3(\mathrm{SL}_n(\mathbb{Z});\mathbb{Z}) \to H^3(\Gamma_n(p);\mathbb{Z})$ ($n \geq 11$) is a 2-torsion problem and we can use in cohomology the argument explained in Section 1 for homology, since $p \neq 2$: we get easily the injectivity of the restriction $H^3(\mathrm{SL}_n(\mathbb{Z});\mathbb{Z}[1/p]) \to H^3(\Gamma_n(p);\mathbb{Z}[1/p])$; therefore, for all odd primes p, it follows from $H^3(\mathrm{SL}_n(\mathbb{Z});\mathbb{Z}) \cong H^3(\mathrm{SL}_n(\mathbb{Z});\mathbb{Z}[1/p])$ that $\mathrm{res}: H^3(\mathrm{SL}_n(\mathbb{Z});\mathbb{Z}) \to H^3(\Gamma_n(p);\mathbb{Z})$ is injective.

More interesting is the study of $\mathrm{res}: H^4(\mathrm{SL}_n(\mathbb{Z});\mathbb{Z}) \to H^4(\Gamma_n(p);\mathbb{Z})$. We have proved in [2] that for $n \geq 9$, $H^4(\mathrm{SL}_n(\mathbb{Z});\mathbb{Z})$ is a cyclic group of order 24, generated by the second Chern class $c_2(\mathrm{SL}_n(\mathbb{Z}))$ which is defined as follows: the inclusion $\varrho: \mathrm{SL}_n(\mathbb{Z}) \hookrightarrow \mathrm{GL}_n(\mathbb{C})$ induces a homomorphism $\varrho^*: H^*(B\mathrm{GL}_n(\mathbb{C});\mathbb{Z}) = \mathbb{Z}[c_1,c_2,\ldots,c_n] \to H^*(\mathrm{SL}_n(\mathbb{Z});\mathbb{Z})$ where $\deg c_i = 2i$ and we define $c_i(\mathrm{SL}_n(\mathbb{Z})) := \varrho^*(c_i) \in H^{2i}(\mathrm{SL}_n(\mathbb{Z});\mathbb{Z})$ for $i \geq 1$. We also define Chern classes of $\Gamma_n(p)$ by $c_i(\Gamma_n(p)) := \mathrm{res}(c_i(\mathrm{SL}_n(\mathbb{Z}))) \in H^{2i}(\Gamma_n(p);\mathbb{Z})$. In order to study $\mathrm{res}: H^4(\mathrm{SL}_n(\mathbb{Z});\mathbb{Z}) \to H^4(\Gamma_n(p);\mathbb{Z})$

for $p \neq 2$ we look at $c_2(\Gamma_n(p))$.

If p is a prime ≥ 5, then we have shown that $c_2(\Gamma_n(p)) = 0$ for $n \geq 2$ [2, Satz 3.2]; therefore the restriction is zero (for $n \geq 9$). If $p = 3$ we know from [2, Satz 3.3] that $3c_2(\Gamma_n(3)) = 0$ for $n \geq 2$. (These results are also consequence of [5].) It remains to determine if $c_2(\Gamma_n(3))$ is 0 or an element of order 3 in $H^4(\Gamma_n(3); \mathbb{Z})$. This question is more difficult because it is a 3-torsion problem and $p = 3$. Therefore we cannot use the method developed in Section 1; in particular the (co)homology stability fails and we cannot apply Corollary 1.3.

In order to solve this problem (Theorem 3.2) we start with the exact sequence $\Gamma_n(3) \rightarrowtail \mathrm{SL}_n(\mathbb{Z}) \xrightarrow{\pi_n} \mathrm{SL}_n(\mathbb{F}_3)$ where π_n denotes the surjective homomorphism induced by the reduction mod 3. For $n \geq 2$ we define U_n as the subgroup of $\mathrm{SL}_n(\mathbb{F}_3)$ consisting of all upper triangular matrices with 1's on the diagonal (U_n is a 3-Sylow subgroup of $\mathrm{SL}_n(\mathbb{F}_3)$). We also define $G_n := \{x \in \mathrm{SL}_n(\mathbb{Z}) \mid \pi_n(x) \in U_n\}$. Obviously the kernel of $\pi_n : G_n \twoheadrightarrow U_n$ is $\Gamma_n(3)$. We have the following commutative diagram (G_n is a pull-back):

$$\begin{array}{ccccc}
\Gamma_n(3) & \rightarrowtail & \mathrm{SL}_n(\mathbb{Z}) & \xrightarrow{\pi_n} & \mathrm{SL}_n(\mathbb{F}_3) \\
\| & & \cup & & \cup \\
\Gamma_n(3) & \rightarrowtail & G_n & \xrightarrow{\pi_n} & U_n
\end{array}$$

We first consider the short exact sequence ($n = 2$)

$$\Gamma_2(3) \rightarrowtail G_2 \xrightarrow{\pi_2} U_2 \cong \mathbb{Z}/3.$$

Lemma 3.1. *For $i \geq 3$ the homomorphism π_2 induces an isomorphism*

$$\pi_2^* : H^i(U_2; \mathbb{Z}) \xrightarrow{\cong} H^i(G_2; \mathbb{Z}).$$

Proof. By [12, p. 505], $\Gamma_2(3)$ is a free group with 3 generators which are given in [7]:

$$a = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, \quad c = \begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix}.$$

The E_2-term $E_2^{i,j} = H^i(U_2; H^j(\Gamma_2(3); \mathbb{Z}))$ of the Hochschild–Serre spectral sequence is zero for $j \geq 2$ since $\Gamma_2(3)$ is free. The group U_2 is cyclic of order 3, generated by $x = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ which acts nontrivially on $\Gamma_2(3) : x^{-1}ax = a$, $x^{-1}bx = c^{-1}$, $x^{-1}cx = a^{-1}bc^{-1}$. The computation of $E_2^{i,1} = H^i(U_2; \mathrm{Hom}(\Gamma_2(3)_{\mathrm{Ab}}, \mathbb{Z}))$ gives the following result: $E_2^{i,1} = 0$ for $i > 0$ and $E_2^{0,1} \cong \mathbb{Z}$. Finally $E_2^{i,0} = H^i(U_2; \mathbb{Z}) \cong H^i(\mathbb{Z}/3; \mathbb{Z})$.

$$E_2^{i,j}: \quad \begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\mathbb{Z} & 0 & 0 & 0 & 0 & 0 \\
\mathbb{Z} & 0 & \mathbb{Z}/3 & 0 & \mathbb{Z}/3 & 0
\end{array}$$

Consequently $\pi_2^*: H^i(U_2;\mathbb{Z}) \to H^i(G_2;\mathbb{Z})$ is an isomorphism for $i \geq 3$.

We are now able to prove the main theorem of this section.

Theorem 3.2. $c_2(\Gamma_n(3)) = 0$ *for* $n \geq 2$.

Proof. (a) For $n \geq 2$ let $c_2(G_n)$ be the image of $c_2(\mathrm{SL}_n(\mathbb{Z}))$ by the restriction $H^4(\mathrm{SL}_n(\mathbb{Z});\mathbb{Z}) \to H^4(G_n;\mathbb{Z})$. Note that the commutativity of

$$\begin{array}{ccc} \Gamma_n(3) & \hookrightarrow & \mathrm{SL}_n(\mathbb{Z}) \\ & \searrow \quad \nearrow & \\ & G_n & \end{array}$$

implies that the restriction $H^4(G_n;\mathbb{Z}) \to H^4(\Gamma_n(3);\mathbb{Z})$ maps $c_2(G_n)$ onto $c_2(\Gamma_n(3))$. The order of $c_2(G_2)$ in $H^4(G_2;\mathbb{Z}) \cong \mathbb{Z}/3$ is equal to 3 because G_2 contains a cyclic group of order 3 generated by $\left(\begin{smallmatrix} -2 & 1 \\ -3 & 1 \end{smallmatrix}\right)$ (cf. [6, Proposition 3.7]). Now let $n \geq 2$ be a given integer and ϕ denote the inclusion $G_2 \hookrightarrow G_n$. The 3-primary component $H^4(\mathrm{SL}(\mathbb{Z});\mathbb{Z})_3$ is cyclic of order 3 generated by $8c_2(\mathrm{SL}(\mathbb{Z}))$ because $c_2(\mathrm{SL}(\mathbb{Z}))$ is of order 24. The composition of restrictions

$$H^4(\mathrm{SL}(\mathbb{Z});\mathbb{Z})_3 \cong \mathbb{Z}/3 \to H^4(G_n;\mathbb{Z})_3 \xrightarrow{\phi^*} H^4(G_2;\mathbb{Z}) \cong \mathbb{Z}/3$$

maps $8c_2(\mathrm{SL}(\mathbb{Z}))$ onto $8c_2(G_2)$, which is also of order 3, and is therefore an isomorphism. We may conclude that $H^4(G_n;\mathbb{Z})_3 \cong \ker \phi^* \oplus A$, where $A \cong \mathbb{Z}/3$ is generated by $8c_2(G_n)$ and $\phi^*(A) \cong H^4(G_2;\mathbb{Z})$.

(b) By Lemma 3.1 it is clear that $8c_2(G_2)$ belongs to the image of $\pi_2^*: H^4(U_2;\mathbb{Z}) \xrightarrow{\cong} H^4(G_2;\mathbb{Z})$; the next step of the proof will establish the same property for $8c_2(G_n)$: there exists an element ω in $H^4(U_n;\mathbb{Z})$ such that $\pi_n^*(\omega) = 8c_2(G_n)$. This follows from the commutative diagram

$$\begin{array}{ccc} H^4(U_n;\mathbb{Z}) & \xrightarrow{\pi_n^*} & H^4(G_n;\mathbb{Z})_3 \cong \ker \phi^* \oplus A \\ \downarrow{\scriptstyle \psi^*} & & \downarrow{\scriptstyle \phi^*} \\ H^4(U_2;\mathbb{Z}) & \xrightarrow[\cong]{\pi_2^*} & H^4(G_2;\mathbb{Z}) \cong \mathbb{Z}/3 \end{array}$$

where ψ^* is induced by the inclusion $U_2 \overset{\psi}{\hookrightarrow} U_n$. There exists an obvious homomorphism $\chi: U_n \twoheadrightarrow U_2$ such that $\chi \circ \psi$ is the identity; this implies that ψ^* is surjective. Therefore $\phi^* \circ \pi_n^*$ is also surjective which proves the existence of an element $\omega \in H^4(U_n;\mathbb{Z})$ such that $\pi_n^*(\omega) = 8c_2(G_n)$.

(c) The composition

$$H^4(U_n;\mathbb{Z}) \xrightarrow{\pi_n^*} H^4(G_n;\mathbb{Z}) \xrightarrow{\mathrm{res}} H^4(\Gamma_n(3);\mathbb{Z})$$

is zero since $\Gamma_n(3) \rightarrowtail G_n \xrightarrow{\pi_n} U_n$ is exact. We get $8c_2(\Gamma_n(3)) = \mathrm{res}(8c_2(G_n)) = \mathrm{res} \circ \pi_n^*(\omega) = 0$. The proof is then complete because we have shown (cf. [2, Satz 3.3]) that $3c_2(\Gamma_n(3)) = 0$ for $n \geq 2$.

Since $c_2(\mathrm{SL}_n(\mathbb{Z}))$ generates $H^4(\mathrm{SL}_n(\mathbb{Z}); \mathbb{Z}) \cong \mathbb{Z}/24$ for $n \geq 9$ the following corollary is an immediate consequence of the vanishing of $c_2(\Gamma_n(p))$ for all odd primes.

Corollary 3.3. *For all odd primes p the restriction homomorphism*

$$\mathrm{res}: H^4(\mathrm{SL}_n(\mathbb{Z}); \mathbb{Z}) \rightarrow H^4(\Gamma_n(p); \mathbb{Z})$$

is zero for $n \geq 9$.

Remark. This assertion is not true for $p=2$: since $\Gamma_n(2)$ contains a cyclic subgroup of order 2, the order of $c_2(\Gamma_n(2))$ is a positive multiple of 2; therefore the image of $\mathrm{res}: H^4(\mathrm{SL}_n(\mathbb{Z}); \mathbb{Z}) \rightarrow H^4(\Gamma_n(2); \mathbb{Z})$ contains a cyclic subgroup of order 2 for $n \geq 2$.

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